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Jacobi-Perron Algorithms, Bi-Orthogonal Polynomials and Inverse Scattering Problems

By

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§ 0. Introduction

It is well-known that the spectral theory of a Jacobi matrix is essentially equivalent to the theory of orthogonal polynomials with respect to its spectral density on a real interval. This is also related to the Padé approximation of a Stieltjes integral by rational functions which gives us the inverse spectral theory of Jacobi matrices (See [19] and [20]).

On the other hand several people have shown that the equations of Toda lattices are described by Lax type formalism of Jacobi matrices and related to its spectral densities (See [2], [6], [7], [8] and [10]).

It seems to be interesting to ask whether these facts can be generalized to higher order linear difference operators. The purpose of this note is to construct 3rd order linear difference operators by Jacobi-Perron algorithms as a generalization of continued fractions (See [5]) and to prove an equivalence between linear evolution equations of spectral densities and Lax type equations (See Theorem 1 and 2).

§ 1. Jacobi-Perron Algorithms

Let two Radon measures $\mu_0(d\zeta)$, $\mu_1(d\zeta)$ be given on a compact set Γ in $\boldsymbol{C}$. We consider the Stieltjes integrals

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$$\omega_j(z) = \int_\Gamma \frac{\mu_j(d\zeta)}{z-\zeta}, \qquad 0 \leqq j \leqq 1, \tag{1.1}$$

which are holomorphic with respect to $z \in \boldsymbol{C} - \Gamma$. Then the functions $\omega_j(z)$ have Laurent expansions at the infinity as follows:

$$\omega_j(z) = \sum_{\nu=0}^{\infty} c_{j,\nu} z^{-\nu-1}, \tag{1.2}$$

where $c_{j,\nu}$ denote the moments:

$$c_{j,\nu} = \int_\Gamma \zeta^\nu \mu_j(d\zeta). \tag{1.3}$$

We assume the following "*regularity condition*" holds for $\mu_0(d\zeta)$ and $\mu_1(d\zeta)$:

($\mathscr{H}$.1) *All the determinants of order* $2k$ *and* $2k+1$ *for the sequence* $\{c_{0,\nu}, c_{1,\nu}\}_{0 \leqq \nu < \infty}$

$$\det \begin{bmatrix} (c_{0,i+j})_{0 \leqq i \leqq k-1, 0 \leqq j \leqq 2k-1} \\ (c_{1,i+j})_{0 \leqq i \leqq k-1, 0 \leqq j \leqq 2k-1} \end{bmatrix}, \qquad k \geqq 1, \tag{1.4}$$

$$\det \begin{bmatrix} (c_{0,i+j})_{0 \leqq i \leqq k, 0 \leqq j \leqq 2k} \\ (c_{1,i+j})_{0 \leqq i \leqq k-1, 0 \leqq j \leqq 2k} \end{bmatrix}, \qquad k \geqq 0, \tag{1.5}$$

are different from zero for $k \geqq 0$. *These will be denoted by* D_{2k} *and* D_{2k+1} *respectively. We put* D_0 *to be equal to* 1.

We can put

$$\omega_0 = \frac{u_1}{w_1}, \ \omega_1 = \frac{v_1}{w_1}, \tag{1.6}$$

such that u_1, v_1, w_1 are described as Laurent series as follows:

$$\begin{cases} u_1 = 1, \\ v_1 = \xi_0 + \dfrac{\xi_1}{z} + \dfrac{\xi_2}{z^2} + \cdots, \\ w_1 = \eta_{-1} z + \eta_0 + \dfrac{\eta_1}{z} + \dfrac{\eta_2}{z^2} + \cdots, \end{cases} \tag{1.7}$$

where $\eta_{-1} = 1/c_{0,0}$ and $\xi_0 = c_{1,0}/c_{0,0}$ are well-defined. We apply the Jacobi-Perron algorithm of 3rd order (see [5]) to (1.7) at the infinity $z = \infty$ in the following manner:

$$\begin{cases} \dfrac{v_1}{u_1} = \alpha_2 + \dfrac{u_2}{w_2}, \\ \dfrac{w_1}{u_1} = \beta_2 z + \beta_2' + \dfrac{v_2}{w_2}, \end{cases} \tag{1.8}$$

where $u_2/w_2=O(z^{-1})$ and $v_2/w_2=O(z^{-1})$. Namely we consider a projective transformation

$$(1.9)\qquad \begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & \alpha_2 \\ 0 & 1 & \beta_2 z+\beta_2' \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix},$$

between the points $u_1:v_1:w_1$ and $u_2:v_2:w_2$ in $\boldsymbol{C}P^2$. u_2/w_2 and v_2/w_2 are also expressed by Laurent series:

$$(1.10)\qquad \begin{cases} \dfrac{u_2}{w_2}=\displaystyle\sum_{\nu=0}^{\infty}\frac{c'_{0,\nu}}{z^{\nu+1}}, \\ \dfrac{v_2}{w_2}=\displaystyle\sum_{\nu=0}^{\infty}\frac{c'_{1,\nu}}{z^{\nu+1}}. \end{cases}$$

Then we can define the similar determinants D'_{2k} and D'_{2k+1} as D_{2k} and D_{2k+1} respectively for the sequence $\{c'_{0,\nu},\ c'_{1,\nu}\}_{0\leqq\nu<\infty}$. First we show

Lemma 1.

$$(1.11)_{2k}\qquad D'_{2k}=c_{0,0}^{-3k-1}D_{2k+1},$$

$$(1.11)_{2k+1}\qquad D'_{2k+1}=(-1)^k c_{0,0}^{-3k-2}D_{2k+2},$$

for $k\geqq 0$.

Proof. The relation (1.8) implies the following identities:

$$(1.12)\qquad \alpha_2 c_{0,n}+\sum_{\nu=0}^{n-1}c'_{0,\nu}c_{0,n-1-\nu}=c_{1,n},\quad n\geqq 0,$$

$$(1.13)\qquad \beta_2 c_{0,n}+\beta_2' c_{0,n-1}+\sum_{\nu=0}^{n-2}c'_{1,\nu}c_{0,n-\nu}=0,\ n\geqq 0.$$

Then we have firstly

$$(1.14)\qquad \alpha_2=\frac{c_{1,0}}{c_{0,0}},\ \beta_2=\frac{1}{c_{0,0}},\ \beta_2'=\frac{-c_{0,1}}{c_{0,0}^2}.$$

Successive uses of the relations $(1.12)\sim(1.14)$ give the formula

$$(1.15)\qquad D'_{2k}=\det\begin{pmatrix} \dfrac{c_{1,s+r+1}-\alpha_2 c_{0,s+r+1}}{c_{0,0}}-\displaystyle\sum_{\nu=0}^{r-1}\frac{c'_{0,\nu}c_{0,s+r-\nu}}{c_{0,0}}, & {\scriptstyle 0\leqq r\leqq k-1 \atop 0\leqq s\leqq 2k-1} \\ \dfrac{-\beta_2 c_{0,s+r+2}-\beta_2' c_{0,s+r+1}}{c_{0,0}}-\displaystyle\sum_{\nu=0}^{r-1}\frac{c'_{1,\nu}c_{0,s+r-\nu}}{c_{0,0}}, & {\scriptstyle 0\leqq r\leqq k-1 \atop 0\leqq s\leqq 2k-1} \end{pmatrix}$$

$$=\frac{1}{c_{0,0}}\det\left(\begin{array}{c|l} c_{0,0} & c_{0,1}\quad c_{0,2}\qquad\qquad c_{0,2k} \\ \hline c'_{0,k-1} & \dfrac{c_{1,s+r+1}-\alpha_2 c_{0,s+r+1}}{c_{0,0}}-\displaystyle\sum_{\nu=0}^{r-2}\frac{c'_{0,\nu}c_{0,s+r-\nu}}{c_{0,0}},\ {\scriptstyle 0\leqq r\leqq k-1 \atop 0\leqq s\leqq 2k-1} \\ c'_{1,k-1} & \dfrac{-\beta_2 c_{0,s+r+2}-\beta_2' c_{0,s+r+1}}{c_{0,0}}-\displaystyle\sum_{\nu=0}^{r-2}\frac{c'_{1,\nu}c_{0,s+r-\nu}}{c_{0,0}},\ {\scriptstyle 0\leqq r\leqq k-1 \atop 0\leqq s\leqq 2k-1} \end{array}\right),$$

for $c'_{0,-1}=0$, which is equal to $c_{0,0}^{-3k-1}D_{2k+1}$. We have thus proved $(1.11)_{2k}$. $(1.11)_{2k+1}$ can be proved similarly.

