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On the Structure of Arithmetically Buchsbaum Curves in P^3_k

By

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Introduction

Let X be an equidimensional complete subscheme of P^3_k of dimension one. X will be called a curve throughout this paper. Let $\mathcal{I}$ be the sheaf of ideals of X and $I := \bigoplus_{\nu\geqq 0} \mathrm{H}^0(\mathbf{P}^3_k, \mathcal{I}(\nu)) \subset R := k[x_1, x_2, x_3, x_4]$. We know a kind of general structure theorem for the ideal I and its free resolution [1], which enables us to enter into a detailed study of some special classes of curves. As a first attempt, we investigate arithmetically Buchsbaum curves, which are characterized by the following property [13]:

$$\mathrm{H}^1_*(\mathcal{I}) := \bigoplus_{\nu\in \boldsymbol{Z}} \mathrm{H}^1(\mathbf{P}^3_k, \mathcal{I}(\nu)) \quad \text{is annihilated by} \quad \mathfrak{m} := (x_1, x_2, x_3, x_4)R\,.$$

When $\mathrm{H}^1_*(\mathcal{I})=0$, the curve is arithmetically Cohen-Macaulay and is studied thoroughly in [9]. So our concern is centered on the case where $\mathrm{H}^1_*(\mathcal{I})\neq 0$ and $\mathfrak{m}\mathrm{H}^1_*(\mathcal{I})=0$. We give structure theorems for the ideal I and for the free resolution of the R-module $\mathrm{H}^0_*(\mathcal{O}_X) := \bigoplus_{\nu\geqq 0} \mathrm{H}^0(\mathbf{P}^3_k, \mathcal{O}_X(\nu))$, then use them to consider small deformations in $\mathbf{P}^3_k$ of those curves.

Let us explain the content of each section.

Section 1. The results of [1; Section 3] are sometimes inconvenient, because it involves unnecessary procedure, that is, we have to take beforehand an ideal J such that R/J is Cohen-Macaulay. We give up this procedure and make simple modification of [1; Proposition 3.1] to define a numerical invariant "basic sequence" of an arbitrary homogeneous ideal $I\subset R$ such that $\dim R/I=2$ and $\mathrm{depth}_{\mathfrak{m}} R/I \geqq 1$ (Proposition 1.3, Definition 1.4), which extend "caractère numérique" of [9]. It is a sequence of integers $(a\,;\, \nu_1, \cdots, \nu_a\,;\, \nu_{a+1}, \cdots . \nu_{a+b})$ consisting of the degrees of the special generators of I.

Section 2. The structure of the module $\mathrm{H}^1_*(\mathcal{I})$ is important in every case, and we mentioned the relations between the matrix λ_3 (see [1; Section 3]) and

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$H^1_*(\mathcal{J})$. In particular, we find that $(\nu_{a+1}, \cdots, \nu_{a+b})$ a part of the basic sequence of I reflects a certain property of $H^1_*(\mathcal{J})$ (Proposition 2.4). Then the free resolution for $H^0_*(\mathcal{O}_X)$ is computed as an R-module in a simple case (Proposition 2.8 and 2.9).

Section 3. With the use of the results of Sections one and two we reach a structure theorem for the ideals defining arithmetically Buchsbaum curves in $\mathbf{P}^3_k$. This theorem is stated in the language of Proposition 1.3 (Theorems 3.1, 3.2 and Corollary 3.3).

Theorem. *Let X be an arithmetically Buchsbaum curve with basic sequence $(a\,;\nu_1, \cdots, \nu_a\,;\nu_{a+1}, \cdots, \nu_{a+b})$. Then*

1) $\dim_{R/\mathfrak{m}} H^1_*(\mathcal{J}) = b$

2) *There exist $i_1, \cdots, i_{2b}$ $(1 \leqq i_1 < \cdots < i_{2b} \leqq a)$ such that $(\nu_{i_1}, \cdots, \nu_{i_{2b}}) = (\nu_{a+1}, \cdots, \nu_{a-b}, \nu_{a+1}, \cdots, \nu_{a+b})$ up to a permutation.*

3) *$a \geqq 2b$, that is, the minimal degree of the surfaces containing X is larger than or equal to $2 \cdot \dim_{R/\mathfrak{m}} H^1_*(\mathcal{J})$.*

4) *$I = f_0 k(0) \oplus \bigoplus_{i=1}^{a} f_i k(1) \oplus \bigoplus_{j=1}^{b} f_{a+j} k(2)$, where*

$$\lambda_2 = \begin{pmatrix} U_{01} & U_{02} & 0 \\ & & 0 \\ U_1 & U_2 & x_3 1_b \\ & & x_4 1_b \\ U_{21} & U_3 & x_2 1_b \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 0 \\ -x_3 1_b \\ -x_4 1_b \\ -x_2 1_b \\ U_3 \end{pmatrix}, \quad \lambda_2 \lambda_3 = 0$$

and

$$f_i = (-1)^i \det \begin{pmatrix} U_{01} & U_{02} \\ U_1 & U_2 \\ U_{21} & U_3 \end{pmatrix} \left({}^{i}\right) \Big/ \det U_3 \quad (0 \leqq i \leqq a+b).$$

