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The Order of the Attaching Class in the Suspended Quaternionic Quasi-Projective Space

Dedicated to Professor Minoru Nakaoka on his 60th birthday

By

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§ 0. Introduction

In this note, F denotes the field of the complex numbers C or the field of the quaternions H. We denote by FP^n the F-projective space of nF-dimensions and by $Q_n(F)$ the quasi-F-projective space. $G_n(F)$ denotes the unitary group $U(n)$ or the symplectic group $Sp(n)$ according as F is C or H. Let d be the dimension of F over the field of the real numbers R and S^{dn-1} the unit sphere in F^n. Let $T'_n: S^{dn-2} \to G_{n-1}(F)$ be the characteristic map for the normal form of the principal $G_{n-1}(F)$-bundle over S^{dn-1}. Then, as is well known ([2], [3] and [9]), $\operatorname{Im} T'_n = Q_{n-1}(F)$, precisely, the following diagram commutes:

$$\begin{array}{ccc} S^{dn-2} & \xrightarrow{\quad T_n \quad} & Q_{n-1}(F) \\ & {\scriptstyle T'_n}\searrow \quad \swarrow{\scriptstyle j_{n-1}} & \\ & G_{n-1}(F) & \end{array},$$

where j_{n-1} is the canonical reflection map. $Q_n(F) = Q_{n-1}(F) \cup_{T_n} e^{dn-1}$ and $Q_n(C) = E(CP^{n-1}_+)$, where $E(\ )$ denotes the reduced suspension and CP^{n-1}_+ a disjoint union of CP^{n-1} and {one point}.

Let $\omega_{n-1} = \omega_{n-1}(F)$ be the homotopy class of T_n and $p: Q_n(C) \to Q_n(C)/Q_1(C) = ECP^{n-1}$ the collapsing map. In the previous paper [6], we proved that the k-th suspension $E^k(p_*\omega_n(C))$ is of order $n!$ for $k \geqq 0$.

The purpose of this note is to examine the order of $E^k\omega_{n-1}(H)$.

Let α be an element of a homotopy group $\pi_n(\ )$ and $E^\infty\alpha \in \pi_n^S(\ )$ the stable element of α. $o(\beta)$ denotes the order of β. Then, our result is the following

Theorem. i) $o(E^k\omega_{n-1}(H)) = 2\cdot(2n-1)!$ *for* $k \geqq 0$ *if* n *is even.*
ii) $o(E^\infty\omega_{n-1}(H)) = (2n-1)!$ *if* n *is odd.*

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Our method is essentially to use the K-theory. To examine $o(\omega_{n-1}(H))$, we use Toda's theorem about the generator of $\pi_{2n-1}(U(n))$ [6] and the group structure of $\pi_{4n+2}(Sp(n))$ [4]. To determine the lower bound of $o(E^k\omega_{n-1}(H))$, we use the standard method of D. M. Segal [7] from the unstable viewpoint, exactly, we use the Hurewicz homomorphism $h: \pi_{k+4n-1}(E^kQ_n(H)) \to H_{k+4n-1}(E^kQ_n(H); Z)$. A powerful tool is Toda-Kozima's map $\tilde{i}_n: Q_n(H) \to Q_{2n}(C)$ [8].

In the stable case, our result overlaps with Corollary 4 of [5] and Theorem 5.8 of [9].

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§1. Determination of $o(\omega_{n-1}(H))$ for Even n

First we recall the definition of the quasi-projective space and the reflection map. $S(F^n)$ denotes the unit sphere in F^n. $Q_n(F)$ is the space obtained from $S(F^n)\times S(F)$ by imposing the equivalence relation: $(u, q)\sim(ug, g^{-1}qg)$ for $g\in S(F)$ and collapsing $S(F^n)\times\{1\}$ to a point. The reflection map $j_n=j_n(F): Q_n(F)\to G_n(F)$ is defined as follows:

$$j_n([u, q])(v)=v+u(q-1)\langle u, v\rangle$$

for $u\in S(F^n)$, $q\in S(F)$ and $v\in F^n$, where $\langle u, v\rangle=\sum_{k=1}^{n}\bar{u}_k v_k$ for $u=(u_1, \cdots, u_n)$ and $v=(v_1, \cdots, v_n)$.

