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Global Realization of Strongly Pseudoconvex CR Manifolds

Dedicated to Professor S. Nakano on his 60th birthday

By

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Introduction

Let S be a real hypersurface in a complex manifold M. Then, $T_M^{1,0}$, the holomorphic tangent bundle of M, determines an integrable subbundle $T'_S := T_M^{1,0}|_S \cap (T_S \otimes \mathbf{C}) \subset T_S \otimes \mathbf{C}$. Modelled on (S, T'_S), a CR manifold (a Cauchy-Riemann manifold) is defined as a pair (X, T'_X) consisting of a differentiable manifold X and a subbundle T'_X of $T_X \otimes \mathbf{C}$ satisfying the following two conditions:

(i) $T'_X \cap \bar{T}'_X = 0$,
(ii) T'_X is closed under the Poisson bracket (integrability condition).

Newlander-Nirenberg's theorem [9] says that a CR manifold is nothing but a complex manifold if $\dim X = \mathrm{rank}_{\mathbf{R}} T'_X$. Thus, an interesting problem arises concerning the realizability, or imbeddability, of CR manifolds as submanifolds of complex manifolds. Boutet de Monvel [1] showed that s. p. c. manifolds (cf. Section 1) are holomorphically imbeddable into some $\mathbf{C}^N$, provided that $\dim X \geqq 5$. Recently Kuranishi [7] proved that *locally* every s. p. c. manifold is imbeddable as a real hypersurface of the ball in $\mathbf{C}^n$, provided that $\dim X \geqq 9$.

Our result is as follows.

Theorem. *Let (X, T'_X) be a compact s. p. c. manifold of dimension $\geqq 5$. Then, (X, T'_X) is realizable as a hypersurface of a complex manifold.*

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Our proof is quite different from Kuranishi's argument and based upon Boutet de Monvel's imbedding theorem which assures the existence of a realization $X\subset\mathbf{C}^N$ for sufficiently large N. We apply then Tanaka's stability theorem in [10] to perform a finite number of bumps on X and obtain an s. p. c. manifold $\hat{X}$ which is the boundary of a complex manifold containing X as a hypersurface.

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1. Preliminaries

We recall briefly Boutet de Monvel's imbedding theorem and Tanaka's stability theorem.

Let (X, T'_X) be a compact CR manifold of dimension $2n-1$, and let $T_X\otimes\mathbf{C}=T'_X\oplus\bar{T}'_X\oplus F$ be a decomposition with $F=\bar{F}$.

Definition 1. (X, T'_X) is called a strongly pseudoconvex CR manifold, or shortly an s. p. c. manifold, if $\operatorname{rank}_{\mathbf{C}}F=1$ and for any local frame $\{v_1, \cdots, v_{n-1}\}$ of T'_X, we can choose a local frame $\{\theta\}$ of F with $\bar{\theta}=\theta$ such that the $(n-1)\times(n-1)$ matrix (c_{ij}) defined by $\sqrt{-1}c_{ij}\theta\equiv[v_i, v_j]$ $(\mathrm{mod}\ T'_X\oplus\bar{T}'_X)$ is positive definite.

We note that the above condition is satisfied by a real hypersurface S in a complex manifold M if and only if for any point $p\in S$ there exist a neighbourhood U in M and a C^∞ function φ on U such that $S\cap U=\{\varphi=0\}$, $d\varphi\neq 0$, and $\partial\bar{\partial}\varphi>0$.

A $\mathbf{C}$-valued function f defined on an open set $V\subset X$ is said to be holomorphic if f is of class C^∞ and $\bar{v}f=0$ for any section v of T'_X over V. We denote by $\mathcal{O}_X$ the sheaf over X of the germs of holomorphic functions.

Theorem 1. *Let (X, T'_X) be an s. p. c. manifold of dimension $2n-1$. If $n\geqq 3$, there exist an integer N and holomorphic functions f_i, $i=1, 2, \cdots, N$, such that the map $F:=(f_1, \cdots, f_N)$ gives a C^∞ imbedding of X into $\mathbf{C}^N$.*

Proof. See Boutet de Monvel [1].

Corollary 1. *Every s.p.c. manifold of dimension $\geqq 5$ is isomorphic to a CR submanifold of a complex number space.*

Proof. Let $F: X \hookrightarrow \mathbf{C}^N$ be an imbedding by holomorphic functions. Then, for any point $x \in X$, $F_*(T'_{X,x}) \subset T^{1,0}_{\mathbf{C}^N, F(x)} \cap (T_{F(X)} \otimes \mathbf{C})$. Since rank $T'_X = n-1$, this inclusion is an equality.

