Publ. RIMS, Kyoto Univ.
20 (1984), 595-598

A Remark on the Segal-Becker Theorem

Dedicated to Professor Minoru Nakaoka on his 60th birthday

By

Kazumoto KOZIMA*

§1. Introduction

Let CP^∞ be the infinite dimensional complex projective space and BU the classifying space of stable complex vector bundles. Then there is the natural inclusion $j: CP^\infty \hookrightarrow BU$ and the structure map of the infinite loop space structure defined by the Bott periodicity $\xi: Q(BU) \longrightarrow BU$ where $Q(\ ) = \mathrm{Colim}_n \Omega^n \Sigma^n(\ )$. Let $\lambda: Q(CP^\infty) \longrightarrow BU$ be the composition $\xi \circ Q(j)$. The results of Segal [8] and Becker [2] show us that there exists a map $s: BU \longrightarrow Q(CP^\infty)$ such that $\lambda \circ s = \mathrm{id}$.

The main result of this paper is to show that one can take s satisfying that $s \circ j \simeq$ inclusion: $CP^\infty \longrightarrow Q(CP^\infty)$.

To show this, we will use the results of Brumfiel-Madsen [4] for the evaluation of the transfer map.

§2. The Construction of the Splitting

Let $U(n)$ be the unitary group and T^n its maximal torus. Let NT^n be the normaliser of T^n in $U(n)$. We also define homogeneous spaces of $U(2n)$:

$$E_n = U(2n)/U(n) \text{ and } E'_n = U(n+1)/U(n).$$

Then the construction of splitting in [2] can be reformulated as follows.

Let $r: Q(X_+) \longrightarrow Q(X)$ be the map induced by the canonical projection and $a: Q(X) \longrightarrow Q(X_+)$ the right adjoint of r. Let $t_n: E_n/U(n)_+$

Communicated by N. Shimada, May 10, 1983.
* Department of Mathematics, Kyoto University, Kyoto 606, Japan.

$\longrightarrow Q(E_n/NT^n_+)$ the Becker-Gottlieb transfer ([2], [3]) associated with the smooth fiber bundle

$$U(n)/NT^n \longrightarrow E_n/NT^n \longrightarrow E_n/U(n).$$

E_n/T^n has the action of $NT^n/T^n = \Sigma_n$ which sends eT^n to enT^n where $e \in E_n$ and $n \in NT^n$. $(X)^n$ is also a Σ_n-space by the permutation of the coordinates.

Since the elements of E_n can be considered as the n-frames in $\boldsymbol{C}^{2n}$, we define a Σ_n-equivariant map

$$h_n\colon\ E_n/T^n \longrightarrow (CP^{2n-1})^n$$

by corresponding each vector to its representative element in CP^{2n-1}.

Also, since $E_n/T^n \longrightarrow E_n/NT^n$ is a principal Σ_n-bundle, there is a Σ_n-equivariant map

$$c_n\colon\ E_n/T^n \longrightarrow E\Sigma_n$$

which covers the classifying map of this principal bundle where $E\Sigma_n$ is the contractible free Σ_n-space. Thus we obtain a map

$$k_n = (c_n \times h_n)/\Sigma_n\colon\ E_n/NT^n \longrightarrow (E\Sigma_n \times (CP^{2n-1})^n)/\Sigma_n.$$

There is also the Barratt-Quillen map

$$w_n\colon\ (E_n \times (X)^n)/\Sigma_n \longrightarrow Q(X_+).$$

Notice that the composition $X \xrightarrow{i_1} (E\Sigma_n \times (X)^n)/\Sigma_n \xrightarrow{w_n} Q(X_+)$ is homotopic to the composition $X \xrightarrow{\text{incl.}} Q(X) \xrightarrow{a} Q(X_+)$ where i_1 is the map defined by the equation

$$i_1(x) = (*_{E_n},\ (x,\ *_X,\ *_X, \cdots,\ *_X)) \text{ for } x \in X.$$

So the following Lemma is clear.

Lemma 2.1. *The composition*

$$CP^{n-1} = E'_n/S^1 \longrightarrow E_n/NT^n \xrightarrow{w_n \circ k_n} Q(CP^{2n-1})$$

is homotopic to the composition $CP^{n-1} \longrightarrow CP^{2n-1}_+ \xrightarrow{\text{incl}} Q(CP^{2n-1}_+)$.

Remark. One can easily show that the composition $w_n \circ k_n\colon E_n/NT^n \longrightarrow Q(CP^{2n-1}_+)$ agrees with the composition of the Kahn-Priddy pre-transfer $t\colon E_n/NT^n \longrightarrow Q(E_n/NT^{n-1} \times S^1_+)$ associated with the n-fold covering $E_n/NT^{n-1} \times S^1 \longrightarrow E_n/NT^n$ and the map $Q(E_n/NT^{n-1} \times S^1_+) \longrightarrow Q(CP^{2n-1}_+)$ which is induced from the quotient map. (Compare

[6], [7].)