Section 4. We know [14; (2.6) Theorem] that, for an arbitrary R-module M of finite length, there exists a nonsingular irreducible curve X such that $H^1_*(\mathcal{J}) \cong M$ up to a shift in grading. But the basic sequences or the detailed structure of such curves is not known in general. In view of this we prove the existence of integral arithmetically Buchsbaum curves with a special basic sequence $(a\,;\underbrace{n, \cdots, n}_{a}\,;\underbrace{n, \cdots, n}_{b})$ for arbitrary a, b, n satisfying $n \geqq a \geqq 2b$ (Theorem 4.4), to supply manageable examples of arithmetically Buchsbaum curves. They are, however, not in general verified to be nonsingular as yet (see Remark 4.10).

Section 5. Finally, applying the results of the previous sections, we try to find irreducible components of $\mathrm{Hilb}(\mathbf{P}^3_k)$ whose general points correspond to arithmetically Buchsbaum curves. It consists of computing flat deformations of the ring R/I and of the R-module $H^0_*(\mathcal{O}_X)$, so that the cases which cannot be treated by this method are left as problems. These irreducible components are expressed in terms of the basic sequences of the curves corresponding to the general points of them (Theorem 5.11).

Notation

1. k denotes an infinite field with arbitrary characteristic except in Section four.

2. Let A be a commutative ring and x_1, x_2, x_3, x_4 indeterminates over A. We set

$$A(0)=A[x_1, x_2, x_3, x_4], \qquad A(1)=A[x_2, x_3, x_4],$$

$$A(2)=A[x_3, x_4], \qquad A(3)=A[x_4].$$

3. Let U be an arbitrary matrix. We denote by $U\begin{pmatrix} i_1 \cdots i_p \\ j_1 \cdots j_q \end{pmatrix}$ the matrix obtained by deleting the i_1-th, i_2-th, $\cdots$, i_p-th rows and j_1-th, j_2-th, $\cdots$, j_q-th columns from U.

4. Let U be a matrix of homogeneous polynomials with coefficients in a ring A. We denote by $\Delta(U)$ the matrix of integers whose (i, j)-component is $\deg(u_{ij})$, where u_{ij} is the (i, j)-component of U.

5. For a matrix $U=(u_1, \cdots, u_n)$ in a ring B with n columns $u_1, \cdots, u_n$ and for a subring B' of B, we make the following convention:

$$\operatorname{Im}^{B'}(U)=\{\sum_{i=1}^{n} b_i u_i \mid b_i \in B' \quad 1 \leqq i \leqq n\}$$

and we denote this set by $U\cdot(B')^n$ if and only if the columns $u_1, \cdots, u_n$ are linearly independent over B'.

6. Let $C=\bigoplus_{\nu\in Z} C_\nu$ be a graded ring, $\bar{n}=(n_1, \cdots, n_r)$ a sequence of integers, and l an integer. We set

$$C[\bar{n}]=\bigoplus_{i=1}^{r} C[n_i], \qquad C[\bar{n}+l]=\bigoplus_{i=1}^{r} C[n_i+l],$$

where $C[m]$ denotes the graded module such that $C[m]_\nu=C_{\nu+m}$ for an integer m. See [1; Notation] for the symbol $\bigoplus$.

7. For a coherent sheaf of modules $\mathcal{F}$ on $\mathbf{P}_k^3$ we will often write $\mathrm{H}_*^i(\mathbf{P}_k^3, \mathcal{F})$ or $\mathrm{H}_*^i(\mathcal{F})$ to denote the graded module $\bigoplus_{\nu\in Z} \mathrm{H}^i(\mathbf{P}_k^3, \mathcal{F}(\nu))$.

8. 1_p denotes the $p\times p$ identity matrix.

9. $\boldsymbol{Z}_0=\{\nu\in \boldsymbol{Z} \mid \nu\geqq 0\}$.

§1. Definition of the Basic Sequence of a Homogeneous Ideal in $k[x_1, x_2, x_3, x_4]$

In this paper R always denotes the polynomial ring $k[x_1, x_2, x_3, x_4]$ and $\mathfrak{m}$ its maximal ideal $(x_1, x_2, x_3, x_4)R$. For a graded module M, M_ν denotes the set of homogeneous elements of M of degree ν as usual. We begin with some modification of the results of [1; Section 3].