Since D_{2k} and D_{2k+1}, $k\geqq 0$, are all different from zero, D'_{2k} and D'_{2k+1}, $k\geqq 0$, are also different from zero. In particular we have $c'_{0,0}\neq 0$. This makes it possible to repeat the substitution (1.8) for $(v_2/u_2, w_2/u_2)$ instead of $(v_1/u_1, w_1/u_1)$ and so on. Therefore the regularity property allows us to consider successively the n-th Jacobi-Perron transform

$$(1.16)\qquad \begin{bmatrix} u_n \\ v_n \\ w_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & \alpha_{n+1} \\ 0 & 1 & \beta_{n+1}z+\beta'_{n+1} \end{bmatrix} \begin{bmatrix} u_{n+1} \\ v_{n+1} \\ w_{n+1} \end{bmatrix},$$

for $n\geqq 2$. By the same reason, we can also see all the coefficients $\beta_n=D_{n+1}/D_n$, $n\geqq 2$, are different from zero. If we take $\omega_0=v_0/u_0$, $\omega_1=w_0/u_0$, then u_0, v_0, w_0 are linearly related to u_n, v_n, w_n as follows:

$$(1.17)\qquad \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} p_n(z) & p_{n+1}(z) & p_{n+2}(z) \\ p'_n(z) & p'_{n+1}(z) & p'_{n+2}(z) \\ p''_n(z) & p''_{n+1}(z) & p''_{n+2}(z) \end{bmatrix} \begin{bmatrix} u_n \\ v_n \\ w_n \end{bmatrix}, \qquad n\geqq 3,$$

where $p_n(z)$, $p'_n(z)$, $p''_n(z)$ denote polynomials of exact degree $n-3$, $n-4$, $n-4$ respectively. The matrix in the right hand side will be abbreviated by $A_n(z)$. Note that $\det A_n(z)=1$, so that $A_n(z)$ and ${}^tA_n(z)^{-1}$ lie in $GL_3(\boldsymbol{C}[z])$. Therefore ${}^tA_n(z)^{-1}$ can be written

$$(1.18)\qquad {}^tA_n(z)^{-1} = \begin{bmatrix} \tilde{p}_n(z) & \tilde{q}_n(z) & \tilde{r}_n(z) \\ \tilde{p}'_n(z) & \tilde{q}'_n(z) & \tilde{r}'_n(z) \\ \tilde{p}''_n(z) & \tilde{q}''_n(z) & \tilde{r}''_n(z) \end{bmatrix},$$

where $\tilde{p}_n$, $\tilde{q}_n$, $\tilde{r}_n$, $\tilde{p}'_n$, $\tilde{q}'_n$, $\tilde{r}'_n$, $\tilde{p}''_n$, $\tilde{q}''_n$, $\tilde{r}''_n$ denote polynomials. This algorithm to compute sequences of polynomials $p_n(z)$, $p'_n(z)$, $p''_n(z)$ and $\tilde{p}_n(z)$, $\tilde{p}'_n(z)$, $\tilde{p}''_n(z)$ is nothing else than the usual Jacobi-Perron algorithm with respect to Laurent series.

Proposition 2. (i) *Each polynomial* $\varphi_n=p_n$, p'_n, p''_n *satisfies the difference equation*:

$$(1.19)\qquad \varphi_{n+3}=\varphi_n+\alpha_{n+1}\varphi_{n+1}+(\beta_{n+1}z+\beta'_{n+1})\varphi_{n+2}, \quad n\geqq 1,$$

(ii) *Each triple* $(\tilde{\varphi}_n, \tilde{\psi}_n, \tilde{\chi}_n)=(\tilde{p}_n, \tilde{q}_n, \tilde{r}_n)$, $(\tilde{p}'_n, \tilde{q}'_n, \tilde{r}'_n)$, $(\tilde{p}''_n, \tilde{q}''_n, \tilde{r}''_n)$

satisfies the difference equations

$$\left\{\begin{array}{l}\tilde{\varphi}_{n+1}=-\alpha_{n+1}\tilde{\varphi}_n+\tilde{\psi}_n,\\ \tilde{\psi}_{n+1}=-(\beta_{n+1}z+\beta'_{n+1})\tilde{\varphi}_n+\tilde{\chi}_n,\\ \tilde{\chi}_{n+1}=\tilde{\varphi}_n.\end{array}\right. \tag{1.20}$$

Proof. In fact these follow from the obvious recurrence equation for $A_n(z)$:

$$A_{n+1}(z)=A_n(z)\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & \alpha_{n+1}\\ 0 & 1 & \beta_{n+1}z+\beta'_{n+1}\end{bmatrix}. \tag{1.21}$$

Since all β_n are different from zero, we can see from (1.20) that the degrees of $\tilde{p}_n, \tilde{q}_n, \tilde{r}_n, \tilde{p}'_n, \tilde{q}'_n, \tilde{r}'_n, \tilde{p}''_n, \tilde{q}''_n, \tilde{r}''_n$ are exactly equal to $k-2$, $k-1$, $k-2$, $k-1$, k, $k-1$, $k-1$, $k-1$, $k-2$ if $n=2k$ and equal to $k-1$, k, $k-1$, k, k, $k-1$, $k-1$, k, $k-1$ if $n=2k+1$ respectively.

Proposition 3. (i) *For a pair of measures $\mu_0(d\zeta)$, $\mu_1(d\zeta)$ on Γ which have the regularity property ($\mathscr{H}$.1), the Jacobi-Perron algorithm can be carried out up to any stage. We have the following Padé approximation for $\omega_0(z)$, $\omega_1(z)$ at the infinity $z=\infty$:*

$$\left\{\begin{array}{l}\omega_0-p'_{2k}/p_{2k}=O(z^{-3k+3}),\\ \omega_1-p''_{2k}/p_{2k}=O(z^{-3k+4}),\quad k\geqq 1,\end{array}\right. \tag{1.22}$$

$$\left\{\begin{array}{l}\omega_0-p'_{2k+1}/p_{2k+1}=O(z^{-3k+2}),\\ \omega_1-p''_{2k+1}/p_{2k+1}=O(z^{-3k+2}),\quad k\geqq 1.\end{array}\right. \tag{1.23}$$

(ii) *The function*

$$u_n/u_0=\tilde{p}_n+\tilde{p}'_n\omega_0+\tilde{p}''_n\omega_1,\quad n\geqq 1, \tag{1.24}$$

has the property

$$u_n/u_0=O(z^{-n}). \tag{1.25}$$

Proof. First we want to prove (1.24). Since $u_{n+1}=-\alpha_{n+1}u_n+v_n$ and $v_n/u_n=\alpha_{n+1}+O(z^{-1})$, we have

$$\frac{u_{n+1}}{u_n}=O(z^{-1}). \tag{1.26}$$

This shows (1.25) by induction on n. Next we want to prove the first part of (1.22). The left hand side is equal to

$$(1.27)\quad \frac{v_0}{u_0}-\frac{p'_{2k}}{p_{2k}}=\frac{p'_{2k}u_{2k}+p'_{2k+1}v_{2k}+p'_{2k+2}w_{2k}}{p_{2k}u_{2k}+p_{2k+1}v_{2k}+p_{2k+2}w_{2k}}-\frac{p'_{2k}}{p_{2k}}$$
$$=\frac{(p_{2k}p'_{2k+1}-p'_{2k}p_{2k+1})v_{2k}+(p_{2k}p'_{2k+2}-p'_{2k}p_{2k+2})w_{2k}}{u_0p_{2k}}$$
$$=\frac{\tilde{r}''_{2k}v_{2k}-\tilde{q}''_{2k}w_{2k}}{u_0p_{2k}}$$
$$=\frac{u_{2k}\left(\tilde{r}''_{2k}\dfrac{v_{2k}}{u_{2k}}-\tilde{q}''_{2k}\dfrac{w_{2k}}{u_{2k}}\right)}{u_0p_{2k}}.$$

On the other hand, we have that $u_{2k}/u_0=O(z^{-2k})$, $v_{2k}/u_{2k}=O(1)$, $w_{2k}/u_{2k}=O(z)$, deg $p_{2k}=2k-3$, deg $\tilde{r}''_{2k}=k-2$, deg $\tilde{q}''_{2k}=k-1$. Thus,

$$(1.28)\quad \frac{v_0}{u_0}-\frac{p'_{2k}}{p_{2k}}=O(z^{-3k+3}).$$

For the second part we have

$$(1.29)\quad \frac{w_0}{u_0}-\frac{p''_{2k}}{p_{2k}}=\frac{-\tilde{r}'_{2k}v_{2k}+\tilde{q}'_{2k}w_{2k}}{u_0p_{2k}}$$
$$=O(z^{-3k+4}).$$

(1.23) can be proved similarly.

