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Fourier Integral Operators in Gevrey Class on $\boldsymbol{R}^n$ and the Fundamental Solution for a Hyperbolic Operator

By

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Introduction

Consider a hyperbolic operator

$$L=D_t^m+\sum_{j=0}^{m-1}\sum_{|\alpha|\leqq m-j} a_{j,\alpha}(t, x)D_x^\alpha D_t^j \quad \text{on } [0, T] \tag{1}$$

with constant multiplicity, where $a_{j,\alpha}(t, x)$ are functions in the Gevrey class of order $d(>1)$, that is, they satisfy

$$|\partial_t^k\partial_x^\beta a_{j,\alpha}(t, x)|\leqq CM^{-(k+|\beta|)}k!^d\beta!^d \quad \text{for } (t, x)\in[0, T]\times R_x^n.$$

The purpose of the present paper is to construct the fundamental solution $E_0(t, s)$ of the Cauchy problem

$$\begin{cases} Lu=0 & t>0, \\ \partial_t^j u(0)=g_j, & j=0, 1,\cdots, m-1, \end{cases} \tag{2}$$

and obtain the result on the propagation of singularities for a solution $u(t)$ of (2).

To investigate the above problem we introduce the following symbol classes as subclasses of a symbol class S^m studied in [12]. In the following we tacitly use the notation in [12].

Definition (S). i) We say that a symbol $p(x, \xi)$ $(\in S^m)$ belongs to a class $S_{G(d)}^m$ if

$$|p_{(\beta)}^{(\alpha)}(x, \xi)|\leqq CM^{-(|\alpha|+|\beta|)}\alpha!^d\beta!^d\langle\xi\rangle^{m-|\alpha|} \tag{3}$$

hold with constants C and M independent of α and β.

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ii) We say that a symbol $p(x, \xi)$ belongs to a class $S^m_{G(d,1)}$ if $p(x, \xi)$ belongs to $S^m_{G(d)}$ and satisfies for constants C, M and μ independent of α, β

$$(4) \qquad |p^{(\alpha)}_{(\beta)}(x, \xi)| \leqq CM^{-(|\alpha|+|\beta|)}\alpha!\beta!^d\langle\xi\rangle^{m-|\alpha|} \quad \text{for} \quad |\xi| \geqq \mu.$$

iii) We say that a symbol $p(x, \xi)\,(\in S^{-\infty})$ belongs to a class $\mathscr{R}_{G(d)}$ if for any α there exists a constant C_α such that

$$(5) \qquad |p^{(\alpha)}_{(\beta)}(x, \xi)| \leqq C_\alpha M^{-(|\beta|+N)}\beta!^d N!^d\langle\xi\rangle^{-|\alpha|-N}$$

hold for any β and N with a constant M independent of α, β and N.

Definition (**T**). Let Δ be a subset of an Euclidian space $R^{\tilde{n}}_{\tilde{t}}$. We say that a symbol $p(\tilde{t}, x, \xi)$ in $\Delta\times R^{2n}_{x,\xi}$ belongs to $M^l_{\tilde{t}}(S^m_{G(d)})$ if for any α and β $p^{(\alpha)}_{(\beta)}(\tilde{t}, x, \xi)$ is a C^l-function and for any fixed $\tilde{t}\in\Delta$ the symbol $\partial^{\tilde{\gamma}}_{\tilde{t}}p(\tilde{t}, x, \xi)$ $(|\tilde{\gamma}|\leqq l)$ satisfies (3) with C and M independent also of $\tilde{t}$. We also set $M_{\tilde{t}}(S^m_{G(d)})=\bigcap_l M^l_{\tilde{t}}(S^m_{G(d)})$.

In the same way we define the classes $M^l_{\tilde{t}}(S^m_{G(d,1)})$, $M^l_{\tilde{t}}(\mathscr{R}_{G(d)})$ and $M_{\tilde{t}}(\mathscr{R}_{G(d)})$, which correspond to $S^m_{G(d,1)}$ and $\mathscr{R}_{G(d)}$. Using these symbol classes we reduce the problem (2) to the problem

$$(6) \qquad \begin{cases} \mathscr{L}_o U=0, \\ U(0)=G \end{cases}$$

for the perfectly diagonalized operator

$$(7) \qquad \mathscr{L}_o=D_t-\begin{pmatrix} \lambda_1(t, X, D_x)\mathscr{I}_{l_1} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \cdot \\ & & & \lambda_r(t, X, D_x)\mathscr{I}_{l_r} \end{pmatrix} + \begin{pmatrix} B_1(t) & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & B_r(t) \end{pmatrix} + R_o(t)$$

under the condition that (1) is a hyperbolic operator with constant multiplicity (c.f. Proposition 3.4). Here, $\lambda_j(t, x, \xi)$ belong to $M_t(S^1_{G(d)})$, $\mathscr{I}_l$ is an identity matrix, $B_j(t)$ are $l_j\times l_j$ matrices of pseudo-differential operators with symbols in $M_t(S^\sigma_{G(d)})$ $(0\leqq\sigma\leqq(r-1)/r)$ and

$R_o(t)$ is a matrix of pseudo-differential operators with symbols in $\mathscr{R}_{G(d)}$. Note that from (5), for any t, $R_o(t)$ maps a class $\mathscr{E}'$ of distributions with compact supports to a class γ^d of functions in the Gevrey class of order d. This result shows that in order to study the problem (2) for (1) it is sufficient to construct the fundamental solution $E(t, s)$ for the operator

$$(8)\qquad \mathscr{L}=D_t-\lambda(t, X, D_x)+b(t, X, D_x) \quad \text{on } [0, T]$$

with $\lambda(t, x, \xi)\in M_t^0(S_{G(d)}^1)$ and $b(t, x, \xi)\in M_t^0(S_{G(d)}^\sigma)\,(0\leqq\sigma\leqq 1/d)$. So, what we have to do is the construction of the fundamental solution $E(t, s)$ for $\mathscr{L}$.

Now, we give our main theorem in this paper.

Theorem 1. *Assume $\lambda(t, x, \xi)\in M_t^0(S_{G(d)}^1)$ is real-valued and $b(t, x, \xi)\in M_t^0(S_{G(d)}^\sigma)$ for some $0\leqq\sigma\leqq 1/d$ (<1). Then, the fundamental solution $E(t, s)$ of (8) can be written in the form*

$$(9)\qquad E(t, s)=\{I+\sum_{\nu=1}^{\infty} W_\nu(t, s)\}I_\phi(t, s)+R(t, s) \quad \text{for } 0\leqq t,\ s\leqq T_o$$

for a small T_o. In (9) $\sum_{\nu=1}^{\infty} W_\nu(t, s)$ is a series of pseudo-differential operators $W_\nu(t, s)$ with symbols $w_\nu(t, s; x, \xi)$ satisfying for $j=0, 1$

$$(10)\qquad \begin{aligned}&|\partial_t^j\partial_\xi^\alpha D_x^\beta w_\nu(t, s; x, \xi)|\\ &\leqq (C_O^\nu|t-s|^\nu\nu!^{-1})M^{-(|\alpha|+|\beta|)}\alpha!^d\beta!^d\langle\xi\rangle^{\nu\sigma-|\alpha|}\end{aligned}$$

with constants C_O and M independent of α, β, and ν; $I_\phi(t, s)$ is a Fourier integral operator with the phase function $\phi(t, s; x, \xi)$, where $\phi(t, s; x, \xi)$ is a solution of

$$(11)\qquad \begin{cases}\partial_t\phi=\lambda(t, x, \nabla_x\phi),\\ \phi|_{t=s}=x\cdot\xi;\end{cases}$$

and $R(t, s)$ is a pseudo-differential operator with symbol $r(t, s; x, \xi)$ in $M_{t,s}^1(\mathscr{R}_{G(d)})$.

Since the symbol of $R(t, s)$ belongs to $M_{t,s}^1(\mathscr{R}_{G(d)})$, $R(t, s)$ maps $\mathscr{E}'$ to γ^d for any fixed t, s. Hence we can call $R(t, s)$ a regularizer. From Theorem 1 and Proposition 1.3 we easily obtain

Theorem 2. *Assume that $\lambda(t, x, \xi)$ in (8) is homogeneous for large*

$|\xi|$. *Then, for a solution* $u(t)$ *of*

$$(12)\qquad \begin{cases} \mathscr{L}u(t)=0 & t>0, \\ u(0)=g, \end{cases}$$

we have

$$(13)\quad \mathrm{WF}_{G(d)}(u(t))=\{(q(t, y, \eta), \rho p(t, y, \eta));\ (y, \eta)\in \mathrm{WF}_{G(d)}(g) \text{ for large } |\eta|, \rho>0\},$$

where $\mathrm{WF}_{G(d)}(u)$ *is a wave front set of* u *in the Gevrey class of order* d *(see Definition* 1.2) *and* $\{q(t, y, \eta), p(t, y, \eta)\}$ *is a solution of*

$$(14)\qquad \begin{cases} \dfrac{dq}{dt}=-\nabla_\xi\lambda(t, q, p), \quad \dfrac{dp}{dt}=\nabla_x\lambda(t, q, p), \\ q|_{t=0}=y, \quad p|_{t=0}=\eta. \end{cases}$$

This result is also obtained by Mizohata [17]. He has showed it by using the energy method, not by constructing the fundamental solution. For the parametrix of $\mathscr{L}$ Lascar reported in [15] that he constructed it, but the author has not known his detailed proof. Another result concerning the construction of the parametrix is reported in [2] and the propagation of singularities for a solution of (2) is studied in [19] and [25].

From Theorems 1, 2 and Proposition 3.4 we obtain

Corollary 3. *In* (1) *we assume*

$$\tau^m+\sum_{j=0}^{m-1}\sum_{|\alpha|=j} a_{j,\alpha}(t, x)\xi^\alpha\tau^j=\prod_{j=1}^{\kappa}(\tau-\lambda_j(t, x, \xi))^{m_j} \quad \text{for } |\xi|\geqq 1$$

and $\sigma\equiv\max_j\{(m_j-1)/m_j\}\leqq 1/d$. *Then, the fundamental solution* $E_o(t, s)$ *can be constructed in the form*

$$(15)\qquad E_o(t, s)=\sum_{j=1}^{\kappa}\sum_{\nu=0}^{\infty} W_{j,\nu}(t, s)I_{\phi_j}(t, s)+R(t, s),$$

where $W_{j,\nu}(t, s)$, $I_{\phi_j}(t, s)$ *and* $R(t, s)$ *satisfy the similar properties to those in Theorem 1. Moreover, let* $\{q_j(t, y, \eta), p_j(t, y, \eta)\}$ *be a solution of* (14) *with* $\lambda=\lambda_j$. *Then, we have for a solution of* (2)

$$(16)\qquad \bigcup_{j=0}^{m-1}\mathrm{WF}_{G(d)}(\partial_t^j u(t)) =\bigcup_{j=1}^{\kappa}\{(q_j(t, y, \eta), \rho p_j(t, y, \eta));\ (y, \eta)\in\bigcup_{j=0}^{m-1}\mathrm{WF}_{G(d)}(g_j) \text{ for large } |\eta|, \rho>0\}.$$

In Section 3 we study the above result under a weaker condition: There exist regularly hyperbolic operators $L_1, L_2, \cdots, L_r$ such that L has a form

$$(17) \qquad L=L_1L_2\cdots L_r+\sum_{j=0}^{m-q} a'_j(t, X, D_x)D_t^j$$

with $a'_j(t, x, \xi)\in M_t(S_{G(d)}^{m-q-j})$ $(1\leqq q\leqq r)$. This formulation is based on the work in [16], where the authors proved γ^d-well-posedness for $d\leqq r/(r-q)$ in the case that $a'_j(t, X, D_x)$ are differential operators. The number r/q is called the irregularity in [8]. For the case of constant multiplicity, Ohya [22] also proved the γ^d-well-posedness and in [5] Ivrii gave the necessary and sufficient condition for (1) to be γ^d-well-posed. Under Ivrii's condition we can also reduce (2) for (1) to (6) for (7) and get Corollary 3.

The construction of the fundamental solution $E(t, s)$ (the proof of Theorem 1) is performed by the way employed in [13], [14] and [23]. There, the authors construct $E(t, s)$ for the C^∞-case by using the successive approximation after solving an eiconal equation (11). The key point of their proof is obtaining a sharp estimate of multi-products of Fourier integral operators. Since we assume $d>1$, we can use cut functions in the Gevrey class and can improve their estimate to the Gevrey class. This enables us to prove Theorem 1. In [6] and [7] Kajitani has constructed the fundamental solution for a hyperbolic system with coefficients in the Gevrey class of order d by solving transport equations and using the asymptotic sum of amplitude functions. His fundamental solution $E(t, s)$ has the form similar to (15) and the regularizer $R(t, s)$ in his $E(t, s)$ is an integral operator with a kernel in the Gevrey class of order $2d-1$. So, from his $E(t, s)$ we get (16) in the case $d=1$, but we cannot obtain (16) for the case $d>1$. In our construction, since we do not solve transport equations and hence we do not use the asymptotic sum, the regularizer $R(t, s)$ becomes an integral operator with a kernel in the Gevrey class of order d and get (16) for the case $d>1$.

The outline of the present paper is the following: In Section 1 we give a class of Fourier integral operators and a result on wave front sets. In Section 2, after showing the result on products of Fourier integral operators, conjugate Fourier integral operators and pseudo-differential operators we obtain a sharp estimate of multi-products of Fourier integral operators. Since we need tedious calculation to obtain the former results, we devote their proofs to Section 4. In Section 3 we prove Theorem 1 and show the way of reducing the problem (2) to the problem (6).

In Section 5 we prove a sharp estimate of symbols of multi-products of pseudo-differential operators, which is also used in Section 2. For the proof we follow the discussions in Section 1 of [23]. There, to obtain the key estimate we divide the multi-product of $\nu+1$ pseudo-differential operators into 2^{ν} terms by using cut functions depending on a parameter ε (see (1.57) of [23]). But, in our case we cannot use such a decomposition since we cannot find a suitable ε to obtain our estimate, especially to find a suitable "radius of convergence". So, we employ a different method of the division into 2^{ν} terms. Then, we obtain the desired estimate for our case.

The final section, Section 6, is devoted to the proof of Proposition 3.4 on the perfect diagonalization. This is a version of the one in [10] for the Gevrey class. Since we use the asymptotic sum for products of pseudo-differential operators, we use the class $S^m_{G(d,1)}$, not the class $S^m_{G(d)}$, and the discussion in [1]. Then, the discussions in [10] work well and we can obtain Proposition 3.4.

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§ 1. Definitions and Wave Front Sets

Throughout this paper the constant d denotes a number larger than 1. To define Fourier integral operators we will introduce a class $\mathscr{P}_{G(d)}(\tau)$ of phase functions as follows:

Definition 1.1. Let $0\leqq\tau<1$. We say that a phase function $\phi(x, \xi)$ belongs to a class $\mathscr{P}_{G(d)}(\tau)$ if $\phi(x, \xi)$ belongs to $\mathscr{P}(\tau)$ and for $J(x, \xi)\equiv\phi(x, \xi)-x\cdot\xi$ the estimate

$$(1.1)\qquad |J^{(\alpha)}_{(\beta)}(x, \xi)|\leqq\tau M^{-(|\alpha|+|\beta|)}\alpha!^d\beta!^d\langle\xi\rangle^{1-|\alpha|}$$

hold for a constant M independent of α, β. We also set

$$\mathscr{P}_{G(d)}=\bigcup_{0\leqq\tau<1}\mathscr{P}_{G(d)}(\tau).$$

Remark 1. We say that for $\phi_\theta\in\mathscr{P}_{G(d)}(\tau_\theta)$ $(0\leqq\tau_\theta<1)$ the set $\{\phi_\theta\}_{\theta\in\Theta}$ is bounded in $\mathscr{P}_{G(d)}$ if $\tau_\theta\leqq\tilde{\tau}_0$ for a constant $\tilde{\tau}_0(<1)$ independent of θ

and $J_\theta(x, \xi) \equiv \phi_\theta(x, \xi) - x\cdot\xi$ satisfies (1.1) with $\tau = \tau_\theta$ and a constant M independent of θ.

Remark 2. In the same way we define bounded sets in $S^m_{G(d)}$ and $\mathscr{R}_{G(d)}$ as follows: We say that for $p_\theta \in S^m_{G(d)}$ the set $\{p_\theta\}_{\theta\in\Theta}$ is bounded in $S^m_{G(d)}$ if we can take constants C and M in (3) independent also of the parameter $\theta \in \Theta$, and we say that for $p_\theta \in \mathscr{R}_{G(d)}$ the set $\{p_\theta\}_{\theta\in\Theta}$ is bounded in $\mathscr{R}_{G(d)}$ if we can take constants C_α and M in (5) independent also of $\theta \in \Theta$.

Let $\phi(x, \xi)$ be a phase function in $\mathscr{P}_{G(d)}$. Then, a Fourier integral operator $P_\phi = p_\phi(X, D_x)$ with the phase function $\phi(x, \xi)$ and a symbol $\sigma(P_\phi) = p(x, \xi)$ in $S^m_{G(d)}$ is defined by

$$P_\phi u(x) = O_s\text{-}\iint e^{i(\phi(x,\xi)-x'\cdot\xi)} p(x, \xi) u(x') dx' đ\xi \qquad \text{for } u \in \mathscr{S}, \tag{1.2}$$

where $\mathscr{S}$ is the Schwartz space of rapidly decresing functions on R^n and the right hand side of (1.2) is the oscillatory integral defined in [12] (Chap. 10). Following [12] we denote the set of such Fourier integral operators by $\mathcal{S}^m_{G(d),\phi}$. If $\phi = x\cdot\xi$, the set $\mathcal{S}^m_{G(d),\phi}$ is the one of pseudo-differential operators. In this case we write $\mathcal{S}^m_{G(d),\phi}$ simply by $\mathcal{S}^m_{G(d)}$. Similally, we define a Fourier integral operator P_ϕ with the phase function $\phi(x, \xi)$ and a symbol $p(x, \xi) \in \mathscr{R}_{G(d)}$ by (1.2) and denote a class of such Fourier integral operators by $\mathscr{R}_{G(d),\phi}$. Corresponding to this class we write a class of pseudo-differential operators as $\mathscr{R}_{G(d)} = \{p(X, D_x); p(x, \xi) \in \mathscr{R}_{G(d)}\}$, since no confusion occurs between the class of symbols and that of pseudo-differential operators. Remark that the following holds: If $p(x, \xi)$ belongs to $\mathscr{R}_{G(d)}$ and a real symbol $J(x, \xi)$ satisfies (1.1) then $e^{iJ(x,\xi)}\, p(x, \xi)$ also belongs to $\mathscr{R}_{G(d)}$. This fact shows that $\mathscr{R}_{G(d),\phi} = \mathscr{R}_{G(d)}$ for all $\phi \in \mathscr{P}_{G(d)}$, which corresponds to (2.6) of [9]. So, we may use mainly the class $\mathscr{R}_{G(d)}$ among the class $\mathscr{R}_{G(d),\phi}$, $\phi \in \mathscr{P}_{G(d)}$. Denote by $\gamma^d(M)$ the class of functions $u(x)$ satisfying

$$|\partial_x^\alpha u(x)| \leqq CM^{-|\alpha|}\alpha!^d \quad (x \in R^n)$$

and denote $\gamma^d = \bigcup_{M>0} \gamma^d(M)$. Then, the operator in $\mathscr{R}_{G(d)}$ maps a class $\mathscr{E}'$ of distributions with compact supports to a class γ^d. In this sense, we call the operators in $\mathscr{R}_{G(d)}$ regularizers.

The following definition coincides with that of $\mathrm{WF}_L(u)$ for $L=\{L_k=(k+1)^d\}$ in [4].

Definition 1.2. Let $u\in\mathscr{E}'$. We say that a point (x°, ξ°) of $T^*(R^n)\setminus\{0\}$ does not belong to the wave front set $\mathrm{WF}_{G(d)}(u)$ $(\subset T^*(R^n)\setminus\{0\})$ of u, if there exist a conic neighborhood Γ of ξ° and a function $\chi(x)$ in γ^d with $\chi(x^\circ)\neq 0$ such that the Fourier transform $\mathscr{F}[\chi u](\xi)$ of $\chi(x)u(x)$ satisfies for any N

$$(1.3)\qquad |\xi|^N|\mathscr{F}[\chi u](\xi)|\leqq CM^{-N}N!^d \quad \text{for } \xi\in\Gamma$$

with constants C and M indepedent of N.

Concerning the wave front set $\mathrm{WF}_{G(d)}(u)$ of the Gevrey class the following holds.

Proposition 1.3. *Let a phase function $\phi(x, \xi)\in\mathscr{P}_{G(d)}$ be homogeneous for large $|\xi|$. Then, for a Fourier integral operator P_ϕ with a symbol $p(x, \xi)$ in $S^m_{G(d)}$ the relation*

$$(1.4)\qquad \mathrm{WF}_{G(d)}(P_\phi u) \subset\{(x, \rho\nabla_x\phi(x, \xi))\,;\,(\nabla_\xi\phi(x, \xi), \xi)\in\mathrm{WF}_{G(d)}(u) \ \textit{for large } |\xi|,\ \rho>0\}$$

holds for $u\in\mathscr{E}'$.