Let $z=x+jy\in H$, where $x, y\in C$. By regarding $x\in C$ as $x+j0\in H$, we have the injection $C\subset H$. Obviously, this induces the canonical maps $i_n: Q_n(C)\to Q_n(H)$ and $i'_n: U(n)\to Sp(n)$. From the definition, the following diagram commutes:

$$\begin{array}{ccc} Q_n(C) & \xrightarrow{i_n} & Q_n(H) \\ \downarrow j_n & & \downarrow j_n \\ U(n) & \xrightarrow{i'_n} & Sp(n). \end{array} \tag{1.1}$$

In the complex case, we can define the reduced reflection map [6]:

$$\tilde{j}_n=\tilde{j}_n(C): ECP^{n-1}\cong Q_n(C)/Q_1(C) \longrightarrow U(n)/U(1)\cong SU(n).$$

By abuse of notation, we often use the same letter j_n for the reduced case.

Lemma 1.2. i) *If n is even, $j_{n*}: \pi_{4n-1}(Q_n(H))\to\pi_{4n-1}(Sp(n))$ is an epimorphism.*
ii) *If n is odd, $\mathrm{Im}\, j_{n*}=a\pi_{4n-1}(Sp(n))$, where $a=1$ or 2.*

Proof. Let $p: Q_{2n}(C)\to Q_{2n}(C)/Q_1(C)\cong ECP^{2n-1}$ be the collapsing map, $k: Q_n(H)\to Q_{2n}(H)$ and $k': Sp(n)\to Sp(2n)$ the inclusion maps, respectively. Then, by (1.1), the following diagram commutes for $r=4n-1$:

$$\begin{array}{ccccccc}
\pi_r(ECP^{2n-1}) & \xleftarrow{p_*} & \pi_r(Q_{2n}(C)) & \xrightarrow{i_{2n*}} & \pi_r(Q_{2n}(H)) & \xleftarrow{k_*} & \pi_r(Q_n(H)) \\
\downarrow \tilde{j}_{2n}(C)_* & & \downarrow j_{2n}(C)_* & & \downarrow j_{2n*} & & \downarrow j_{n*} \\
\pi_r(SU(2n)) & = & \pi_r(U(2n)) & \xrightarrow{i'_{2n*}} & \pi_r(Sp(2n)) & \xleftarrow{k'_*} & \pi_r(Sp(n))\,.
\end{array}$$

p_* is an epimorphism since $Q_{2n}(C)\simeq ECP^{2n-1}\vee S^1$. By Theorem 4.1 of [6], $\tilde{j}_{2n}(C)_*$ is an epimorphism. So, $j_{2n}(C)_*$ is an epimorphism. k_* and k'_* are isomorphisms respectively. As is well known, i'_{2n*} is an isomorphism if n is even and $\operatorname{Im} i'_{2n*}=2\pi_{4n-1}(Sp(2n))$ if n is odd. Therefore, the above commutative diagram leads us to the assertion. This completes the proof.

Proposition 1.3. i) $o(\omega_{n-1})=2\cdot(2n-1)!$ *for even* n.
ii) $o(\omega_{n-1})=a\cdot(2n-1)!$ *for odd* n, *where* a *is the same number as in Lemma* 1.2.