Lemma 1.1. *Let* $f, g \in k[x_1, x_2]$ $(\deg f \leqq \deg g)$ *be homogeneous polynomials such that* $\dim k[x_1, x_2]/(f, g)k[x_1, x_2]=0$ *and suppose*

$$E(f, g)=\{(a, 0)+\boldsymbol{Z}_0^2\} \cup \bigcup_{i=1}^{a} \{(a-i, \beta_i)+0\times\boldsymbol{Z}_0\},$$

where $E(f, g)$ *denotes the generical monoideal associated with the ideal* $(f, g) \cdot k[x_1, x_2]$ *and* a, β_i $(1\leqq i\leqq a)$ *are positive integers (see* [16; *p.* 282] *and* [4]). *Then* $a=\deg f$ *and the sequence of integers* $(a-1+\beta_1, a-2+\beta_2, \cdots, \beta_a)$ *is equal to* $(\deg g, \deg g+1, \cdots, \deg g+a-1)$ *up to a permutation.*

Proof. Suppose

$$(f, g)k[x_1, x_2]=f_0k[x_1, x_2] \oplus \bigoplus_{i=1}^{a} f_ik[x_2]$$

with $\deg f_0=a$, $a\leqq \deg f_1\leqq \cdots \leqq \deg f_a$ and $(\deg f_1, \deg f_2, \cdots, \deg f_a)=(a-1+\beta_1, a-2+\beta_2, \cdots, \beta_a)$ up to a permutation (see [1; Example 2.7]). The degree of the (i, j)-component of the matrix of relations $\begin{bmatrix} U_{01} \\ U_1 \end{bmatrix}$ among $f_0, f_1, \cdots, f_a$, computed by [1; Theorem 1.6] is $\deg f_j+1-\deg f_i$ $(0\leqq i\leqq a, 1\leqq j\leqq a)$, so that

$$\Delta\left(\begin{bmatrix} U_{01} \\ U_1 \end{bmatrix}\right)=\left[\begin{array}{ccc} \cdots & * & \cdots \\ 1 & & \\ & \ddots & \\ & & 1 \end{array}\right]\begin{array}{l} \}1 \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\}a \end{array}$$

where the entries situated in $*$ are all positive. It is therefore necessary and sufficient for the ideal $f_0k[x_1, x_2] \oplus \bigoplus_{i=1}^{a} f_ik[x_2]$ to be a complete intersection that $\deg f=a$, $\deg g=\deg f_1$ and that $\operatorname{rank}_k U_1\binom{1}{a}(\operatorname{mod}(x_1, x_2)k[x_1, x_2])=a-1$. This is possible if and only if $(\deg f_1, \deg f_2, \cdots, \deg f_a)=(\deg g, \deg g+1, \cdots, \deg g+a-1)$, which proves our assertion. Q.E.D.

Lemma 1.2. *Let* J *be a homogeneous ideal in* $k[x_1, x_2]$ *such that* $\dim k[x_1, x_2]/J=0$ *and* $f_0, f_1, \cdots, f_a$ *be those generators of* J *described in* [1; *Example* 2.7], *namely*

$$J=f_0k[x_1, x_2] \oplus \bigoplus_{i=1}^{a} f_ik[x_2]$$

with $\deg f_0=a$, $a\leqq \deg f_1\leqq \deg f_2\leqq \cdots \leqq \deg f_a$. *Suppose* $\dim k[x_1, x_2]/(f_0, h)k[x_1, x_2]=0$ *for a homogeneous polynomial* $h\in J_p$ $(p\geqq 1)$. *Then* $\deg f_a\leqq p+a-1$.

Proof. Let $E(J)$ be the generical monoideal associated with J and $E(f_0, h)$ the generical monoideal associated with the ideal $(f_0, h)k[x_1, x_2]$. We know

$$E(J)=\{(a,\ 0)+\boldsymbol{Z}_0^2\}\cup\bigcup_{i=1}^{a}\{(a-i,\ \beta_i)+0\times\boldsymbol{Z}_0\}$$

$$E(f_0,\ h)=\{(a,\ 0)+\boldsymbol{Z}_0^2\}\cup\bigcup_{i=1}^{a}\{(a-i,\ \beta_i')+0\times\boldsymbol{Z}_0\}\ ,$$

where a, β_i, β_i' $(1\leqq i\leqq a)$ are positive integers. Since $E(f_0,\ h)\subset E(J)$ by definition, we have $\beta_i\leqq\beta_i'$ and $a-i+\beta_i\leqq a-i+\beta_i'$ for $1\leqq i\leqq a$. The sequence of integers $(a-1+\beta_i',\ a-2+\beta_i',\ \cdots,\ \beta_a')$, on the other hand, coincides with $(p,\ p+1,\ \cdots.\ p+a-1)$ up to a permutation by Lemma 1.1, hence $a-i+\beta_i\leqq p+a-1$ for $1\leqq i\leqq a$ and $\deg f_a\leqq p+a-1$. Q.E.D.

The following proposition is a modification of [1; Section 3] which forms the basis for this paper.