Proof. We may assume $u\in\mathscr{S}$ by the similar result for the C^∞-case. Suppose that the points (x°, ξ°) and (y°, η°) satisfy $\xi^\circ=\nabla_x\phi(x^\circ, \eta^\circ)$, $y^\circ=\nabla_\xi\phi(x^\circ, \eta^\circ)$, $(y^\circ, \eta^\circ)\notin\mathrm{WF}_{G(d)}(u)$ and $|\eta^\circ|\geqq C_1$. From the definition there exist a function $\chi_1(x)$ in γ^d satisfying $\chi_1=1$ in a neighborhood of y° and a conic neighborhood Γ_1 of η° such that (1.3) holds with $\chi=\chi_1$ and $\Gamma=\Gamma_1$. Take $\psi(\xi)\in S^0_{G(d)}$ satisfying $\mathrm{supp}\,\psi\subset\Gamma_1$ and $\psi=1$ in a conic neighborhood of η°, and take a function $\chi_2(x)$ in γ^d such that $\chi_2(x^\circ)\neq 0$ and $\mathrm{supp}\chi_2\subset\{x\,;\,\chi_1(\nabla_\xi\phi(x, \eta))=1 \textit{ for all } \eta\}$. Using these functions we divide $\mathscr{F}[\chi_2P_\phi u](\xi)$ into three parts:

$$(1.5)\qquad \mathscr{F}[\chi_2P_\phi u](\xi) =\int e^{i(-x\cdot\xi+\phi(x,\eta))}\chi_2(x)p(x, \eta)\psi(\eta)\mathscr{F}[\chi_1u](\eta)\,d\!\!\!{}^-\eta dx +\int e^{i(-x\cdot\xi+\phi(x,\eta))}\chi_2(x)p(x, \eta)(1-\psi(\eta))\mathscr{F}[\chi_1u](\eta)\,d\!\!\!{}^-\eta dx$$

$$+\int e^{-ix\cdot\xi}\chi_2(x)\{O_s\text{-}\iint e^{i(\phi(x,\eta)-y\cdot\eta)}p(x,\eta)(1-\chi_1(y))u(y)dyd\!\!{}^-\eta\}dx$$
$$\equiv f_1(\xi)+f_2(\xi)+f_3(\xi).$$

For a fixed α we estimate $\xi^\alpha f_j(\xi)$, $j=1, 2, 3$, individually. First, we estimate $\xi^\alpha f_1(\xi)$. Note that we can write

$$e^{-i\phi(x,\xi)}D_x^\beta e^{i\phi(x,\xi)}=\sum_{k=0}^{|\beta|}\phi_{\beta,k}(x,\xi) \tag{1.6}$$

with the property

$$|\partial_\xi^\alpha\partial_x^\delta\phi_{\beta,k}(x,\xi)|\leqq CM^{-(|\beta|+|\alpha|+|\delta|)}\beta!^d\alpha!^d\delta!^d k!^{-d}\langle\xi\rangle^{k-|\alpha|}. \tag{1.7}$$

Using (1.6) we write

$$\xi^\alpha f_1(\xi)=\sum_{\alpha'+\alpha''=\alpha}\sum_{k=0}^{|\alpha'|}\binom{\alpha}{\alpha'}\iint e^{i(-x\cdot\xi+\phi(x,\eta))}\phi_{\alpha',k}(x,\eta)$$
$$\times D_x^{\alpha''}(\chi_2(x)p(x,\eta))\psi(\eta)F[\chi_1 u](\eta)d\!\!{}^-\eta dx.$$

Then, from (1.3) for $u(x)$ we have

$$|\xi^\alpha f_1(\xi)|\leqq CM^{-|\alpha|}\alpha!^d \tag{1.8}$$

if we take new constants C and M independent of α. Next, we estimate $\xi^\alpha f_2(\xi)$. If we take an appropriate conic neighborhood Γ_2 of ξ°, the relation

$$|\xi-\nabla_x\phi(x,\eta)|\geqq C(|\xi|+|\eta|)\quad\text{for } \xi\in\Gamma_2\ (c>0)$$

holds on the support of the integrand of $f_2(\xi)$. So, if we set $L_1=i|\xi-\nabla_x\phi(x,\eta)|^{-2}(\xi-\nabla_x\phi(x,\eta))\cdot\nabla_x$ we have from $L_1 e^{i(-x\cdot\xi+\phi(x,\eta))}=e^{i(-x\cdot\xi+\phi(x,\eta))}$

$$\xi^\alpha f_2(\xi)=\xi^\alpha\int e^{i(-x\cdot\xi+\phi(x,\eta))}(L_1{}^t)^{|\alpha|}\{\chi_2(x)p(x,\eta)\}$$
$$\times(1-\psi(\eta))\mathscr{F}[\chi_1 u](\eta)d\!\!{}^-\eta dx.$$

Hence, we have

$$|\xi^\alpha f_2(\xi)|\leqq CM^{-|\alpha|}\alpha!^d\quad\text{for } \xi\in\Gamma_2. \tag{1.9}$$

For $f_3(\xi)$ we write

$$\xi^\alpha f_3(\xi)=\sum_{\alpha'+\alpha''+\delta=\alpha}\frac{\alpha!}{\alpha'!\alpha''!\delta!}\sum_{k=0}^{|\delta|}\int e^{-ix\cdot\xi}\chi_{2(\alpha')}(x)$$
$$\times\{O_s\text{-}\iint e^{i(\phi(x,\eta)-y\cdot\eta)}\phi_{\delta,k}(x,\eta)p_{(\alpha'')}(x,\eta)$$
$$\times(1-\chi_1(y))u(y)dyd\!\!{}^-\eta\}dx$$

with $\phi_{\beta,k}(x,\xi)$ in (1.6)-(1.7). Note that on $\operatorname{supp}\chi_2(x)(1-\chi_1(y))$ the inequality $|\nabla_\xi\phi(x,\eta)-y|\geqq C_0>0$ holds. Hence, setting

$L_2=-i|\nabla_\xi\phi(x,\ \eta)-y|^{-2}(\nabla_\xi\phi(x,\ \eta)-y)\cdot\nabla_\eta$ we write

$$\begin{aligned}\xi^\alpha f_3(\xi)=\sum_{\alpha'+\alpha''+\delta=\alpha}\frac{\alpha!}{\alpha'!\alpha''!\delta!}\sum_{k=0}^{|\delta|}\int e^{-ix\cdot\xi}\chi_{2(\alpha')}(x)\\ \times\{O_s\text{-}\iint e^{i(\phi(x,\eta)-y\cdot\eta)}(L_2{}^t)^{k+l(m)+n+1}\{\phi_{\delta,k}(x,\ \eta)\\ \times p_{(\alpha'')}(x,\ \eta)\}(1-\chi_1(y))u(y)dyd\!\!\!{}^-\eta\}dx,\end{aligned}$$

where $l(m)=[\max(m,\ 0)]$. This implies

$$|\xi^\alpha f_3(\xi)|\leqq CM^{-|\alpha|}\alpha!^d. \tag{1.10}$$

Consequently, we have for any α

$$|\xi^\alpha\mathscr{F}[\chi_2P_\phi u](\xi)|\leqq CM^{-|\alpha|}\alpha!^d\quad\text{for }\xi\in\Gamma_2$$

from (1.5) and (1.8)–(1.10). This means $(x^\circ,\ \xi^\circ)\notin\mathrm{WF}_{G(d)}(P_\phi u)$.

Q. E. D.

§ 2. Multi-Products of Fourier Integral Operators

In this section we will obtain a sharp estimate for the symbols of multi-products of Fourier integral operators. For simplicity we denote for $\phi\in\mathscr{P}_{G(d)}$

$$L^m_{G(d)}(\phi)=\{p^0_\phi(X,D_x)+\tilde{p}_\phi(X,D_x)\ ;\ p^0(x,\ \xi)\in S^m_{G(d)},\ \tilde{p}(x,\ \xi)\in\mathscr{R}_{G(d)}\},$$

that is, symbolically $L^m_{G(d)}(\phi)=\mathcal{S}^m_{G(d),\phi}+\mathscr{R}_{G(d),\phi}$. If $\phi(x,\ \xi)=x\cdot\xi$ we denote $L^m_{G(d)}(\phi)$ simply by $L^m_{G(d)}$.

For a sequence $\{\phi_j\}$ of phase functions $\phi_j(x,\ \xi)\in\mathscr{P}_{G(d)}(\tau_j)$ we consider multi-products

$$\tilde{Q}_{\nu+1}=P_{1,\phi_1}P_{2,\phi_2}\cdots P_{\nu+1,\phi_{\nu+1}} \tag{2.1}$$

of Fourier integral operators P_{j,ϕ_j} in $L^\sigma_{G(d)}(\phi_j)$ with $\sigma\geqq0$. We put following assumptions:

(A.1) There exists a small constant τ^0 such that $\sum_{j=1}^\infty\tau_j\leqq\tau^0$. If we set $J_j(x,\ \xi)=\phi_j(x,\ \xi)-x\cdot\xi$, $\{J_j/\tau_j\}$ is bounded in $S^1_{G(d)}$.

(A.2) If we write $P_{j,\phi_j}=p^0_{j,\phi_j}(X,\ D_x)+\tilde{p}_{j,\phi_j}(X,\ D_x)\in\mathcal{S}^\sigma_{G(d),\phi_j}+\mathscr{R}_{G(d),\phi_j}$ the set $\{p^0_j(x,\ \xi)\}$ is bounded in $S^\sigma_{G(d)}$ and $\{\tilde{p}_j(x,\ \xi)\}$ is bounded in $\mathscr{R}_{G(d)}$.

The result we want to show in this section is the following:

Theorem 2.1. *We assume* (A.1) *and* (A.2). *Then, the multi-product* (2.1) *of Fourier integral operators* P_{j,ϕ_j} *is a Fourier integral operator* $Q_{\nu+1,\Phi_{\nu+1}}$ *in* $L_{G(d)}^{(\nu+1)\sigma}(\Phi_{\nu+1})$ *with a phase function* $\Phi_{\nu+1}(x, \xi)$ *in* $\mathscr{P}_{G(d)}$ *and is represented by the form*

$$Q_{\nu+1,\Phi_{\nu+1}}=q_{\nu+1}^{0}(X, D_x)I_{\Phi_{\nu+1}}+\tilde{q}_{\nu+1}(X, D_x)I_{\Phi_{\nu+1}} \tag{2.2}$$

for the symbols $q_{\nu+1}^{0}(x, \xi)$ *and* $\tilde{q}_{\nu+1}(x, \xi)$ *satisfying*

$$|q_{\nu+1}^{0}{}_{(\beta)}^{(\alpha)}(x, \xi)| \leqq C_0^{\nu} M^{-(|\alpha|+|\beta|)} \alpha!^d \beta!^d \langle\xi\rangle^{(\nu+1)\sigma-|\alpha|}, \tag{2.3}$$

$$|\tilde{q}_{\nu+1}{}_{(\beta)}^{(\alpha)}(x, \xi)| \leqq C_0^{\nu} C_\alpha M^{-(|\beta|+N)} \beta!^d (N+[\nu\sigma])!^d \langle\xi\rangle^{-|\alpha|-N} \text{ for any } N, \tag{2.4}$$

where the constants C_0 *and* M *are independent of* ν, α, β, N *and the constant* C_α *is independent of* ν, β *and* N.

In (2.2) the operator I_ϕ for $\phi\in\mathscr{P}_{G(d)}$ is the Fourier integral operator with the phase function $\phi(x, \xi)$ and the symbol 1.

We will prove Theorem 2.1 after some preparations. First, we give the product formulae between Fourier integral operators, conjugate Fourier integral operators and pseudo-differential operators, whose proofs are given in Section 4.

Proposition 2.2. *The following inclusion formulae hold.*

$$S_{G(d)}^{m}\cdot S_{G(d),\phi}^{m'}\subset L_{G(d)}^{m+m'}(\phi), \tag{2.5}$$

$$S_{G(d),\phi}^{m}\cdot S_{G(d)}^{m'}\subset L_{G(d)}^{m+m'}(\phi), \tag{2.6}$$

$$\mathscr{R}_{G(d)}\cdot L_{G(d)}^{m}(\phi)\subset\mathscr{R}_{G(d)}, \quad L_{G(d)}^{m}(\phi)\cdot\mathscr{R}_{G(d)}\subset\mathscr{R}_{G(d)}. \tag{2.7}$$

Remark 1. It is easy to see from (2.5)−(2.7)

$$L_{G(d)}^{m}\cdot L_{G(d)}^{m'}(\phi)\subset L_{G(d)}^{m+m'}(\phi), \tag{2.8}$$

$$L_{G(d)}^{m}(\phi)\cdot L_{G(d)}^{m'}\subset L_{G(d)}^{m+m'}(\phi). \tag{2.9}$$

Remark 2. The inclusion mapping (2.5) is bounded in the following sense: "Assume $\{p_\theta\}_{\theta\in\Theta}$ is bounded in $S_{G(d)}^{m}$, $\{p'_\theta\}_{\theta\in\Theta}$ is bounded in $S_{G(d)}^{m'}$ and $\{\phi_\theta\}_{\theta\in\Theta}$ is bounded in $\mathscr{P}_{G(d)}$. Denote $p_\theta(X, D_x)p'_{\theta,\phi_\theta}(X, D_x)$ $=q_{\theta,\phi_\theta}^{0}(X, D_x)+\tilde{q}_{\theta,\phi_\theta}(X, D_x)$ with $q_\theta^0(x, \xi)\in S_{G(d)}^{m+m'}$ and $\tilde{q}_\theta(x, \xi)\in\mathscr{R}_{G(d)}$. Then, $\{q_\theta^0\}_{\theta\in\Theta}$ and $\{\tilde{q}_\theta\}_{\theta\in\Theta}$ are bounded in $S_{G(d)}^{m+m'}$ and $\mathscr{R}_{G(d)}$ respectively." In the same sense, the inclusion mappings in (2.6) and (2.7) are bounded.

Denote by I_{ϕ^*} the conjugate Fourier integral operator with the phase function $\phi(x, \xi)\in\mathscr{P}_{G(d)}$ and the symbol 1. Then, we have

Proposition 2.3. *The following relations hold.*

$$L^m_{G(d)}(\phi)\cdot I_{\phi^*}\subset L^m_{G(d)}, \quad I_{\phi^*}\cdot L^m_{G(d)}(\phi)\subset L^m_{G(d)}. \tag{2.10}$$

Remark. The inclusion mappings in (2.10) are bounded in the similar sense to Remark 2 of Proposition 2.2.

For the multi-products of phase functions we have

Proposition 2.4. *Let* $\phi_j(x, \xi)\in\mathscr{P}_{G(d)}(\tau_j)$, $j=1, 2, \cdots$. *Assume* (A.1). *Then, the multi-product* $\Phi_{\nu+1}(x, \xi)=\phi_1\#\cdots\#\phi_{\nu+1}(x, \xi)$ *defined in* [14] *belongs to* $\mathscr{P}_{G(d)}(c_o\bar{\tau}_{\nu+1})$ $(\bar{\tau}_{\nu+1}=\tau_1+\cdots+\tau_{\nu+1})$ *with some constant* c_o *independent of* ν.

Proof. Let $\{X^j_\nu, \Xi^j_\nu\}^\nu_{j=1}(x, \xi)$ be the solution of

$$\begin{cases} X^j_\nu=\nabla_\xi\phi_j(X^{j-1}_\nu, \Xi^j_\nu), \\ \Xi^j_\nu=\nabla_x\phi_{j+1}(X^j_\nu, \Xi^{j+1}_\nu), \quad j=1, \cdots, \nu \quad (X^0_\nu=x, \ \Xi^{\nu+1}_\nu=\xi). \end{cases} \tag{2.11}$$

Then, by the induction on N we can prove from the method of the proof of Theorem 1.7′ in [14]

$$\begin{aligned}\sum_{1\leqq|\alpha+\beta|\leqq N}[M^{|\alpha|+|\beta|-1}/(\alpha!\beta!(|\alpha+\beta|-1)!^{d-1})\\ \times\sum_{j=1}^{\nu}\{\langle\xi\rangle^{|\alpha|}||\partial^\alpha_\xi\partial^\beta_x(X^j_\nu-X^{j-1}_\nu)||\\ +\langle\xi\rangle^{-1+|\alpha|}||\partial^\alpha_\xi\partial^\beta_x(\Xi^j_\nu-\Xi^{j+1}_\nu)||\}]\leqq C_1\bar{\tau}_{\nu+1}\end{aligned}$$

for constants C_1 and M independent of α, β and ν. This implies

$$\begin{aligned}\sum_{j=1}^{\nu}\{\langle\xi\rangle^{|\alpha|}||\partial^\alpha_\xi\partial^\beta_x(X^j_\nu-X^{j-1}_\nu)||+\langle\xi\rangle^{-1+|\alpha|}||\partial^\alpha_\xi\partial^\beta_x(\Xi^j_\nu-\Xi^{j+1}_\nu)||\}\\ \leqq C_1\bar{\tau}_{\nu+1}M^{-(|\alpha|+|\beta|)}\alpha!^d\beta!^d.\end{aligned} \tag{2.12}$$

Since $\Phi_{\nu+1}(x, \xi)=\phi_1\#\cdots\#\phi_{\nu+1}(x, \xi)$ is defined by

$$\begin{aligned}\Phi_{\nu+1}(x, \xi)=\sum_{j=1}^{\nu}(\phi_j(X^{j-1}_\nu, \Xi^j_\nu)-X^j_\nu\cdot\Xi^j_\nu)+\phi_{\nu+1}(X^\nu_\nu, \xi)\\ (X^0_\nu=x),\end{aligned} \tag{2.13}$$

we get $\Phi_{\nu+1}(x, \xi)\in\mathscr{P}_{G(d)}(c_o\bar{\tau}_{\nu+1})$ with an appropriate constant c_o.

Q. E. D.

Proposition 2.5. *Let* $\phi_j(x, \xi)\in\mathscr{P}_{G(d)}(\tau_j)$, $j=1, 2$. *Assume* $\tau_1+\tau_2$

is small enough. Then, we have

$$I_{\phi_1} I_{\phi_2} \in L^0_{G(d)}(\phi_1 \# \phi_2). \tag{2.14}$$

Remark. For $\phi_{j,\theta} \in \mathscr{P}_{G(d)}(\tau_{j,\theta})$, $j=1, 2$, we denote $I_{\phi_{1,\theta}} I_{\phi_{2,\theta}} = p^0_{\theta,\Phi_\theta}(X, D_x) + \tilde{p}_{\theta,\Phi_\theta}(X, D_x)$ $(\Phi_\theta = \phi_{1,\theta} \# \phi_{2,\theta})$. Then, if $\tau_{1,\theta} + \tau_{2,\theta} \leqq \tau_o$ for a τ_o independent of θ and the sets $\{\phi_{j,\theta}\}_{\theta\in\Theta}$ $(j=1, 2)$ are bounded in $\mathscr{P}_{G(d)}$, the sets $\{p^0_\theta\}_{\theta\in\Theta}$ and $\{\tilde{p}_\theta\}_{\theta\in\Theta}$ of the corresponding symbols $p^0_\theta(x, \xi)$ and $\tilde{p}_\theta(x, \xi)$ are bounded in $S^0_{G(d)}$ and $\mathscr{R}_{G(d)}$, respectively.

We postpone the proof of this proposition to Section 4.

For the multi-products of pseudo-differential operators we have

Theorem 2.6. *Let* $P_j = p^0_j(X, D_x) + \tilde{p}_j(X, D_x) \in L^\sigma_{G(d)}$ $(j=1, 2, \cdots)$ *with* $\sigma \geqq 0$. *Assume*

$$|p^{0(\alpha)}_{j(\beta)}(x, \xi)| \leqq C_o M^{-(|\alpha|+|\beta|)} \alpha!^d \beta!^d \langle\xi\rangle^{\sigma-|\alpha|}, \tag{2.15}$$

$$|\tilde{p}^{(\alpha)}_{j(\beta)}(x, \xi)| \leqq C_\alpha M^{-(|\beta|+N)} \beta!^d N!^d \langle\xi\rangle^{-|\alpha|-N} \tag{2.16}$$

for any N, *where the constants* C_o *and* M *are independent of* j, α, β, N *and the constant* C_α *is independent of* j, β, N. *Then, the multi-product* $Q_{\nu+1} = P_1 P_2 \cdots P_{\nu+1}$ *has the form* $Q_{\nu+1} = q^0_{\nu+1}(X, D_x) + \tilde{q}_{\nu+1}(X, D_x)$ *with the properties*

$$|q^0_{\nu+1}{}^{(\alpha)}_{(\beta)}(x, \xi)| \leqq A^\nu C_o^{\nu+1} M_1^{-(|\alpha|+|\beta|)} \alpha!^d \beta!^d \langle\xi\rangle^{(\nu+1)\sigma-|\alpha|}, \tag{2.17}$$

$$|\tilde{q}_{\nu+1}{}^{(\alpha)}_{(\beta)}(x, \xi)| \leqq A^\nu \tilde{C}_o^{\nu+1} C'_\alpha M_1^{-(|\beta|+N)} \beta!^d (N+[\nu\sigma])!^d \langle\xi\rangle^{-|\alpha|-N} \quad \text{for any } N. \tag{2.18}$$

Here, A *and* M_1 *are constants determined only by the dimension* n *and* M, $\tilde{C}_o = \max_{|\alpha|\leqq n+1} \{C_\alpha, C_o\}$ *and the constants* C'_α *are determined by* n, α *and* C_α. *All the constants* A, M_1 *and* C'_α *are independent of* ν.

Since the proof is rather long, we will give it in Section 5. As in [23] we get two corollaries.

Corollary 2.7 (cf. Theorem 3 of [23]). *Let* $P = p^0(X, D_x) + \tilde{p}(X, D_x) \in L^0_{G(d)}$ *with*

$$\begin{cases} |p^{0(\alpha)}_{(\beta)}(x, \xi)| \leqq C_o M^{-(|\alpha|+|\beta|)} \alpha!^d \beta!^d \langle\xi\rangle^{-|\alpha|}, \\ |\tilde{p}^{(\alpha)}_{(\beta)}(x, \xi)| \leqq C_\alpha M^{-(|\beta|+N)} \beta!^d N!^d \langle\xi\rangle^{-|\alpha|-N} \quad \text{for any } N. \end{cases} \tag{2.19}$$

Assume that C_o *and* $\max_{|\alpha|\leqq n+1} C_\alpha$ *are small enough. Then the inverse* Q *of*

$I-P$ exists in $L^0_{G(d)}$ and is represented by the Neumann series $Q=\sum_{\nu=0}^{\infty} P^{\nu}$.

Corollary 2.8 (cf. Proposition 2.2 of [23]). *Let $\phi(x, \xi)\in\mathscr{P}_{G(d)}(\tau)$. If τ is small enough, there exist pseudo-differential operators R and R' in $L^0_{G(d)}$ such that*

$$\begin{cases} I_{\phi}RI_{\phi^*}=I, \\ I_{\phi^*}R'I_{\phi}=I. \end{cases} \tag{2.20}$$

Now, we are prepared to *prove Theorem* 2.1. Since we can prove it by the parallel way to the proof of Theorem 1 in [23], we will give only the sketch of the proof. Set $\Phi_j=\phi_1\#\phi_2\#\cdots\#\phi_j$. Then, as in Proposition 2.3 and Lemma 2.10 in [23] we can find, by using Proposition 2.2, Proposition 2.3, Proposition 2.5 and Corollary 2.8, pseudo-differential operators P'_j in $L^{\sigma}_{G(d)}$ such that

$$\begin{cases} P_{1,\phi_1}=P'_1I_{\phi_1}, \\ I_{\Phi_{j-1}}P_{j,\phi_j}=P'_jI_{\Phi_j} \quad (j\geqq 2\,;\,\Phi_1=\phi_1). \end{cases}$$

From this we have

$$\begin{aligned} \tilde{Q}_{\nu+1} &= P'_1(I_{\phi_1}P_{2,\phi_2})P_{3,\phi_3}\cdots P_{\nu+1,\phi_{\nu+1}} \\ &= P'_1P'_2(I_{\Phi_2}P_{3,\phi_3})P_{4,\phi_4}\cdots P_{\nu+1,\phi_{\nu+1}} \\ &= \cdots\cdots\cdots \\ &= P'_1P'_2\cdots P'_{\nu}(I_{\Phi_{\nu}}P_{\nu+1,\phi_{\nu+1}}) \\ &= P'_1P'_2\cdots P'_{\nu+1}I_{\Phi_{\nu+1}}. \end{aligned}$$

Combining this with Theorem 2.6 and (2.8) we get the theorem.