Conversely, under the condition ($\mathscr{H}$.1) the triples (p_n, p'_n, p''_n) or $(\tilde{p}_n, \tilde{p}'_n, \tilde{p}''_n)$ are the unique ones except for constant factors such that they satisfy (1.22), (1.23) or (1.25) among polynomials of the same degrees as (p_n, p'_n, p''_n) or $(\tilde{p}_n, \tilde{p}'_n, \tilde{p}''_n)$. Namely the triples (p_n, p'_n, p''_n) or $(\tilde{p}_n, \tilde{p}'_n, \tilde{p}''_n)$ are "*perfect systems*" in the sense of K. Mahler (See [13] and [14]). In fact (1.22) and (1.23) imply

$$(1.22)'\quad \begin{cases} \omega_0p_{2k}-p'_{2k}=O(z^{-k}), \\ \omega_1p_{2k}-p''_{2k}=O(z^{-k+1}), \end{cases}$$

$$(1.23)'\quad \begin{cases} \omega_0p_{2k+1}-p'_{2k+1}=O(z^{-k}), \\ \omega_1p_{2k+1}-p''_{2k+1}=O(z^{-k}), \end{cases}$$

which give linear equations with respect to the coefficients of p_{2k}, p'_{2k}, p''_{2k} or p_{2k+1}, p'_{2k+1}, p''_{2k+1}. The determinants coincide with (1.4) and (1.5). Owing to ($\mathscr{H}$.1), (1.22)′ and (1.23)′ determine uniquely the polynomials $p_{2k}(z)$ and $p_{2k+1}(z)$ except for constant factors. These give us the following formulae:

$$(1.30)\qquad p_{2k}(z)=\begin{vmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,2k-3} \\ c_{0,1} & c_{0,2} & \cdots & c_{0,2k-2} \\ \vdots & \vdots & & \vdots \\ c_{0,k-2} & c_{0,k-1} & \cdots & c_{0,3k-5} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,2k-3} \\ \vdots & \vdots & & \vdots \\ c_{1,k-3} & c_{1,k-2} & \cdots & c_{1,3k-6} \\ 1 & z & \cdots & z^{2k-3} \end{vmatrix},$$

$$(1.31)\qquad p_{2k+1}(z)=\begin{vmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,2k-2} \\ c_{0,1} & c_{0,2} & \cdots & c_{0,2k-1} \\ \vdots & \vdots & & \vdots \\ c_{0,k-2} & c_{0,k-1} & \cdots & c_{0,3k-4} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,2k-2} \\ \vdots & \vdots & & \vdots \\ c_{1,k-2} & c_{1,k-1} & \cdots & c_{1,3k-4} \\ 1 & z & \cdots & z^{2k-2} \end{vmatrix}.$$

These are rewritten in integral form by using the relation (1.3) as follows:

$$(1.32)\qquad p_{2k}(z)=\sum_{\substack{i_1<\cdots<i_{k-1}\\ j_1<\cdots<j_{k-2}\\ (i_1,\cdots,i_{k-1})\amalg(j_1,\cdots,j_{k-2})=(1,2,\cdots,2k-3)}}\int_\Gamma (z-z_1)(z-z_2)\cdots(z-z_{2k-3})$$
$$\cdot\prod_{1\leq i<j\leq 2k-3}(z_j-z_i)\cdot\prod_{1\leq\alpha<\beta\leq k-1}(z_{i_\beta}-z_{i_\alpha})\cdot\prod_{1\leq\gamma<\delta\leq k-2}(z_{j_\delta}-z_{j_\gamma})$$
$$\cdot\mu_0(dz_{i_1})\cdots\mu_0(dz_{i_{k-1}})\cdot\mu_1(dz_{j_1})\cdots\mu_1(dz_{j_{k-2}}),$$

$$(1.33)\qquad p_{2k+1}(z)=\sum_{\substack{i_1<\cdots<i_{k-1}\\ j_1<\cdots<j_{k-1}\\ (i_1,\cdots,i_{k-1})\amalg(j_1,\cdots,j_{k-1})=(1,\cdots,2k-2)}}\int_\Gamma (z-z_1)(z-z_2)\cdots(z-z_{2k-2})$$
$$\cdot\prod_{1\leq i<j\leq 2k-2}(z_j-z_i)\cdot\prod_{1\leq\alpha<\beta\leq k-1}(z_{i_\beta}-z_{i_\alpha})\cdot\prod_{1\leq\gamma<\delta\leq k-1}(z_{j_\delta}-z_{j_\gamma})$$
$$\cdot\mu_0(dz_{i_1})\cdots\mu_0(dz_{i_{k-1}})\mu_1(dz_{j_1})\cdots\mu_1(dz_{j_{k-1}}).$$

Owing to (1.1), the left hand side of (1.22)′ is equal to

$$(1.34)\qquad \int_\Gamma\frac{p_{2k}(z)\mu_0(d\zeta)}{z-\zeta}-p'_{2k}(z)$$
$$=\underbrace{\int_\Gamma\frac{(p_{2k}(z)-p_{2k}(\zeta))\mu_0(d\zeta)}{z-\zeta}-p'_{2k}(z)}_{\text{polynomial part}}+\int_\Gamma\frac{p_{2k}(\zeta)\mu_0(d\zeta)}{z-\zeta}.$$

Therefore we have

$$(1.35)\qquad p'_{2k}(z)=\int_\Gamma \frac{p_{2k}(z)-p_{2k}(\zeta)}{z-\zeta}\mu_0(d\zeta),$$

$$(1.36)\qquad O(z^{-k})=\int_\Gamma \frac{p_{2k}(\zeta)\mu_0(d\zeta)}{z-\zeta}.$$

Similarly

$$(1.37)\qquad p''_{2k}(z)=\int_\Gamma \frac{p_{2k}(z)-p_{2k}(\zeta)}{z-\zeta}\mu_1(d\zeta),$$

$$(1.38)\qquad O(z^{-k+1})=\int_\Gamma \frac{p_{2k}(\zeta)\mu_1(d\zeta)}{z-\zeta},$$

$$(1.39)\qquad p'_{2k+1}(z)=\int_\Gamma \frac{p_{2k+1}(z)-p_{2k+1}(\zeta)}{z-\zeta}\mu_0(d\zeta),$$

$$(1.40)\qquad O(z^{-k})=\int_\Gamma \frac{p_{2k+1}(\zeta)\mu_0(d\zeta)}{z-\zeta},$$

$$(1.41)\qquad p''_{2k+1}(z)=\int_\Gamma \frac{p_{2k+1}(z)-p_{2k+1}(\zeta)}{z-\zeta}\mu_1(d\zeta),$$

$$(1.42)\qquad O(z^{-k})=\int_\Gamma \frac{p_{2k+1}(\zeta)\mu_1(d\zeta)}{z-\zeta}.$$

In particular from (1.36), (1.38), (1.40) and (1.42) the following bi-orthogonality holds:

$$(1.43)\qquad \int_\Gamma \zeta^n p_{2k}(\zeta)\mu_0(d\zeta)=0,\qquad 0\leqq n\leqq k-2,$$

$$(1.44)\qquad \int_\Gamma \zeta^n p_{2k}(\zeta)\mu_1(d\zeta)=0,\qquad 0\leqq n\leqq k-3,$$

$$(1.45)\qquad \int_\Gamma \zeta^n p_{2k+1}(\zeta)\mu_0(d\zeta)=0,\qquad 0\leqq n\leqq k-2,$$

$$(1.46)\qquad \int_\Gamma \zeta^n p_{2k+1}(\zeta)\mu_1(d\zeta)=0,\qquad 0\leqq n\leqq k-2.$$

(1.19) can be rewritten as follows:

$$(1.19)'\qquad z\varphi_{n+2}=-\frac{\varphi_n}{\beta_{n+1}}-\frac{\alpha_{n+1}}{\beta_{n+1}}\varphi_{n+1}-\frac{\beta'_{n+1}}{\beta_{n+1}}\varphi_{n+2}+\frac{\varphi_{n+3}}{\beta_{n+1}}.$$

Namely the operators of multiplication by z gives a matrix representation as follows:

$$(1.47)\qquad \begin{aligned} &z(\varphi_3, \varphi_4, \varphi_5, \cdots) \\ &=(\varphi_3, \varphi_4, \varphi_5, \cdots) \end{aligned} \begin{pmatrix} -\frac{\beta_2'}{\beta_2} & -\frac{\alpha_3}{\beta_3} & -\frac{1}{\beta_4} & 0 & & \\ \frac{1}{\beta_2} & -\frac{\beta_3'}{\beta_3} & -\frac{\alpha_4}{\beta_4} & -\frac{1}{\beta_5} & 0 & \\ 0 & \frac{1}{\beta_3} & -\frac{\beta_4'}{\beta_4} & -\frac{\alpha_5}{\beta_5} & -\frac{1}{\beta_6} & \\ & \ddots & \ddots & \ddots & \ddots & \\ & 0 & & & & \\ & & \ddots & & & \end{pmatrix}.$$

The matrix in the right hand side will be denoted by L. Then the pair $(\beta_{n+1}\tilde{p}_n', \beta_{n+1}\tilde{p}_n'')$ satisfy the adjoint difference equation:

$$(1.48)\qquad z\begin{bmatrix} \beta_2\tilde{p}_1' & \beta_2\tilde{p}_1'' \\ \beta_3\tilde{p}_2' & \beta_3\tilde{p}_2'' \\ \vdots & \vdots \end{bmatrix} = L\begin{bmatrix} \beta_2\tilde{p}_1' & \beta_2\tilde{p}_1'' \\ \beta_3\tilde{p}_2' & \beta_3\tilde{p}_2'' \\ \vdots & \vdots \end{bmatrix}.$$

Let E be the Banach space consisting of continuous functions on Γ which can be approximated uniformly by polynomials. Then the operator L defines a bounded operator on E. We denote by E^* the dual space of E. Then $e_{n-3}=p_n(\zeta)=p_n(L)e_0$, $n\geqq 3$, or $e_{n-1}^*=(\beta_{n+1}\tilde{p}_n', \beta_{n+1}\tilde{p}_n'')=\beta_{n+1}\tilde{p}_n'(L)e_0^*+\beta_{n+1}\tilde{p}_n''(L)e_1^*$, $n\geqq 1$, give dual bases each other of E or E^*:

$$(1.49)\qquad \int_\Gamma p_{n+2}(\zeta)\,(\tilde{p}_m'(\zeta)\mu_0(d\zeta)+\tilde{p}_m''(\zeta)\mu_1(d\zeta))=0,$$

for $n\neq m$, n, $m\geqq 1$. We can also prove the following formulae:

Proposition 4. *For* $n\geqq 1$,

$$(1.50)\qquad \tilde{p}_n(z)=\int_\Gamma \frac{\tilde{p}_n'(z)-\tilde{p}_n'(\zeta)}{z-\zeta}\mu_0(d\zeta)+\int_\Gamma \frac{\tilde{p}_n''(z)-\tilde{p}_n''(\zeta)}{z-\zeta}\mu_1(d\zeta),$$

and

$$(1.51)\qquad u_n(z)/u_0(z)=\int_\Gamma \frac{\tilde{p}_n'(\zeta)\mu_0(d\zeta)}{z-\zeta}+\int_\Gamma \frac{\tilde{p}_n''(\zeta)\mu_1(d\zeta)}{z-\zeta}.$$

These imply that the matrix elements of the resolvent of L, $(e_m^*, (z-L)^{-1}e_n)$ are described by the integral representation

$$(1.52)\qquad (e_m^*, (z-L)^{-1}e_n)=\int_\Gamma \frac{p_{n+3}(\zeta)\,(\tilde{p}_{m+1}'(\zeta)\mu_0(d\zeta)+\tilde{p}_{m+1}''(\zeta)\mu_1(d\zeta))}{z-\zeta},$$

m, $n\geqq 0$. In particular,

$$(1.53)\qquad \begin{cases}\omega_0=(e_0^*,(z-L)^{-1}e_0),\\ \omega_1=(e_1^*,(z-L)^{-1}e_0).\end{cases}$$

The operator L has thus been reconstructed from the spectral densities $\mu_0(d\zeta)$, $\mu_1(d\zeta)$ such that L coincides with the multiplication of the variable z with respect to the bi-orthogonal system of polynomials $p_{n+3}(z)$ and $(\tilde{p}'_{n+1}(z), \tilde{p}''_{n+1}(z))$. This result can be summarized in the following way.

Theorem 1. *Under the regularity condition* ($\mathscr{H}$.1), *we can uniquely construct by Jacobi-Perron algorithm a bounded operator L on the Banach space E such that its spectral densities coincide with the pair $\mu_0(d\zeta)$, $\mu_1(d\zeta)$ on Γ.*

§ 2. Spectral Densities and Lax Type Equations

We want to show that Lax type equations for the operators constructed in §1 are equivalent to linear differential equations for their spectral densities.

Let $\omega=\{\omega_{\sigma\tau}(\zeta, d\zeta)\}_{0\leqq\sigma\leqq N-1,\ 0\leqq\tau\leqq M-1}$, $\zeta\in\boldsymbol{C}$, be a matrix valued continuous 1-form in a neighbourhood of a rectifiable curve Γ in $\boldsymbol{C}$. Let $e_i=\{e_i^{(\tau)}(\zeta),\ 0\leqq\tau\leqq M-1\}$, $0\leqq i<\infty$, and $e_j^*=\{e_j^{*(\sigma)}(\zeta),\ 0\leqq\sigma\leqq N-1\}$, $0\leqq j<\infty$, be two linearly independent systems of continuous functions on Γ such that the following bi-orthogonal relations hold:

$$(2.1)\qquad \begin{aligned}\delta_{ij}&=\langle e_j^*|\omega|e_i\rangle\\ &=\sum_{\sigma,\tau}\int_\Gamma e_j^{*(\sigma)}(\zeta)\omega_{\sigma\tau}(\zeta,\ d\zeta)\,e_i^{(\tau)}(\zeta).\end{aligned}$$

We assume further that there exist two suitable Banach spaces E and E^* which are generated by the dual bases $\{e_i=\langle e_i^{(\tau)}\rangle,\ 0\leqq i<\infty\}$ and $\{e_i^*=\langle e_i^{*(\sigma)}\rangle,\ 0\leqq i<\infty\}$ respectively. The duality is denoted by

$$(2.2)\qquad \begin{aligned}&E^*\times E\mapsto\boldsymbol{C}\\ &(x^*,\ y)\to(x^*,\ y)_\omega=\langle x^*|\omega|y\rangle,\end{aligned}$$

such that the right hand side of (2.1) is equal to $\langle e_j^*|\omega|e_i\rangle$. We assume further that e_i, e_j^* and ω depend differentiably on time t in a neighbourhood U of the origin and satisfy the following order preserving properties:

($\mathscr{H}$.2) *$\dot{e}_i$ and $\dot{e}_j^*$ are linear combinations of $e_{i'}$, $i \geqq i'$, and $e_{j'}^*$, $j \geqq j'$, respectively.*

We consider linear differential equation for $\omega(\zeta, d\zeta)$ of the following type:

$$(\mathscr{E}_1) \quad \dot{\omega}(\zeta, d\zeta) \left(=\frac{d\omega}{dt}(\zeta, d\zeta)\right)=\omega(\zeta, d\zeta)f(t, \zeta)=f^*(t, \zeta)\omega(\zeta, d\zeta)$$

for suitable continuous square matrices $f(t, \zeta)=(f_{\alpha\beta})$ and $f^*(t, \zeta)=(f^*_{\alpha\beta})$ on Γ of order M and N respectively. Let L be the operator on E such that its spectral density matrix coincides with $\omega(\zeta, d\zeta)$:

$$(2.3) \qquad (e_j^*, Le_i)_\omega=\langle e_j^* | \omega L | e_i\rangle=\sum_{\sigma,\tau}\int_\Gamma \zeta e_j^{*(\sigma)}(\zeta)\omega_{\sigma\tau}(\zeta, d\zeta)e_i^{(\tau)}(\zeta).$$

Further we define the operators $f(t, L)$ and $f^*(t, L^*)$ respectively as follows:

$$(2.4) \qquad (e_j^*, f(t, L)e_i)_\omega=\sum_{\sigma,\rho,\tau}\int_\Gamma e_j^{*(\sigma)}(\zeta)\omega_{\sigma\rho}(\zeta, d\zeta)f_{\rho\tau}(t, \zeta)e_i^{(\tau)}(\zeta),$$

$$(2.5) \qquad (f^*(t, L^*)e_j^*, e_i)_\omega=\sum_{\sigma,\rho,\tau}\int_\Gamma e_j^{*(\sigma)}(\zeta)f^*_{\sigma\rho}(t, \zeta)\omega_{\rho\tau}(\zeta, d\zeta)e_i^{(\tau)}(\zeta).$$

Definition. *Let A (or A^*) be a bounded operator on E (or E^*) defined by an infinite matrix $(a_{ij})_{0 \leqq i\, j<\infty}$ with respect to the bases $\{e_i\}$ and $\{e_j^*\}$ such that $(e_j^*, Ae_i)=a_{ij}$. Then $A^{(+)}$, $A^{(-)}$ and $A^{(0)}$ are defined as follows:*

$$(2.6) \qquad (e_j^*, A^{(+)}e_i)=\begin{cases} a_{ij} & \textit{if} \quad i>j \\ \frac{1}{2}a_{ii} & \textit{if} \quad i=j \\ 0 & \textit{if} \quad i<j \end{cases}$$

$$(2.7) \qquad (e_j^*, A^{(0)}e_i)=\begin{cases} 0 & \textit{if} \quad i\neq j \\ \frac{1}{2}a_{ii} & \textit{if} \quad i=j \end{cases}$$

and $A^{(-)}=A-A^{(+)}$. Remark $(A^{(+)})^*=(A^*)^{(-)}$, $(A^{(-)})^*=(A^*)^{(+)}$.

Then we have the following theorem:

Theorem 2. *Suppose that the measures $\omega_{\sigma\tau}(\zeta, d\zeta)$ are non-degenerate and normalized such that we can choose $e_i(\zeta)$, $e_j^*(\zeta)$ as follows:*

$$(2.8)\qquad \begin{cases} e_i(\zeta)={}^t(0,\ \cdots,\ 0,\ \underset{(1+i)\text{-}th}{1},\ 0,\ \cdots,\ 0), & 0\leqq i\leqq M-1, \\ e_j^*(\zeta)=(0,\ \cdots,\ 0,\ \underset{(1+j)\text{-}th}{1},\ 0,\ \cdots,\ 0), & 0\leqq j\leqq N-1. \end{cases}$$

Then the differential equation ($\mathscr{E}_1$) *is equivalent to the Lax type nonlinear equations*:

$$(\mathscr{E}_2)\qquad \dot{L}=[f(t,\ L)^{(+)},\ L],$$

$$(\mathscr{E}_2^*)\qquad \dot{L}^*=[f^*(t,\ L^*)^{(+)},\ L^*].$$

Proof. We firstly show that ($\mathscr{E}_1$) implies ($\mathscr{E}_2$) and ($\mathscr{E}_2^*$). By Leibniz rule, the differentiation of (2.1) with respect to t gives

$$(2.9)\qquad 0=\langle \dot{e}_j^*\ |\omega|\ e_i\rangle+\langle e_j^*\ |\dot{\omega}\ |e_i\rangle+\langle e_j^*\ |\ \omega|\ \dot{e}_i\rangle.$$