Proof. Let $p:(Q_n(H), Q_{n-1}(H))\rightarrow(S^{4n-1}, *)$ be the collapsing map. We consider the natural homomorphism between the exact sequences for $r=4n-1$:

$$\begin{array}{ccccccc}
\pi_r(Q_n(H)) & \xrightarrow{j'_*} & \pi_r(Q_n(H), Q_{n-1}(H)) & \xrightarrow{\partial} & \pi_{r-1}(Q_{n-1}(H)) & \longrightarrow & \pi_{r-1}(Q_n(H)) \\
\downarrow j_{n*} & & \downarrow p_* & & \downarrow j_{n-1*} & & \downarrow j_{n*} \\
\pi_r(Sp(n)) & \xrightarrow{p'_*} & \pi_r(S^{4n-1}) & \xrightarrow{\Delta'} & \pi_{r-1}(Sp(n-1)) & \longrightarrow & \pi_{r-1}(Sp(n))\,,
\end{array}$$

where the mappings are canonical and ∂ and Δ' are the connecting homomorphisms.

As is well known, $\pi_{4n-1}(Sp(n))\approx Z$, $\pi_{4n-2}(Sp(n))\approx 0$ and $\pi_m(S^m)=\{\iota_m\}\approx Z$. By the Blakers-Massey theorem [1], p_* is an isomorphism. By the definition, $\omega_{n-1}=\Delta(\iota_{4n-1})$, where $\Delta=\partial\circ p_*^{-1}$. So, by Theorem 2.2 of [4], j_{n-1*} is an epimorphism and the following holds:

(1.4) $\quad \pi_{4n-2}(Sp(n-1))=\{j_{n-1*}\omega_{n-1}\}\approx Z_{b\cdot(2n-1)!}$, where $b=1$ for odd n and $b=2$ for even n.

By the exactness of the upper sequence, $o(\omega_{n-1})$ is equal to the order of the cokernel of j'_*. Hence, by (1.4), Lemma 1.2 and by the above commutative diagam, we have the assertion. This completes the proof.

By inspecting the above proof, we have the following

Proposition 1.5. $j_{n*}:\pi_{4n-1}(Q_n(H))\rightarrow\pi_{4n-1}(Sp(n))$ *is an epimorphism if and only if* $o(\omega_{n-1})=b\cdot(2n-1)!$, *where* b *is the same number as in* (1.4).

§2. Some Fundamental Facts

For $n\geqq 0$, X_n denotes a connected finite CW-complex such that $X_0=\{*\}$ and $X_n=e^0\cup e^{r_1}\cup\cdots\cup e^{r_n}$ for $n\geqq 1$. Here $r=r_n=dn-\varepsilon$ with $\varepsilon=0$ or 1 and $d-\varepsilon\geqq 2$.

$\theta_{n-1}: S^{r-1} \to X_{n-1}$ denotes the attaching map, and so $X_n = X_{n-1} \cup_{\theta_{n-1}} e^r$. For example, $X_n = FP^n (d=2$ or 4 and $\varepsilon = 0)$ and $X_n = Q_n(H) (d=4$ and $\varepsilon = 1)$.

Let $p: X_n \to X_n / X_{n-1} = S^r$ and $p': (X_n, X_{n-1}) \to (S^r, *)$ be the collapsing maps. Let $\partial: \pi_{r+m}(E^m X_n, E^m X_{n-1}) \to \pi_{r+m-1}(E^m X_{n-1})$ be the connecting homomorphism. Then, $(E^m p')_*: \pi_{r+m}(E^m X_n, E^m X_{n-1}) \to \pi_{r+m}(S^{r+m})$ is an isomorphism for $m \geqq 0$ [1], and we define a homomorphism $\Delta: \pi_{r+m}(S^{r+m}) \to \pi_{r+m-1}(E^m X_{n-1})$ by the composition $\partial \circ (E^m p')_*^{-1}$. By the definition, $\Delta(\iota_{r+m}) = E^m \theta_{n-1}$, where the same letter is used for a mapping and its homotopy class.

Let $h = h_m: \pi_{r+m}(E^m X_n) \to H_{r+m}(E^m X_n; Z) \approx Z$ for $m \geqq 0$ be the Hurewicz homomorphism and $h(n, m)$ the non-negative integer such that $\operatorname{Im} h = h(n, m) H_{r+m}(E^m X_n; Z)$. Then we have the following

Lemma 2.1. $o(E^m \theta_{n-1}) = h(n, m)$.