§ 3. Fundamental Solution for a Hyperbolic Operator

We will construct the fundamental solution for $\mathscr{L}$ of (8). For the proof we will solve an eiconal equation

$$\begin{cases} \partial_t\phi=\lambda(t,\ x,\ \nabla_x\phi), \\ \phi\,|_{t=s}=x\cdot\xi. \end{cases} \tag{3.1}$$

Proposition 3.1. *Assume that $\lambda(t, x, \xi)$ is a real symbol in $M^0_t(S^1_{G(d)})$. Then, there exists a constant T_0 such that the solution $\phi(t, s\,; x, \xi)$ exists uniquely in $\{(t, s)\,;\, 0\leqq t,\ s\leqq T_0\}$ and belongs to $\mathscr{P}_{G(d)}(\tilde{c}_o|t-s|)$ for a* *constant $\tilde{c}_o$ independent of t and s.*

Proof. We follow the proof of Theorem 3.1 in [9] combined with the idea in §1 of Chap. XI of [24]. Let $\{q, p\}(t, s; y, \eta)$ be a solution of

$$\begin{cases} \dfrac{dq}{dt} = -\nabla_\xi \lambda(t, q, p), & \dfrac{dp}{dt} = \nabla_x \lambda(t, q, p) \\ q|_{t=s} = y, & p|_{t=s} = \eta. \end{cases}$$

Then, by the method of the proof of Lemma 3.1 in [9] we obtain

$$|\partial_\eta^\alpha \partial_y^\beta (q-y)| \leqq C|t-s| M^{-(|\alpha|+|\beta|)} \alpha!^d \beta!^d \langle\xi\rangle^{-|\alpha|}, \tag{3.2}$$

$$|\partial_\eta^\alpha \partial_y^\beta (p-\eta)| \leqq C|t-s| M^{-(|\alpha|+|\beta|)} \alpha!^d \beta!^d \langle\xi\rangle^{1-|\alpha|} \tag{3.3}$$

if $0 \leqq t, s \leqq T_0'$ for a small T_0'. From (3.2) there exists an inverse function $Y(t, s; x, \xi)$ of $x = q(t, s; Y, \xi)$ for $0 \leqq t, s \leqq T_0$ if $T_0 (\leqq T_0')$ is small, and it satisfies

$$|\partial_\xi^\alpha \partial_x^\beta (Y-x)| \leqq C|t-s| M^{-(|\alpha|+|\beta|)} \alpha!^d \beta!^d \langle\xi\rangle^{-|\alpha|}. \tag{3.4}$$

Set $\psi(t, s; x, \xi) = p(t, s; Y(t, s; x, \xi), \xi)$ and

$$\phi(t, s; x, \xi) = x \cdot \xi + \int_s^t \lambda(\theta, x, \psi(\theta, s; x, \xi)) d\theta.$$

Then, $\phi(t, s; x, \xi)$ is a solution of (3.1) by the similar discussions in §1 of Chap. XI of [24] and it belongs to $\mathscr{P}_{G(d)}(\tilde{c}_o|t-s|)$ with a constant $\tilde{c}_o$ independent of t and s. Q. E. D.

Now, we *prove Theorem* 1. Let $I_\phi(t, s)$ be the Fourier integral operator with the phase function $\phi(t, s; x, \xi)$ and the symbol 1. Operate $\mathscr{L}$ in (8) to $I_\phi(t, s)$. Then, we can prove by the similar way to the proof of (2.5)

$$\mathscr{L} I_\phi(t, s) = P_\phi(t, s) \tag{3.5}$$

with $P_\phi(t, s)$ in $L^\sigma_{G(d)}(\phi(t, s))$ for any t and s. Now, we seek $E(t, s)$ in the form

$$E(t, s) = I_\phi(t, s) + \int_s^t I_\phi(t, \theta) W(\theta, s) d\theta. \tag{3.6}$$

Then, $W(t, s)$ must satisfy

$$P_\phi(t, s) - iW(t, s) + \int_s^t P_\phi(t, \theta) W(\theta, s) d\theta = 0. \tag{3.7}$$

Set

$$(3.8)\qquad \begin{cases} W_1(t,\ s)=-iP_\phi(t,\ s), \\ W_\nu(t,\ s)=-i\int_s^t P_\phi(t,\ \theta)W_{\nu-1}(\theta,\ s)d\theta \quad (\nu\geqq 2). \end{cases}$$

Then, in the formal sense $W(t,\ s)=\sum_{\nu=1}^{\infty} W_\nu(t,\ s)$ is a solution of (3.7). From (3.8) W_ν(t, s) for $\nu\geqq 2$ has the form

$$(3.9)\qquad W_\nu(t_0,\ s)=(-i)^\nu\int_s^{t_0}\cdots\int_s^{t_{\nu-2}} P_\phi(t_0,\ t_1)P_\phi(t_1,\ t_2)\cdots \times P_\phi(t_{\nu-1},\ s)dt_{\nu-1}\cdots dt_1.$$

Hence, substituting $W(t,\ s)=\sum_{\nu=1}^{\infty} W_\nu(t,\ s)$ with (3.9) into (3.6), the fundamental solution $E(t,\ s)$ can be written formally in the form

$$(3.10)\qquad E(t,\ s)=I_\phi(t,\ s)-i\int_s^t I_\phi(t,\ t_0)P_\phi(t_0,\ s)dt_0 + \sum_{\nu=2}^{\infty}(-i)^\nu\int_s^t\int_s^{t_0}\cdots\int_s^{t_{\nu-2}} I_\phi(t,\ t_0)P_\phi(t_0,\ t_1)\cdots \times P_\phi(t_{\nu-1},\ s)dt_{\nu-1}\cdots dt_0.$$

In what follows we give the precise meaning for (3.10). From 3° of Theorem 2.3 in [14] we have $\phi(t,\ t_0)\#\phi(t_0,\ t_1)\#\cdots\#\phi(t_{\nu-1},\ s) =\phi(t,\ s)$. Hence, if $T_o'(\leqq T_o)$ is small, from Theorem 2.1 there exist symbols $w_\nu^0(t,\ \tilde{t}^{\nu-1},\ s\,;\ x,\ \xi)$ in $M^0_{(t,\tilde{t}^{\nu-1},s)}(S^{\nu\sigma}_{G(d)})$ and $\tilde{w}_\nu(t,\ \tilde{t}^{\nu-1},\ s\,;\ x,\ \xi)$ in $M^0_{(t,\tilde{t}^{\nu-1},s)}(\mathscr{R}_{G(d)})$ $(\tilde{t}^{\nu-1}=(t_0,\ t_1,\ \cdots,\ t_{\nu-1}))$ such that for $j=0,\ 1$

$$|\partial_t^j\partial_\xi^\alpha D_x^\beta w_\nu^0(t,\ \tilde{t}^{\nu-1},\ s\,;\ x,\ \xi)|\leqq C_0^\nu M^{-(|\alpha|+|\beta|)}\alpha!^d\beta!^d\langle\xi\rangle^{\nu\sigma-|\alpha|},$$
$$|\partial_t^j\partial_\xi^\alpha D_x^\beta \tilde{w}_\nu(t,\ \tilde{t}^{\nu-1},\ s\,;\ x,\ \xi)|\leqq C_0^\nu C_\alpha M^{-(|\beta|+N)}\beta!^d N!^d \nu!^{\sigma d}\langle\xi\rangle^{-|\alpha|-N} \quad \text{for any } N$$

and

$$I_\phi(t,\ t_0)P_\phi(t_0,\ t_1)\cdots P_\phi(t_{\nu-1},\ s) =w_\nu^0(t,\ \tilde{t}^{\nu-1},\ s\,;\ X,\ D_x)I_\phi(t,\ s)+\tilde{w}_\nu(t,\ \tilde{t}^{\nu-1},\ s\,;\ X,\ D_x)I_\phi(t,\ s).$$

Here, we have applied Theorem 2.1 noting that the order of the above operator becomes $\nu\sigma$ because the order of $I_\phi(t,\ s)$ is zero. Define

$$w_\nu(t,\ s\,;\ x,\ \xi)=(-i)^\nu\int_s^t\int_s^{t_0}\cdots\int_s^{t_{\nu-2}} w_\nu^0(t,\ \tilde{t}^{\nu-1},\ s\,;\ x,\ \xi)dt_{\nu-1}\cdots dt_0 \quad (t_{-1}=t)$$

and

$$\tilde{r}_\nu(t,\ s\,;\ x,\ \xi)=(-i)^\nu\int_s^t\int_s^{t_0}\cdots\int_s^{t_{\nu-2}} \tilde{w}_\nu(t,\ \tilde{t}^{\nu-1},\ s;\ x,\ \xi)dt_{\nu-1}\cdots dt_0.$$

Then, they satisfy for $j=0,\ 1$

$$|\partial_t^j\partial_\xi^\alpha D_x^\beta w_\nu(t,\ s\,;\ x,\ \xi)|$$

$$\leqq (C_O^\nu |t-s|^\nu \nu!^{-1}) M^{-(|\alpha|+|\beta|)} \alpha!^d \beta!^d \langle\xi\rangle^{\nu\sigma-|\alpha|},$$
$$|\partial_t^j \partial_\xi^\alpha D_x^\beta \tilde{r}_\nu(t, s; x, \xi)|$$
$$\leqq (C_O^\nu |t-s|^\nu \nu!^{\sigma d-1}) C_\alpha M^{-(|\beta|+N)} \beta!^d N!^d \langle\xi\rangle^{-|\alpha|-N} \quad \text{for any } N.$$

Take $T_o(\leqq T_o')$ such that $T_o C_O < 1$. Then, the series $\tilde{r}(t, s; x, \xi) = \sum_{\nu=1}^\infty \tilde{r}_\nu(t, s; x, \xi)$ converges and we can see the sum $\tilde{r}(t, s; x, \xi)$ belongs to $M^1_{t,s}(\mathscr{R}_{G(d)})$ if $0 \leqq t, s \leqq T_o$. Note that the series $\sum_{\nu=1}^\infty w_\nu(t, s; X, D_x)$ has a meaning as an operator from γ^d into itself if $\sigma d < 1$ and as an operator from $\gamma^d(M)$ into γ^d for a small M if $\sigma d = 1$. Summing up the above results, we obtain Theorem 1.

Remark. In the expression (9) of Theorem 1 we set $R(t, s) = \tilde{r}(t, s; X, D_x) I_\phi(t, s)$. This belongs to $\mathscr{R}_{G(d), \phi(t,s)}$. But from $\mathscr{R}_{G(d), \phi(t,s)} = \mathscr{R}_{G(d)}$ we see that $R(t, s)$ belongs to $\mathscr{R}_{G(d)}$.

Next, we consider a hyperbolic system

$$(3.11) \quad \mathscr{L} = D_t - \begin{pmatrix} \lambda_1(t, X, D_x) & & 0 \\ & \ddots & \\ 0 & & \lambda_l(t, X, D_x) \end{pmatrix} + (b_{jk}(t, X, D_x))$$

with the diagonal principal part, where $\lambda_j(t, x, \xi)$ are real symbols in $M_t^0(S^1_{G(d)})$ and $b_{jk}(t, x, \xi)$ belong to $M_t^0(S^\sigma_{G(d)})$ $(0 \leqq \sigma \leqq 1/d)$. Let $\phi_j(t, s; x, \xi)$ be a solution of (3.1) with $\lambda = \lambda_j$. Then, we have by the similar discussions for the proof of Theorem 1

Theorem 3.2. *The fundamental solution $E(t, s)$ for (3.11) can be represented in the form*

$$(3.12) \quad E(t, s) = \sum_{m=1}^{l} E_{0,m,\phi_m}(t, s) + \sum_{\nu=1}^\infty \sum_{\mu \in \Pi_{\nu+1}} \int_s^t \int_s^{t_0} \cdots \int_s^{t_{\nu-2}} E_{\nu,\mu,\Phi_\mu}(t, t_0, \cdots, t_{\nu-1}, s) dt_{\nu-1} \cdots dt_0 + R(t, s) \quad (t_{-1} = t)$$

when $0 \leqq t, s \leqq T_o$. Here, T_o is a small constant, $\Pi_{\nu+1} = \{\mu = (m_1, \cdots, m_{\nu+1})$; $m_j = 1, \cdots, l\}$ and $\Phi_\mu(t, t_0, \cdots, t_{\nu-1}, s) = \phi_{m_1}(t, t_0) \# \phi_{m_2}(t_0, t_1) \# \cdots \# \phi_{m_{\nu+1}}(t_{\nu-1}, s)$

for $\mu=(m_1, \cdots, m_{\nu+1})\in\Pi_{\nu+1}$. *In* (3.12) $E_{\nu,\mu,\Phi_\mu}(t, t_0, \cdots, t_{\nu-1}, s)$ *for* $\mu\in\Pi_{\nu+1}$ *is a Fourier integral operator with symbol* $e_{\nu,\mu}(t, t_0, \cdots, t_{\nu-1}, s; x, \xi)$ *satisfying*

$$|\partial_\xi^\alpha D_x^\beta e_{\nu,\mu}|\leqq C_0^\nu M^{-(|\alpha|+|\beta|)}\alpha!^d\beta!^d\langle\xi\rangle^{\nu\sigma-|\alpha|}$$

and $R(t, s)$ *is a regularizer with a symbol in* $\mathscr{R}_{G(d)}$.

From (3.12) we can investigate the propagation of singularities for a solution $U(t)$ of the Cauchy problem

$$(3.13)\qquad \begin{cases} \mathscr{L}U(t)=0 & 0<t\leqq T_o, \\ U|_{t=0}=G. \end{cases}$$

The details are left in the future. The author is not convinced that the similar result to (3.34) in [13] holds.

In the remainder of this section we give a method of reducing a Cauchy problem

$$(3.14)\qquad \begin{cases} Lu=0, \\ \partial_t^j u|_{t=0}=h_j, \quad j=0, 1, \cdots, m-1 \end{cases}$$

for a hyperbolic operator

$$(3.15)\qquad L=D_t^m+\sum_{j=0}^{m-1}\sum_{|\alpha|+j\leqq m} a_{j,\alpha}(t, x)D_x^\alpha D_t^j \quad \text{with } a_{j,\alpha}(t, x)\in M_t(\gamma^d)$$

to a Cauchy problem (3.13) for a hyperbolic system $\mathscr{L}$ of the form (3.11). Here, $a(t, x)\in M_t(\gamma^d)$ means that it satisfies $|\partial_t^j D_x^\alpha a(t, x)|\leqq C_j M_j^{-|\alpha|}\alpha!^d$ for constants C_j and M_j independent of α. We assume that there exist regularly hyperbolic operators $L_1, L_2, \cdots, L_r$ such that L has a form

$$(3.16)\qquad L=L_1L_2\cdots L_r+\sum_{j=0}^{m-q}a_j'(\mathrm{t}, X, D_x)D_t^j \quad \text{for } a_j'\in M_t(S_{G(d)}^{m-q-j}),$$

where q is an integer satisfying $1\leqq q\leqq r$.

Proposition 3.3. *Set* $\sigma=(r-q)/r$. *Then, there exists a hyperbolic system* $\mathscr{L}$ *of the form* (3.11) *with* $b_{jk}(t, x, \xi)$ *in* $M_t^0(S_{G(d)}^\sigma)$ *such that the Cauchy problem* (3.14) *can be reduced to an equivalent Cauchy problem* (3.13).

In the following we disregard the contribution of regularizers and the equality means that it holds modulo regularizers in $\mathscr{R}_{G(d)}$.

Proof of Proposition 3.3. We divide the proof into two steps.

(I) Denote the order of L_k by s_k and let $\lambda_{k,j}(t, x, \xi)$, $j=1, \cdots, s_k$, be characteristic roots of L_k. We may assume $\lambda_{k,j}(t, x, \xi)\in M_t(S^1_{G(d)})$ by multiplying a cut function with respect to ξ if necessary. Since L_k is a regularly hyperbolic operator, there exist $\lambda'_{k,j}(t, x, \xi)\in M_t(S^0_{G(d)})$ such that

$$\begin{aligned} L_k=(D_t-\lambda_{k,1}(t, X, D_x)-\lambda'_{k,1}(t, X, D_x))\cdots \\ \times(D_t-\lambda_{k,s_k}(t, X, D_x)-\lambda'_{k,s_k}(t, X, D_x)) \\ +\sum_{j=0}^{s_k-1} b_{k,j}(t, X, D_x)D_t^j \end{aligned}$$

with $b_{k,j}\in M_t(S^{-m}_{G(d)})$. Denote $\bar{s}_0=0$, $\bar{s}_k=s_1+\cdots+s_k$ and

$$\partial_j=D_t-\lambda_{k,j-\bar{s}_{k-1}}(t, X, D_x)-\lambda'_{k,j-\bar{s}_{k-1}}(t, X, D_x) \quad \text{if } \bar{s}_{k-1}<j\leqq\bar{s}_k.$$

Then, L has the form

$$L=\partial_1\partial_2\cdots\partial_m+\sum_{j=2}^{m} b_j(t, X, D_x)\partial_j\cdots\partial_m+\sum_{j=0}^{m-q}\tilde{b}_j(t, X, D_x)D_t^j \tag{3.17}$$

with $b_j(t, x, \xi)\in M_t(S^0_{G(d)})$ and $\tilde{b}_j(t, x, \xi)\in M_t(S^{m-q-j}_{G(d)})$. Set $\Pi=\{0\}\cup\bigcup_{r'=1}^{r}\Pi_{r'}$ for

$$\begin{aligned} \Pi_r&=\{J=(j_1, \cdots, j_k)\,;\ \bar{s}_{r-1}+1\leqq j_1<\cdots<j_k\leqq m,\ k\leqq s_r-1\} \\ \Pi_{r'}&=\{J=(j_1, \cdots, j_k)\,;\ \bar{s}_{r'-1}+1\leqq j_1<\cdots<j_k\leqq m, \\ &\qquad \bar{s}_{r'+1}\leqq k\leqq\bar{s}_{r'}-1,\quad \mu_{r'}(J)\geqq 1\geqq\mu_{r'+1}(J)\geqq\cdots\geqq\mu_r(J)\} \\ &\qquad\qquad (1\leqq r'<r) \end{aligned}$$

and denote the number of elements in Π by l, where $\bar{s}_{r'}=s_{r'}+\cdots+s_r$ and $\mu_{r'}(J)=s_{r'}-$(*the number of elements in*: $\{j_1, \cdots j_k\}\cap\{\bar{s}_{r'-1}+1, \cdots, \bar{s}_{r'}\}$) for $J=(j_1, \cdots, j_k)$. Then, by the method of [20] (see also §III in Appendix of [12]) we can prove that L in (3.17) has the form

$$L=\partial_1\partial_2\cdots\partial_m+\sum_{j=2}^{r} b'_j(t, X, D_x)\partial_j\cdots\partial_m+\sum_{J\in\Pi'} b_J(t, X, D_x)\partial^J \tag{3.18}$$

with $b'_j\in M_t(S^0_{G(d)})$ $(1\leqq j\leqq q)$, $b'_j\in M_t(S^{j-q-1}_{G(d)})$ $(q<j\leqq r)$ and $b_J\in M_t(S^{r-q}_{G(d)})$, where $\Pi'=\{(j_1, \cdots, j_k)\in\Pi\,; k\leqq m-r\}$, $\partial^J=\partial_{j_1}\cdots\partial_{j_k}$ for $J=(j_1, \cdots, j_k)$ and $\partial^J=I$ for $J=0$. In fact, first we prove

$$\sum_{j=0}^{m-q}\tilde{b}_j(t, X, D_x)D_t^j=\sum_{J\in\Pi_r, |J|=s_r-1}\Big(\sum_{j=0}^{\bar{s}_{r-1}-q+1}\tilde{b}^1_{J,j}(t, X, D_x)D_t^j\Big)\partial^J$$

$$+\sum_{J\in\Pi_r} b_J^1(t,\ X,\ D_x)\partial^J$$

with $\tilde{b}_{J,j}^1\in M_t(S_{G(d)}^{m-|J|-q-j})$ and $b_J^1\in M_t(S_{G(d)}^{r-q})$, where $|J|$ denotes the length k of $J=(j_1,\ \cdots,\ j_k)$; and next we prove

$$\sum_{J\in\Pi_r,\ |J|=s_r-1}\Big(\sum_{j=0}^{\bar{s}_{r-1}-q+1}\tilde{b}_{J,j}^1(t,\ X,\ D_x)D_t^j\Big)\partial^J$$
$$=\sum_{r'=1}^{2}\ \sum_{J\in\Pi_{r-1},\ |J|=\bar{s}_{r-1}-r'}\Big(\sum_{j=0}^{\bar{s}_{r-2}-q+1}\tilde{b}_{J,j}^2(t,\ X,\ D_x)D_t^j\Big)\partial^J$$
$$+\sum_{J\in\Pi_{r-1}\cup\Pi_r} b_J^2(t,\ X,\ D_x)\partial^J$$

with $\tilde{b}_{J,j}^2\in M_t(S_{G(d)}^{m-|J|-q-j})$ and $b_J^2\in M_t(S_{G(d)}^{r-q})$. Repeating this method we can prove (3.18).

(II) Suppose that a function $u\equiv u(t,\ x)$ satisfies $Lu=0$. Set for $J\in\Pi$

$$\begin{cases} u_0=\Lambda^{(r-1)\sigma}u & (J=0),\\ u_J=\Lambda^{(r-1)\sigma}\partial^J u & \text{for } |J|\leqq m-r,\\ u_J=\Lambda^{(r-1-k)\sigma}\partial^J u & \text{for } |J|=m-r+k \quad (1\leqq k\leqq r-1),\end{cases}$$

where $\Lambda=\langle D_x\rangle$. Then, by applying the method of §3 in [11] the l-dimensional vector $U=(u_J)_{J\in\Pi}$ satisfies $\mathscr{L}U=0$ for a system $\mathscr{L}$ of the form (3.11). In this way we reduce the problem (3.14) to a problem (3.13). The fact that (3.14) and (3.13) are equivalent is verified by the method in [18] and [11]. This concludes the proof.

Q. E. D.

From this proposition and Theorem 3.2 we can prove that (3.14) is γ^d-correct (or γ^d-well-posed) in the sense of [5] if $d\leqq r/(r-q)$ and we can investigate the propagation of singularities for a solution of the Cauchy problem (3.14). In the case of constant multiplicity we can improve Proposition 3.3 as follows.

Proposition 3.4. *Assume that the operator* (3.15) *is a hyperbolic operator with constant multiplicity and its coefficients* $a_{j,\alpha}(t,\ x)$ *satisfy*

(3.19) $\quad |\partial_t^k D_x^\beta a_{j,\alpha}(t,\ x)|\leqq CM^{-(k+|\beta|)}k!^d\beta!^d$ *for* $(t,\ x)\in[0,\ T]\times R_x^n$.

Then, the problem (3.14) *is reduced to the problem* (3.13) *for a perfectly diagonalized operator* $\mathscr{L}$ *of the form* (7) *in the Introduction. Moreover, if the operator* (3.15) *satisfies* (3.16), *then the lower order terms* $B_j(t)$

in (7) *are pseudo-differential operators of order* σ *with* $\sigma=(r-q)/r$.

We will prove this proposition in Section 6. We note that from Proposition 3.4 we obtain Corollary 3 with $\sigma=\max_j\{(m_j-1)/m_j\}$ replaced by $\sigma=(r-q)/r$ if (3.15) satisfies (3.16).