Corollary 2. *Every s.p.c. manifold of dimension $\geqq 5$ is locally realizable as a hypersurface.*

Proof. Immediate from Corollary 1.

Definition 2. A complex manifold with boundary $\bar{M}$ is a C^∞ manifold with boundary of dimension $2n$ with a system of coordinate patches

$$\phi_i : U_i \rightrightarrows \{(z_1, \cdots, z_n) \in \mathbf{C}^n \mid \textstyle\sum_{k=1}^n |z_k|^2 < 1,\ r_i(z_1, \cdots, z_n) \geqq 0\},$$

where r_i is a C^∞ real valued function on the ball with $dr \neq 0$ everywhere, such that $\phi_j \phi_i^{-1}$ is holomorphic on $\phi_i(U_i \cap U_j) \setminus \{r_i = 0\}$.

We denote by ∂M the boundary of $\bar{M}$ and put $\partial M = \bar{M} \setminus M$. Clearly, ∂M has a canonical structure of CR-manifold.

Proposition 2. *Let $X \subset \mathbf{C}^N$ be a compact $(2n-1)$-dimensional C^∞ submanifold. Suppose X is s.p.c. with respect to the induced CR structure and that $n \geqq 2$. Then, there exists a unique analytic subvariety W in $\mathbf{C}^N \setminus X$ whose closure $\bar{W}$ in $\mathbf{C}^N$ is compact and satisfies $\bar{W} \setminus W = X$. Moreover, Sing W, the set of singular points of W, consists of a finite set of points and $\bar{W} \setminus$Sing W is a complex manifold with boundary.*

Proof. The reader is referred to Kuranishi [12], section 2.

In virtue of Hironaka's desingularization theorem [4], we obtain from $\bar{W}$, by a finite succession of blowing ups, a complex manifold with boundary $\bar{M}$. We shall call $\bar{M}$ the associated complex manifold of X.

Let $C^{0,q}(\bar{M})$ denote the C^∞ $(0, q)$-forms on $\bar{M}$ and let $C^{0,q}_0(\bar{M}) := \{f \in C^{0,q}(\bar{M});\ f|_{\partial M} = 0\}$. We set

$$\begin{aligned}
&Z^{0,q}(\bar{M}) = \{f \in C^{0,q}(\bar{M});\ \bar{\partial} f = 0\},\\
&B^{0,q}(\bar{M}) = \bar{\partial} C^{0,q-1}(\bar{M}).\\
&H^{0,q}(\bar{M}) = Z^{0,q}(\bar{M})/B^{0,q}(\bar{M}),\\
&Z_0^{0,q}(\bar{M}) = \{f \in Z^{0,q}(\bar{M});\ f|_X = 0\}\\
&B_0^{0,q}(\bar{M}) = \bar{\partial} C_0^{0,q-1}(\bar{M}) \cap C_0^{0,q}(\bar{M}),\\
&H_0^{0,q}(\bar{M}) = Z_0^{0,q}(\bar{M})/B_0^{0,q}(\bar{M}).
\end{aligned}$$

Let $C^{0,q}(X)$ denote the C^∞ sections of the bundle $\overset{q}{\wedge} \bar{T}'^*_X \longrightarrow X$. We set

$$\begin{aligned}
&Z^{0,q}(X) = \{f \in C^{0,q}(X);\ \langle df, v\rangle = 0 \text{ for } v \in \overset{q+1}{\wedge} \bar{T}'^*_X\},\\
&B^{0,q}(X) = \{f \in C^{0,q}(X);\\
&\qquad f = \overset{q}{\wedge} \bar{T}'^*_X\text{-part of } dg, \text{ for some } g \in C^{0,q-1}(X)\},
\end{aligned}$$

and

$$H^{0,q}(X) = Z^{0,q}(X)/B^{0,q}(X).$$

We shall be allowed simply to refer [11] and [2] concerning the properties of $H^{0,q}(\bar{M})$, $H_0^{0,q}(\bar{M})$, and $H^{0,q}(X)$. Namely we have, under the situation that $\partial \bar{M} = X$,

Pıoposition 3 (cf. [2] and [11] Proposition 6.6).

$$H_0^{0,q}(\bar{M}) = 0, \quad \textit{for } q \leqq n-1.$$

Corollary 3. $H^{0,q}(\bar{M}) \cong H^{0,q}(X)$ *if* $q < n-1$.