Proof. $j: (E^m X_n, *) \to (E^m X_n, E^m X_{n-1})$ denotes the inclusion. Then, we consider the commutative diagram:

$$\begin{array}{ccccc} \pi_{r+m}(E^m X_n) & \xrightarrow{j_*} & \pi_{r+m}(E^m X_n, E^m X_{n-1}) & \xrightarrow{\partial} & \pi_{r+m-1}(E^m X_{n-1}) \\ \downarrow h & & \downarrow h' & & \\ H_{r+m}(E^m X_n; Z) & \xrightarrow{j_*} & H_{r+m}(E^m X_n, E^m X_{n-1}; Z), & & \end{array}$$

where h' denotes the relative Hurewicz homomorphism and the upper sequence is exact. From the cell structure of X_n, the lower j_* is an isomorphism. By the relative Hurewicz theorem, h' is an isomorphism. This completes the proof.

According to [8], a representative element of $Q_n(H)$ can be taken as $(x+jy, e^{i\pi t})$, where $x, y \in C^n$ satisfying $x+jy \in S(H^n)$ and $0 \leqq t \leqq 1$. Toda and Kozima defined $\tilde{t}_n: Q_n(H) \to Q_{2n}(C)$ by the equation

$$\tilde{t}_n[(x+jy, e^{i\pi t})] = [(x \oplus y, e^{2i\pi t})].$$

We define $t_n: Q_n(H) \to ECP^{2n-1}$ by the composition $p \circ \tilde{t}_n$, where $p: Q_{2n}(C) \to ECP^{2n-1}$ is the collapsing map. From the definition, the following diagram commutes for $k < n$:

$$(2.2) \qquad \begin{array}{ccc} Q_k(H) & \xrightarrow{t_k} & ECP^{2k-1} \\ \downarrow i & & \downarrow i' \\ Q_n(H) & \xrightarrow{t_n} & ECP^{2n-1}, \end{array}$$

where i and i' are the canonical inclusions.

The following lemma is a reduced version of Proposition 2.5 of [8].

Lemma 2.3 (Toda-Kozima). *The following diagram commutes up to homotopy:*

$$\begin{array}{ccc} Q_n(H) & \xrightarrow{t_n} & ECP^{2n-1} \\ \downarrow j_n & c & \downarrow j_{2n} \\ Sp(n) & \longrightarrow & SU(2n), \end{array}$$

where c is the complexification map.

Let $p: Q_n(H) \to Q_n(H)/Q_{n-1}(H) = S^{4n-1}$ for $n \geqq 1$ and $p': ECP^{2n-1} \to ECP^{2n-1}/ECP^{2n-3} \simeq S^{4n-3} \vee S^{4n-1}$ for $n \geqq 2$ be the collapsing maps. Then, by (2.2), there exists a mapping $t'_n: S^{4n-1} \to S^{4n-3} \vee S^{4n-1}$ for $n \geqq 2$ such that the following diagram commutes:

$$\begin{array}{ccc} Q_n(H) & \xrightarrow{t_n} & ECP^{2n-1} \\ \downarrow p & t'_n & \downarrow p' \\ S^{4n-1} & \longrightarrow & S^{4n-3} \vee S^{4n-1}. \end{array} \tag{2.4}$$

Let $p_2: S^{4n-3} \vee S^{4n-1} \to S^{4n-1}$ for $n \geqq 2$ be the projection map. Then, we have the following

Lemma 2.5. $\deg t_1 = -1$ *and* $\deg(p_2 t'_n) = (-1)^n$ *for* $n \geqq 2$.

Proof. We define $g_n: S(H^n) \to S(C^{2n})$ by the equation

$$g_n(x+jy) = x \oplus y$$

for $x, y \in C^n$. It is clear that g_n is a homeomorphism and $\deg g_n = (-1)^n$. By Lemma 2.3, $t_1 \simeq g_1$ and $p_2 t'_n \simeq g_n$ for $n \geqq 2$. This completes the proof.