Next, we turn to the problem studied in [3] and [21]. Consider a regularly hyperbolic operator

$$L=D_t^2-\sum_{j,k}b_{jk}^0(t,\ x)D_{x_j}D_{x_k}+b(t,\ x)D_t+\sum_j b_j(t,\ x)D_{x_j} \\ +c(t,\ x)\quad \text{on } [0,\ T] \tag{3.20}$$

with continuous coefficients. We assume

$$b_{jk}^0(t,\ x),\ b(t,\ x),\ b_j(t,\ x),\ c(t,\ x)\in\gamma^d \text{ for any fixed } t, \tag{3.21}$$

$$|D_x^\alpha(b_{jk}^0(t,\ x)-b_{jk}^0(s,\ x))|\leqq C|t-s|^\kappa M^{-|\alpha|}\alpha!^d\quad (t,\ s\in[0,\ T]), \tag{3.22}$$

$$|\sum_{j,k}b_{jk}^0(t,\ x)\xi_j\xi_k|\geqq\delta|\xi|^2\quad (\delta>0). \tag{3.23}$$

We show Proposition 3.3 with $\sigma=(r-q)/r$ replaced by $\sigma=1-\kappa$. We may assume (3.22) holds for all $t,\ s\in R^1$. Take an even function $\chi(t)\in\gamma^d$ such that $\int\chi(t)dt=1$ and $\chi=0$ if $|t|\geqq 1/2$. We approximate $b_{jk}^0(t,\ x)$ by

$$b_{jk}(t,\ x,\ \xi)=\int\chi((t-s)\langle\xi\rangle)b_{jk}^0(s,\ x)ds\cdot\langle\xi\rangle.$$

Then, from the evenness of χ the symbols $b_{jk}(t,\ x,\ \xi)$ belong to $M_t^0(S_{G(d)}^0)\cap M_t^1(S_{G(d)}^\sigma)$ and

$$b_{jk}(t,\ x,\ \xi)-b_{jk}^0(t,\ x)\in M_t^0(S_{G(d)}^{-\kappa})$$

hold. Denote

$$\lambda(t,\ x,\ \xi)=\{\sum_{j,k}b_{jk}(t,\ x,\ \xi)\xi_j\xi_k\}^{\frac{1}{2}}\chi'(\xi)$$

where $\chi'(\xi)$ is a function in γ^d satisfying $\chi'=1$ for $|\xi|\geqq 2$ and $\chi'=0$ for $|\xi|\leqq 1$. Then, the operator L can be written in the form

$$L=D_t^2-\lambda(t,\ X,\ D_x)^2+b(t,\ X)(D_t-\lambda(t,\ X,\ D_x))+\tilde{b}(t,\ X,\ D_x)$$

with $\tilde{b}(t,\ x,\ \xi)\in M_t^0(S_{G(d)}^{1+\sigma})$. Note that $\partial_t\lambda(t,\ x,\ \xi)\in M_t^0(S_{G(d)}^{1+\sigma})$. Then, L has the form

$$L=(D_t+\lambda(t,\ X,\ D_x)+b(t,X))(D_t-\lambda(t,\ X,\ D_x))+\tilde{b}'(t,X,D_x)$$

with $\tilde{b}'(t,\ x,\ \xi)\in M_t^0(S_{G(d)}^{1+\sigma})$. For a function $u=u(t,\ x)$ we set $u_1=\Lambda u$

and $u_2=(D_t-\lambda(t, X, D_x))u$. Then, if u satisfies $Lu=0$, $U={}^t(u_1, u_2)$ satisfies $\mathscr{L}_1U=0$ for a system

$$\mathscr{L}_1=D_t-\begin{bmatrix} \lambda(t, X, D_x) & \Lambda \\ 0 & -\lambda(t, X, D_x) \end{bmatrix}+\begin{bmatrix} [\lambda(t, X, D_x), \Lambda]\Lambda^{-1} & 0 \\ \tilde{b}'(t, X, D_x)\Lambda^{-1} & b(t, X) \end{bmatrix}.$$

$\mathscr{L}_1$ has a lower order term in $M_t^0(S^\sigma_{G(d)})$. Next, we set

$$N(t)=\begin{bmatrix} 1 & n(t, X, D_x) \\ 0 & 1 \end{bmatrix}$$

for a symbol $n(t, x, \xi)\in M_t^0(S^0_{G(d)})\cap M_t^1(S^\sigma_{G(d)})$ satisfying $n(t, x, \xi)=-\langle\xi\rangle/(2\lambda(t, x, \xi))$ for $|\xi|\geqq 2$. Then, using $N(t)^{-1}=\begin{bmatrix} 1 & -n(t, X, D_x) \\ 0 & 1 \end{bmatrix}$ and $\partial_t(\sigma(N(t)))\in M_t^0(S^\sigma_{G(d)})$ we obtain

$$\mathscr{L}_1N(t)=N(t)\mathscr{L} \tag{3.24}$$

with

$$\mathscr{L}=D_t-\begin{bmatrix} \lambda(t, X, D_x) & 0 \\ 0 & -\lambda(t, X, D_x) \end{bmatrix}+(b_{jk}(t, X, D_x)) \quad (b_{jk}\in M_t^0(S^\sigma_{G(d)})). \tag{3.25}$$

In this way, we can prove Proposition 3.3 with $\mathscr{L}$ in (3.25).

§ 4. Calculus of Products of Fourier Integral Operators

The end of this section is to prove Proposition 2.2, Proposition 2.3 and Proposition 2.5. To begin with, we prove

Lemma 4.1. i) *Suppose that a double symbol* $r(x, \xi, x', \xi')$ *satisfies*

$$|r^{(\alpha,\alpha')}_{(\beta,\beta')}(x, \xi, x', \xi')|\leqq CM^{-(|\alpha|+|\alpha'|+|\beta|+|\beta'|)} \times\alpha!^d\alpha'!^d\beta!^d\beta'!^d\langle\xi'\rangle^{m-|\alpha|-|\alpha'|} \tag{4.1}$$

with constants C *and* M *independent of* α, α', β *and* β'. *Then, the simplified symbol*

$$r_L(x, \xi)=O_s\text{-}\iint e^{-iy\cdot\eta}r(x, \xi+\eta, x+y, \xi)dy\,đ\eta$$

of $r(x, \xi, x', \xi')$ *belongs to* $S^m_{G(d)}$.

ii) *Suppose that a double symbol* $r(x, \xi, x', \xi')$ *satisfies*

$$|r^{(\alpha,\alpha')}_{(\beta,\beta')}(x, \xi, x', \xi')|\leqq C_{\alpha,\alpha'}M^{-(|\beta|+|\beta'|+N)} \times\beta!^d\beta'!^dN!^d\langle\xi'\rangle^{-|\alpha|-|\alpha'|-N} \quad \textit{for any } N \tag{4.2}$$

with a constant M independent of α, α', β, β', N and a constant $C_{\alpha,\alpha'}$ independent of β, β' and N. Then, the simplified symbol $r_L(x, \xi)$ of $r(x, \xi, x', \xi')$ belongs to $\mathscr{R}_{G(d)}$.

Proof. First we assume (4.1). Differentiating $r_L(x, \xi)$ with respect to x and ξ, we have

$$r_{L(\beta)}^{(\alpha)}(x, \xi)=\sum_{\substack{\alpha'+\alpha''=\alpha \\ \beta'+\beta''=\beta}}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}\iint e^{-iy\cdot\eta}r_{(\beta',\beta'')}^{(\alpha',\alpha'')}(x, \xi+\eta, x+y, \xi)dy\,d\!\!\!^{-}\eta.$$

Here and in what follows we often omit the notion "O_s-". Using (6.10) of Chap. 1 in [12] we obtain

$$|r_{L(\beta)}^{(\alpha)}(x, \xi)|\leqq CM^{-(|\alpha|+|\beta|)}\alpha!^d\beta!^d\langle\xi\rangle^{m-|\alpha|}$$

for new constants C and M indepedent of α and β. This shows $r_L(x, \xi)\in S_{G(d)}^m$. Similarly, we can prove ii). Q. E. D.

Now, we prove Proposition 2.2.

Proof of Proposition 2.2. Let $p_1(X, D_x)\in S_{G(d)}^m$ and $p_{2,\phi}(X, D_x)\in S_{G(d),\phi}^{m'}$. Then, from Theorem 2.2. of Chap. 10 in [12] the product $p_1(X, D_x)p_{2,\phi}(X, D_x)$ is a Fourier integral operator with a symbol $q(x, \xi)$ defined by

$$q(x, \xi')=O_s\text{-}\iint e^{i\psi}p_1(x, \xi)\,p_2(x', \xi')dx'd\!\!\!^{-}\xi,$$

where $\psi=x\cdot\xi-x'\cdot\xi+\phi(x', \xi')-\phi(x, \xi')$. Take a function $\chi(\xi)$ in γ^d satisfying

$$0\leqq\chi\leqq1,\quad \chi=1\ (|\xi|\leqq2/5),\quad \chi=0\ (|\xi|\geqq1/2) \tag{4.3}$$

and divide $q(x, \xi)$ into two terms:

$$q(x, \xi)=q_O(x, \xi)+q_\infty(x, \xi), \tag{4.4}$$

where

$$q_O(x, \xi')=\iint e^{i\psi}p_1(x, \xi)\chi((\xi-\xi')/\langle\xi'\rangle)p_2(x', \xi')dx'd\!\!\!^{-}\xi,$$
$$q_\infty(x, \xi')=\iint e^{i\psi}p_1(x, \xi)(1-\chi((\xi-\xi')/\langle\xi'\rangle))p_2(x', \xi')dx'd\!\!\!^{-}\xi.$$

Denote $\tilde{\nabla}_x\phi(x, \xi, x')=\int_0^1\nabla_x\phi(x'+\theta(x-x'), \xi)d\theta$. Then, ψ has the form $\psi=(x-x')\cdot(\xi-\tilde{\nabla}_x\phi(x, \xi', x'))$. Hence, using the change of variables:

$y=x'-x$, $\eta=\xi-\tilde{\nabla}_x\phi(x, \xi', x')$ we have

$$q_O(x, \xi')=\iint e^{-iy\cdot\eta}p_1(x, \eta+\tilde{\nabla}_x\phi(x, \xi', x+y))$$
$$\times\chi((\eta+\tilde{\nabla}_x\phi(x, \xi', x+y)-\xi')/\langle\xi'\rangle)p_2(x+y, \xi')dy đ\eta.$$

This formula shows that $q_O(x, \xi)$ is the simplified symbol of a double symbol

$$q_O'(x, \xi, x', \xi')$$
$$=[p_1(x, \zeta)\chi((\zeta-\xi')/\langle\xi'\rangle)p_2(x', \xi')]_{\zeta=\xi-\xi'+\tilde{\nabla}_x\phi(x,\xi',x')},$$

which satisfies (4.1). Hence, from i) of Lemma 4.1 we obtain $q_O(x, \xi)\in S^{m+m'}_{G(d)}$.

Now, we prove $q_{\infty,\phi}(X, D_x)\in\mathscr{R}_{G(d),\phi}$. This result is equivalent to $\tilde{q}_\infty(X, D_x)\in\mathscr{R}_{G(d)}$ if we set $\tilde{q}_\infty(x, \xi)=q_\infty(x, \xi)e^{iJ(x,\xi)}$ for $J(x, \xi)=\phi(x, \xi)-x\cdot\xi$. So, we may prove $\tilde{q}_\infty(x, \xi)\in\mathscr{R}_{G(d)}$. It has the form

$$\tilde{q}_\infty(x, \xi')=\iint e^{i\tilde{\phi}}p_1(x, \xi)(1-\chi((\xi-\xi')/\langle\xi'\rangle))p_2(x', \xi')dx'đ\xi,$$

where $\tilde{\phi}=x\cdot\xi-x'\cdot\xi+\phi(x', \xi')-x\cdot\xi'$. Set $\tilde{\Psi}_\alpha(x', \xi; x, \xi')=e^{-i\tilde{\phi}}\partial^\alpha_{\xi'}e^{i\tilde{\phi}}$. Then, it satisfies

$$|D^\beta_x D^\delta_{x'}\partial^\gamma_\xi\tilde{\Psi}_\alpha|\leqq C_{\alpha,\gamma}M^{-(|\beta|+|\delta|)}\beta!^d\delta!^d\langle x-x'\rangle^{|\alpha|}.$$

Using this and $\nabla_x\tilde{\phi}=\xi-\xi'$ we have

$$\tilde{q}_{\infty}{}^{(\alpha)}_{(\beta)}(x, \xi')=\sum_{\alpha'+\alpha''=\alpha}\binom{\alpha}{\alpha'}D^\beta_x\iint e^{i\tilde{\phi}}\tilde{\Psi}_{\alpha'}(x', \xi; x, \xi')$$
$$\times p_1(x, \xi)\partial^{\alpha''}_{\xi'}\{(1-\chi((\xi-\xi')/\langle\xi'\rangle))p_2(x',\xi')\}dx'đ\xi$$
$$=\sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''+\beta'''=\beta}}\binom{\alpha}{\alpha'}\frac{\beta!}{\beta'!\beta''!\beta'''!}\iint e^{i\tilde{\phi}}(\xi-\xi')^{\beta'}D^{\beta''}_x\tilde{\Psi}_{\alpha'}(x', \xi; x, \xi')$$
$$\times p_{1(\beta''')}(x, \xi)\partial^{\alpha''}_{\xi'}\{(1-\chi((\xi-\xi')/\langle\xi'\rangle))p_2(x', \xi')\}dx'đ\xi.$$

Set $L_1=-i|-\xi+\nabla_x\phi(x', \xi')|^{-2}(-\xi+\nabla_x\phi(x', \xi'))\cdot\nabla_{x'}$ and $L_2=(1+|x-x'|^2)^{-1}(1-i(x-x')\cdot\nabla_\xi)$. Then, integrating by parts we have

$$\tilde{q}_{\infty}{}^{(\alpha)}_{(\beta)}(x, \xi')=\sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''+\beta'''=\beta}}\binom{\alpha}{\alpha'}\frac{\beta!}{\beta'!\beta''!\beta'''!}\tilde{q}^{\alpha',\alpha''}_{\infty,\beta',\beta'',\beta'''}(x, \xi') \tag{4.5}$$

with

$$\tilde{q}^{\alpha',\alpha''}_{\infty,\beta',\beta'',\beta'''}(x, \xi')=\iint e^{i\tilde{\phi}}({}^tL_1)^l({}^tL_2)^{n+1+|\alpha'|}\{(\xi-\xi')^{\beta'} \tag{4.6}$$
$$\times D^{\beta''}_x\tilde{\Psi}_{\alpha'}(x', \xi; x, \xi')p_{1(\beta''')}(x, \xi)$$
$$\times\partial^{\alpha''}_{\xi'}\{(1-\chi((\xi-\xi')/\langle\xi'\rangle))p_2(x', \xi')\}\}dx'đ\xi.$$

In (4.6) we take $l=|\beta'|+|\alpha'|+N+n+1+[\max(m, 0)+\max(m', 0)]$.

Then, we obtain from $|-\xi+\nabla_x\phi(x, \xi')|\geqq C|\xi-\xi'|\geqq C'\langle\xi'\rangle$

$$(4.7)\qquad |\tilde{q}_{\infty}{}^{(\alpha)}_{(\beta)}(x, \xi')|\leqq C_\alpha M^{-(|\beta|+N)}\beta!^d N!^d\langle\xi'\rangle^{-|\alpha|-N} \quad \text{for any } N.$$

This shows $q_\infty\in\mathscr{R}_{G(d)}$. Consequently, we have proved (2.5). Similarly, we can prove (2.6). The above proof is also valid for the proof of (2.7) if we use Lemma 4. 1-ii) instead of using Lemma 4. 1-i). This concludes the proof of Proposition 2.2. Q. E. D.

Remark. Throughout this section we denote by $\chi(\xi)$ a function in γ^d satisfying (4. 3).

Using the method of the proof of (2.12) we can improve Lemma 2.1 in [23] as follows.

Lemma 4.2. *Let* $\phi(x, \xi)$ *belong to* $\mathscr{P}_{G(d)}$. *Then, we have the following:*

i) *The inverse function* $\xi=\tilde{\nabla}_x\phi^{-1}(x, \eta, x')$ *of* $\eta=\int_0^1\nabla_x\phi(x'+\theta(x-x'), \xi)d\theta$ *satisfies* (2. 3)-a), (2. 3)-b) *in* [23] *and*

$$(4.8)\qquad \begin{aligned}&|\partial_\eta^\alpha D_x^\beta D_{x'}^{\beta'}\tilde{\nabla}_x\phi^{-1}(x, \eta, x')|\\ &\quad\leqq CM^{-(|\alpha|+|\beta|+|\beta'|)}\alpha!^d\beta!^d\beta'!^d\langle\eta\rangle^{1-|\alpha|}.\end{aligned}$$

ii) *The inverse function* $x'=\tilde{\nabla}_\xi\phi^{-1}(\xi, y', \xi')$ *of* $y'=\int_0^1\nabla_\xi\phi(x', \xi'+\theta(\xi-\xi'))d\theta$ *satisfies*

$$(4.9)\qquad \begin{aligned}&|\partial_\xi^\alpha\partial_{\xi'}^{\alpha'}D_{y'}^{\beta'}(\tilde{\nabla}_\xi\phi^{-1}(\xi, y', \xi')-y')|\\ &\quad\leqq CM^{-(|\alpha|+|\alpha'|+|\beta'|)}\alpha!^d\alpha'!^d\beta'!^d,\end{aligned}$$

$$(4.10)\qquad \begin{aligned}&|\partial_\xi^\alpha\partial_{\xi'}^{\alpha'}D_{y'}^{\beta'}\{\chi((\xi-\xi')/\langle\xi'\rangle)(\tilde{\nabla}_\xi\phi^{-1}(\xi, y', \xi')-y')\}|\\ &\quad\leqq CM^{-(|\alpha|+|\alpha'|+|\beta'|)}\alpha!^d\alpha'!^d\beta'!^d\langle\xi'\rangle^{-|\alpha|-|\alpha'|}.\end{aligned}$$

Now, we prove Proposition 2.3.

Proof of Proposition 2.3. Let $p_\phi(X, D_x)$ be a Fourier integral operator in $S^m_{G(d),\phi}$. Then, by (1. 29)-(1. 30) in Chap. 10 of [12] $p_\phi(X, D_x)I_{\phi*}$ is a pseudo-differential operator with a symbol

$$q(x, \xi)=O_s\text{-}\iint e^{-iy\cdot\eta}q'(x, \xi+\eta, x+y)dy\,d\!\!{}^-\eta$$

for

$$q'(x,\ \xi,\ x')=\{p(x,\ \zeta)\ |\det\frac{\partial}{\partial\xi}\tilde{\nabla}_x\phi(x,\ \zeta,\ x')|^{-1}\}_{\zeta=\tilde{\nabla}_x\phi^{-1}(x,\xi,x')}$$

where for a vector $f={}^t(f_1,\ \cdots,\ f_n)$ of functions $f_j(x,\ \xi)$ $\frac{\partial}{\partial\xi}f$ denotes $\left(\partial f_j/\partial\xi_k\ \begin{matrix} j\downarrow 1,\ \cdots,\ n \\ k\rightarrow 1,\ \cdots,\ n\end{matrix}\right)$. Write

$$q(x,\ \xi)=\iint e^{-iy\cdot\eta}\chi(\eta/\langle\xi\rangle)q'(x,\ \xi+\eta,\ x+y)\,dy\,d\!\!\!{}^-\eta$$
$$+\iint e^{-iy\cdot\eta}(1-\chi(\eta/\langle\xi\rangle))q'(x,\ \xi+\eta,\ x+y)\,dy\,d\!\!\!{}^-\eta.$$

Then, using Lemma 4.2-i) we see that the first term belongs to $S^m_{G(d)}$ because of Lemma 4.1-i) and the second term belongs to $\mathscr{R}_{G(d)}$ if we use the integration by parts. This shows $S^m_{G(d),\phi}\cdot I_{\phi^*}\subset L^m_{G(d)}$. Next, we assume $p_\phi(X,\ D_x)\in\mathscr{R}_{G(d),\phi}$. Then, the product $p_\phi(X,\ D_x)I_{\phi^*}$ is a pseudo-differential operator $\tilde{q}(X,\ D_x,\ X')$ with a symbol $\tilde{q}(x,\ \xi,\ x')$ $=p(x,\ \xi)e^{i(J(x,\xi)-J(x',\xi))}$. Since $\tilde{q}(x,\ \xi,\ x')$ satisfies

$$|\tilde{q}^{(\alpha)}_{(\beta,\beta')}(x,\ \xi,\ x')|\leqq C_\alpha M^{-(|\beta|+|\beta'|+N)}\beta!^d\beta'!^dN!^d\langle\xi\rangle^{-|\alpha|-N}$$
for any N,

we can prove that its simplified symbol $\tilde{q}_L(x,\ \xi)$ belongs to $\mathscr{R}_{G(d)}$. These results show the first formula of (2.10). In the same way we can prove the second formula of (2.10) by using Lemma 4.2-ii).

Q. E. D.

The remainder of this section is devoted to the proof of Proposition 2.5. We divide it into three steps.