Proposition 1.3. *Let I be a homogeneous ideal in R such that* $\dim R/I=2$ *and* $\operatorname{depth}_{\mathfrak{m}} R/I\geqq 1$. *After a suitable change of variables by a linear transformation,* $\dim R/I+(x_3,\ x_4)R=0$ *and there exist homogeneous polynomials f_i $(0\leqq i\leqq a+b,$ $a=\deg f_0,\ b\geqq 0)$ which have the following properties.*

1) *There exist positive integers a, β_i $(1\leqq i\leqq a)$ such that $f_0-x_1^a$, $f_i-x_1^{a-i}x_2^{\beta_i}$ $(1\leqq i\leqq a)\in N_E$ and f_{a+j} $(1\leqq j\leqq b)$ are in $(x_3,\ x_4)N_E$, where*

$$N_E=\bigoplus_{i=1}^{a}\bigoplus_{j=1}^{\beta_i-1}x_1^{a-i}x_2^{j}k(2)\ .$$

2) *Put*

$$\bar{I}=\{\bar{f}\in k[x_1,\ x_2]\,|\,\bar{f}=f(x_1,\ x_2,\ 0,\ 0)\ \textit{for some}\ f\in I\}$$

and $\bar{f}_i=f_i(x_1,\ x_2,\ 0,\ 0)$ $(0\leqq i\leqq a)$. *Then*

$$\begin{cases}\bar{I}=\bar{f}_0k[x_1,\ x_2]\oplus\bigoplus_{i=1}^{a}\bar{f}_ik[x_2]\\[2ex] k[x_1,\ x_2]=\bar{I}\oplus\bigoplus_{i=1}^{a}\bigoplus_{j=1}^{\beta_i-1}x_1^{a-1}x_2^{j}\cdot k\end{cases}$$

3) $$\begin{cases}I=f_0k(0)\oplus\bigoplus_{i=1}^{a}f_ik(1)\oplus\bigoplus_{j=1}^{b}f_{a+j}k(2)\\ R=I\oplus N_I\end{cases}$$

where N_I is a graded submodule of N_E as a $k(3)$-module.

4) $$\begin{cases}x_1N_E\subset f_0k(1)\oplus\bigoplus_{i=1}^{a}f_ik(1)\oplus\bigoplus_{j=1}^{b}f_{a+j}k(2)\oplus N_I\\[2ex] x_2N_E\subset\bigoplus_{i=1}^{a}f_ik(2)\oplus\bigoplus_{j=1}^{b}f_{a+j}k(2)\oplus N_I\end{cases}$$

5) *R/I has a free resolution described in* [1; *Corollary* 3.5], *where*

$$\lambda_2=\begin{bmatrix} U_{01} & U_{02} & 0 \\ U_1 & U_2 & U_4 \\ U_{21} & U_3 & U_5 \end{bmatrix}$$

with U_{01}, U_{02} *and* U_i $(1\leqq i\leqq 5)$ *satisfying the conditions* [1; *Corollary* 3.5.2)-3)-4)] *and with a matrix* U_{21} *of homogeneous polynomials of* $k(2)$.

Proof. After a change of variables by a linear transformation we may assume that x_4 is R/I-regular and that $\dim R/I+(x_3, x_4)R=0$. Observe that $E(\bar{I})=\{(a, 0)+\boldsymbol{Z}_0^2\}\cup\bigcup_{i=1}^{a}\{(a-i, \beta_i)+0\times\boldsymbol{Z}_0\}$ with positive integers a, $\beta_1, \cdots \beta_a$. Set N_E as in 1). By generalized Weierstrass preparation theorem applied to $\bar{I}$ there exist homogeneous polynomials $\bar{f}_i\in\bar{I}$ $(0\leqq i\leqq a)$ with $\deg\bar{f}_0=a$, $\deg\bar{f}_i=a-i+\beta_i$ $(1\leqq i\leqq a)$ satisfying the condition $\bar{f}_0-x_1^a$, $\bar{f}_i-x_1^{a-i}x_2^{\beta_i}\in N_E$ $(1\leqq i\leqq a)$, and such that we have 2) (see [7; Satz 4], [1; Theorem 2.3]). Suppose $f_i'(x_1, x_2, 0, 0)=\bar{f}_i$ $(0\leqq i\leqq a)$ with homogeneous polynomials $f_i'\in I$. Let $\mathring{I}$ denote the subset $f_0'k(0)+\sum_{i=1}^{a}f_i'k(1)$ of I. We see by the proof of [6; (1.2.7)] that this expression is in fact a direct sum, namely we have $\mathring{I}=f_0'k(0)\oplus\bigoplus_{i=1}^{a}f_i'k(1)$ and $R=\mathring{I}\oplus N_E$. Then the proof of [1; Proposition 3.3] goes well with J replaced by $\mathring{I}$ which in general is not an ideal. In this way there exist homogeneous polynomials f_i $(0\leqq i\leqq a+b)$ such that 2), 3) and 4) hold. The proof of 5) is the same as that of [1; Corollary 3.5]. It remains to prove 1). Since $f_0-x_1^a$, $f_i-x_1^{a-i}x_2^{\beta_i}$ $(1\leqq i\leqq a)$, f_{a+j} $(1\leqq j\leqq b)\in N_E$ is clear by the proof of [1; Proposition 3.3], we have only to show $f_{a+j}\in(x_3, x_4)R$, that is $\bar{f}_{a+j}=\bar{f}_{a+j}(x_1, x_2, 0, 0)=0$ for $1\leqq j\leqq b$. This, however is obvious, because, if $\bar{f}_{a+j}\neq 0$, we would have $\operatorname{lex}\bar{f}_{a+j}\in E(\bar{I})\cap(\boldsymbol{Z}_0^2\backslash E(\bar{I}))$, which is impossible. Q.E.D.