Hereafter the same letter is often used for a mapping and its homotopy class. Let $\gamma_n = \gamma_n(F): S(F^{n+1}) \to FP^n$ be the projection map. Let $i: ECP^{2n-1} \to ECP^{2n}$ be the inclusion map. Then, we have the following

Proposition 2.6. $(-1)^{n+1} E\gamma_{2n}(C) = i t_n \omega_n(H)$.

Proof. By (2.2) and (2.4), the following diagram commutes for $r = 4n+3$:

$$\begin{array}{ccccc} \pi_r(S^{4n+3}) & \xleftarrow{p_*} & \pi_r(Q_{n+1}(H), Q_n(H)) & \xrightarrow{\partial} & \pi_{r-1}(Q_n(H)) \\ \downarrow t'_{n+1*} & p'_* & \downarrow t_{n+1*} & \partial' & \downarrow t_{n*} \\ \pi_r(S^{4n+1} \vee S^{4n+3}) & \longleftarrow & \pi_r(ECP^{2n+1}, ECP^{2n-1}) & \longrightarrow & \pi_{r-1}(ECP^{2n-1}) \\ \downarrow p_{2*} & p''_* & \downarrow i'_* & \partial'' & \downarrow i_* \\ \pi_r(S^{4n+3}) & \longleftarrow & \pi_r(ECP^{2n+1}, ECP^{2n}) & \longrightarrow & \pi_{r-1}(ECP^{2n}), \end{array}$$

where the mappings are canonical.

p_* and p''_* are isomorphisms [1]. We note that $\omega_n(H) = \partial p_*^{-1}(\iota_{4n+3})$ and $E\gamma_{2n}(C) = \partial'' p''^{-1}_*(\iota_{4n+3})$. So, by Lemma 2.5 and the above commutative diagram, we have the assertion. This completes the proof.

Remark 1. Owing to Proposition 2.6, it suffices to take $(-1)^{n+1}t_n\omega_n$ as λ_{2n} in Proposition 6.5. ii) of [6]. By Theorem 1.2 of [6] and Proposition 1.3, $o(\lambda_{2n})=(2n+1)!$ or $2\cdot(2n+1)!$. In the last section, we shall show that $o(\lambda_4)=5!$ (cf. Lemma 11.1 of [6]).

§3. Determination of the Lower Bound of $o(E^m\omega_{n-1}(H))$

Let $v\in\tilde{K}(CP^{2n-1})$ be the stable isomorphism class of the canonical line bundle over CP^{2n-1}. We denote by $I_C:\tilde{K}(\quad)\to\tilde{K}(E^2\quad)$ the Bott periodicity isomorphism. The following lemma is well known (cf. Lemma 2.2 of [8]).

Lemma 3.1. *$I_C(v)\in\tilde{K}(E^2CP^{2n-1})$ is represented by the adjoint of the composite of the canonical maps:*

$$ECP^{2n-1}\xrightarrow{j_{2n}}SU(2n)\xrightarrow{i}U(2n)\xrightarrow{k}\Omega BU(2n),$$

where k is the homotopy equivalence.

Hereafter, Z or the rational number field Q is taken as the coefficients of the homology or cohomology groups, unless otherwise stated.

Let $\mathrm{ch}^k:\tilde{K}(\quad)\to H^{2k}(\quad;Q)$ be the k-th Chern character and $\mathrm{ch}=\sum_k\mathrm{ch}^k$ the total Chern character. Let $\sigma:\tilde{H}^i(E\quad)\to\tilde{H}^{i-1}(\quad)$ be the suspension isomorphism. Then, as is well known, the following diagram commutes:

(3.2)
$$\begin{array}{ccc}\tilde{K}(CP^{2n-1}) & \xrightarrow{I_C} & \tilde{K}(E^2CP^{2n-1})\\ \downarrow\mathrm{ch} & & \downarrow\mathrm{ch}\\ H^*(CP^{2n-1};Q) & \xrightarrow{\sigma^{-2}} & H^*(E^2CP^{2n-1};Q).\end{array}$$