(I) We follow the proof of Proposition 2.8 in [23]. Set $\Phi(x,\ \xi)$ $=\phi_1\#\phi_2(x,\ \xi)$ and set

$$\begin{aligned}\psi\equiv\psi(x,\ x'\ ;\ \xi,\ \xi')\\ =\phi_1(x,\ \xi)-x'\cdot\xi+\phi_2(x',\ \xi')-\Phi(x,\ \xi').\end{aligned}$$

Then, $I_{\phi_1}I_{\phi_2}$ is a Fourier integral operator with the phase function $\Phi(x,\ \xi)$ and a symbol $p(x,\ \xi)$ defined by

$$p(x,\ \xi')=O_s\text{-}\iint e^{i\psi}dx'\,d\!\!\!{}^-\xi. \tag{4.11}$$

Set $\chi_\infty(\xi,\ \xi')=1-\chi((\xi-\xi')/\langle\xi'\rangle)$ and consider

$$p_\infty(x,\ \xi')=O_s\text{-}\iint e^{i\psi}\chi_\infty(\xi,\ \xi')\,dx'\,d\!\!\!{}^-\xi.$$

For $\psi' = \phi_1(x, \xi) + \phi_2(x', \xi') - \Phi(x, \xi')$ we have

$$\partial_{\xi'}^{\alpha} D_x^{\beta} e^{i\psi'} = e^{i\psi'} \sum_{k=0}^{|\beta|} \Psi_{k;\alpha,\beta}(x', \xi; x, \xi') \tag{4.12}$$

with symbols $\Psi_{k;\alpha,\beta}(x', \xi; x, \xi')$ satisfying

$$\begin{aligned} &|\partial_{\xi}^{\gamma} D_{x'}^{\delta} \Psi_{k;\alpha,\beta}| \\ &\leqq C_{\alpha,\gamma} M^{-(|\beta|+|\delta|)} \beta!^d \delta!^d k!^{-d} \langle x - x' \rangle^{|\alpha|} \langle \xi - \xi' \rangle^k. \end{aligned} \tag{4.13}$$

Hence, following the way of (2.34)−(2.36) in [23] we obtain

$$p_{\infty}{}_{(\beta)}^{(\alpha)}(x, \xi') = \sum_{k=0}^{|\beta|} \iint e^{i\psi} p_{\infty,(k,\alpha,\beta)}(x', \xi; x, \xi') dx' đ\xi \tag{4.14}$$

for symbols $p_{\infty,(k,\alpha,\beta)}(x', \xi; x, \xi')$ satisfying

$$\begin{aligned} &|\partial_{\xi}^{\gamma} D_{x'}^{\delta} p_{\infty,(k,\alpha,\beta)}| \\ &\leqq C_{\alpha,\gamma} M^{-(|\beta|+|\delta|)} \beta!^d \delta!^d k!^{-d} \langle x - x' \rangle^{|\alpha|} \langle \xi - \xi' \rangle^k. \end{aligned} \tag{4.15}$$

On supp $p_{\infty,(k,\alpha,\beta)}$ the inequality $|\xi - \xi'| \geqq (2/5)\langle \xi' \rangle$ holds. This implies

$$|\nabla_{x'}\psi| \geqq \frac{1}{6} |\xi - \xi'| \geqq \frac{1}{15} \langle \xi' \rangle \tag{4.16}$$

if $\tau_2 < 1/3$. Set

$$\begin{cases} L_1 = -i |\nabla_{x'}\psi|^{-2} \nabla_{x'}\psi \cdot \nabla_{x'}, \\ L_2 = (1 + |\nabla_{\xi}\psi|^2)^{-1}(1 - i\nabla_{\xi}\psi \cdot \nabla_{\xi}). \end{cases}$$

Then, by the integration by parts we have

$$p_{\infty}{}_{(\beta)}^{(\alpha)}(x, \xi') = \sum_{k=0}^{|\beta|} \iint e^{i\psi} ({}^tL_1)^{n+1+k+N} ({}^tL_2)^{n+1+|\alpha|} p_{\infty,(k,\alpha,\beta)}(x', \xi; x, \xi') dx' đ\xi.$$

Note $1 + |\nabla_{\xi}\psi| \geqq C\langle x - x' \rangle$. Then, we obtain from (4.15) and (4.16)

$$|p_{\infty}{}_{(\beta)}^{(\alpha)}(x, \xi')| \leqq C_{\alpha} M^{-(|\beta|+N)} \beta!^d N!^d \langle \xi' \rangle^{-N} \quad \text{for any } N.$$

This implies

$$p_{\infty}(x, \xi) \in \mathscr{R}_{G(d)}. \tag{4.17}$$

(II) For $\chi_O(\xi, \xi') = \chi((\xi - \xi')/\langle \xi' \rangle)$ we consider

$$p_O(x, \xi') = O_s\text{-}\iint e^{i\psi} \chi_O(\xi, \xi') dx' đ\xi.$$

Let $\{X, \Xi\}(x, \xi)$ be the solution of

$$\begin{cases} X = \nabla_{\xi}\phi_1(x, \Xi), \\ \Xi = \nabla_x \phi_2(X, \xi). \end{cases} \tag{4.18}$$

Using a change of variables: $x'=X(x, \xi')+y$, $\xi=\Xi(x, \xi')+\eta$, we write

$$p_O(x, \xi')=O_s\text{-}\iint e^{-i\tilde{\phi}(y,\eta;x,\xi')}\tilde{\chi}_O(\eta; x, \xi')dy\,đ\eta. \tag{4.19}$$

Here,

$$\tilde{\chi}_O(\eta; x, \xi)=\chi((\Xi(x, \xi)+\eta-\xi)/\langle\xi\rangle) \tag{4.20}$$

and

$$\begin{aligned}\tilde{\phi}\equiv\tilde{\phi}(y, \eta; x, \xi) &= -\phi(x, X(x, \xi)+y; \Xi(x, \xi)+\eta, \xi)\\ &=y\cdot\eta-\{\phi_1(x, \Xi+\eta)-X\cdot\eta-\phi_1(x, \Xi)\}\\ &\quad-\{\phi_2(X+y, \xi)-y\cdot\Xi-\phi_2(X, \xi)\}.\end{aligned} \tag{4.21}$$

As in the proof of Theorem 2. 3 in [13] we divide $p_O(x, \xi)$ into two terms

$$p_O(x, \xi)=p_{O,O}(x, \xi)+p_{O,\infty}(x, \xi); \tag{4.22}$$

$$\begin{aligned}p_{O,O}(x, \xi)=O_s\text{-}\iint e^{-i\tilde{\phi}}\tilde{\chi}_O(\eta; x, \xi)\chi((\langle\xi\rangle^2|y|^2\\ +|\eta|^2)/(\varepsilon\langle\xi\rangle)^2)dy\,đ\eta,\end{aligned} \tag{4.23}$$

$$\begin{aligned}p_{O,\infty}(x, \xi)=O_s\text{-}\iint e^{-i\tilde{\phi}}\tilde{\chi}_O(\eta; x, \xi)\\ \times(1-\chi((\langle\xi\rangle^2|y|^2+|\eta|^2)/(\varepsilon\langle\xi\rangle)^2))dy\,đ\eta,\end{aligned} \tag{4.24}$$

where a positive constant ε $(\leqq 1/2)$ is determined in the step (III). In this step we will prove

$$p_{O,\infty}\in S^0_{G(d)}\cap\mathscr{R}_{G(d)}. \tag{4.25}$$

On the support of the integrand of (4.24) we have

$$|\partial_\xi^\alpha D_x^\beta\partial_\eta^\gamma D_y^\delta\tilde{\phi}|\leqq CM^{-(|\alpha|+|\beta|+|\gamma|+|\delta|)}\alpha!^d\beta!^d\gamma!^d\delta!^d(\langle\xi\rangle|y|+|\eta|)\langle\xi\rangle^{-|\alpha|-|\gamma|}$$
$$(|\alpha|+|\beta|+|\gamma|+|\delta|\geqq 1)$$

because of

$$\begin{aligned}\tilde{\phi}=y\cdot\eta-\{\int_0^1\nabla_\xi\phi_1(x, \Xi+\theta\eta)d\theta\cdot\eta-X\cdot\eta\}\\ -\{\int_0^1\nabla_x\phi_2(X+\theta y, \xi)d\theta\cdot y-y\cdot\Xi\}.\end{aligned}$$

Hence, $p_{O,\infty}{}^{(\alpha)}_{(\beta)}(x, \xi)$ can be written in the form

$$p_{O,\infty}{}^{(\alpha)}_{(\beta)}(x, \xi)=\sum_{k=0}^{|\alpha+\beta|}\iint e^{-i\tilde{\phi}}p_{O,\infty,(k,\alpha,\beta)}(y, \eta; x, \xi)dy\,đ\eta \tag{4.26}$$

for symbols $p_{O,\infty,(k,\alpha,\beta)}(y, \eta; x, \xi)$ satisfying

$$(4.27)\qquad |\partial_\eta^\gamma D_y^\delta p_{O,\infty,(k,\alpha,\beta)}| \leqq CM^{-(|\alpha|+|\beta|+|\gamma|+|\delta|)}\alpha!^d\beta!^d\gamma!^d\delta!^d k!^{-d} \times (\langle\xi\rangle|y|+|\eta|)^k\langle\xi\rangle^{-|\alpha|-|\gamma|}.$$

Here, we have used the similar discussions to the one in (4.12)—(4.15). From (2.45) in [23] we have

$$(4.28)\qquad \langle\xi\rangle^2|\nabla_\eta\tilde{\phi}|^2+|\nabla_y\tilde{\phi}|^2 \geqq \frac{1}{18}(\langle\xi\rangle|y|+|\eta|)^2 \geqq \frac{1}{45}\varepsilon^2\langle\xi\rangle^2 \qquad \text{on } \operatorname{supp} p_{O,\infty,(k,\alpha,\beta)}.$$

Set $L_3=i(\langle\xi\rangle^2|\nabla_\eta\tilde{\phi}|^2+|\nabla_y\tilde{\phi}|^2)^{-1}(\langle\xi\rangle^2\nabla_\eta\tilde{\phi}\cdot\nabla_\eta+\nabla_y\tilde{\phi}\cdot\nabla_y)$ and integrate each term in (4.26) by parts. Then, we have for any N

$$(4.29)\qquad p_{O,\infty}{}^{(\alpha)}_{(\beta)}(x,\xi)=\sum_{k=0}^{|\alpha+\beta|}\iint e^{-i\tilde{\phi}}(L_3^t)^{2n+1+k+N}p_{O,\infty,(k,\alpha,\beta)}(y,\eta;x,\xi)\,dy\,d\!\!{}^-\!\eta.$$

From (4.27)—(4.29) we have

$$|p_{O,\infty}{}^{(\alpha)}_{(\beta)}(x,\xi)|\leqq CM^{-(|\alpha|+|\beta|+N)}\alpha!^d\beta!^d N!^d\langle\xi\rangle^{-|\alpha|-N},$$

which shows (4.25).

(III) In this step we consider (4.23). The following lemma originates from the idea in the III)-step of the proof of Theorem 2.3 in [13].

Lemma 4.3. *Assume* $\tau_1+\tau_2\leqq 1/4$. *Then, there exist matrices* $F(y,\eta;x,\xi)$, $F'(y,\eta;x,\xi)$ *and* $G(y,\eta;x,\xi)$ *such that*

$$(4.30)\qquad \begin{cases} |\partial_\xi^\alpha D_x^\beta\partial_\eta^\gamma D_y^\delta F|\leqq CM^{-(|\alpha|+|\beta|+|\gamma|+|\delta|)}\alpha!^d\beta!^d\gamma!^d\delta!^d\langle\xi\rangle^{-1-|\alpha|-|\gamma|}, \\ |\partial_\xi^\alpha D_x^\beta\partial_\eta^\gamma D_y^\delta F'|\leqq CM^{-(|\alpha|+|\beta|+|\gamma|+|\delta|)}\alpha!^d\beta!^d\gamma!^d\delta!^d\langle\xi\rangle^{1-|\alpha|-|\gamma|}, \\ |\partial_\xi^\alpha D_x^\beta\partial_\eta^\gamma D_y^\delta G|\leqq CM^{-(|\alpha|+|\beta|+|\gamma|+|\delta|)}\alpha!^d\beta!^d\gamma!^d\delta!^d\langle\xi\rangle^{-|\alpha|-|\gamma|} \end{cases}$$

when $|\eta|\leqq\langle\xi\rangle/2$ *and*

$$(4.31)\qquad \tilde{\phi}=(y+F\eta)\cdot(F'y+G\eta)$$

holds for $\tilde{\phi}$ *in* (4.21). *Moreover, there exists a constant* ε $(\leqq 1/2)$ *such that*

$$(4.32)\qquad D(y,\eta;x,\xi)\equiv\det\begin{pmatrix}\dfrac{\partial}{\partial y}(y+F\eta) & \dfrac{\partial}{\partial\eta}(y+F\eta)\\ \dfrac{\partial}{\partial y}(F'y+G\eta) & \dfrac{\partial}{\partial\eta}(F'y+G\eta)\end{pmatrix}\geqq C_1>0$$

holds when $|y|+|\eta|\langle\xi\rangle^{-1}\leqq\varepsilon$.

Admitting this lemma for the moment we continue the proof of Proposition 2.5. In (4.23) we take a constant ε as the one in the above lemma. Then, by (4.32) we can change the variables from (y, η) to $(z, \zeta)=(y+F\eta, F'y+G\eta)$. Let $y=Y(z, \zeta; x, \xi)$ and $\eta=H(z, \zeta; x, \xi)$ be the inverse function of $z=y+F(y, \eta; x, \xi)\eta$, $\zeta=F'(y, \eta; x, \xi)y+G(y, \eta; x, \xi)\eta$. Then, from (4.23) we have

$$p_{O,O}(x, \xi)=O_s\text{-}\iint e^{-iz\cdot\zeta}\{\tilde{\chi}_O(\eta; x, \xi)\chi((\langle\xi\rangle^2|y|^2+|\eta|^2)/(\varepsilon\langle\xi\rangle)^2)\times D(y, \eta; x, \xi)^{-1}\}_{(y,\eta)=(Y,H)(z,\zeta;x,\xi)}dz\,đ\zeta.$$

This shows that $p_{O,O}(x, \xi)$ is the simplified symbol of

$$p'_{O,O}(x, \xi, x', \xi')=\{\tilde{\chi}_O(\eta; x, \xi')\chi((\langle\xi'\rangle^2|y|^2+|\eta|^2)/(\varepsilon\langle\xi'\rangle)^2)\times D(y, \eta; x, \xi')^{-1}\}_{(y,\eta)=(Y,H)(x'-x,\xi-\xi';x,\xi')}.$$

Since $p'_{O,O}(x, \xi, x', \xi')$ satisfies (4.1), we obtain

$$p_{O,O}(x, \xi)\in S^0_{G(d)}. \tag{4.33}$$

Combining (4.33) with (4.17) and (4.25) we obtain (2.14). This concludes the proof of Proposition 2.5.

Proof of Lemma 4.3. Set

$$\begin{cases} B\equiv B(\eta; x, \xi)=\int_0^1(1-\theta)\dfrac{\partial}{\partial\eta}\nabla_\eta\phi_1(x, \Xi+\theta\eta)d\theta, \\ B'\equiv B'(y; x, \xi)=\int_0^1(1-\theta)\dfrac{\partial}{\partial x}\nabla_x\phi_2(X+\theta y, \xi)d\theta. \end{cases}$$

Then, from (4.21) and (4.18) we have

$$\tilde{\phi}=y\cdot\eta-B\eta\cdot\eta-B'y\cdot y.$$

Hence, the equation (4.31) holds when F, F' and G satisfy

$$\begin{cases} {}^tF'=-B', \\ {}^tG+{}^tFF'=\mathscr{I}, \\ {}^tGF=-B, \end{cases} \tag{4.34}$$

where $\mathscr{I}$ is the identity matrix.

Denote the norm $\max_k\sum_{j=1}^n|a_{jk}|$ of a matrix $A=(a_{jk})$ by $\|A\|$. Note that B and B' are symmetric and

$$\|B\|\leqq 2\tau_1\langle\xi\rangle^{-1}, \quad \|B'\|\leqq\tau_2\langle\xi\rangle \tag{4.35}$$

if $|\eta|\leqq\langle\xi\rangle/2$. In the following we assume that the inequality $|\eta|\leqq$

$\langle\xi\rangle/2$ always holds. Set

$$F=-B\sum_{l=0}^{\infty}\frac{(2l)!}{(l+1)!l!}(B'B)^{l}. \tag{4.36}$$

Then, since $2\tau_1\tau_2<1/4$ the series (4.36) converges from (4.35) and satisfies

$$FB'F+F+B=0.$$

Using this matrix F we set

$$\begin{cases} F'=-B', \\ G=\mathscr{I}+B'F. \end{cases} \tag{4.37}$$

Then, the matrices F, F' and G satisfy (4.34). From (4.36) and (4.37) we also get (4.30) and

$$\begin{cases} \|F\|\leqq 3\tau_1\langle\xi\rangle^{-1}, \\ \|F'\|\leqq\tau_2\langle\xi\rangle, \\ \|G-\mathscr{I}\|\leqq\tau_1. \end{cases}$$

Hence, we obtain (4.32) if $\tau_1+\tau_2\leqq 1/4$ and $|y|+|\eta|\langle\xi\rangle^{-1}\leqq\varepsilon$ hold for a small constant ε. Summing up, we get matrices F, F' and G satisfying (4.30)—(4.32). Q. E. D.

§ 5. Multi-Products of Pseudo-Differential Operators

In this section we prove Theorem 2.6. First, we consider

$$Q_{\nu+1}=p_1^0(X, D_x)p_2^0(X, D_x)\cdots p_{\nu+1}^0(X, D_x) \tag{5.1}$$

for pseudo-differential operators $p_j^0(X, D_x)$ in $S_{G(d)}^{\sigma}$.

Proposition 5.1. *Assume that* $p_j^0(x, \xi)$ *satisfy* (2.15) *with constants* C_o *and* M *independent of* j, α *and* β. *Then, the same result as in Theorem* 2.6 *holds with* $\tilde{C}_o=C_o$.

Proof. We follow the discussions in Section 1 of [23]. We divide the proof into three steps.

(I) It is well-known that the symbol $q_{\nu+1}(x, \xi)$ of $Q_{\nu+1}$ has the form

$$q_{\nu+1}(x, \xi)=O_s\text{-}\iint e^{-i\psi}\prod_{j=1}^{\nu+1}p_j^0(x+y^{j-1}, \xi+\eta^j)d\tilde{y}^{\nu}d\tilde{\eta}^{\nu} \tag{5.2}$$
$$(y^0=\eta^{\nu+1}=0),$$

where

$$(5.3)\qquad \phi=\sum_{j=1}^{\nu} y^j\cdot(\eta^j-\eta^{j+1})=\sum_{j=1}^{\nu}(y^j-y^{j-1})\cdot\eta^j \quad (y^0=\eta^{\nu+1}=0)$$

and $d\tilde{y}^\nu=dy^1\cdots dy^\nu$, $đ\tilde{\eta}^\nu=đ\eta^1\cdots đ\eta^\nu$ for $\tilde{y}^\nu=(y^1, \cdots, y^\nu)$, $\tilde{\eta}^\nu=(\eta^1, \cdots, \eta^\nu)$. For $j\leqq\nu$ we set

$$(5.4)\quad p_j'(x, \xi, x')=(1+|x-x'|^2)^{-(n+1)}(1+i(x-x')\cdot\nabla_\xi)^{n+1}p_j^0(x, \xi).$$

Then, from Lemma 1.5 of [23] we have

$$Q_{\nu+1}=p_1'(X, D_x, X')\cdots p_\nu'(X, D_x, X')p_{\nu+1}(X, D_x),$$

which implies

$$(5.5)\qquad q_{\nu+1}(x, \xi)=O_s\text{-}\iint e^{-i\phi}\prod_{j=1}^{\nu}p_j'(x+y^{j-1}, \xi+\eta^j, x+y^j)\times p_{\nu+1}^0(x+y^\nu, \xi)\,d\tilde{y}^\nu đ\tilde{\eta}^\nu.$$

From (5. 4) we also have

$$(5.6)\qquad |p'^{(\alpha)}_{j(\beta,\beta')}(x, \xi, x')|\leqq C_o A_1 M^{-(|\alpha|+|\beta|+|\beta'|)}\alpha!^d\beta!^d\beta'!\times\langle\xi\rangle^{\sigma-|\alpha|}(1+|x-x'|)^{-(n+1)}$$

with new constant A_1 and M independent of j, α, β and β'. Here and in what follows the constant C_o denotes the one in (2.15). For $\chi\in\gamma^d$ satisfying (4.3) we set

$$(5.7)\qquad \chi_0(\xi, \xi')=\chi((\xi-\xi')/\langle\xi'\rangle),\quad \chi_1(\xi, \xi')=1-\chi_0(\xi, \xi').$$

Setting

$$K_\nu=\{\kappa=(k_1, \cdots, k_\nu)\ ;\ k_j=0, 1\},$$

we divide $q_{\nu+1}(x, \xi)$ in (5.5) into 2^ν terms:

$$(5.8)\qquad q_{\nu+1}(x, \xi)=\sum_{\kappa\in K_\nu}q_{\nu+1,(\kappa),L}(x, \xi),$$

where $q_{\nu+1,(\kappa),L}(x, \xi)$ is a simplified symbol of

$$(5.9)\quad q_{\nu+1,(\kappa)}(x^0, \tilde{x}^\nu, \tilde{\xi}^{\nu+1})=\prod_{j=1}^{\nu}\chi_{k_j}(\xi^j, \xi^{\nu+1})\prod_{j=1}^{\nu}p_j'(x^{j-1}, \xi^j, x^j)p_{\nu+1}^0(x^\nu, \xi^{\nu+1})\quad (\tilde{x}^\nu=(x^1, \cdots, x^\nu),\ \tilde{\xi}^{\nu+1}=(\xi^1, \cdots, \xi^{\nu+1})).$$

(II) First, we consider $q_{\nu+1,(\kappa^0),L}(x, \xi)$ for $\kappa^0=(0, 0, \cdots, 0)\in K_\nu$. On $\mathrm{supp}\,q_{\nu+1,(\kappa^0)}(x^0, \tilde{x}^\nu, \tilde{\xi}^{\nu+1})$ the relations $(1/2)\langle\xi^{\nu+1}\rangle\leqq\langle\xi^j\rangle\leqq 2\langle\xi^{\nu+1}\rangle$ hold for any j. Hence, $q_{\nu+1,(\kappa^0)}(x^0, \tilde{x}^\nu, \tilde{\xi}^{\nu+1})$ satisfies

$$(5.10)\quad |\partial_{\xi^1}^{\alpha^1}\cdots\partial_{\xi^{\nu+1}}^{\alpha^{\nu+1}}D_{x^0}^{\beta^0}D_{x^1}^{\beta^1}\cdots D_{x^\nu}^{\beta^\nu}q_{\nu+1,(\kappa^0)}(x^0,\ \tilde{x}^\nu,\ \tilde{\xi}^{\nu+1})| \leqq C_o^{\nu+1}(2^{n+\sigma}A_1)^\nu M^{-(|\tilde{\alpha}^{\nu+1}|+|\beta^0|+|\tilde{\beta}^\nu|)}\tilde{\alpha}^{\nu+1}!^d\beta^0!^d\tilde{\beta}^\nu!^d \times\langle\xi^{\nu+1}\rangle^{(\nu+1)\sigma-|\tilde{\alpha}^{\nu+1}|}\prod_{j=1}^{\nu}(1+|x^{j-1}-x^j|)^{-(n+1)}$$

with a constant M replaced by a smaller constant M, where $|\tilde{\alpha}^{\nu+1}| = |\alpha^1|+\cdots+|\alpha^{\nu+1}|$, $|\tilde{\beta}^\nu| = |\beta^1|+\cdots+|\beta^\nu|$, $\tilde{\alpha}^{\nu+1}! = \alpha^1!\cdots\alpha^{\nu+1}!$ and $\tilde{\beta}^\nu! = \beta^1!\cdots\beta^\nu!$ for $\tilde{\alpha}^{\nu+1}=(\alpha^1,\ \cdots,\ \alpha^{\nu+1})$, $\tilde{\beta}^\nu=(\beta^1,\ \cdots,\ \beta^\nu)$. Differentiating $q_{\nu+1,(\kappa^0),L}(x,\ \xi)$ with respect to x and ξ we have

$$(5.11)\quad \partial_\xi^\alpha D_x^\beta q_{\nu+1,(\kappa^0),L}(x,\ \xi) = \sum_{\alpha,\beta,\nu+1}\frac{\alpha!\beta!}{\tilde{\alpha}^{\nu+1}!\beta^0!\tilde{\beta}^\nu!}q_{\nu+1,(\kappa^0,\tilde{\alpha}^{\nu+1},\beta^0,\tilde{\beta}^\nu),L}(x,\ \xi)\langle\xi\rangle^{(\nu+1)\sigma-|\alpha|}$$

for

$$(5.12)\quad q_{\nu+1,(\kappa^0,\tilde{\alpha}^{\nu+1},\beta^0,\tilde{\beta}^\nu),L}(x,\ \xi) = O_s\text{-}\iint e^{-i\phi}\partial_{\xi^1}^{\alpha^1}\cdots\partial_{\xi^{\nu+1}}^{\alpha^{\nu+1}}D_{x^0}^{\beta^0}D_{x^1}^{\beta^1}\cdots D_{x^\nu}^{\beta^\nu}q_{\nu+1,(\kappa^0)}(x,\ \xi+\eta^1,\ x+y^1,\ \cdots,\ \xi+\eta^\nu,\ x+y^\nu,\ \xi)\langle\xi\rangle^{-(\nu+1)\sigma+|\alpha|}d\tilde{y}^\nu d\tilde{\eta}^\nu.$$

Here, in (5.11) the summation is taken over all $\tilde{\alpha}^{\nu+1}=(\alpha^1,\ \cdots,\ \alpha^{\nu+1})$ and $(\beta^0,\ \tilde{\beta}^\nu)=(\beta^0,\ \beta^1,\ \cdots,\ \beta^\nu)$ satisfying $\alpha^1+\cdots+\alpha^{\nu+1}=\alpha$, $\beta^0+\beta^1+\cdots+\beta^\nu=\beta$. Take one of symbols in (5.11) and denote it simply by

$$r_L(x,\ \xi) = O_s\text{-}\iint e^{-i\phi}r(x,\ \xi+\eta^1,\ x+y^1,\ \cdots,\ \xi+\eta^\nu,\ x+y^\nu,\ \xi)d\tilde{y}^\nu d\tilde{\eta}^\nu,$$

where

$$r(x^0,\ \tilde{x}^\nu,\ \tilde{\xi}^{\nu+1}) = \partial_{\xi^1}^{\alpha^1}\cdots\partial_{\xi^{\nu+1}}^{\alpha^{\nu+1}}D_{x^0}^{\beta^0}\cdots D_{x^\nu}^{\beta^\nu}q_{\nu+1,(\kappa^0)}(x^0,\ \tilde{x}^\nu,\ \tilde{\xi}^{\nu+1})\langle\xi^{\nu+1}\rangle^{-(\nu+1)\sigma+|\alpha|}.$$