According to ($\mathscr{H}$.2), $\dot{e}_i$ and $\dot{e}_j^*$ are described as follows:

$$(2.10)\qquad \dot{e}_i=\sum_{i\geqq i'}\xi_{i'i}e_{i'},$$

$$(2.11)\qquad \dot{e}_j^*=\sum_{j\geqq j'}\eta_{j'j}e_{j'}^*.$$

Therefore from (2.1), (2.10) and (2.11)

$$(2.12)\qquad (e_j^*,\ f(t,\ L)e_i)_\omega+\xi_{ji}=0,\quad i>j,$$

$$(2.13)\qquad (f^*(t,\ L^*)e_j^*,\ e_i)_\omega+\eta_{ij}=0,\quad i<j,$$

and

$$(2.14)\qquad (e_j^*,\ f(t,\ L)e_i)_\omega+\xi_{ii}+\eta_{ii}=0,\quad i=j.$$

By taking $\xi_{ii}=\eta_{ii}$, we can determine uniquely ξ_{ii} as follows:

$$(2.15)\qquad \xi_{ii}=\eta_{ii}=-\frac{1}{2}(e_i^*,\ f(t,\ L)e_i)_\omega.$$

On the other hand

$$\begin{aligned}(2.16)\qquad \dot{\overbrace{(e_j^*,\ Le_i)_\omega}}&=\dot{\overbrace{\langle e_j^*\ |\omega L|\ e_i\rangle}}\\ &=\langle \dot{e}_j^*\ |\omega L|\ e_i\rangle+\langle e_j^*\ |\dot{\omega}L|\ e_i\rangle+\langle e_j^*\ |\omega L|\ \dot{e}_i\rangle\\ &=\sum_{j\geqq j'}\eta_{j'j}\langle e_{j'}^*\ |\omega L|\ e_i\rangle+\langle e_j^*\ |\ \omega f(t,\ L)L|\ e_i\rangle+\sum_{i\geqq i'}\langle e_j^*\ |\ \omega L|\ e_{i'}\rangle\xi_{i'i}\\ &=-\sum_{j>j'\geqq 0}\langle e_j^*\ |\omega f(t,\ L)\ |\ e_{j'}\rangle\langle e_{j'}^*\ |\omega L|\ e_i\rangle+\langle e_j^*\ |\omega f(t,\ L)L|\ e_i\rangle\\ &\quad-\sum_{i>i'\geqq 0}\langle e_j^*\ |\omega L|\ e_{i'}\rangle\langle e_{i'}^*\ |\omega f(t,L)\ |\ e_i\rangle-\frac{1}{2}\langle e_j^*\ |\omega f(t,\ L)\ |\ e_j\rangle\cdot\\ &\quad\cdot\langle e_j^*\ |\omega L|\ e_i\rangle-\frac{1}{2}\langle e_j^*\ |\omega L|\ e_i\rangle\langle e_i^*\ |\omega f(t,\ L)\ |e_i\rangle.\end{aligned}$$

By using the relations of completeness, we have

$$(2.17)\qquad \langle e_j^* |\omega f(t, L) L| e_i\rangle = \sum_{k=0}^{\infty} \langle e_j^* |\omega f(t, L) | e_k\rangle\langle e_k^* |\omega L| e_i\rangle.$$

Therefore

$$\begin{aligned}(2.18)\qquad \overset{\bullet}{\overbrace{(e_j^*, Le_i)_\omega}} &= \sum_{j'>j} \langle e_j^* |\omega f(t, L) | e_{j'}\rangle\langle e_{j'}^* |\omega L| e_i\rangle \\ &\quad - \sum_{i>i'\geqq 0} \langle e_j^* |\omega L| e_{i'}\rangle\langle e_{i'}^* |\omega f(t, L) | e_i\rangle + \frac{1}{2}\langle e_j^* |\omega f(t, L) | e_j\rangle \\ &\qquad\qquad \cdot\langle e_j^* | \omega L| e_i\rangle \\ &\quad -\frac{1}{2}\langle e_j^* |\omega L| e_i\rangle\langle e_i^* |\omega f(t, L) | e_i\rangle \\ &= \sum_{k=0}^{\infty} (e_j^*, f(t, L)^{(+)}e_k)_\omega (e_k^*, Le_i)_\omega - \sum_{k=0}^{\infty} (e_j^*, Le_k)_\omega \\ &\qquad\qquad \cdot (e_k^*, f(t, L)^{(+)}e_i)_\omega \\ &= (e_j^*, [f(t, L)^{(+)}, L]e_i)_\omega.\end{aligned}$$

$(\mathscr{E}_2^*)$ can be deduced similarly.

Conversely, under the condition $(\mathscr{H}.2)$, suppose that $(\mathscr{E}_2)$ or $(\mathscr{E}_2^*)$ holds. Then for $0\leqq i\leqq N-1$, and $0\leqq j\leqq M-1$,

$$(2.19)\qquad \int_\Gamma \frac{\omega_{ij}(\zeta, d\zeta)}{z-\zeta} = \langle e_j^* |\omega(z-L)^{-1}| e_i\rangle = (e_j^*, (z-L)^{-1}e_i)_\omega.$$

By differentiation with respect to t, the left hand side is equal to

$$(2.20)\qquad \int_\Gamma \frac{\dot\omega_{ij}(\zeta, d\zeta)}{z-\zeta},$$

while the right hand side is equal to

$$\begin{aligned}(2.21)\qquad & (e_j^*, \{f(t, L)^{(+)}(z-L)^{-1} - (z-L)^{-1}f(t, L)^{(+)}\}e_i)_\omega \\ &= (e_j^*, (f(t, L) - f(t, L)^{(-)})(z-L)^{-1}e_i)_\omega \\ &\qquad\qquad - (e_j^*, (z-L)^{-1}f(t, L)^{(+)}e_i)_\omega \\ &= (e_j^*, f(t, L)(z-L)^{-1}e_i)_\omega - \sum_{k=0}^{\infty} (f^*(t, L^*)^{(+)}e_j^*, e_k)_\omega \\ &\quad \cdot (e_k^*, (z-L)^{-1}e_i)_\omega - \sum_{k=0}^{\infty} (e_j^*, (z-L)^{-1}e_k)_\omega \cdot (e_k^*, f(t, L)^{(+)}e_i)_\omega.\end{aligned}$$

From the normalization (2.8), $(f^*(t, L^*)^{(+)}e_j^*, e_k)_\omega$ and $(e_k^*, f(t, L)^{(+)} \cdot e_i)_\omega$ vanish for $0\leqq i\leqq M-1$, $0\leqq j\leqq N-1$, $0\leqq k<+\infty$. Therefore (2.21) is equal to

$$(2.22)\qquad (e_j^*, f(t, L)(z-L)^{-1}e_i)_\omega.$$

And we have the equality;

(2.23)
$$\int_\Gamma \frac{\dot{\omega}_{ij}(\zeta,\ d\zeta)}{z-\zeta} = \int_\Gamma \frac{\{\omega(\zeta,\ d\zeta) f(t,\ \zeta)\}_{ij}}{z-\zeta},$$

which implies $(\mathscr{E}_1)$:

(2.24)
$$\dot{\omega}_{ij}(\zeta,\ d\zeta) = \sum_{k=0}^{M-1} \omega_{ik}(\zeta,\ d\zeta) f_{kj}(t,\ \zeta) \text{ on } \Gamma.$$

The theorem has been completely proved.

§ 3. Examples

(1) Pochhammer integrals (see [3]).

Let α, β, γ be complex numbers such that Re α, Re β, Re $\gamma > 0$. The 1-form $\omega(x,\ dx) = x^\alpha |1-x|^\beta |t-x|^\gamma dx$, $t > 1$, gives Radon measures $\mu_0(dx)$ and $\mu_1(dx)$ on each interval $[0,\ 1]$ or $[1,\ t]$. We define the polynomials p_{2k+1} of degree $2k-2$, $k \geqq 1$, and p_{2k} of degree $2k-3$, $k \geqq 2$, respectively by the following formulae. For some constants c_k, c'_k and c''_k

(3.1)
$$c_k\left(\frac{d}{dx}\right)^{k-1}[x^{\alpha+k-1}(x-1)^{\beta+k-1}(x-t)^{\gamma+k-1}]$$
$$= (k-1)!\, x^\alpha (x-1)^\beta (x-t)^\gamma p_{2k+1}(x),$$

and

(3.2)
$$c'_k\left(\frac{d}{dx}\right)^{k-2}[x^{\alpha+k-2}(x-1)^{\beta+k-1}(x-t)^{\gamma+k-2}]$$
$$+ c''_k\left(\frac{d}{dx}\right)^{k-2}[x^{\alpha+k-2}(x-1)^{\beta+k-2}(x-t)^{\gamma+k-1}]$$
$$= (k-2)!\, x^\alpha (x-1)^\beta (x-t)^\gamma p_{2k}(x),$$

where c'_k and c''_k are to be chosen such that

(3.3)
$$c'_k \int_0^1 x^{\alpha+k-2}(x-1)^{\beta+k-1}(x-t)^{\gamma+k-2} dx$$
$$+ c''_k \int_0^1 x^{\alpha+k-2}(x-1)^{\beta+k-2}(x-t)^{\gamma+k-1} dx = 0.$$

We express c'_k and c''_k by

(3.4)
$$c'_k = \rho_k \hat{c}'_k, \quad c''_k = \rho_k \hat{c}''_k,$$

where

(3.5)
$$\begin{cases} \hat{c}'_k = \displaystyle\int_0^1 x^{\alpha+k-2}(x-1)^{\beta+k-2}(x-t)^{\gamma+k-1} dx, \\ \hat{c}''_k = -\displaystyle\int_0^1 x^{\alpha+k-2}(x-1)^{\beta+k-1}(x-t)^{\gamma+k-2} dx, \end{cases}$$

for a suitable constant ρ_k which will be determined later.