We denote by y a generator of $H^2(CP^{2n-1})$. It is also well known that

(3.3)
$$\mathrm{ch}^{2n-1}v=1/(2n-1)!\ y^{2n-1}.$$

Proposition 3.4. *$o(E^m\omega_{n-1})$ is a multiple of $(2n-1)!$ for $m\geqq 0$.*

Proof. The assertion is a direct consequence of Theorem 1.2 of [6] and Proposition 2.6. For the later use, we give another proof for even m.

By (2.4) and Lemma 2.5, $t_n^*:H^{4n-1}(ECP^{2n-1})\to H^{4n-1}(Q_n(H))$ is an isomorphism. So, $y'=t_n^*\sigma^{-1}y^{2n-1}$ is taken as a generator of $H^{4n-1}(Q_n(H))$. We choose a generator x of $H_{4n-1}(Q_n(H))$ satisfying $\langle y',x\rangle=1$, where $\langle\ ,\ \rangle$ denotes the Kronecker index.

Put $o(E^m\omega_{n-1})=k(n)$. Denote by $s:\tilde{H}_i(\quad)\to\tilde{H}_{i+1}(E\quad)$ the suspension isomorphism. Then, by Lemma 2.1, there exists an element $\alpha\in\pi_{m+4n-1}(E^mQ_n(H))$ satisfying $h_m(\alpha)=k(n)s^mx$. By the definition of the Hurewicz homomorphism, $h_m(\alpha)=\alpha_*s^m\xi_n$, where ξ_n denotes a generator of $H_{4n-1}(S^{4n-1})$. So, we have $k(n)=$

$\langle\sigma^{-m}y', \alpha_* s^m \xi_n\rangle = \langle\alpha^*\sigma^{-m}y', s^m\xi_n\rangle$. Choose a generator τ_n of $H^{4n-1}(S^{4n-1})$ satisfying $\langle\tau_n, \xi_n\rangle=1$. Then, we have $\alpha^*\sigma^{-m}y'=k(n)\sigma^{-m}\tau_n$.

Put $m=2t$ and $u=I_C{}^t(Et_n)^*I_C(v)\in\tilde{K}(E^{m+1}Q_n(H))$. Then, by (3.2), (3.3) and by the naturality of the Chern character, we have the following:

$$\sigma\,\mathrm{ch}^{2n+t}(E\alpha)^*u=\alpha^*\sigma^{-m}t_n^*\sigma^{-1}\,\mathrm{ch}^{2n-1}(v)=1/(2n-1)!\ \alpha^*\sigma^{-m}y'.$$

So, we have $\mathrm{ch}^{2n+t}(E\alpha)^*u=k(n)/(2n-1)!\ \sigma^{-m-1}\tau_n$. As is well known, $\mathrm{Im}\,\mathrm{ch}^{2n+t} = H^{4n+m}(S^{4n+m}; Z)$. Hence, $k(n)/(2n-1)!$ is an integer. This completes the proof.

Lemma 3.5. *$(Et_n)^*I_C(v)$ belongs to the image of the complexification homomorphism $c' : \widetilde{KSp}(EQ_n(H))\to\tilde{K}(EQ_n(H))$.*

Proof. By Lemmas 2.3 and 3.1, $u'=(Et_n)^*I_C(v)=(\mathrm{adj}\,(k\circ i\circ j_{2n}(C)))_*(Et_n)=(\mathrm{adj}\,k)_*(Ec)_*(Ej_n(H))$.

Let $\rho_c: BSp(n)\to BU(2n)$ be the mapping induced from $c: Sp(n)\to U(2n)$ and $k': Sp(n)\to\Omega BSp(n)$ be the canonical homotopy equivalence. Then, it is well known that $k\circ c\simeq\Omega\rho_c\circ k'$. So, we have $(\mathrm{adj}\,k)_*(Ec)_*=(\rho_c)_*(\mathrm{adj}\,k')_*$. Hence, $u'=(\rho_c)_*(\mathrm{adj}\,k')_*(Ej_n(H))\in\mathrm{Im}\,c'$. This completes the proof.