Now we are ready to define the splitting s. Let us consider the composition

$$s_n\colon E_n/U(n)\longrightarrow E_n/U(n)_+\xrightarrow{t_n}Q(E_n/NT^n_+)\xrightarrow{Q(w_n\circ k_{n+})}QQ(CP^{2n-1}_+)$$
$$\xrightarrow{\zeta}Q(CP^{2n-1}_+)\xrightarrow{r}Q(CP^{2n-1})$$

where $w_n\circ k_{n+}$ is the pointed extension of $w_n\circ k_n$ and ζ is the structure map of the infinite loop space $Q(CP^{2n-1}_+)$. As in [2] and [9], t_n is compatible with n. So, since all the constructions are compatible with n, by taking the limit, we obtain s: $BU\longrightarrow Q(CP^\infty)$.

§ 3. The Proof of the Main Result

By virtue of (2.1), we have only to prove that the diagram

$$\begin{array}{ccc} E'_n/S^1_+ & \xrightarrow{\text{incl.}} & Q(E'_n/S^1_+) \\ \downarrow & & \downarrow \\ E_n/U(n)_+ & \xrightarrow{t_n} & Q(E_n/NT^n_+) \end{array}$$

commutes up to homotopy where the virtical maps are induced from the inclusion $E'_n\hookrightarrow E_n$.

We need the evaluation of the transfer.

Proposition 3.1. *The following diagram is homotopy commutative;*

$$\begin{array}{ccc} E_n/T^n_+ & \longrightarrow & E_n/NT^n_+ \\ \downarrow & & \downarrow \\ E_n/U(n)_+ & \xrightarrow{t_n} & Q(E_n/NT^n_+) \end{array}$$

where the maps with no name are induced from the canonical projections.

This proposition is a corollary of Brumfiel and Madsen [4]. (See Theorem 3.5 of [4].)

Since the diagram

$$\begin{array}{ccc} E'_n/S^1_+ & \longrightarrow & Q(E'_n/S^1_+) \\ \downarrow & & \downarrow \\ E_n/T^n_+ & \xrightarrow{\text{incl.}} & Q(E_n/T^n_+) \\ \downarrow & & \downarrow \\ E_n/NT^n_+ & \xrightarrow{\text{incl.}} & Q(E_n/NT^n_+) \end{array}$$

commutes up to homotopy, we get the main result:

$$\begin{array}{ccc} CP^{n-1}=E'_n/S^1 & \xrightarrow{\text{incl}} & Q(CP^{n-1}) \\ \downarrow & & \downarrow \\ E_n/U(n) & \xrightarrow{s_n} & Q(CP^{2n-1}) \end{array}$$

commutes up to homotopy.

Thus $s\circ j$ is homotopic to the canonical inclusion as an element of $\lim_n \mathrm{Map}(CP^n, Q(CP^\infty))$. Then $\lambda\circ s\circ j$ is homotopic to j on the finite skeleton. So one can easily show that $\lambda\circ s: BU\longrightarrow BU$ induces identities on the K-homology groups and on the K-cohomology groups, by using the fact that s is an H-map. (See [9].) Thus our s is a splitting.

Let $P^m(\ )$ be the m-th term of the cohomology defined by $Q(CP^\infty)$. Then we have the Milnor exact sequence

$$0\longrightarrow \lim_n^1 P^{-1}(CP^n)\longrightarrow P^0(CP^\infty)\longrightarrow \lim_n P^0(CP^n)\longrightarrow 0.$$

As in [5], one can easily prove that $P^{-1}(CP^n)$ is finite. So $\lim^1$-term vanishes and we have the main theorem:

Theorem 3.2. *The composition*

$$s\circ j:\ CP^\infty \hookrightarrow BU\longrightarrow Q(CP^\infty)$$

is homotopic to the canonical inclusion.

References

[1] Adams, J. F., Infinite loop space, *Annals of mathematics studies*, **90**, Princeton Univ. Press, 1978.

[2] Becker, J. C., Characteristic classes and K-theory, *Lecture Notes in Math.*, **428**, Springer, 1973, 132-143.

[3] Becker, J. C. and Gottlieb, D. H., The transfer map and fibre bundles, *Topology*, **14** (1975), 1-12.

[4] Brumfiel, G. W. and Madsen, I., Evaluation of the transfer and the universal surgery classes, *Inventiones math.*, **32** (1976), 133-169.

[5] Kono, A., A note on the Segal-Becker type splittings, to appear in *J. Math. Kyoto Univ.*

[6] Kahn, D. S. and Priddy, S. B., Applications of the transfer to stable homotopy theory, *Bull. A. M. S.*, **78**(**6**) (1976), 981-987.

[7] Roush, F. W., Transfer in generalized cohomology theories, *Thesis*, Princeton, 1971.

[8] Segal, G. B., The stable homotopy of complex projective space, *Quart. J. Math. Oxford* (2), **24** (1973), 1-5.

[9] Snaith, V. P., Algebraic cobordism and K-theory, *Memoirs of the A. M. S.*, vol. **21**, no. **211**, 1979.