Similarly as in [5], Theorem 3.4.8, we have

Proposition 4. $H^{0,q}(\bar{M}) \cong H^{0,q}(M)$, *for* $q \geqq 1$.

Definition 3. A family $\{X_t\}_{t\in T}$ of CR manifolds is called a differentiable family if there exist a CR manifold $\mathscr{X}$, a C^∞ manifold T and a proper surjective smooth map $\pi: \mathscr{X} \to T$ such that $(X_t, T'_{X_t}) \cong (\pi^{-1}(t), T'_{\mathscr{X}} \cap (T_{\pi^{-1}(t)} \otimes \mathbf{C}))$.

We quote here Tanaka's stability theorem for holomorphic imbeddings.

Theorem 2. *Let* $\{X_t\}_{t\in T}$ *be a differentiable family of s. p. c. manifolds satisfying the following conditions.*

(i) dim $X_t \geqq 5$

(ii) dim $H^{0,1}(X_t)$ *does not depend on t.*

Then, for any $t_0 \in T$ *and holomorphic imbedding* $F_0 : X_{t_0} \hookrightarrow \mathbf{C}^N$, *there exist a neighbourhood* $T' \ni t_0$ *and a* C^∞ *map* $\mathscr{F} : \bigcup_{t \in T'} X_t \longrightarrow \mathbf{C}^N$ *such that* $\mathscr{F}|_{X_{t_0}} = F_0$ *and that* $F|_{X_t}$ *are holomorphic imbeddings for* $t \in T'$.

Proof. See [10], Theorem 9.4.

2. Bumping of S. P. C. Manifolds

Let X be an s. p. c. manifold of dimension $2n-1 \geqq 5$, let $x \in X$ be any point, and let $B(r)$ be the ball $\{(z_1, \cdots, z_n) \in \mathbf{C}^n \mid \sum_{k=1}^n |z_k|^2 < r\}$. Then, by Corollary 2, there exist a neighbourhood $U \ni x$ and an imbedding $F : U \hookrightarrow B(1)$ by holomorphic functions. $F(U)$ is then defined by an equation $\varphi = 0$ on $B(1/2)$, where φ is of class C^∞, $d\varphi \neq 0$ and $\partial\bar{\partial}\varphi > 0$.

Let ρ be a nonnegative C^∞ function on $B(1/2)$ such that supp$\rho \subset B(1/4)$ and $\rho = 1$ on $B(1/8)$. Then, for sufficiently small $\varepsilon > 0$, we have $\partial\bar{\partial}(\varphi - t\rho) > 0$ and $d(\varphi - t\rho) \neq 0$, for any $t \in [-\varepsilon, \varepsilon]$. Choose a C^∞ family of imbeddings $\tilde{F}: U \times [-\varepsilon, \varepsilon] \to B(1)$ such that

(*) $\tilde{F}(x, t) = F(x)$ for $x \in F^{-1}(B(1) \backslash B(1/4))$ and $t \in [-\varepsilon, \varepsilon]$.

(**) $\tilde{F}(U \times \{t\}) \cap B(1/2) = \{\varphi - t\rho = 0\}$.

Using $\tilde{F}$, we define a CR structure $T'_{\mathscr{X}}$ on $\mathscr{X} = X \times (-\varepsilon, \varepsilon)$ as follows:

Let $p : X \times (-\varepsilon, \varepsilon) \longrightarrow X$ and $\pi : X \times (-\varepsilon, \varepsilon) \longrightarrow (-\varepsilon, \varepsilon)$ be the projections. We put $T'_{\mathscr{X},(x,t)} = p^* T'_{X,x}$ for $x \in X \backslash U$, and $T'_{\mathscr{X},(x,t)} = \tilde{F}_{t*}^{-1}(T'_{\tilde{F}(U \times \{t\}), \tilde{F}(x,t)})$ for $x \in U$, where $\tilde{F}_t = \tilde{F}(\ , t)$. Then we have a differentiable family $\{X_t\}$ of s. p. c. manifolds defined by $(\mathscr{X}, T'_{\mathscr{X}})$ and π: $\mathscr{X} \longrightarrow (-\varepsilon, \varepsilon)$. We shall call it a *bumping family* of X with front U. Let $\bar{M}_0$ be a complex manifold with boundary such that $\partial M_0 \cong X_0$. Then, there exist a neighbourhood $W \supset F(U)$ in $B(1)$ and a holomorphic imbedding τ: $W \cap \{\varphi \leqq 0\} \hookrightarrow \bar{M}_0$ such that $\tau = F^{-1}$ on $F(U)$. Thus, for sufficiently small $t > 0$, we can patch $\bar{M}_0$ and $W \cap \{\varphi - t\rho \leqq 0\}$ via the identification $\tau(W \cap \{\varphi \leqq 0\}) \cong W \cap \{\varphi \leqq 0\} \subset W \cap \{\varphi - t\rho < 0\}$, and obtain complex manifolds with boundary $\bar{M}_t$ such that $\partial M_t \cong X_t$. On the other hand, for $t \leqq 0$, $\bar{M}_t := \bar{M}_0 \backslash \{\varphi - t\rho > 0\}$