By partial integration we have the bi-orthogonal relations (1.43) $\sim$(1.46). We want to compute explicitely α_{n+1}, β_{n+1} and β'_{n+1} in (1.47). We put

$$\lambda_0=-k,\ \lambda_1=\alpha+k-1,\ \lambda_2=\beta+k-1,\ \lambda_3=\gamma+k-1, \tag{3.6}$$

and consider the integral of Pochhammer

$$F(\lambda_0, \lambda_1, \lambda_2, \lambda_3)=\oint (y-x)^{\lambda_0}y^{\lambda_1}(y-1)^{\lambda_2}(y-t)^{\lambda_3}dy. \tag{3.7}$$

Then $x^\alpha(x-1)^\beta(x-t)^\gamma p_n(x)$ can be identified with F, except for constant factors:

$$x^\alpha(x-1)^\beta(x-t)^\gamma p_{2k+1}(x)=c_kF(\lambda_0, \lambda_1, \lambda_2, \lambda_3), \tag{3.8}$$

$$x^\alpha(x-1)^\beta(x-t)^\gamma p_{2k}(x)=c'_kF(\lambda_0+1, \lambda_1-1, \lambda_2, \lambda_3-1) +c''_kF(\lambda_0+1, \lambda_1-1, \lambda_2-1, \lambda_3), \tag{3.9}$$

if the integration (3.7) is taken to be the residue around x. A general result about difference systems in [1] shows that $F(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ satisfies a maximal overdetermined linear difference equations with coefficients of rational functions of λ_0, λ_1, λ_2, λ_3. In fact this follows from the following lemma.

Lemma 3.1. *(Stokes formula). The integral*

$$\int(y-x)^{\lambda_0}y^{\lambda_1}(y-1)^{\lambda_2}(y-t)^{\lambda_3}\left\{\frac{d\phi(y)}{dy}+\left(\frac{\lambda_0}{y-x}+\frac{\lambda_1}{y}+\frac{\lambda_2}{y-1}+\frac{\lambda_3}{y-t}\right)\phi(y)\right\}dy \tag{3.10}$$

vanishes for an arbitrary polynomial $\phi(y)$.

We use (3.10) to get an explicit expression of (1.47). α_{n+1}, β_{n+1} and β'_{n+1} in (1.47) can be obtained by solving the following equations:

$$\begin{aligned}&-\frac{c_{k+1}y(y-1)(y-t)}{y-x}+\left\{c'_k\frac{y-x}{y(y-t)}+c''_k\frac{y-x}{y(y-1)}\right\}\\&+\alpha_{2k+1}c_k+(\beta_{2k+1}x+\beta'_{2k+1})\{c'_{k+1}(y-1)+c''_{k+1}(y-t)\}\\&=\frac{d\phi_1(y)}{dy}+\left\{\frac{\lambda_0}{y-x}+\frac{\lambda_1}{y}+\frac{\lambda_2}{y-1}+\frac{\lambda_3}{y-t}\right\}\phi_1(y),\end{aligned} \tag{3.11}$$

for a polynomial $\phi_1(y)$ of degree 3 in case $n=2k$, $k\geqq 2$, and

$$(3.12)\qquad -\frac{c'_{k+2}y(y-1)^2(y-t)+c''_{k+2}y(y-1)(y-t)^2}{y-x}$$
$$+c_k+\alpha_{2k+2}\{c'_{k+1}(y-1)+c''_{k+1}(y-t)\}$$
$$+(\beta_{2k+2}x+\beta'_{2k+2})c_{k+1}\frac{y(y-1)(y-t)}{y-x}$$
$$=\frac{d\phi_2(y)}{dy}+\left\{\frac{\lambda_0}{y-x}+\frac{\lambda_1}{y}+\frac{\lambda_2}{y-1}+\frac{\lambda_3}{y-t}\right\}\phi_2(y),$$

for a polynomial $\phi_2(y)$ of degree 4, in case of $n=2k+1$, $k\geqq 1$.

We can solve (3.11) and (3.12) by elementary computation and have the following result:

Lemma 3.2. (i) *Case* $n=2k$, $k\geqq 2$.

$$(3.13)\qquad \phi_1(y)=-\frac{c_{k+1}}{\lambda_0}y(y-1)(y-t)-\frac{(c'_k+c''_k t)}{\lambda_1 t^2}(y-x)(y-1)(y-t)$$
$$+\frac{c''_k}{\lambda_2(1-t)}(y-x)y(y-t)+\frac{c'_k}{\lambda_3 t^2(t-1)}(y-x)y(y-1),$$

and

$$(3.14)\qquad \frac{(3+\lambda_1+\lambda_2+\lambda_3)}{\lambda_0}c_{k+1}=(3+\lambda_\infty)\left\{-\frac{c'_k+c''_k t}{\lambda_1 t^2}+\frac{c''_k}{\lambda_2(1-t)}\right.$$
$$\left.+\frac{c'_k}{\lambda_3 t^2(t-1)}\right\},$$

where λ_∞ *denotes* $\lambda_0+\lambda_1+\lambda_2+\lambda_3$.

(ii) *Case* $n=2k+1$, $k\geqq 1$.

$$(3.15)\qquad \phi_2(y)=-\frac{(c'_{k+2}+c''_{k+2})y(y-1)(y-t)}{4+\lambda_\infty}\left\{y-\frac{(1+\lambda_2)+(1+\lambda_3)t}{3+\lambda_1+\lambda_2+\lambda_3}\right\},$$

and

$$(3.16)\qquad (3+\lambda_1+\lambda_2+\lambda_3)(4+\lambda_\infty)(c'_{k+1}+c''_{k+1})c_k$$
$$=-(t-1)(c'_{k+2}+c''_{k+2})[c'_{k+1}(1+\lambda_2)\{(1+\lambda_3)(t-1)-(1+\lambda_1)\}$$
$$+c''_{k+1}t(1+\lambda_3)\{(1+\lambda_2)(t-1)+(1+\lambda_1)t\}].$$

(3.14) and (3.16) immediately imply

Lemma 3.3.

$$(3.17)\qquad c_{k+1}=\delta_k\rho_k,\quad c_k=\delta'_k\rho_{k+2},$$

where

$$(3.18)\quad \delta_k=\frac{-k(\alpha+\beta+\gamma+2k)}{(\alpha+\beta+\gamma+3k)}\left\{-\frac{\hat{c}'_k+\hat{c}''_k t}{(\alpha+k-1)t^2}-\frac{\hat{c}''_k}{(\beta+k-1)(t-1)}+\frac{\hat{c}'_k}{(\gamma+k-1)t^2(t-1)}\right\},$$

and

$$(3.19)\quad \delta'_k=\frac{-(t-1)(\hat{c}'_{k+2}+\hat{c}''_{k+2})}{(\alpha+\beta+\gamma+3k)(\alpha+\beta+\gamma+2k+1)(\hat{c}'_{k+1}+\hat{c}''_{k+1})}\cdot[\hat{c}'_{k+1}(\beta+k)\{(\gamma+k)(t-1)-\alpha-k\}+\hat{c}''_{k+1}t(\gamma+k)\{(\beta+k)(t-1)+(\alpha+k)t\}].$$

As a result,

Corollary. *For* $k\geqq 0$

$$(3.20)\quad \begin{cases} c_{3k+1}=\dfrac{\delta_{3k}\cdots\delta_3}{\delta'_{3k-2}\cdots\delta'_1}c_1,\\ c_{3k+2}=\dfrac{\delta_{3k+1}\cdots\delta_4}{\delta'_{3k-1}\cdots\delta'_2}c_2,\\ c_{3k+3}=\dfrac{\delta_{3k+2}\cdots\delta_5}{\delta'_{3k}\cdots\delta'_3}c_3,\end{cases}$$

and

$$(3.21)\quad \rho_{k+2}=\frac{c_{k+3}}{\delta_{k+2}}.$$

Lemma 3.4 (*Characterization of* $\hat{c}'_k$ *and* $\hat{c}''_k$). $\hat{c}'_k$, $\hat{c}''_k$ *satisfy the following recurrence relatlon*:

$$(3.22)\quad (\hat{c}'_{k+1},\ \hat{c}''_{k+1})=(\hat{c}'_k,\ \hat{c}''_k)\begin{bmatrix} a_{11}(k) & a_{12}(k)\\ a_{21}(k) & a_{22}(k)\end{bmatrix},$$

where $a_{11}(k)$, $a_{21}(k)$, $a_{12}(k)$ *and* $a_{22}(k)$, *in view of* (3.6), *denote rational functions of* k *as follows*:

$$(3.23)\quad a_{11}(k)=\frac{\lambda_1 t(t-1)}{\lambda_\infty(\lambda_\infty+1)(\lambda_\infty+2)}\left\{-\lambda_2(1+\lambda_2)+\frac{(\lambda_0+\lambda_2)(\lambda_0+\lambda_2+1)}{t}+\frac{\lambda_0(1+\lambda_1+\lambda_2)}{t-1}\right\},$$

$$(3.24)\quad a_{21}(k)=-\frac{\lambda_2 t^2(t-1)}{\lambda_\infty(\lambda_\infty+1)(\lambda_\infty+2)}\left\{(\lambda_0+\lambda_1)(1+\lambda_2)-\frac{\lambda_1(1+\lambda_0+\lambda_2)}{t}-\frac{\lambda_0^2}{t-1}\right\},$$

$$(3.25)\quad a_{12}(k)=\frac{-\lambda_1 t(t-1)}{\lambda_\infty(\lambda_\infty+1)(\lambda_\infty+2)}\left\{\lambda_2(1+\lambda_0+\lambda_1)-\frac{(\lambda_0+\lambda_2)(1+\lambda_1)}{t}-\frac{\lambda_0^2}{t-1}\right\},$$

and

$$(3.26)\quad a_{22}(k)=\frac{\lambda_2 t^2(t-1)}{\lambda_\infty(\lambda_\infty+1)(\lambda_\infty+2)}\left\{-(\lambda_0+\lambda_1)(\lambda_0+\lambda_1+1)-\frac{\lambda_1(1+\lambda_1)}{t}+\frac{\lambda_0(1+\lambda_1+\lambda_2)}{t-1}\right\}.$$