As is well known, the following diagram commutes:

$$\begin{array}{ccc} \widetilde{KSp}(\quad) & \xrightarrow{c'} & \tilde{K}(\quad) \\ \downarrow I_H & & \downarrow I_C{}^4 \\ \widetilde{KSp}(E^8\quad) & \xrightarrow{c'} & \tilde{K}(E^8\quad), \end{array} \tag{3.6}$$

where I_H denotes the Bott periodicity isomorphism.

Proposition 3.7. *If n is even and $m\equiv 0 \bmod 8$, $o(E^m\omega_{n-1})$ is a multiple of $2\cdot(2n-1)!$.*

Proof. As is well known, the following diagram commutes:

$$\begin{array}{ccc} \widetilde{KSp}(E^{m+1}Q_n(H)) & \xrightarrow{(E\alpha)^*} & \widetilde{KSp}(S^{4n+m}) \\ \downarrow c' & & \downarrow c \\ \tilde{K}(EQ_n(H)) & \xrightarrow{(E\alpha)^*} & \tilde{K}(S^{4n+m}), \end{array}$$

and $\mathrm{Im}\,c=2\tilde{K}(S^{4n+m})$ if n is even. So, by Lemma 3.5, (3.6) and by the proof of Proposition 3.4, $(E\alpha)^*u=(E\alpha)^*I_C{}^t(Et_n)^*I_C(v)\in 2\tilde{K}(S^{4n+m})$ and $\mathrm{ch}^{2n+t}(E\alpha)^*u\in 2H^{4n+m}(S^{4n+m}; Z)$. Therefore, $k(n)/(2n-1)!$ is an even integer. This completes the proof.

Remark 2. By the similar arguments, we have the following for $k\geqq 1$(cf. [7]):

(1) $o(E^k\gamma_{n-1}(C))$ is a multiple of $n!$ for even k.

(2) $o(E^k\gamma_{n-1}(H))$ is a multiple of $(2n)!/2$ for even k. If n is even and $k\equiv$

0 mod 8, $o(E^k\gamma_{n-1}(H))$ is a multiple of $(2n)!$.

§ 4. Proof of the Theorem

To prove ii) of our theorem, we use the following [3]:

Theorem 4.1 (James). *The stunted quasi-projective space $Q_n(F)/Q_{n-k}(F)$ is an S-retract of the factor space $G_n(F)/G_{n-k}(F)$ for $k\leqq n$. In particular, j^S_{n*}: $\pi^S_i(Q_n(H))\rightarrow\pi^S_i(Sp(n))$ is a monomorphism for $i\geqq 0$.*

Now we are ready to prove the theorem. The assertion i) is a direct consequence of Propositions 1.3.i) and 3.7.

By Theorem 4.1, $j^S_{n-1*}: \pi^S_{4n-2}(Q_{n-1}(H))\rightarrow\pi^S_{4n-2}(Sp(n-1))$ is a monomorphism. So, we have $o(E^\infty\omega_{n-1})=o(E^\infty j_{n-1*}\omega_{n-1})$. Therefore, (1.4) and Proposition 3.4 lead us to the assertion. This completes the proof of the theorem.

Remark 3. We can give an improved proof of Theorem 1.2 of [6]. We use the first half of the proof of Theorem 1.2 of [6] and Remark 2.(1). We have

(1) $o(E^k\gamma_{n-1}(C))=n!$ for $k\geqq 1$.

By (1) and Remark 2.(2), we have the following:

(2) If n is even, $o(E^k\gamma_{n-1}(H))=(2n)!$ for $k\geqq 1$.

By Theorem 1.1 of [7] and by Lemma 2.1,

(3) $o(E^\infty\gamma_{n-1}(H))=(2n)!/2$ if n is odd.