Let I, $\bar{I}$, and f_i $(0\leqq i\leqq a+b)$ be as in the proposition above and suppose I is generated by $\bigoplus_{\nu\leqq n}I_\nu$ over R. Then $\bar{I}$ is generated by $\bigoplus_{\nu\leqq n}\bar{I}_\nu$ over $k[x_1, x_2]$ and it follows from Lemma 1.2 that $\max_{1\leqq i\leqq a}\deg\bar{f}_i\leqq n+a-1$. By changing the order if necessary, we may assume $(\deg f_1, \deg f_2, \cdots, \deg f_a)$ is an increasing sequence of integers. Set $\nu_i=\deg f_i$ $(1\leqq i\leqq a)$. We find by the direct sum 1.3.2) that a and this sequence of integers are uniquely determined by $\dim_k\bar{I}_\nu$ $(\nu\geqq 0)$, or rather, since $\max_{1\leqq i\leqq a}\deg\bar{f}_i\leqq n+a-1$, it is uniquely determined by $\dim_k\bar{I}_\nu$ $(0\leqq\nu\leqq n+a-1)$. If the homogeneous coordinates x_1, x_2, x_3, x_4 are chosen generally, $\dim_k\bar{I}_\nu$ $(0\leqq\nu\leqq n+a-1)$ are independent of the choice of coordinates for an ideal I, therefore we can associate with each I uniquely a sequence of integers $(a; \nu_1, \cdots, \nu_a)$ such that $a\leqq\nu_{i-1}\leqq\nu_i$ $(2\leqq i\leqq a)$ where $\nu_i=\deg f_i$. Next put $\nu_{a+j}=\deg f_{a+j}$ $(1\leqq j\leqq b)$. We may assume that $(\deg f_{a+1}, \cdots, \deg f_{a+b})$ is an increasing sequence of integers by changing the order if necessary. Then b and the sequence $(\nu_{a+1}, \cdots,$

$\nu_{a-b})$ are uniquely determined by I, a and $(\nu_1, \cdots, \nu_a)$, because $\dim_k(\bigoplus_{j=1}^{b} f_{a+j}k(2))_\nu = \dim_k I_\nu - \dim_k \mathring{I}_\nu$, where $\mathring{I} = f_0 k(0) \oplus \bigoplus_{i=1}^{a} f_i k(1)$. Thus we are led to the following definition.

Definition 1.4. Let I be a homogeneous ideal in R such that $\dim R/I=2$ and $\operatorname{depth}_{\mathfrak{m}} R/I \geqq 1$. We call the unique sequence of integers $(a; \nu_1, \cdots, \nu_a; \nu_{a+1}, \cdots, \nu_{a+b})$ with $0 < a \leqq \nu_1 \leqq \nu_2 \leqq \cdots \leqq \nu_a$, $b \geqq 0$, $a \leqq \nu_{a+1} \leqq \nu_{a+2} \leqq \cdots \leqq \nu_{a+b}$ described above the basic sequence of I.

Remark 1.5. If $\dim R/I = \operatorname{depth}_{\mathfrak{m}} R/I = 2$, the basic sequence $(a; \nu_1, \cdots, \nu_a)$ defined above corresponds to the 'caractère numérique' $(\nu_a, \nu_{a-1}, \cdots, \nu_1)$ used in [9].

Lemma 1.6. *Let I and f_i $(0 \leqq i \leqq a+b)$ be as in Proposition* 1.3 *and let $(a; \nu_1, \cdots, \nu_a; \nu_{a+1}, \cdots, \nu_{a+b})$ be the basic sequence of I, where $a = \deg f_0$, $\nu_i = \deg f_i$ $(1 \leqq i \leqq a+b)$. In the matrix of relations*

$$\lambda_2 = \begin{bmatrix} U_{01} & U_{02} & 0 \\ U_1 & U_2 & U_4 \\ U_{21} & U_3 & U_5 \end{bmatrix}$$

among $f_0, f_1, \cdots, f_{a+b}$ (see Proposition 1.3.5)), *all entries of U_{02}, U_2 and U_4 are zero* $\bmod (x_3, x_4)R$.