From (5.10) $r(x^0,\ \tilde{x}^\nu,\ \tilde{\xi}^{\nu+1})$ satisfies

$$|D_{x^1}^{\gamma^1}\cdots D_{x^\nu}^{\gamma^\nu}r| \leqq \{C_o^{\nu+1}A_2^\nu M_2^{-(|\alpha|+|\beta|)}\tilde{\alpha}^{\nu+1}!^d\beta^0!^d\tilde{\beta}^\nu!^d\} \times\prod_{j=1}^{\nu}(1+|x^{j-1}-x^j|)^{-(n+1)} \quad \text{for } |\gamma^j|\leqq n+1 \quad (j=1,\ \cdots,\ \nu)$$

for constants A_2 and M_2 independent of α, β and ν. This means that $r(x^0,\ \tilde{x}^\nu,\ \tilde{\xi}^{\nu+1})$ satisfies (1.32) in [23] with $B=C_o^{\nu+1}A_2^\nu M_2^{-(|\alpha|+|\beta|)}\tilde{\alpha}^{\nu+1}!^d \times\beta^0!^d\tilde{\beta}^\nu!^d$, $\delta=0$ and $m_j=0$, $j=1,\ 2,\ \cdots$. Hence, applying Proposition 1.7 in [23] we have

$$|r_L(x,\ \xi)| \leqq A_O{}^\nu B,$$

that is,

$$|q_{\nu+1,(\kappa^0,\tilde{\alpha}^{\nu+1},\beta^0,\tilde{\beta}^\nu),L}(x,\ \xi)| \\ \leqq C_o^{\nu+1}A_O{}^\nu A_2{}^\nu M_2{}^{-(|\alpha|+|\beta|)}\tilde{\alpha}^{\nu+1}!^d\beta^0!^d\tilde{\beta}^\nu!^d.$$

Combining this with (5.11) we get

$$\begin{aligned}&|\partial_\xi^\alpha D_x^\beta q_{\nu+1,(\kappa^0),L}(x,\ \xi)| \\ &\leqq \textstyle\sum_{\alpha,\beta,\nu+1}\dfrac{\alpha!\beta!}{\tilde{\alpha}^{\nu+1}!\beta^0!\tilde{\beta}^\nu!}C_o^{\nu+1}A_O{}^\nu A_2{}^\nu M_2{}^{-(|\alpha|+|\beta|)} \\ &\qquad\times\tilde{\alpha}^{\nu+1}!^d\beta^0!^d\tilde{\beta}^\nu!^d\langle\xi\rangle^{(\nu+1)\sigma-|\alpha|} \\ &\leqq (A_OA_2)^\nu C_o^{\nu+1}(M_2/2)^{-(|\alpha|+|\beta|)}\alpha!^d\beta!^d \\ &\qquad\times\left\{\textstyle\sum_{\alpha,\beta,\nu+1}\left(\dfrac{1}{2}\right)^{\alpha^1}\cdots\left(\dfrac{1}{2}\right)^{\alpha^{\nu+1}}\left(\dfrac{1}{2}\right)^{\beta^0}\cdots\left(\dfrac{1}{2}\right)^{\beta^\nu}\right\} \\ &\qquad\times\langle\xi\rangle^{(\nu+1)\sigma-|\alpha|} \\ &\leqq (2^{2n}A_OA_2)^\nu C_o^{\nu+1}2^{2n}(M_2/2)^{-(|\alpha|+|\beta|)}\alpha!^d\beta!^d\langle\xi\rangle^{(\nu+1)\sigma-|\alpha|}.\end{aligned}$$

Consequently, if we set $q_{\nu+1}^0(x,\xi)=q_{\nu+1,(\kappa^0),L}(x,\ \xi)$, it satisfies (2.17).

(III) We will prove for $\kappa\neq\kappa^0$

$$(5.13)\qquad |\partial_\xi^\alpha D_x^\beta q_{\nu+1,(\kappa),L}(x,\ \xi)| \\ \leqq A^\nu C_o^{\nu+1}C'_\alpha M_1{}^{-(|\beta|+N)}\beta!^d(N+[\nu\sigma])!^d\langle\xi\rangle^{-|\alpha|-N}\quad\text{for any } N$$

with constants A, C'_α and M_1 satisfying the same conditions as in Theorem 2.6. Then, setting $\tilde{q}_{\nu+1}(x,\ \xi)=\sum_{\kappa\in K_\nu,\kappa\neq\kappa^0}\tilde{q}_{\nu+1,(\kappa),L}(x,\ \xi)$ we obtain the desired symbols $q_{\nu+1}^0(x,\ \xi)$ and $\tilde{q}_{\nu+1}(x,\ \xi)$.

In the following we fix $\kappa=(k_1,\ \cdots,\ k_\nu)\in K_\nu$ $(\kappa\neq\kappa^0)$ and prove (5.13). We change the variables in

$$q_{\nu+1,(\kappa),L}(x,\ \xi)=O_s\text{-}\iint\exp\{-i\sum_{j=1}^{\nu}(y^j-y^{j-1})\cdot\eta^j\} \\ \times q_{\nu+1,(\kappa)}(x,\ \xi+\eta^1,\ x+y^1,\ \cdots,\ \xi+\eta^\nu,\ x+y^\nu,\ \xi)\,d\tilde{y}^\nu d\tilde{\eta}^\nu\quad(y^0=0)$$

as $z^j=y^j-y^{j-1}$ $(j=1,\ \cdots,\ \nu;\ y^0=0)$. Then, we have

$$q_{\nu+1,(\kappa),L}(x,\ \xi)=\iint\exp\{-i\sum_{j=1}^{\nu}z^j\cdot\eta^j\} \\ \times q_{\nu+1,(\kappa)}(x,\ \xi+\eta^1,\ x+\bar{z}^1,\ \cdots,\ \xi+\eta^\nu,\ x+\bar{z}^\nu,\ \xi)\,d\tilde{z}^\nu d\tilde{\eta}^\nu,$$

where $\bar{z}^j=z^1+\cdots+z^j$ and $d\tilde{z}^\nu=dz^1\cdots dz^\nu$. Take a sequence $\{m_j\}_{j=1}^\infty$ and a sequence $\{\mu_j\}_{j=1}^\infty$ of non-negative integers μ_j satisfying

$$(5.14)\qquad \begin{cases} m_j+\mu_j=\sigma, & j=1,\ 2,\ \cdots,\ \nu+1,\\ |\sum_{j'=1}^{j}m_{j'}|\leqq\dfrac{1}{2} & \text{for any } j.\end{cases}$$

Set $J^0=\{j;\ k_j=0\}\cup\{\nu+1\}$, $J^1=\{j\,;\ k_j=1\}$, $j^0=\max\{j\,;\ j\in J^1\}$ and $l=\sum_{j\in J^0}\mu_j+[\bar{m}_{\nu+1}+1]$. Then, using the integration by parts we have

$$
\begin{aligned}
q_{\nu+1,(\kappa),L}(x,\ \xi) &= \iint \exp\{-i\sum_{j=1}^{\nu} z^j\cdot\eta^j\} \prod_{j\in J^1}\{-i\,|\eta^j|^{-2}\eta^j\cdot\nabla_{z^j}\}^{\mu_j} \\
&\qquad \cdot\{-i\,|\eta^{j^0}|^{-2}\eta^{j^0}\cdot\nabla_{z^{j^0}}\}^{l+N} q_{\nu+1,(\kappa)}(x, \\
&\qquad \xi+\eta^1,\ x+\bar{z}^1,\ \cdots,\ \xi+\eta^\nu,\ x+\bar{z}^\nu,\ \xi)\, d\bar{z}^\nu d\bar{\eta}^\nu \\
&\equiv \iint \exp\{-i\sum_{j=1}^{\nu} z^j\cdot\eta^j\}\, q'_{\nu+1,(\kappa)}(x,\ \xi+\eta^1, \\
&\qquad x+\bar{z}^1,\ \cdots,\ \xi+\eta^\nu,\ x+\bar{z}^\nu,\ \xi)\, d\bar{z}^\nu d\bar{\eta}^\nu \\
&\qquad \times\langle\xi\rangle^{-N-\bar{m}_{\nu+1}}
\end{aligned}
$$

where $\bar{m}_{\nu+1}=m_1+\cdots+m_{\nu+1}$ and

$$
\begin{aligned}
& q'_{\nu+1,(\kappa)}(x^0,\ \tilde{x}^\nu,\ \tilde{\xi}^{\nu+1}) \\
&= \prod_{j\in J^1}\{-i\,|\xi^j-\xi^{\nu+1}|^{-2}(\xi^j-\xi^{\nu+1})\cdot(\nabla_{x^j}+\cdots+\nabla_{x^\nu})\}^{\mu_j} \\
&\quad \cdot\{-i\,|\xi^{j^0}-\xi^{\nu+1}|^{-2}(\xi^{j^0}-\xi^{\nu+1})\cdot(\nabla_{x^{j^0}}+\cdots+\nabla_{x^\nu})^{l+N} \\
&\quad \cdot q_{\nu+1,(\kappa)}(x^0,\ \tilde{x}^\nu,\ \tilde{\xi}^{\nu+1})\langle\xi^{\nu+1}\rangle^{N+\bar{m}_{\nu+1}}.
\end{aligned}
$$

Note that on $\operatorname{supp} q'_{\nu+1,(\kappa)}(x^0,\ \tilde{x}^\nu,\ \tilde{\xi}^{\nu+1})$ we have

$$
\begin{cases}
\dfrac{1}{2}\langle\xi^{\nu+1}\rangle\leqq\langle\xi^j\rangle\leqq 2\langle\xi^{\nu+1}\rangle & \text{if } j\in J^0, \\[2mm]
|\xi^j-\xi^{\nu+1}|\geqq\dfrac{2}{5}\langle\xi^{\nu+1}\rangle & \text{if } j\in J^1.
\end{cases}
$$

Hence from (5.9) and (5.6) we have for $|\gamma^j|\leqq n+1$ $(j=1,\ \cdots,\ \nu)$

$$
\begin{aligned}
& |\partial_{x^1}^{\gamma^1}\cdots\partial_{x^\nu}^{\gamma^\nu} q'_{\nu+1,(\kappa)}(x^0,\ \tilde{x}^\nu,\ \tilde{\xi}^{\nu+1})| \\
&\leqq C_o^{\nu+1}A_2^{\nu}M_2^{-(\nu+N)}(\sum_{j\in J^1}\mu_j+l+N)!^d \\
&\qquad \times\prod_{j=1}^{\nu+1}\langle\xi^j\rangle^{m_j}\prod_{j\in J^1}\{\langle\xi^j\rangle\,|\xi^j-\xi^{\nu+1}|^{-1}\}^{\mu_j} \\
&\qquad \times\prod_{j\in J^0}\langle\xi^j\rangle^{\mu_j}\langle\xi^{\nu+1}\rangle^{N+\bar{m}_{\nu+1}}|\xi^{j^0}-\xi^{\nu+1}|^{-l-N} \\
&\qquad \times\prod_{j=1}^{\nu}(1+|x^{j-1}-x^j|)^{-(n+1)} \\
&\leqq C_o^{\nu+1}A_2^{\nu}M_2^{-(\nu+N)}5^{(\nu+1)\sigma+N+1}(N+[(\nu+1)\sigma]+1)!^d\prod_{j=1}^{\nu+1}\langle\xi^j\rangle^{m_j} \\
&\qquad \times\prod_{j=1}^{\nu}(1+|x^{j-1}-x^j|)^{-(n+1)} \\
&\leqq C_o^{\nu+1}A_3^{\nu}C'_0M_2^{-N}5^N2^{dN}(N+[\nu\sigma])!^d\prod_{j=1}^{\nu+1}\langle\xi^j\rangle^{m_j} \\
&\qquad \times\prod_{j=1}^{\nu}(1+|x^{j-1}-x^j|)^{-(n+1)}
\end{aligned}
$$

with constants A_2, A_3, C'_0 and M_2 independent of ν and N. This means that $q'_{\nu+1,(\kappa)}(x^0, \tilde{x}^\nu, \tilde{\xi}^{\nu+1})$ satisfies (1.32) in [23] with $B=A_3^\nu \times C_o^{\nu+1}C'_0M_2^{-N}5^N2^{dN}(N+[\nu\sigma])!^d$, $\delta=0$ and a sequence $\{m_j\}$ defined by (5.14). Hence applying Proposition 1.7 in [23] we obtain

$$|q_{\nu+1,(\kappa),L}(x, \xi)| \leqq \{A_O{}^\nu A_3{}^\nu C_o{}^{\nu+1}C'_0M_2{}^{-N}5^N2^{dN}(N+[\nu\sigma])!^d\langle\xi\rangle^{\bar{m}_{\nu+1}}\}\langle\xi\rangle^{-N-\bar{m}_{\nu+1}}.$$

Here, we remark that in [23] we have used only the condition "$|\sum_{j=1}^{k} m_j|\leqq\frac{1}{2}$ for any k" for the proof of Proposition 1.7 in [23] instead of using a stronger condition (1.12) in [23]. So, we obtain (5.13) with constants $A=A_OA_3$ and $M_1=2^{-d}M_2/5$ for the case $\alpha=\beta=0$. Similarly we obtain (5.13) for the other cases. This proves Proposition 5.1. Q. E. D.

Next, we consider a multi-product

$$p_1(X, D_x)p_2(X, D_x)\cdots p_{\nu+1}(X, D_x) \tag{5.15}$$

of pseudo-differential operators $p_j(X, D_x)$, assuming that at least one factor $p_l(X, D_x)$ belongs to $\mathscr{R}_{G(d)}$.

Proposition 5.2. *In* (5.15) *we assume that*

$$|p_{j(\beta)}^{(\alpha)}(x, \xi)|\leqq C_\alpha M^{-|\beta|}\beta!^d\langle\xi\rangle^{\sigma-|\alpha|} \tag{5.16}$$

and there exists a number $l\in\{1, 2, \cdots, \nu+1\}$ such that $p_l(x, \xi)$ satisfies

$$|p_{l(\beta)}^{(\alpha)}(x, \xi)|\leqq C_\alpha M^{-(|\beta|+N)}\beta!^dN!^d\langle\xi\rangle^{-|\alpha|-N} \quad \textit{for any } N, \tag{5.17}$$

where M is independent of α, β, j, N and C_α is independent of β, j, N. Set $\tilde{C}_o=\max_{|\alpha|\leqq n+1} C_\alpha$. Then, the symbol $\tilde{q}_{\nu+1}(x, \xi)$ of multi-product (5.15) *satisfies* (2.18).

Proof. As in (5.4) we set for $j(\leqq\nu)$

$$p'_j(x, \xi, x')=(1+|x-x'|^2)^{-(n+1)}(1+i(x-x')\cdot\nabla_\xi)^{n+1}p_j(x, \xi).$$

Then, we have

$$\begin{aligned}\tilde{q}_{\nu+1}(x, \xi)&=\iint e^{-i\phi}\prod_{j=1}^{\nu}p'_j(x+y^{j-1}, \xi+\eta^j, x+y^j)\\&\qquad\times p_{\nu+1}(x+y^\nu, \xi)\,d\tilde{y}^\nu d\tilde{\eta}^\nu\\&=\sum_{\kappa\in K_\nu}\tilde{q}_{\nu+1,(\kappa),L}(x, \xi)\qquad(y^0=0)\end{aligned}$$

for

$$\tilde{q}_{\nu+1,(\kappa),L}(x,\ \xi)=\iint e^{-i\psi}\prod_{j=1}^{\nu}\chi_{k_j}(\xi+\eta^j,\ \xi)$$
$$\times\prod_{j=1}^{\nu}p'_j(x+y^{j-1},\ \xi+\eta^j,\ x+y^j)$$
$$\times p_{\nu+1}(x+y^\nu,\ \xi)\,d\tilde{y}^\nu đ\eta^\nu.$$

Here, ψ is a function in (5.3) and $\chi_k(\xi,\ \xi')$, $k=0,\ 1$, are defined by (5.7). Note that the following holds: There exists a constant A_1 independent of j, α, β, β' and N such that we have for constants C''_α with $C''_0=1$

$$|p'^{(\alpha)}_{j(\beta,\beta')}(x,\ \xi,\ x')|\leqq A_1\tilde{C}_oC''_\alpha M^{-(|\beta|+|\beta'|)}\beta!^d\beta'!\langle\xi\rangle^{\sigma-|\alpha|}$$
$$\times(1+|x-x'|)^{-(n+1)}$$

and

$$|p'^{(\alpha)}_{l(\beta,\beta')}(x,\ \xi,\ x')|\leqq A_1\tilde{C}_oC''_\alpha M^{-(|\beta|+|\beta'|+N)}\beta!^d\beta'!N!^d\langle\xi\rangle^{-|\alpha|-N}$$
$$\times(1+|x-x'|)^{-(n+1)}$$

if $p_l(x,\ \xi)$ satisfies (5.17). Hence, by the same way as in (III) in the proof of the preceding proposition we obtain

$$(5.18)\qquad |\partial^\alpha_\xi D^\beta_x\tilde{q}_{\nu+1,(\kappa),L}(x,\ \xi)|$$
$$\leqq A^\nu\tilde{C}_o^{\nu+1}C'_\alpha M_1^{-(|\beta|+N)}\beta!^d(N+[\nu\sigma])!^d\langle\xi\rangle^{-|\alpha|-N}\quad\text{for any } N$$

for $\kappa\neq\kappa^0$. It is clear that $\tilde{q}_{\nu+1,(\kappa^0),L}(x,\ \xi)$ also satisfies (5.18). This shows that the symbol $\tilde{q}_{\nu+1}(x,\ \xi)$ of (5.15) satisfies (2.18).

Q. E. D.

Now, we prove Theorem 2.6.

Proof of Theorem 2.6. Assume that $P_j=p^0_j(X,\ D_x)+\tilde{p}_j(X,\ D_x)$ for $p^0_j(x,\ \xi)$ and $\tilde{p}_j(x,\ \xi)$ satisfying (2.15) and (2.16). Write $Q_{\nu+1}=P_1P_2\cdots P_{\nu+1}$ as

$$(5.19)\qquad \begin{aligned}Q_{\nu+1}=&P^0_1P^0_2\cdots P^0_{\nu+1}\\&+\tilde{P}_1P^0_2\cdots P^0_{\nu+1}\\&+P_1\tilde{P}_2P^0_3\cdots P^0_{\nu+1}\\&+\cdots\cdots\\&+P_1P_2\cdots P_{\nu-1}\tilde{P}_\nu P^0_{\nu+1}\\&+P_1P_2\cdots P_\nu\tilde{P}_{\nu+1},\end{aligned}$$

where $P^0_j=p^0_j(X,\ D_x)$ and $\tilde{P}_j=\tilde{p}_j(X,\ D_x)$. Note that $p_j(x,\ \xi)\,(=\sigma(P_j))=p^0_j(x,\ \xi)+\tilde{p}_j(x,\ \xi)$ satisfies (5.16). Hence, applying Proposition 5.1 to the first term in (5.19) and Proposition 5.2 to the other terms in

(5.19), we obtain $Q_{\nu+1}=q^0_{\nu+1}(X, D_x)+\tilde{q}_{\nu+1}(X, D_x)$ for the symbols $q^0_{\nu+1}(x, \xi)$ and $\tilde{q}_{\nu+1}(x, \xi)$ satisfying (2.17) and (2.18). This concludes the proof. Q. E. D.

§ 6. Perfect Diagonalization

In this section we give a proof of Proposition 3.4 by showing a method of the perfect diagonalization for a first order hyperbolic system with constant multiplicity. We will begin with giving the product formula for pseudo-differential operators with symbols in $S^m_{G(d,1)}$.

Proposition 6.1. *Let $p_j(x, \xi)$ belong to $S^{m_j}_{G(d,1)}$ $(j=1, 2)$. Then, there exist symbols $q^0(x, \xi)$ in $S^{m_1+m_2}_{G(d,1)}$ and $\tilde{q}(x, \xi)$ in $\mathscr{R}_{G(d)}$ such that for $P_j=p_j(X, D_x)$ the equation*

$$P_1P_2=q^0(X, D_x)+\tilde{q}(X, D_x) \tag{6.1}$$

holds. Moreover, there exist constants C, M and μ such that

$$\begin{aligned}|\partial^\alpha_\xi D^\beta_x(q^0(x, \xi)-\sum_{|\gamma|<N}\frac{1}{\gamma!}p_1^{(\gamma)}(x, \xi)p_{2(\gamma)}(x, \xi))| \\ \leqq CM^{-(|\alpha|+|\beta|+N)}\alpha!\beta!^dN!^d\langle\xi\rangle^{m_1+m_2-N-|\alpha|} \\ \text{for } |\xi|\geqq\mu\end{aligned} \tag{6.2}$$

hold for any N.

We will prove Proposition 6.1 after some preparations. Let μ^0 be a constant satisfying for $j=1, 2$

$$|p_{j(\beta)}^{(\alpha)}(x, \xi)|\leqq CM^{-(|\alpha|+|\beta|)}\alpha!\beta!^d\langle\xi\rangle^{m_j-|\alpha|} \quad \text{for } |\xi|\geqq\mu^0. \tag{6.3}$$

Define

$$q_\nu(x, \xi)=\sum_{|\gamma|=\nu}\frac{1}{\gamma!}p_1^{(\gamma)}(x, \xi)p_{2(\gamma)}(x, \xi). \tag{6.4}$$

Then, $q_\nu(x, \xi)$ satisfy for new constants C and M independent of α, β and ν

$$|q_{\nu(\beta)}^{(\alpha)}(x, \xi)|\leqq CM^{-(|\alpha|+|\beta|+\nu)}\alpha!\beta!^d\nu!^d\langle\xi\rangle^{m_1+m_2-\nu-|\alpha|} \quad \text{for } |\xi|\geqq\mu^0. \tag{6.5}$$

Taking account of (6.5) we investigate the following lemma.

Lemma 6.2. *Assume that* $q_\nu(x, \xi) \in S^{m-\nu}_{G(d,1)}$ $(\nu=0, 1, \cdots)$ *satisfy*

$$(6.6) \qquad |q^{(\alpha)}_{\nu(\beta)}(x, \xi)| \leqq CM^{-(|\alpha|+|\beta|+\nu)}\alpha!\beta!^d\nu!^d\langle\xi\rangle^{m-\nu-|\alpha|} \quad for \quad |\xi| \geqq \mu$$

with constants C, M *and* μ *independent of* α, β *and* ν. *Then, there exists a symbol* $q(x, \xi)$ *in* $S^m_{G(d,1)}$ *such that the inequalities*

$$(6.7) \qquad |\partial^\alpha_\xi D^\beta_x(q(x, \xi) - \sum_{\nu=0}^{N-1} q_\nu(x, \xi))| \leqq C_1 M_1^{-(|\alpha|+|\beta|+N)}\alpha!\beta!^d N!^d\langle\xi\rangle^{m-N-|\alpha|} \quad for \quad |\xi| \geqq \mu+1$$

hold for any N, *where the constants* C_1 *and* M_1 *are independent of* α, β *and* N.