In this case, the Adams spectral sequence is used for the 2-primary stable homotopy of quaternionic and complex projective spaces [7].

§ 5. An Example

An open problem is to determine the order of $\omega_n(H)$ completely. The author hopes that an affirmative answer is given to the following

Conjecture. $o(\omega_{n-1}(H))=(2n-1)!$ *if n is odd.*

In this section, we determine the group structure of $\pi_{10}(Q_2(H))$ and we show that the conjecture is true for $n=3$. We use the following: $\pi_{11}(S^{10})\approx Z_2$, $\pi_{10}(S^7)=\{\nu_7\}\approx Z_{24}$, $\pi_{11}(S^7)\approx 0$, $\pi_9(S^3)\approx Z_3$ and $\pi_{10}(S^3)\approx Z_{15}$.

Example. $\pi_{10}(Q_2(H))\approx Z_{5!}\oplus Z_2$ *and* $o(\omega_2(H))=5!$.

Proof. Let $p: (Q_2(H), S^3)\rightarrow(S^7, *)$ be the collapsing map. Then, $p_*: \pi_7(Q_2(H), S^3)\rightarrow\pi_7(S^7)$ is an isomorphism [1]. We choose a generator α of $\pi_7(Q_2(H), S^3)\approx Z$ such that $p_*\alpha=\iota_7$.

$Sp(2)$ is regarded as the cell complex $Q_2(H)\cup e^{7,3}$. Let $p': (Sp(2), Q_2(H))\rightarrow(S^{10}, *)$ be the collapsing map. Then, $p'_*: \pi_n(Sp(2), Q_2(H))\rightarrow\pi_n(S^{10})$ is an isomorphism for $n\leqq 11$ [1].

We consider the following commutative diagram:

$$
\begin{array}{ccccccccccc}
 & & & & & & \pi_{11}(Sp(2), S^3) \approx 0 & & & & \\
 & & & & & & \downarrow & & & & \\
 & & & & \pi_{11}(Sp(2), Q_2(H)) & = & \pi_{11}(Sp(2), Q_2(H)) \approx Z_2 & & & & \\
 & & & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial'} & & & & \\
 & & \pi_{10}(S^3) & \xrightarrow{i_*} & \pi_{10}(Q_2(H)) & \xrightarrow{j_*} & \pi_{10}(Q_2(H), S^3) & \xrightarrow{\partial''} & \pi_9(S^3) \approx Z_3 & & \\
 & & \| & & \downarrow{\scriptstyle j_{2*}} & & \downarrow{\scriptstyle p_*} & & \| & & \\
\pi_{11}(S^7) & \longrightarrow & \pi_{10}(S^3) & \xrightarrow{i'_*} & \pi_{10}(Sp(2)) & \xrightarrow{p'_*} & \pi_{10}(S^7) & \xrightarrow{\Delta'} & \pi_9(S^3) & \longrightarrow & \pi_9(Sp(2)), \\
\wr & & \wr & & \downarrow \quad \wr & & \downarrow \quad \wr & & & & \wr \\
0 & & Z_{15} & & 0 \quad Z_{5!} & & \pi_{10}(Sp(2), Q_2(H)) \quad \pi_{10}(Sp(2), S^3) & & & & 0 \\
 & & & & & & \wr & & & & \\
 & & & & & & Z & & & &
\end{array}
$$

where the mappings are canonical and the horizontal and perpendicular sequences are exact respectively.

p_* is a split epimorphism since $p_*(\alpha\nu_7)=\nu_7$. So, we have $\pi_{10}(Q_2(H), S^3)\approx Z_{24}\oplus Z_2$. By the commutativity of the above diagram, i_* is a monomorphism and ∂'' is an epimorphism. Therefore, by the upper horizontal sequence, $\pi_{10}(Q_2(H))\approx Z_{5!}\oplus Z_2$. Hence, by Proposition 1.3. ii), we have $o(\omega_2)=5!$. This completes the proof.

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