Proof. Let ${}^t(g_0, g_1, \cdots, g_a, h_1, \cdots, h_{i-1}, x_1+h_i, h_{i+1}, \cdots, h_b)$ be the i-th column of $\begin{bmatrix} U_{02} \\ U_2 \\ U_3 \end{bmatrix}$ $(1 \leqq i \leqq b)$. By the very definition

$$\sum_{i=0}^{a} g_i f_i + \sum_{j=1}^{b} h_j f_{a+j} + x_1 f_{a+i} = 0 .$$

When this equation is considered in the ring $R/(x_3, x_4)R = k[x_1, x_2]$, we get $\sum_{i=0}^{a} \bar{g}_i \bar{f}_i = 0$, since $f_{a+j} \in (x_3, x_4)R$ for $1 \leqq j \leqq b$ by Proposition 1.3.1). From this follows $g_i \equiv 0 \bmod (x_3, x_4)R$ $(0 \leqq i \leqq a)$ by Proposition 1.3.2). The assertion for U_{02}, U_2 is proved in this way and we find similarly that all entries of U_4 are zero $\bmod (x_3, x_4)R$. Q.E.D.

The following proposition is a minor modification of [1; Theorem 3.7.1)] which is in fact a corollary of [2; Theorem 3.1].

Proposition 1.7. *Let μ_{ij} $(0 \leqq i \leqq a+b,\ 1 \leqq j \leqq a+2b)$, ν_i $(0 \leqq i \leqq a+b)$ be integers satisfying* [1; (3.4)] *and $0 \leqq i \leqq \sum_{i=1}^{a} (\nu_i + i - a)$. Let*

$$\lambda_2=\begin{bmatrix} U_{01} & U_{02} & 0 \\ U_1 & U_2 & U_4 \\ U_{21} & U_3 & U_5 \end{bmatrix}$$

be a matrix of homogeneous polynomials of R with $\Delta(\lambda_2)=(\mu_{ij})_{0\leq i\leq a+b,\,1\leq j\leq a+2b}$ *which satisfies the conditions* [1; *Corollary* 3.5.2.α)-β)-γ)] *and such that the entries of* U_{21} *are in* $k(2)$. *Suppose* $\lambda_2\lambda_3=0$ *with* $\lambda_3=\begin{bmatrix} -U_4 \\ -U_5 \\ U_3 \end{bmatrix}$. *Then* $\det W_1\binom{i}{\,}$ (*resp.* $\det W_2\binom{i}{\,}$) *is divisible by* $\det U_3$ (*resp.* $\det U_5$) *for* $0\leqq i\leqq a+b$, *where*

$$W_1=\begin{bmatrix} U_{01} & U_{02} \\ U_1 & U_2 \\ U_{21} & U_3 \end{bmatrix} \qquad W_2=\begin{bmatrix} U_{01} & 0 \\ U_1 & U_4 \\ U_{21} & U_5 \end{bmatrix}.$$

Proof. The formulae (3), (4) and (5) in the proof of [1; Theorem 3.7.2)] hold in the present case as well and the sequence

$$0\longrightarrow R^b\xrightarrow{\lambda_3}R^{a+2b}\xrightarrow{\lambda_2}R^{a+b+1}$$

is exact. We then use [2; Theorem 3.1 (a)] with $n=2$, $P_2=R^b$, $P_1=R^{a+2b}$, $P_0=R^{a+b+1}$, $f_2=\lambda_3$ and $f_1=\lambda_2$, and get the following commutative triangle:

$$\begin{tikzcd}
\overset{b}{\bigwedge}(R^*)^{a+2b}\cong\overset{a+b}{\bigwedge}R^{a+2b} \arrow[rr, "\overset{a+b}{\bigwedge}\lambda_2"] \arrow[dr, "a_2^*"'] & & \overset{a+b}{\bigwedge}R^{a+b+1} \\
& R \arrow[ur, "a_1"'] &
\end{tikzcd}$$

where

$$a_2=\overset{b}{\wedge}\lambda_3:\ R=\overset{b}{\wedge}R^b\longrightarrow\overset{b}{\wedge}R^{a+2b}.$$

The assertion follows from this immediately. Q.E.D.

Remark 1.8. In the situation of the previous proposition put $f_i=(-1)^i\det W_1\binom{i}{\,}/\det U_3=(-1)^i\det W_2\binom{i}{\,}/\det U_5$ up to units and suppose $ht(f_0,\cdots,f_{a+b})R\geqq 2$. Then the statement of [1; Theorem 3.7.2)] concerning the ideal $I=(f_0,\cdots,f_{a+b})R$ certainly holds in the present case.

Remark 1.9. The Hilbert polynomial $P(\nu)$ of Proj R/I is

$$P(\nu)=\left\{\sum_{i=1}^{a}\nu_i-\frac{1}{2}a(a-1)-b\right\}\nu$$
$$+\frac{1}{6}a(a-1)(a-5)-\sum_{i=1}^{a}\frac{1}{2}\nu_i(\nu_i-3)+\sum_{j=1}^{b}\nu_{a+j}-b$$

where $(a; \nu_1, \cdots, \nu_a; \nu_{a-1}, \cdots, \nu_{a+b})$ is the basic sequence of I.