are complex manifolds with boundary, if $|t|$ is sufficiently small, and we have $\partial M_t \cong X_t$, too. Fix ε so that we have a family of compact complex manifolds with boundary $\{\bar{M}_t\}$ with $\partial M_t \cong X_t$ for $|t| < \varepsilon$.

Let A be the maximal compact analytic set of M_0, then we have $H^q(M_0, \mathcal{O}_{M_0}) \cong H^q(A, \mathcal{O}_{M_0 | A})$ for $q \geqq 1$. (cf. Narasimhan [8]). From the definition of M_t, choosing ε smaller if necessary we may assume that the maximal compact analytic sets of M_t and their neighbourhoods are analytically equivalent for all t. Hence we can suppose $H^q(M_0, \mathcal{O}_{M_0}) \cong H^q(M_t, \mathcal{O}_{M_t})$ for $q \geqq 1$. Thus, combining Proposition 4 and Corollary 3 with this fact, we have

Proposition 5. $H^{0,1}(X_t) \cong H^{0,1}(X)$ *for* $|t| < \varepsilon$.

3. Proof of Theorem

We shall now prove that every compact s. p. c. manifold of dimension $\geqq 5$ is realizable as a hypersurface of a complex manifold. Let the notations be as in Section 2. By Theorem 1 we may assume that X is a CR submanifold of $\mathbf{C}^N$. We take a finite covering $\{V_i\}_{i=1}^m$ of X by open sets of $\mathbf{C}^N$ and systems of affine coordinates $(z_{i1}, \cdots, z_{iN})$ of $\mathbf{C}^N$ so that the projections $p_i: (z_{i1}, \cdots, z_{iN}) \longrightarrow (z_{i1}, \cdots, z_{in})$ define holomorphic imbeddings φ_i of $\overline{V_i \cap X}$ into $\overline{B(1)} \subset \mathbf{C}^n$. We may assume moreover that $\cup \varphi_i^{-1}(B(1/8)) = X$. Let $\{X_t\}$ be a bumping family of X with front $V_1 \cap X$. Then, by Proposition 5, we can apply Theorem 2 and obtain a differentiable family of holomorphic imbeddings $\Phi_1: \bigcup_{-\delta<t<\delta} X_t \longrightarrow \mathbf{C}^N$ for sufficiently small δ. Thus, there exist $t_1 > 0$ such that $\cup p_i^{-1}(B(1/8)) \cap V_i \supset \Phi_1(X_{t_1})$ and that p_i are one-to-one immersions on neighbourhoods in $\Phi_1(X_{t_1})$ of $\overline{V_i \cap \Phi_1(X_{t_1})}$. Let $\{\bar{M}_t\}$ be the corresponding family of complex manifolds with boundary. Then we have $\bar{M}_{t_1} \supset X$. Inductively we can define Φ_k and obtain a CR manifold $X_{t_1 \cdots t_k}$ as a member of a bumping family of $X_{t_1 \cdots t_{k-1}}$ with front $\Phi_{k-1}^{-1}(V_k \cap \Phi_{k-1}(X_{t_1 \cdots t_{k-1}}))$. *H*ere, $t_a > 0$, $a = 1, 2, \cdots, k$. Let $\bar{M}_{t_1 \cdots t_k}$ be the complex manifold with boundary corresponding to $X_{t_1 \cdots t_k}$. Clearly $M_{t_1 \cdots t_m} \supset X$ for sufficiently small $t_1, \cdots, t_m$.

Q. E. D.

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