Proof. In view of (3.5), $\hat{c}_k'$ and $\hat{c}_k''$ are equal to $F(0, \lambda_1-1, \lambda_2-1, \lambda_3)$ and $-F(0, \lambda_1-1, \lambda_2, \lambda_3-1)$ respectively. Then the relation (3.22) implies that there exist polynomials $\psi_1(x)$ and $\psi_2(x)$ of degree 2 such that

$$(3.27)\quad x-t=\frac{a_{11}(k)}{x(x-1)}+\frac{a_{21}(k)}{x(x-t)}+\frac{d\psi_1(x)}{dx}+\left(\frac{\lambda_1}{x}+\frac{\lambda_2}{x-1}+\frac{\lambda_3}{x-t}\right)\psi_1(x),$$

and

$$(3.28)\quad x-1=\frac{a_{12}(k)}{x(x-1)}+\frac{a_{22}(k)}{x(x-t)}+\frac{d\psi_2(x)}{dx}+\left(\frac{\lambda_1}{x}+\frac{\lambda_2}{x-1}+\frac{\lambda_3}{x-t}\right)\psi_2(x).$$

$a_{11}(k)$, $a_{21}(k)$, $a_{12}(k)$ and $a_{22}(k)$ can be determined as in Lemma 3.2 by explicit computation of these equations.

Proposition 4. (i) *α_{2k+1}, β_{2k+1} and β'_{2k+1} are rational functions of c_k, c_{k+1}, c_k', c_k'', c_{k+1}', c_{k+1}'', α, β, γ, k and t.*

(ii) *α_{2k+2}, β_{2k+2} and β'_{2k+2} are rational functions of c_k, c_{k+1}, c_{k+1}', c_{k+1}'', c_{k+2}', c_{k+2}'', α, β, γ, k and t. Here c_k, ρ_k, $\hat{c}_k'$ and $\hat{c}_k''$ are obtained by the formulae* (3.9), (3.21) *and* (3.4) *respectively and $\hat{c}_k'$ and $\hat{c}_k''$ satisfy the recurrence relation* (3.22).

The spectrum of L coincides with the intervals $[0, t]$. The supports of its spectral densities $\mu_0(d\zeta)$ and $\mu_1(d\zeta)$ are equal to $[0, 1]$ and $[1, t]$ respectively. $\mu_0(d\zeta)$ and $\mu_1(d\zeta)$ satisfy a linear differential equation as function of t:

$$(\dot{\mu}_0(d\zeta),\ \dot{\mu}_1(d\zeta)) = (\mu_0(d\zeta),\ \mu_1(d\zeta))\frac{\gamma}{t-\zeta}. \tag{3.29}$$

According to Theorem 2 in §2, we have

Proposition 5.

$$\dot{L} = \left[L,\ \left(\frac{\gamma}{t-L}\right)^{(+)}\right]. \tag{3.30}$$

(2) Periodic case (See [18]).

We assume further that the matrix elements of L have period m:

$$\alpha_{n+m} = \alpha_n,\quad \beta_{n+m} = \beta_n,\quad \beta'_{n+m} = \beta'_n, \tag{3.31}$$

for $n \geqq l \geqq 2$. The matrix L can be extended to a periodic matrix $\tilde{L}$ such that (3.11) holds for non-positive integers. Then ${}^t(u_l,\ v_l,\ w_l)$ satisfies the characteristic equation

$$[y - A_{l,m}(z)]\begin{bmatrix} u_l \\ v_l \\ w_l \end{bmatrix} = 0, \tag{3.32}$$

where $A_{l,m}$ denotes the product

$$\begin{bmatrix} & & 1 \\ 1 & & \alpha_{l+1} \\ & 1 & \beta_{l+1}z + \beta'_{l+1} \end{bmatrix} \cdots \begin{bmatrix} & & 1 \\ 1 & & \alpha_{l+m} \\ & 1 & \beta_{l+m}z + \beta'_{l+m} \end{bmatrix}, \tag{3.33}$$

and y is an eigenvalue of $A_{l,m}(z)$. y satisfies an integral algebraic equation

$$y^3 - (a_0 + a_1 z + \cdots + a_{2k}z^{2k})y^2 + (b_0 + b_1 z + \cdots + b_k z^k)y - 1 = 0, \tag{3.34}$$

or

$$y^3 - (a_0 + a_1 z + \cdots + a_{2k+1}z^{2k+1})y^2 + (b_0 + b_1 z + \cdots + b_k z^k)y - 1 = 0, \tag{3.35}$$

according as $m = 2k$ or $2k+1$. We denote by $\mathscr{L}_m$ the algebraic curve defined by (3.34) or (3.35). Then we can make use of the same method as in [17] and prove the following:

Proposition 6. *Under the assumption that*

$$\begin{cases} a_{2k}(b_k^2 - 4a_{2k}) \neq 0 & \text{for } m = 2k, \quad (3.36) \\ a_{2k+1} \neq 0 & \text{for } m = 2k+1, \quad (3.37) \end{cases}$$

the genus of $\mathscr{L}_m$ *is equal to* $3k-2$ *for* $m=2k$ *or* $3k$ *for* $m=2k+1$.
i) *The divisor of* y *in* $\mathscr{L}_m$ *is written as follows*

$$(3.38)\qquad (y)=-2kP+k(P'+P'') \qquad for\ m=2k,$$
$$(3.39)\qquad (y)=-(2k+1)P+(2k+1)P' \qquad for\ m=2k+1,$$

where P, P', P'' $(m=2k)$ *or* P, P' $(m=2k+1)$ *denote effective divisors of degree* 1 *lying over* $z=\infty$:

$$(z)+P+P'+P''\geqq 0,$$
or
$$(z)+P+P'\geqq 0.$$

ii) u_{n+l} *and* $\left(\frac{v_{n+l}}{u_{n+l}}, \frac{w_{n+l}}{u_{n+l}}\right)$ *are meromorphic functions on* $\mathscr{L}_m$ *whose divisors satisfy*

$$(3.40)\qquad (u_{2j+l})=j(2P-P'-P'')+\mathscr{D}_{2j+l}-\mathscr{D}_l,$$

$$(3.41)\qquad \begin{cases} \left(\dfrac{v_{2j+l}}{u_{2j+l}}\right)=-P'-P''+\mathscr{D}'_{2j-1+l}-\mathscr{D}_{2j+l}, \\ \left(\dfrac{w_{2j+l}}{u_{2j+l}}\right)=-P+\mathscr{D}_{2j-1+l}-\mathscr{D}_{2j+l}, \end{cases}$$

$$(3.42)\qquad (u_{2j+1+l})=(2j+1)P-j(P'+P'')+\mathscr{D}_{2j+1+l}-\mathscr{D}_l,$$

$$(3.43)\qquad \begin{cases} \left(\dfrac{v_{2j+1+l}}{u_{2j+1+l}}\right)=\mathscr{D}'_{2j+1-l}-\mathscr{D}_{2j+1+l}, \\ \left(\dfrac{w_{2j+1+l}}{u_{2j+1+l}}\right)=-P+P'+P''+\mathscr{D}_{2j+l}-\mathscr{D}_{2j+1+l}, \end{cases}$$

in case $m=2k$, *or*

$$(3.44)\qquad (u_{n+l})=n(P-P')+\mathscr{D}_{n+l}-\mathscr{D}_l,$$

$$(3.45)\qquad \begin{cases} \left(\dfrac{v_{n+l}}{u_{n+l}}\right)=-P'+\mathscr{D}'_{n+l}-\mathscr{D}_{n+l}, \\ \left(\dfrac{w_{n+l}}{u_{n+l}}\right)=P'-P+\mathscr{D}_{n-1+l}-\mathscr{D}_{n+l}, \end{cases}$$

in case $m=2k+1$.