§2. Free Resolution for the $k[x_1, x_2, x_3, x_4]$-Module $H^0_*(\mathcal{O}_X)$ of a Curve X in P^3_k

In this paper we mean by a curve an equidimensional complete scheme over a field k of dimension one. Let X be a curve in P^3_k and $\mathcal{J}$ its sheaf of ideals. Set $I = H^0_*(P^3_k, \mathcal{J}) \subset R$. Then $\dim R/I = 2$, $\operatorname{depth}_m R/I \geqq 1$ and the basic sequence of I is defined, which we call the basic sequence of X. Let λ_2, λ_3 be as in Propositions 1.3 and 1.7. Since $\mathcal{O}_{x,X} = \mathcal{O}_{x,P^3_k}/\mathcal{J}_{x,P^3_k}$ is Cohen-Macaulay for every $x \in X$,

(2.1.1) $I(\lambda_3)$ contains an R-sequence of length four or $I(\lambda_3) = R$ (see [1; (3.5.5)']).

A. P. Rao in [14] and E. Sernesi in [11] both describe a connection between the free resolution of the module $H^1_*(P^3_k, \mathcal{J})$ and that of R/I itself. We will discuss the same subject in the spirit of [1; Section 2].

Let M be a graded module over R with finite length and let $y_j = \sum_{i=1}^{4} a_{ij}x_i$ $(j = 3, 4)$ be two elements of R_1 algebraically independent over k, where $(a_{ij}) \in k^8$. M is then an $S := k[y_3, y_4]$-module of finite length and has a free resolution of length two

$$(2.1.2) \qquad 0 \longrightarrow S[-\bar{\varepsilon}^2] \xrightarrow{G} S[-\bar{\varepsilon}^1] \xrightarrow{H} S[-\bar{\varepsilon}^0] \longrightarrow M \longrightarrow 0,$$

where $\bar{\varepsilon}^0 = (\varepsilon^0_1, \cdots, \varepsilon^0_p)$, $\bar{\varepsilon}^1 = (\varepsilon^1_1, \cdots, \varepsilon^1_q)$ and $\bar{\varepsilon}^2 = (\varepsilon^2_1, \cdots, \varepsilon^2_r)$ are sequences of integers, by Auslander-Buchsbaum's theorem. By local duality [8] $\operatorname{Ext}^2_S(M, S) \cong \operatorname{Hom}_k(M, k)[2]$, so taking the duals of (2.1.2) we get a free resolution of $\operatorname{Hom}_k(M, k)$:

$$(2.1.3) \quad 0 \longrightarrow S[\bar{\varepsilon}^0 - 2] \xrightarrow{{}^tH} S[\bar{\varepsilon}^1 - 2] \xrightarrow{{}^tG} S[\bar{\varepsilon}^2 - 2] \longrightarrow \operatorname{Hom}_k(M, k) \longrightarrow 0.$$

Lemma 2.2. *Suppose the free resolution* (2.1.2) *of M is minimal. Then the integers p, ε^0_i $(1 \leqq i \leqq p)$, q, ε^1_i $(1 \leqq i \leqq q)$, r, ε^2_i $(1 \leqq i \leqq r)$ are independent of y_3, y_4 for general $(a_{ij}) \in k^8$.*

Proof. Suppose (2.1.2) is a minimal free resolution. Since $M_\nu = 0$ for all but a finite number of ν, $\dim_k(M_\nu / \bigoplus_{\mu \geqq 1} S_\mu M_{\nu-\mu})$ does not change for each ν, when $(a_{ij}) \in k^8$ varies in a certain Zarisky open set of k^8. It follows from this that p and ε^0_i $(1 \leqq i \leqq p)$ are uniquely determined by M and independent of y_3, y_4 for general (a_{ij}). Similarly, since (2.1.3) is minimal as well, r and ε^2_i $(1 \leqq i \leqq r)$ are also uniquely determined by $\operatorname{Hom}_k(M, k)$ or rather by M itself and independent of y_3, y_4 for general (a_{ij}). We have

$$\dim_k S[-\bar{\varepsilon}^1]_\nu = \dim_k S[-\bar{\varepsilon}^2]_\nu + \dim_k S[-\bar{\varepsilon}^0]_\nu - \dim_k M_\nu,$$

whence the uniqueness of q, ε_i^1 $(1 \leqq i \leqq q)$ follows. Q.E.D.