Proof. Let $\{s_\nu\}_{\nu=0}^\infty$ be a sequence of complex numbers satisfying

$$(6.8) \qquad \|\{s_\nu\}\|^2 \equiv \sum_\nu |s_\nu|^2 M_2^{\nu}\nu!^{-2(d+1)} < \infty$$

for a constant M_2. Then, from the discussions in [1], pp. 314–317, we can find a function $\psi(t)$ such that the inequalities

$$\begin{cases} |\partial^k_t\psi(t)| \leqq C\|\{s_\nu\}\|M_3^{-k}k!\,|t|^{-k} \quad (t\neq 0), \\ \left|\partial^k_t\left(\psi(t) - \sum_{\nu=0}^{N-1}\frac{t^\nu}{\nu!}s_\nu\right)\right| \leqq C\|\{s_\nu\}\|M_3^{-(k+N)}k!N!^d\,|t|^{N-k} \quad (t\neq 0) \end{cases}$$

hold with constants C and M_3 independent of k, N and $\{s_\nu\}$. Apply this result to a sequence

$$s_\nu(x, \xi) = q_\nu(x, \xi)\langle\xi\rangle^\nu\nu!$$

with parameters x and ξ. Note that from (6.6) we have for new constants C and M

$$\|\{\partial^\alpha_\xi D^\beta_x s_\nu(x, \xi)\}\| \leqq CM^{-(|\alpha|+|\beta|)}\alpha!\beta!^d\langle\xi\rangle^{m-|\alpha|} \quad \text{for} \quad |\xi| \geqq \mu$$

if we take an appropriate constant M_2 in (6.8). Then, we can find a function $\psi(t; x, \xi)$ satisfying

$$(6.9) \qquad |\partial^k_t\partial^\alpha_\xi D^\beta_x\psi(t; x, \xi)| \leqq CM_4^{-(k+|\alpha|+|\beta|)}k!\alpha!\beta!^d\langle\xi\rangle^{m-|\alpha|}\,|t|^{-k} \quad \text{for} \quad |\xi| \geqq \mu,\ t\neq 0,$$

$$(6.10) \qquad |\partial^k_t\partial^\alpha_\xi D^\beta_x\{\psi(t; x, \xi) - \sum_{\nu=0}^{N-1}\frac{t^\nu}{\nu!}s_\nu(x, \xi)\}| \leqq CM_4^{-(k+|\alpha|+|\beta|+N)}k!\alpha!\beta!^d N!^d\langle\xi\rangle^{m-|\alpha|}\,|t|^{N-k} \quad \text{for} \quad |\xi| \geqq \mu, \quad t\neq 0.$$

Take a function $\chi_\mu(\xi)$ in γ^d satisfying $\chi_\mu=1$ for $|\xi| \geqq \mu+1$ and $\chi_\mu=0$ for $|\xi| \leqq \mu$. Then, the desired symbol $q(x, \xi)$ is obtained by setting

$$q(x, \xi) = \psi(\langle\xi\rangle^{-1}; x, \xi)\chi_\mu(\xi),$$

since the property $q(x, \xi)\in S^{m}_{G(d,1)}$ follows from (6.9) and the property (6.7) follows from (6.10). Q. E. D.

Now, we prove Proposition 6.1.

Proof of Proposition 6.1. For a sequence $\{q_\nu(x, \xi)\}$ of symbols defined by (6.4) we apply Lemma 6.2. Then, we can find a symbol $q^0(x, \xi)$ in $S^{m_1+m_2}_{G(d,1)}$ satisfying (6.2). Note that for the symbol $q(x, \xi)$ defined by

$$q(x, \xi)=O_s\text{-}\iint e^{-iy\cdot\eta}p_1(x, \xi+\eta)p_2(x+y, \xi)dy\,đ\eta$$

we have

$$P_1P_2=q(X, D_x).$$

So, the proof is completed if we prove that the symbol

$$\tilde{q}(x, \xi)=q(x, \xi)-q^0(x, \xi)$$

belongs to $\mathscr{R}_{G(d)}$.

Let $\chi(\xi)$ be the function in γ^d satisfying (4.3). We write $\tilde{q}(x, \xi)$ as

$$\tilde{q}(x, \xi)=\tilde{r}(x, \xi)+\tilde{r}'(x, \xi), \tag{6.11}$$

where

$$\tilde{r}(x, \xi)=O_s\text{-}\iint e^{-iy\cdot\eta}p_1(x, \xi+\eta)\chi(\eta/\langle\xi\rangle)p_2(x+y, \xi)dy\,đ\eta-q^0(x, \xi), \tag{6.12}$$

$$\tilde{r}'(x, \xi)=O_s\text{-}\iint e^{-iy\cdot\eta}p_1(x, \xi+\eta)(1-\chi(\eta/\langle\xi\rangle))p_2(x+y, \eta)dy\,đ\eta. \tag{6.13}$$

It is easy to see

$$\tilde{r}'(x, \xi)\in\mathscr{R}_{G(d)} \tag{6.14}$$

from the similar discussions in the proof of Proposition 2.2 in §4. For the proof of

$$\tilde{r}(x, \xi)\in\mathscr{R}_{G(d)} \tag{6.15}$$

we fix an integer N and divide $\tilde{r}(x, \xi)$ into three terms

$$\begin{aligned}\tilde{r}(x, \xi)=\Big[&\sum_{|\gamma|\leq N}\frac{1}{\gamma!}p_1^{(\gamma)}(x, \xi)p_{2(\gamma)}(x, \xi)\\&+\sum_{|\gamma|<N}\sum_{|\delta|=1}\frac{1}{\gamma!}\int_0^1(1-\theta)^{|\gamma|}\Big\{\iint e^{-iy\cdot\eta}p_1^{(\gamma)}(x, \xi+\eta)\end{aligned} \tag{6.16}$$

$$\begin{aligned}&\times\chi^{(\delta)}(\eta/\langle\xi\rangle)\langle\xi\rangle^{-1}p_{2(\gamma+\delta)}(x+\theta y,\ \xi)\,dy\,d\!\!{}^-\eta\}\,d\theta\\ &+N\sum_{|\gamma|=N}\frac{1}{\gamma!}\int_0^1(1-\theta)^{N-1}\{\iint e^{-iy\cdot\eta}p_1{}^{(\gamma)}(x,\ \xi+\eta)\chi(\eta/\langle\xi\rangle)\\ &\qquad\times p_{2(\gamma)}(x+\theta y,\ \xi)\,dy\,d\!\!{}^-\eta\}\,d\theta]-q^0(x,\ \xi)\\ &\equiv\tilde{r}_N^1(x,\ \xi)+\tilde{r}_N^2(x,\ \xi)+\tilde{r}_N^3(x,\ \xi),\end{aligned}$$

where

$$\begin{aligned}\tilde{r}_N^1(x,\ \xi)&=\sum_{|\gamma|<N}\frac{1}{\gamma!}p_1{}^{(\gamma)}(x,\ \xi)p_{2(\gamma)}(x,\ \xi)-q^0(x,\ \xi),\\ \tilde{r}_N^2(x,\ \xi)&=\sum_{|\gamma|<N}\sum_{|\delta|=1}\frac{1}{\gamma!}\int_0^1(1-\theta)^{|\gamma|}\{\iint e^{-iy\cdot\eta}p_1{}^{(\gamma)}(x,\ \xi+\eta)\\ &\qquad\times\chi^{(\delta)}(\eta/\langle\xi\rangle)\langle\xi\rangle^{-1}p_{2(\gamma+\delta)}(x+\theta y,\ \xi)\,dy\,d\!\!{}^-\eta\}\,d\theta,\\ \tilde{r}_N^3(x,\ \xi)&=N\sum_{|\gamma|=N}\frac{1}{\gamma!}\int_0^1(1-\theta)^{N-1}\{\iint e^{-iy\cdot\eta}p_1{}^{(\gamma)}(x,\ \xi+\eta)\\ &\qquad\times\chi(\eta/\langle\xi\rangle)p_{2(\gamma)}(x+\theta y,\xi)\,dy\,d\!\!{}^-\eta\}\,d\theta.\end{aligned}$$

Then, from (6.2) we have

$$|\tilde{r}^{1\,(\alpha)}_{N\,(\beta)}(x,\ \xi)|\leqq CM^{-(|\alpha|+|\beta|+N)}\alpha!\beta!^dN!^d\langle\xi\rangle^{m_1+m_2-N-|\alpha|}\quad\text{for}\quad|\xi|\geqq\mu. \tag{6.17}$$

In the integrand of each term in $\tilde{r}_N^2(x,\ \xi)$ the inequality $(2/5)\langle\xi\rangle\leqq|\eta|\leqq\langle\xi\rangle/2$ holds. Hence, integrating by parts we have

$$\begin{aligned}\tilde{r}_N^2(x,\ \xi)=\sum_{|\gamma|<N}\sum_{|\delta|=1}\frac{1}{\gamma!}\int_0^1(1-\theta)^{|\gamma|}&\{\iint e^{-iy\cdot\eta}p_1{}^{(\gamma)}(x,\ \xi+\eta)\\ &\times\chi^{(\delta)}(\eta/\langle\xi\rangle)\langle\xi\rangle^{-1}\\ &\times(-i|\eta|^{-2}\eta\cdot\nabla_y)^{N-|\gamma|-1}p_{2(\gamma+\delta)}(x+\theta y,\ \xi)\cdot dy\,d\!\!{}^-\eta\}\,d\theta,\end{aligned}$$

which implies

$$|\tilde{r}^{2\,(\alpha)}_{N\,(\beta)}(x,\ \xi)|\leqq CM^{-(|\alpha|+|\beta|+N)}\alpha!^d\beta!^dN!^d\langle\xi\rangle^{m_1+m_2-N-|\alpha|}\quad\text{for}\quad|\xi|\geqq\mu' \tag{6.18}$$

if we take a constant $\mu'(\geqq\mu)$ satisfying

$$|\xi+\eta|\geqq\mu^0\quad\text{when}\quad|\xi|\geqq\mu'\ \text{and}\ |\eta|\leqq\langle\xi\rangle/2 \tag{6.19}$$

for the constant μ^0 in (6.3). Using (6.19) and (6.3) we also have

$$|\tilde{r}^{3\,(\alpha)}_{N\,(\beta)}(x,\ \xi)|\leqq CM^{-(|\alpha|+|\beta|+N)}\alpha!^d\beta!^dN!^d\langle\xi\rangle^{m_1+m_2-N-|\alpha|}\quad\text{for}\quad|\xi|\geqq\mu'. \tag{6.20}$$

Summing up (6.16), (6.17), (6.18) and (6.20) we obtain for $|\xi|\geqq\mu'$

$$|\tilde{r}^{(\alpha)}_{(\beta)}(x,\ \xi)|\leqq CM^{-(|\alpha|+|\beta|+N)}\alpha!^d\beta!^dN!^d\langle\xi\rangle^{m_1+m_2-N-|\alpha|}\quad\text{for any}\ N. \tag{6.21}$$

From the definition (6.12) the inequalities (6.21) hold also for $|\xi| \leqq \mu'$. This proves (6.15). Q. E. D.

Remark. We will write $q^0(x, \xi)$ satisfying (6.1) and (6.2) as $\sigma_0(P_1P_2)$.

Now, we turn to the *proof of Proposition* 3.4. In the following we use the symbol class $\Gamma_t^d(S^m_{G(d,1)})$ defined by the following: We say that a symbol $p(t, x, \xi)$ belongs to $\Gamma_t^d(S^m_{G(d,1)})$ if $p(t, x, \xi)$ belongs to $\bigcap_l M_t^l(S^m_{G(d,1)})$ and satisfies

$$|\partial_t^k p^{(\alpha)}_{(\beta)}(t, x, \xi)| \leqq CM^{-(k+|\alpha|+|\beta|)} k!^d \alpha! \beta!^d \langle\xi\rangle^{m-|\alpha|}$$
$$\text{for } (t, x, \xi) \in [0, T] \times R^{2n}_{x,\xi}, \ |\xi| \geqq \mu$$

with constants C, M and μ independent of k, α and β. Since the operator L of (3.15) is of constant multiplicity, in view of (3.19) we may assume that its distinct characteristic roots $\lambda_1(t, x, \xi), \cdots, \lambda_h(t, x, \xi)$ satisfy $\lambda_j(t, x, \xi) \in \Gamma_t^d(S^1_{G(d,1)})$ and

$$(6.22) \qquad |\lambda_j(t, x, \xi) - \lambda_k(t, x, \xi)| \geqq C\langle\xi\rangle \quad (C>0,\ j \neq k)$$

by modifying them in $[0, T] \times R^n_x \times \{|\xi| \leqq 1\}$. Assume (3.16). Then, using Proposition 6.1 and the discussions in the proof of Proposition 3.3 we can reduce the problem (3.14) to the problem (3.13) with

$$(6.23) \qquad \mathscr{L} = D_t - \mathscr{D}(t) + \left(B_{jk}(t) \ \begin{matrix} j \downarrow 1, \cdots, h \\ k \rightarrow 1, \cdots, h \end{matrix}\right) + R(t),$$

where

$$(6.24) \qquad \mathscr{D}(t) = \begin{pmatrix} \lambda_1(t, X, D_x)\mathscr{I}_{l_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_h(t, X, D_x)\mathscr{I}_{l_h} \end{pmatrix}$$
$$(l_1 + \cdots + l_h = l),$$

$B_{jk}(t)$ are $l_j \times l_k$ matrices of pseudo-differential operators with symbols in $\Gamma_t^d(S^\sigma_{G(d,1)})$, $\sigma = (r-q)/r$, and $R(t)$ is a regularizer in $\mathscr{R}_{G(d)}$, which means that $R(t)$ is an $l \times l$ matrix of pseudo-differential operators with symbols in $M_t(\mathscr{R}_{G(d)})$. So, the proof of Proposition 3.4 is completed if we apply the following theorem.

Theorem 6.3. *Let $\mathscr{L}$ be a hyperbolic operator of the form* (6.23), *where $\mathscr{D}(t)$ is defined by* (6.24) *with real symbols $\lambda_j(t, x, \xi)$ in $\Gamma_t^d(S_{G(d,1)}^1)$ satisfying* (6.22), *$B_{jk}(t)$ are $l_j \times l_k$ matrices of pseudo-differential operators with symbols in $\Gamma_t^d(S_{G(d,1)}^\sigma)$ $(0 \leqq \sigma \leqq 1/d)$ and $R(t)$ is a reqularizer in $\mathscr{R}_{G(d)}$. Then, there exists a matrix $P(t)$ of pseudo-differential operators with symbols in $\Gamma_t^d(S_{G(d,1)}^{\sigma-1})$ such that*

$$\mathscr{L}(\mathscr{I}+P(t))=(\mathscr{I}+P(t))\mathscr{L}_o \tag{6.25}$$

holds for a perfectly diagonalized operator

$$\mathscr{L}_o=D_t-\mathscr{D}(t)+\begin{pmatrix} B_1(t) & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & B_h(t) \end{pmatrix}+R_o(t) \tag{6.26}$$

and the inverse $Q(t)$ of $\mathscr{I}+P(t)$ exists with $Q(t) \in L_{G(d)}^0$ for any t, where in (6.26) *$B_j(t)$ are $l_j \times l_j$ matrices of pseudo-differential operators with symbols in $\Gamma_t^d(S_{G(d,1)}^\sigma)$ and $R_o(t)$ is a regularizer in $\mathscr{R}_{G(d)}$.*

For the proof we will give two lemmas.

Lemma 6.4. *Let $\mathscr{L}$ be a hyperbolic operator of the form* (6.23). *Then, under the assumptions in Theorem* 6.3 *there exists a matrix $P^1(t)$ of pseudo-differential operators with symbols in $\Gamma_t^d(S_{G(d,1)}^{\sigma-1})$ such that for a matrix $B^1(t)$ with $\sigma(B^1(t)) \in \Gamma_t^d(S_{G(d,1)}^0)$ and a regularizer $R^1(t)$ in $\mathscr{R}_{G(d)}$*

$$\mathscr{L}(\mathscr{I}+P^1(t))=(\mathscr{I}+P^1(t))\mathscr{L}_1+B^1(t)+R^1(t) \tag{6.27}$$

holds with $\mathscr{L}_1$ of the form

$$\mathscr{L}_1=D_t-\mathscr{D}(t)+F^o(t),$$

where

$$F^o(t)=\mathrm{diag}[F_1^o(t), \cdots, F_h^o(t)]$$

for $l_j \times l_j$ matrices $F_j^o(t)$ of pseudo-differential operators with symbols in $\Gamma_t^d(S_{G(d,1)}^\sigma)$. Here, $\mathrm{diag}[F_1^o(t), \cdots, F_h^o(t)]$ *means*

$$\mathrm{diag}[F_1^o(t), \cdots, F_h^o(t)]=\begin{pmatrix} F_1^o(t) & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & F_h^o(t) \end{pmatrix}.$$

Proof. We construct $F^{o}(t)$ and $P^{1}(t)$ satisfying

$$(6.28)\quad \begin{aligned}&\sigma(F^{o}(t))+\sigma(\mathscr{D}(t))\sigma(P^{1}(t))-\sigma(P^{1}(t))\sigma(\mathscr{D}(t))\\ &=\sigma(B(t))+\sigma(B(t))\sigma(P^{1}(t))-\sigma(P^{1}(t))\sigma(F^{o}(t))\end{aligned}$$

modulo $\Gamma_{t}^{d}(S_{G(d,1)}^{0})$, where $B(t)=(B_{jk}(t))$. Then, from Proposition 6.1 the property (6.28) yields (6.27).

Define pseudo-differential operators $F^{[0]}(t)$ and $P^{[0]}(t)$ by

$$F^{[0]}(t)=\mathrm{diag}[B_{11}(t),\ \cdots,\ B_{hh}(t)]$$

and

$$\sigma(P^{[0]}(t))=\left((\sigma(P^{[0]}(t))_{jk}\ \begin{matrix} j\rightarrow 1, & \cdots, & h\\ k\downarrow 1, & \cdots, & h\end{matrix}\right)$$

with

$$\begin{cases}\sigma(P^{[0]}(t))_{jj}=0,\\ \sigma(P^{[0]}(t))_{jk}=\dfrac{1}{\lambda_{j}(t)-\lambda_{k}(t)}\sigma(B_{jk}(t))\quad (j\neq k).\end{cases}$$

Here, we denote the $(j,\ k)$ blocks of $\sigma(P^{[0]}(t))$ by $\sigma(P^{[0]}(t))_{jk}$. Then, $\sigma(F^{[0]}(t))$ belongs to $\Gamma_{t}^{d}(S_{G(d,1)}^{\sigma})$ and $\sigma(P^{[0]}(t))$ belongs to $\Gamma_{t}^{d}(S_{G(d,1)}^{\sigma-1})$. Hence, if $\sigma\leqq 1/2$, the property (6.28) holds by setting $F^{o}(t)=F^{[0]}(t)$ and $P^{1}(t)=P^{[0]}(t)$. This proves the lemma in the case of $\sigma\leqq 1/2$. For the case of $\sigma>1/2$ we take an integer N satisfying $N\geqq(2\sigma-1)/(1-\sigma)$ and define $F^{[\nu]}(t)$ and $P^{[\nu]}(t)$ $(\nu=1,\ \cdots,\ N)$ inductively by $\sigma(F^{[\nu]}(t))=\mathrm{diag}[\sigma(F^{[\nu]}(t))_{11},\ \cdots,\ \sigma(F^{[\nu]}(t))_{hh}]$ with

$$\sigma(F^{[\nu]}(t))_{jj}=\sum_{k=1}^{r}\sigma(B_{jk}(t))\sigma(P^{[\nu-1]}(t))_{kj}\qquad (\in\Gamma_{t}^{d}(S_{G(d,1)}^{\sigma-(1-\sigma)\nu}))$$

and $\sigma(P^{[\nu]}(t))=(\sigma(P^{[\nu]}(t))_{jk})$ with

$$\begin{cases}\sigma(P^{[\nu]}(t))_{jj}=0,\\ \sigma(P^{[\nu]}(t))_{jk}=\dfrac{1}{\lambda_{j}(t)-\lambda_{k}(t)}\{\sum_{k'=1}^{h}\sigma(B_{jk'}(t))\sigma(P^{[\nu-1]}(t))_{k'k}\\ \qquad +\sum_{\nu'+\nu''=\nu-1}\sigma(P^{[\nu']}(t))_{jk}\sigma(F^{[\nu'']}(t))_{kk}\\ \qquad (\in\Gamma_{t}^{d}(S_{G(d,1)}^{-(1-\sigma)(\nu+1)})),\quad j\neq k.\end{cases}$$

Then, we obtain the desired operators $F^{o}(t)$ and $P^{1}(t)$ by setting

$$\begin{cases}F^{o}(t)=F^{[0]}(t)+F^{[1]}(t)+\cdots+F^{[N]}(t),\\ P^{1}(t)=P^{[0]}(t)+P^{[1]}(t)+\cdots+P^{[N]}(t).\end{cases}$$

Q. E. D.

Lemma 6.5. *Let $P^{1}(t)$ be as in the preceding lemma. Then, there* *exists a matrix* $Q^1(t)$ *of pseudo-differential operators with symbols in* $\Gamma_t^d(S^0_{G(d,1)})$ *such that*

(6.29) $$(\mathscr{I}+P^1(t))Q^1(t)=\mathscr{I}+R(t)$$

holds for a regularizer $R(t)$ *in* $\mathscr{R}_{G(d)}$.

Proof. Since $\sigma<1$, there exists a constant μ such that

$$|\det(\mathscr{I}_l+\sigma(P^1(t))(x,\ \xi)|\geqq C>0 \quad \text{for } |\xi|\geqq\mu.$$

Take a function $\chi_\mu(\xi)$ in γ^d satisfying $\chi_\mu(\xi)=0$ if $|\xi|\leqq\mu$ and $\chi_\mu(\xi)=1$ if $|\xi|\geqq\mu+1$. Define matrices $q^{[\nu]}(t, x, \xi)\,(\nu=0, 1, \cdots)$ of symbols by

$$\begin{cases} q^{[0]}(t,\ x,\ \xi)=(\mathscr{I}_l+\sigma(P^1(t))(x,\ \xi))^{-1}\chi_\mu(\xi), \\ q^{[\nu]}(t,\ x,\ \xi)=-\sum\limits_{\substack{\nu'+|\gamma|=\nu \\ \nu'<\nu}}\dfrac{1}{\gamma!}q^{[0]}(t,\ x,\ \xi)\,\partial_\xi^\gamma\sigma(P^1(t))(x,\ \xi) \\ \qquad\qquad\qquad\qquad\times D_x^\gamma q^{[\nu']}(t,\ x,\ \xi)\quad(\nu\geqq1). \end{cases}$$

Then, $q^{[\nu]}(t, x, \xi)$ belongs to $\Gamma_t^d(S^{-\nu}_{G(d,1)})$ and satisfy

$$\begin{aligned} &|\partial_t^k\partial_\xi^\alpha D_x^\beta q^{[\nu]}(t,\ x,\ \xi)| \\ &\quad\leqq CM^{-(k+|\alpha|+|\beta|+\nu)}k!^d\alpha!\beta!^d\nu!^d\langle\xi\rangle^{-\nu-|\alpha|}\quad \text{for } |\xi|\geqq\mu+1 \end{aligned}$$

with constants C and M independent of k, α, β and ν. Hence, applying Lemma 6.2 we can find $q^1(t, x, \xi)$ in $\Gamma_t^d(S^0_{G(d,1)})$ satisfying for constants C_1, M_1 and μ_1

(6.30) $$\begin{aligned} &|\partial_t^k\partial_\xi^\alpha D_x^\beta(q^1(t,\ x,\ \xi)-\sum_{\nu=0}^{N-1}q^{[\nu]}(t,\ x,\ \xi))| \\ &\quad\leqq C_1M_1^{-(k+|\alpha|+|\beta|+N)}k!^d\alpha!\beta!^dN!^d\langle\xi\rangle^{-N-|\alpha|}\quad \text{for } |\xi|\geqq\mu_1. \end{aligned}$$

Using (6.30) and Proposition 6.1 we get (6.29) with $Q^1=q^1(t, X, D_x)$ by the usual method. Q. E. D.