Conversely

Proposition 7. *Suppose that an arbitrary sequence of complex numbers* $\{a_0, a_1, \cdots, a_{2k}, b_0, \cdots, b_k\}$ *for* $m=2k$ *or* $\{a_0, \cdots, a_{2k+1}, b_0, \cdots, b_k\}$ *for* $m=2k+1$ *is given such that* (3.36) *or* (3.37) *holds. We choose* (u_l, v_l, w_l) *for* $l\geqq 2$ *such that*

$$(3.46)\qquad \begin{cases} \left(\dfrac{v_l}{u_l}\right) = -P' - P'' + \mathscr{D}'_{l-1} - \mathscr{D}_l, \\ \left(\dfrac{w_l}{u_l}\right) = -P + \mathscr{D}_{l-1} - \mathscr{D}_l, \quad m = 2k, \end{cases}$$

or

$$(3.47)\qquad \begin{cases} \left(\dfrac{v_l}{u_l}\right) = -P' + \mathscr{D}'_l - \mathscr{D}_l. \\ \left(\dfrac{w_l}{u_l}\right) = P' - P + \mathscr{D}_{l-1} - \mathscr{D}_l, \quad m = 2k+1, \end{cases}$$

where $\mathscr{D}_l$ *denotes a regular effective divisor on* $\mathscr{L}_m$. *Then we have the periodic Jacobi-Perron algorithm for the pair* $(v_l/u_l, w_l/u_l)$ *of period* $2k$ *or* $2k+1$ *such that*

$$(3.48)\qquad y u_{2k+l} = u_l,$$

$$(3.49)\qquad \frac{v_{2k+l}}{u_{2k+l}} = \frac{v_l}{u_l}, \quad \frac{w_{2k+l}}{u_{2k+l}} = \frac{w_l}{u_l},$$

or

$$(3.50)\qquad y u_{2k+1+l} = u_l,$$

$$(3.51)\qquad \frac{v_{2k+l+1}}{u_{2k+l+1}} = \frac{v_l}{u_l}, \quad \frac{w_{2k+l+1}}{u_{2k+l+1}} = \frac{w_l}{u_l}.$$

In particular we consider the case $m=2$. Then $\mathscr{L}_2$ is an algebraic curve defined by the equation where we put $y = h^{-1}$:

$$(3.52)\qquad F(z, h, h^{-1}) = \begin{vmatrix} -\dfrac{\beta'_2}{\beta_2} - \dfrac{h}{\beta_2} & \dfrac{1}{\beta_3} - \dfrac{\alpha_3}{\beta_3} h \\ -\dfrac{\alpha_2}{\beta_2} + \dfrac{h^{-1}}{\beta_2} & -z - \dfrac{\beta'_3}{\beta_3} - \dfrac{h}{\beta_3} \end{vmatrix},$$

and h, h', h'' be three roots of it. If $|h| \neq 1$, $|h'| \neq 1$, $|h''| \neq 1$, we may assume that there are only two cases, (i) $|h| < 1$, $|h'| > 1$, $|h''| > 1$ or (ii) $|h| < 1$, $|h'| < 1$, $|h''| > 1$. For $|z| \gg 1$, case (i) occurs. There exist three linearly independent quasi-periodic solutions for (1.47)

$$\varphi^{(0)} = \{\varphi_i^{(0)}\}, \ \varphi^{(1)} = \{\varphi_i^{(1)}\}, \ \varphi^{(2)} = \{\varphi_i^{(2)}\},$$

such that

$$(3.53)\qquad \begin{cases} \varphi_{n+2}^{(0)} = h \varphi_n^{(0)}, \\ \varphi_{n+2}^{(1)} = h' \varphi_n^{(1)}, \\ \varphi_{n+2}^{(2)} = h'' \varphi_n^{(2)}. \end{cases}$$

Consequently $\varphi^{(0)}\in l^2[0,\ \infty)$ and none of non-trivial linear combinations of $\varphi^{(1)}$ and $\varphi^{(2)}$ belong to $l^2[0,\ \infty)$. We have

$$\varphi_n^{(0)}=O(z^{-n-1}), \tag{3.54}$$

$$\varphi_n^{(0)}=\tilde{p}_{n+1}(z)+\varphi_0^{(0)}\tilde{p}'_{n+1}(z)+\varphi_1^{(0)}\tilde{p}''_{n+1}(z), \tag{3.55}$$

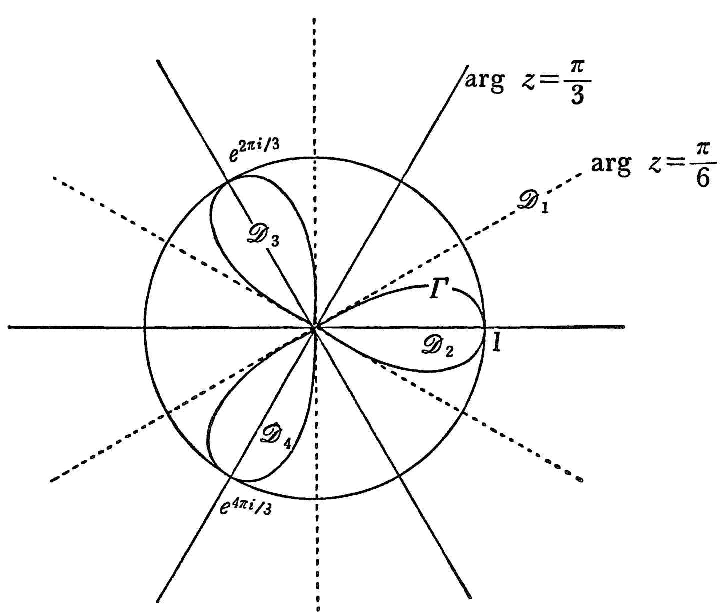
for $n\geqq 0$. $\varphi_n^{(0)}$ coincides with u_{n+1} in (1.24). Suppose further that *the algebraic equation* (3.34) *defines a Galois covering over* $\boldsymbol{C}P^1$. Then by choosing a suitable variable for z we may assume that h is equal to

$$h=z+\sqrt[3]{1-z^3} \tag{3.56}$$

and the matrix L has the following expression

$$L=\frac{1}{e^{2\pi i/3}-1}\begin{pmatrix} h'(c) & e^{2\pi i/3}h'(c)h''(c) & -1 & 0 & & \\ 1 & -h'(c) & -e^{2\pi i/3}h'(c)h(c) & e^{2\pi i/3} & 0 & \\ 0 & -e^{2\pi i/3} & \cdot & \cdot & \cdot & \cdot \\ & 0 & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \tag{3.57}$$

where $c\in\boldsymbol{C}$. We denote by Γ the set of all $z\in\boldsymbol{C}$ such that $|h|=1$ or $|h'|=1$ or $|h''|=1$. Then Γ is a real algebraic curve having branch points 1, $e^{2\pi i/3}$, $e^{4\pi i/3}$, and $\boldsymbol{C}-\Gamma$ has four components which we denote by $\mathscr{D}_1$, $\mathscr{D}_2$, $\mathscr{D}_3$ and $\mathscr{D}_4$ (see the figure).



Proposition 8. *The spectrum of* L *consists of* $\bar{\mathscr{D}}_2$, $\bar{\mathscr{D}}_3$ *and* $\bar{\mathscr{D}}_4$. *The interior part of* $\boldsymbol{C}-\Gamma$, $\mathscr{D}_2\cup\mathscr{D}_3\cup\mathscr{D}_4$ *coincides with the residual spectrum of* L.

Proof. We denote by $\begin{bmatrix} 0, & 1, & 2 \\ i, & j, & k \end{bmatrix}$ and $\begin{bmatrix} a, & b \\ i, & j \end{bmatrix}$ the determinants

$$\begin{vmatrix} \varphi_i^{(0)} & \varphi_j^{(0)} & \varphi_k^{(0)} \\ \varphi_i^{(1)} & \varphi_j^{(1)} & \varphi_k^{(1)} \\ \varphi_i^{(2)} & \varphi_j^{(2)} & \varphi_k^{(2)} \end{vmatrix}, \quad \begin{vmatrix} \varphi_i^{(a)} & \varphi_j^{(a)} \\ \varphi_i^{(b)} & \varphi_j^{(b)} \end{vmatrix},$$

respectively. Then the resolvent $(z-L)^{-1}$ can be described as the Green kernel $G(i, j|z)$, $i, j\geqq 2$, as follows:

$$(3.58)\qquad G(i, j|z)=\begin{cases} \beta_j\varphi_i^{(0)}\begin{bmatrix} 1, & 2 \\ j, & j+1 \end{bmatrix}\Big/\begin{bmatrix} 0, & 1, & 2 \\ j-1, & j, & j+1 \end{bmatrix}, & i\geqq j, \\ \dfrac{\beta_j\left\{\varphi_i^{(1)}\begin{bmatrix} 0, & 2 \\ j, & j+1 \end{bmatrix}-\varphi_i^{(0)}\begin{bmatrix} 0, & 1 \\ j, & j+1 \end{bmatrix}\right\}}{\begin{bmatrix} 0, & 1, & 2 \\ j-1, & j, & j+1 \end{bmatrix}}, & i<j. \end{cases}$$

If z lies in the resolvent set of L, then $(z-L)^{-1}$ and $(z-L^*)^{-1}$ must be bounded operators, so that

$$(3.59)\qquad \begin{cases} \sum_{j=0}^{\infty} |G(i_0, j|z)|^2<\infty, \\ \sum_{i=0}^{\infty} |G(i, j_0|z)|^2<\infty, \end{cases}$$

for fixed i_0 and j_0. From (3.53), we must have $|h|<1$, $|h'|>1$ and $|h''|>1$. This implies the proposition.

Remark. In Examples 1, the matrix L has only continuous spectrum while in Example 2, L has residual spectrum. It seems to be interesting to give a criterion whether L of (1.47) has any residual spectrum or not.

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