Define $B^2(t)=\left(B^2_{jk}(t)\ \begin{matrix} j\downarrow 1, & \cdots, & h \\ k\rightarrow 1, & \cdots, & h\end{matrix}\right)$ by

$$B^2(t)=Q^1(t)B^1(t).$$

Then, from (6.27) we obtain

(6.31) $$\mathscr{L}(\mathscr{I}+P^1(t))=(\mathscr{I}+P^1(t))\mathscr{L}_2+R^2(t)$$

for a regularizer $R^2(t)$ in $\mathscr{R}_{G(d)}$, where $\mathscr{L}_2$ denotes the operator

(6.32) $$\mathscr{L}_2=D_t-\mathscr{D}(t)+F^o(t)+B^2(t).$$

In (6.32) we may assume $\sigma(B^2(t))\in\Gamma_t^d(S^0_{G(d,1)})$ by means of Proposition 6.1. Moreover, replacing $F^o(t)$ by

$$F^o(t)+\mathrm{diag}[B^2_{11}(t),\ \cdots,\ B^2_{hh}(t)],$$

we may assume that in (6.32) the diagonal blocks of $B^2(t)$ are zero. Now, we are prepared to prove Theorem 6.3. The following discussions originate from those in [10] and in §IV of Appendix in [12].

Proof of Theorem 6.3. First, we construct $P^2(t)$ and $F(t)$ with $\sigma(P^2(t))\in\Gamma^d_t(S^{-1}_{G(d,1)})$ and $\sigma(F(t))\in\Gamma^d_t(S^{-1}_{G(d,1)})$ such that

$$\mathscr{L}_2(\mathscr{I}+P^2(t))=(\mathscr{I}+P^2(t))(D_t-\mathscr{D}(t)+F^o(t)+F(t))+R^3(t) \tag{6.33}$$

holds, where the blocks of $F(t)$ are zero except diagonal blocks and $R^3(t)$ is a regularizer in $\mathscr{R}_{G(d)}$. Set

$$\Sigma=\{\{P^{[\nu]}(t)\}^\infty_{\nu=0};\ \sigma(P^{[\nu]}(t))=\left((\sigma(P^{[\nu]}(t))_{jk}\ \begin{matrix} j\to 1,\ \cdots,\ h\\ k\to 1,\ \cdots,\ h\end{matrix}\right),$$
$$\sigma(P^{[\nu]}(t))_{jj}=0,\quad \sigma(P^{[\nu]}(t))_{jk}\in\Gamma^d_t(S^{-1-\nu}_{G(d,1)})\ (j\neq k)\},$$

Here, the notation $\sigma(P^{[\nu]}(t))_{jk}$ has the same meaning as in the proof of Lemma 6.4. For $\{P^{[\nu]}(t)\}^\infty_{\nu=0}\in\Sigma$ we define $\{F^{[\nu]}(t)\}$ and $\{A^{[\nu]}(t)\}$ as follows:

$$\sigma(F^{[\nu]}(t))=\mathrm{diag}[\sigma(F^{[\nu]}(t))_{11},\ \cdots,\ \sigma(F^{[\nu]}(t))_{hh}] \tag{6.34}$$

with

$$\sigma(F^{[\nu]}(t))_{jj}=\sum_{k=1}^{h}\sum_{|\gamma|+\nu'=\nu}\frac{1}{\gamma!}\partial^\gamma_\xi\sigma(B^2_{jk}(t))D^\gamma_x\sigma(P^{[\nu']}(t))_{kj}\ (\in\Gamma^d_t(S^{-1-\nu}_{G(d,1)})) \tag{6.35}$$

and

$$\sigma(A^{[0]}(t))=\sigma(B^2(t)), \tag{6.36}$$

$$\sigma(A^{[\nu]}(t))=(\sigma(A^{[\nu]}(t))_{jk})\quad(\nu\geqq 1) \tag{6.37}$$

with

$$\begin{cases}\sigma(A^{[\nu]}(t))_{jj}=0,\\ \sigma(A^{[\nu]}(t))_{jk}\\ \quad =\displaystyle\sum_{k'=1}^{h}\sum_{|\gamma|+\nu'=\nu-1}\frac{1}{\gamma!}\partial^\gamma_\xi(\sigma(B^2_{jk'}(t)))D^\gamma_x(\sigma(P^{[\nu']}(t))_{k'k})\\ \quad\quad +D_t(\sigma(P^{[\nu-1]}(t))_{jk}\\ \quad\quad -\displaystyle\sum_{|\gamma|+\nu'+\nu''=\nu-2}\frac{1}{\gamma!}\partial^\gamma_\xi(\sigma(P^{[\nu']}(t))_{jk})D^\gamma_x(\sigma(F^{[\nu'']}(t))_{kk})\\ \quad\quad -\displaystyle\sum_{\substack{|\gamma|+\nu'=\nu\\ |\gamma|\geqq 1}}\frac{1}{\gamma!}\partial^\gamma_\xi\{\lambda_j(t)\mathscr{I}_{l_j}-\sigma(F^o_j(t))\}D^\gamma_x(\sigma(P^{[\nu']}(t))_{jk})\end{cases} \tag{6.38}$$

$$+\sum_{\substack{|\gamma|+\nu'=\nu \\ |\gamma|\geqq 1}} \frac{1}{\gamma!}\partial_\xi^\gamma(\sigma(P^{[\nu']}(t))_{jk})D_x^\gamma\{\lambda_k(t)\,\mathscr{I}_{l_k}-\sigma(F_k^o(t))\}$$

$$(\in\Gamma_t^d(S_{G(d.1)}^{-\nu}))\quad (j\neq k),$$

regarding the third term in the right hand side of (6.38) as zero in case $\nu=1$. Next, we solve equations for unknowns $\sigma(\tilde{P}^{[\nu]}(t))=(\sigma(\tilde{P}^{[\nu]}(t))_{jk})$ $(j,\ k=1,\ \cdots,\ h)$:

(6.39) $\sigma(\tilde{P}^{[\nu]}(t))_{jj}=0,$

(6.40) $$\sigma(\tilde{P}^{[\nu]}(t))_{jk}-\left\{\frac{1}{\lambda_j(t)-\lambda_k(t)}\sigma(F_j^o(t))\right\}\sigma(\tilde{P}^{[\nu]}(t))_{jk} +\sigma(\tilde{P}^{[\nu]}(t))_{jk}\left\{\frac{1}{\lambda_j(t)-\lambda_k(t)}\sigma(F_k^o(t))\right\} =\frac{1}{\lambda_j(t)-\lambda_k(t)}\sigma(A^{[\nu]}(t))_{jk}\quad \text{for large } |\xi|\quad (j\neq k),$$

where $A^{[\nu]}(t)$ are defined by (6.36)—(6.38). Set

$$\begin{cases} g_{jk}(t,\ x,\ \xi)=\dfrac{1}{\lambda_j(t)-\lambda_k(t)}\sigma(F_j^o(t)), \\ h_{jk}(t,\ x,\ \xi)=-\dfrac{1}{\lambda_j(t)-\lambda_k(t)}\sigma(F_k^o(t)) \quad (j\neq k). \end{cases}$$

From (6.22) and $\sigma(F_j^o(t))\in\Gamma_t^d(S_{G(d.1)}^\sigma)$ for $\sigma<1$ there exists a constant μ such that the matrix norms $\|g_{jk}(t,\ x,\ \xi)\|$ and $\|h_{jk}(t,\ x,\ \xi)\|$ of $g_{jk}(t,\ x,\ \xi)$ and $h_{jk}(t,\ x,\ \xi)$ satisfy

$$\|g_{jk}(t,\ x,\ \xi)\|\leqq 1/4,\quad \|h_{jk}(t,\ x,\ \xi)\|\leqq 1/4\quad \text{for } |\xi|\geqq\mu.$$

Hence, the solutions $\sigma(\tilde{P}^{[\nu]}(t))_{jk}$ of (6.40) are given by

(6.41) $$\sigma(\tilde{P}^{[\nu]}(t))_{jk}=\sum_{\kappa=0}^{\infty}\sum_{\kappa'=0}^{\kappa}\binom{\kappa}{\kappa'}g_{jk}(t)^{\kappa'} \times\left(\frac{1}{\lambda_j(t)-\lambda_k(t)}\sigma(A^{[\nu]}(t))_{jk}\right)h_{jk}(t)^{\kappa-\kappa'}$$

for $|\xi|\geqq\mu$. We modify $\sigma(\tilde{P}^{[\nu]}(t))_{jk}$ in $[0,\ T]\times R_x^n\times\{|\xi|\leqq\mu+1\}$ such that $\sigma(\tilde{P}^{[\nu]}(t))_{jk}$ belong to $\Gamma_t^d(S_{G(d.1)}^{-1-\nu})$. Then, we obtain a solution $\{\sigma(\tilde{P}^{[\nu]}(t))\}$ of (6.39)—(6.40) in Σ.

Now, we define a mapping $\mathscr{T}$ from Σ to Σ by $\mathscr{T}(\{P^{[\nu]}(t)\})=\{\tilde{P}^{[\nu]}(t)\}$. From the definition the operator $\tilde{P}^{[\nu]}(t)$ is determined only by $P^{[0]}(t),\ \cdots,\ P^{[\nu-1]}(t)$. So, by the induction on ν we can see that the fixed point $\{P_o^{[\nu]}(t)\}$ of $\mathscr{T}$ exists uniquely in Σ. Assume that the fixed point $\{P_o^{[\nu]}(t)\}$ of $\mathscr{T}$ satisfies

(6.42) $$|\partial_t^k\partial_\xi^\alpha D_x^\beta\sigma(P_o^{[\nu]}(t))| \leqq CM^{-(k+|\alpha|+|\beta|+\nu)}k!^d\alpha!\beta!^d\nu!^d\langle\xi\rangle^{-1-\nu-|\alpha|}\quad \text{for } |\xi|\geqq\mu$$

with constants C, M and μ independent of k, α, β and ν. Then, from Lemma 6.2 there exists a symbol $p^2(t, x, \xi)$ in $\Gamma_t^d(S_{G(d,1)}^{-1})$ such that

$$\begin{aligned}&|\partial_t^k\partial_\xi^\alpha D_x^\beta(p^2(t, x, \xi)-\sum_{\nu=0}^{N-1}\sigma(P_o^{[\nu]}(t))(x, \xi))|\\ &\quad\leqq CM^{-(k+|\alpha|+|\beta|+N)}k!^d\alpha!\beta!^dN!^d\langle\xi\rangle^{-1-N-|\alpha|}\quad\text{for } |\xi|\geqq\mu\end{aligned}$$

with new constants C, M and μ. Then, setting $P^2(t)=p^2(t, X, D_x)$ and

$$F(t)=\operatorname{diag}[F_1(t), \cdots, F_h(t)]$$

with $\sigma(F_j(t))=$ *the* (j, j) *block of* $\sigma_c(B^2(t)P^2(t))$, we have

$$\begin{aligned}&F(t)+(\mathscr{D}(t)-F^o(t))P^2(t)-P^2(t)(\mathscr{D}(t)-F^o(t))\\ &\quad=B^2(t)+B^2(t)P^2(t)+P_t^2(t)-P^2(t)F(t)-R^3(t),\end{aligned}$$

for a regularizer $R^3(t)$ in $\mathscr{R}_{G(d)}$, where $P_t^2(t)$ is a matrix of pseudo-differential operators with symbol $D_t\sigma(P^2(t))$. This is nothing else but (6.33).

In order to prove that the fixed point $\{P_o^{[\nu]}(t)\}$ of $\mathscr{T}$ satisfies (6.42), we define following [1] a formal norm $|||\{q^{[\nu]}(t)\}, M|||^{(m)}$ for a sequence $\{q^{[\nu]}(t)\}_{\nu=0}^{\infty}$ with $q^{[\nu]}(t)\equiv q^{[\nu]}(t, x, \xi)\in\Gamma_t^d(S_{G(d,1)}^{m-\nu})$ by

$$\begin{aligned}(6.43)\quad &|||\{q^{[\nu]}(t)\}, M|||^{(m)}\\ &\quad=\sum_{k,\alpha,\beta,\nu}C_{k,\alpha,\beta}^{\nu}\sup_{\substack{t,x\\|\xi|\geqq\mu}}\{||\partial_t^k\partial_\xi^\alpha D_x^\beta q^{[\nu]}(t, x, \xi)||\langle\xi\rangle^{-m+|\alpha|+\nu}\}\\ &\qquad\times M^{k+|\alpha|+|\beta|+2\nu},\end{aligned}$$

where μ is some constant,

$$(6.44)\qquad C_{k,\alpha,\beta}^{\nu}=\frac{2(2n)^{-\nu}\nu!}{(|\alpha|+\nu)!(k+|\beta|+\nu)!^d}$$

and $||\cdot||$ denotes the matrix norm. For a symbol $q(t, x, \xi)\in\Gamma_t^d(S_{G(d,1)}^m)$ we denote $|||q(t), M|||^{(m)}=|||\{q^{[\nu]}(t)\}, M|||^{(m)}$ by setting $q^{[0]}(t)=q(t)$ and $q^{[\nu]}(t)=0$ $(\nu\geqq1)$. Then, from the assumptions there exists a constant M such that the following hold if we take μ in (6.43) sufficiently large:

$$(6.45)\qquad |||\sigma(B^2(t)), M|||^{(0)}\leqq C_1,$$

$$(6.46)\qquad |||(\lambda_j(t)-\lambda_k(t))^{-1}, M|||^{(-1)}\leqq C_2\quad(j\neq k),$$

$$(6.47)\qquad \begin{cases}|||g_{jk}(t), M|||^{(0)}\leqq 1/3,\\ |||h_{jk}(t), M|||^{(0)}\leqq 1/3,\end{cases}$$

and

$$(6.48)\qquad \sum_{\substack{k,\alpha,\beta\\|\alpha+\beta|\geqq 1}} C^0_{k,\alpha,\beta} \sup_{\substack{t,x\\|\xi|\geqq\mu}} \{|\partial_t^k\partial_\xi^\alpha D_x^\beta(\lambda_j(t)\mathscr{I}_{l_j}-\sigma(F_j^o(t)))|\langle\xi\rangle^{-1+|\alpha|}\} \times M^{k+|\alpha|+|\beta|-1}\leqq C_3$$

for $C^0_{k,\alpha,\beta}$ defined by (6.44) with $\nu=0$.

Now, suppose that $\{P^{[\nu]}(t)\}\in\Sigma$ satisfies

$$|||\{\sigma(P^{[\nu]}(t))\},\ M|||^{(-1)}<\infty.$$

Let $\{F^{[\nu]}(t)\}_{\nu=0}^{\infty}$, $\{A^{[\nu]}(t)\}_{\nu=0}^{\infty}$ and $\{\tilde{P}^{[\nu]}(t)\}_{\nu=0}^{\infty}$ be defined by (6.34)—(6.40). Then, from Lemma 1.2 of [1] we have

$$(6.49)\qquad |||\{\sigma(F^{[\nu]}(t))\},\ M|||^{(-1)}\leqq C_1|||\{\sigma(P^{[\nu]}(t))\},\ M|||^{(-1)},$$

$$\begin{aligned}(6.50)\qquad &|||\{\sigma(A^{[\nu]}(t))\},\ M|||^{(0)}\\ &\leqq C_1+C_1M^2|||\{\sigma(P^{[\nu]}(t))\},\ M|||^{(-1)}+M|||\{\sigma(P^{[\nu]}(t))\},\ M|||^{(-1)}\\ &\quad +M^4|||\{\sigma(P^{[\nu]}(t))\},\ M|||^{(-1)}|||\{\sigma(F^{[\nu]}(t))\},\ M|||^{(-1)}\\ &\quad +2C_3M|||\{\sigma(P^{[\nu]}(t))\},\ M|||^{(-1)},\end{aligned}$$

and from (6.46)—(6.47) and (6.41)

$$\begin{aligned}(6.51)\qquad &|||\{\sigma(\tilde{P}^{[\nu]}(t))\},\ M|||^{(-1)}\\ &\leqq\sum_{\kappa=0}^{\infty}\sum_{\kappa'=0}^{\kappa}\binom{\kappa}{\kappa'}\left(\frac{1}{3}\right)^{\kappa'}C_2|||\{\sigma(A^{[\nu]}(t))\},\ M|||^{(0)}\left(\frac{1}{3}\right)^{\kappa-\kappa'}\\ &=3C_2|||\{\sigma(A^{[\nu]}(t))\},\ M|||^{(0)}.\end{aligned}$$

Hence, we have

$$\begin{aligned}(6.52)\qquad |||\{\sigma(\tilde{P}^{[\nu]}(t))\},\ M|||^{(-1)}\leqq 3C_1C_2+C_4M|||\{\sigma(P^{[\nu]}(t))\},\ M|||^{(-1)}\\ +C_5M(|||\{\sigma(P^{[\nu]}(t))\},\ M|||^{(-1)})^2\end{aligned}$$

with some constants C_4 and C_5. So, if we set for a constant C^0 larger than $3C_1C_2$

$$\Sigma_o=\{\{P^{[\nu]}(t)\}_{\nu=0}^{\infty}\in\Sigma\,;\ |||\{\sigma(P^{[\nu]}(t))\},\ M|||^{(-1)}\leqq C^0\},$$

we see that the mapping $\mathscr{T}$ maps Σ_o into Σ_o if M is sufficiently small. Moreover, if we go over the proof of (6.52) once again, we see that the restriction of $\mathscr{T}$ to Σ_o is a contraction if we take M satisfying $C_4M+2C_5C^0M<1$. This implies that the fixed point $\{P_o^{[\nu]}(t)\}$ of $\mathscr{T}$ belongs to Σ_o, which means $\{P_o^{[\nu]}(t)\}$ satisfies (6.42). Summing up, we have found $P^2(t)$ and $F(t)$ satisfying (6.33).

In (6.33) we set

$$B^0(t)=F^o(t)+F(t).$$

Then, $B^0(t)$ has a form

$$B^0(t)=\begin{pmatrix} B_1(t) & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & B_h(t) \end{pmatrix}$$

and from (6.31) and (6.33) we have

$$\begin{aligned}(6.53)\quad & \mathscr{L}(\mathscr{I}+P^1(t))(\mathscr{I}+P^2(t)) \\ &= (\mathscr{I}+P^1(t))(\mathscr{I}+P^2(t))(D_t-\mathscr{D}(t)+B^0(t))+R^4(t)\end{aligned}$$

with a regularizer $R^4(t)$ in $\mathscr{R}_{G(d)}$. For a positive number μ we take a function $\chi_\mu(\xi)$ in γ^d satisfying $\chi_\mu(\xi)=0$ if $|\xi|\leqq\mu$ and $\chi_\mu(\xi)=1$ if $|\xi|\geqq\mu+1$ and let $P(t;\mu)$ be the pseudo-differential operator with the symbol

$$\sigma(P(t;\mu))=(\sigma(P^1(t))+\sigma(P^2(t))+\sigma_o(P^1(t)P^2(t)))\chi_\mu(\xi).$$

Then, from (6.53) we have

$$\mathscr{L}(\mathscr{I}+P(t;\mu))=(\mathscr{I}+P(t;\mu))(D_t-\mathscr{D}(t)+B^0(t))+R(t;\mu),$$

for a regularizer $R(t;\mu)$ in $\mathscr{R}_{G(d)}$ depending on a parameter μ. Since the order of $\sigma(P^1(t))+\sigma(P^2(t))+\sigma_o(P^1(t)P^2(t))$ is less than zero, $\sigma(P(t;\mu))$ satisfies the first inequality of (2.19) with an arbitrary small constant C_o if μ tends to the infinity. Therefore, we can take sufficiently large constant μ^0 such that the inverse $Q(t)$ of $\mathscr{I}+P(t;\mu^0)$ exists. Now, we set $P(t)=P(t;\mu^0)$ and $R_o(t)=Q(t)R(t;\mu^0)$. Then, we obtain (6.25). This concludes the proof of Theorem 6.3.

Q. E. D.

Finally, we give some remarks concerning the inverse $Q(t)$ of $\mathscr{I}+P(t)$ in Theorem 6.3. In order to apply Theorem 6.3 to Proposition 3.4 it is sufficient that $Q(t)$ belongs to $L^0_{G(d)}$ for any t. But, we can improve the result "$Q(t)\in L^0_{G(d)}$ for any t" in the following way: "*the inverse* $Q(t)$ *has the form* $Q(t)=q^0(t,X,D_x)+\tilde{q}(t,X,D_x)$ *with symbols* $q^0(t,x,\xi)\in S^0_{G(d,1)}$ *and* $\tilde{q}(t,x,\xi)\in\mathscr{R}_{G(d)}$ *for any* t." This result is proved by applying the following property (*) and the discussions in proving Corollary 2.7 since $\sigma(P(t;\mu))$ belongs to $\Gamma^d_t(S^{\sigma-1}_{G(d,1)})$.

(*) *In Theorem* 2.6 *we assume furthermore that* $p^0_j(x,\xi)$ *belong to*

$S^{\sigma}_{G(d,1)}$ *and satisfy*

$$|p^{0(\alpha)}_{j(\beta)}(x, \xi)| \leqq C_o M^{-(|\alpha|+|\beta|)}\alpha!\beta!^d\langle\xi\rangle^{\sigma-|\alpha|} \quad for \quad |\xi| \geqq \tilde{\mu}$$

with constants C_o, M *and* $\tilde{\mu}$ *independent of* α, β *and* j. *Then, the multi-product* $Q_{\nu+1}=P_1\cdots P_{\nu+1}$ *of* $P_j=p_j(X, D_x)$ *has the form* $Q_{\nu+1}=q^0_{\nu+1}(X, D_x)+\tilde{q}_{\nu+1}(X, D_x)$ *with symbols* $q^0_{\nu+1}(x, \xi)$ *satisfying*

$$|q^0_{\nu+1}{}^{(\alpha)}_{(\beta)}(x, \xi)| \leqq A^{\nu}C_o^{\nu+1}M_1^{-(|\alpha|+|\beta|)}\alpha!\beta!^d\langle\xi\rangle^{(\nu+1)\sigma-|\alpha|} \quad for \quad |\xi| \geqq \tilde{\mu}_1$$

and $\tilde{q}_{\nu+1}(x, \xi)$ *satisfying* (2.18). *Here, the constants* A, M_1 *and* $\tilde{\mu}_1$ *are independent of* ν.

The above discussions also give another proof of Lemma 6.5. In fact, we can prove (6.29) by setting $Q^1(t)=(\mathscr{I}+P^1(t)\chi_{\mu}(D_x))^{-1}$ for large constant μ, where $\chi_{\mu}(\xi)$ is the function used in the last part of the proof of Theorem 6.3.

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