Let X, $\mathcal{J}$ and I as before and $(a\,;\,\bar{\nu}^1\,;\,\bar{\nu}^2)$ its basic sequence, where we have put $\bar{\nu}^1=(\nu_1, \cdots, \nu_a)$ and $\bar{\nu}^2=(\nu_{a+1}, \cdots, \nu_{a+b})$ for the sake of simplicity. $\mathcal{J}$ has a resolution

$$0 \longrightarrow \bigoplus_{j=1}^{b} \mathcal{O}_{\mathbf{P}_k^3}(-\nu_{a+j}-2) \xrightarrow{\lambda_3} \bigoplus_{i=1}^{a} \mathcal{O}_{\mathbf{P}_k^3}(-\nu_i-1) \oplus \{\bigoplus_{j=1}^{b} \mathcal{O}_{\mathbf{P}_k^3}(-\nu_{a+j}-1)\}^2$$
$$\xrightarrow{\lambda_2} \mathcal{O}_{\mathbf{P}_k^3}(-a) \oplus \bigoplus_{i=1}^{a+b} \mathcal{O}_{\mathbf{P}_k^3}(-\nu_i) \xrightarrow{\lambda_1} \mathcal{J} \longrightarrow 0$$

by Proposition 1.2, and the long exact sequences arising from this yield the exact sequence

$$0 \longrightarrow \mathrm{H}_*^1(\mathbf{P}_k^3, \mathcal{J}) \longrightarrow \bigoplus_{j=1}^{b} \mathrm{H}_*^3(\mathcal{O}_{\mathbf{P}_k^3}(-\nu_{a+j}-2))$$
$$\xrightarrow{\lambda_3} \bigoplus_{i=1}^{a} \mathrm{H}_*^3(\mathcal{O}_{\mathbf{P}_k^3}(-\nu_i-1)) \oplus \{\bigoplus_{j=1}^{b} \mathrm{H}_*^3(\mathcal{O}_{\mathbf{P}_k^3}(-\nu_{a+j}-1))\}^2 .$$

We get by this and Serre's duality a resolution

$$R[\bar{\nu}^1-3] \oplus R[\bar{\nu}^2-3]^2 \xrightarrow{{}^t\lambda_3} R[\bar{\nu}^2-2] \longrightarrow \mathrm{Hom}_k(\mathrm{H}_*^1(\mathcal{J}), k) \longrightarrow 0 . \tag{2.3.1}$$

Let us look into $\mathrm{Im}^R({}^t\lambda_3)$ in detail (see Notation 5). ${}^tU_3-x_1 1_b$, ${}^tU_5-x_2 1_b$, ${}^tU_4$ and ${}^tU_{21}$ take their entries in $k(2)$, and ${}^tU_1-x_1 1_a$, ${}^tU_2$ take their entries in $k(1)$. We have therefore by [1; Remark 4.1.1)]

$$R^b={}^tU_3 k(0)^b \oplus {}^tU_5 k(1)^b \oplus k(2)^b , \tag{2.3.2}$$

and by [1; Proposition 1.2]

$$R^{a+2b}=\begin{bmatrix} {}^tU_1 & {}^tU_{21} \\ {}^tU_2 & {}^tU_3 \\ {}^tU_4 & {}^tU_5 \end{bmatrix} k(0)^{a+b} \oplus \{k(1)^{a+b} \oplus k(0)^b\} . \tag{2.3.3}$$

The equation $\lambda_2\lambda_3=0$ implies ${}^t\lambda_3 \begin{bmatrix} {}^tU_1 & {}^tU_{21} \\ {}^tU_2 & {}^tU_3 \\ {}^tU_4 & {}^tU_5 \end{bmatrix} =0$, so we see by (2.3.3)

$$\mathrm{Im}^R({}^t\lambda_3)=\mathrm{Im}\,(k(1)^{a+b} \oplus k(0)^b \xrightarrow{{}^t\lambda_3} R^b) . \tag{2.3.4}$$

Recall ${}^t\lambda_3=[-{}^tU_4 \ \ -{}^tU_5 \ \ {}^tU_3]$, and put $\mathring{U}_5=x_2 1_b-U_5$. Then, for $v=\sum_{i\geqq 0} x_2^i v^{(i)} \in k(1)^a$ with $v^{(i)} \in k(2)^a$, we have

$${}^tU_4 v=\sum_{i\geqq 0} {}^t\mathring{U}_5^i \, {}^tU_4 v^{(i)}+\sum_{i\geqq 1} {}^tU_5(\sum_{j=1}^{i} x_2^{i-j}\, {}^t\mathring{U}_5^{j-1}\, {}^tU_4)v^{(i)} ,$$

hence by (2.3.2) and (2.3.4)

$$\mathrm{Im}^{R}({}^{t}\lambda_{3})={}^{t}U_{3}k(0)^{b}\oplus{}^{t}U_{5}k(1)^{b}\oplus N \tag{2.3.5}$$

where

$$N=\sum_{l=0}\mathrm{Im}^{k(2)}(({}^{t}\mathring{U}_{5})^{l}\,{}^{t}U_{1})\,. \tag{2.3.6}$$

We finally obtain by (2.3.1) and (2.3.5)

$$k(2)[\bar{\nu}^{2}-2]/N\cong\mathrm{Hom}_{k}(\mathrm{H}_{*}^{1}(\mathcal{J}),\ k) \tag{2.3.7}$$

as $k(2)$-modules.

Proposition 2.4. *Let X be a curve in $\mathbf{P}_{k}^{3}$, $\mathcal{J}$ the sheaf of ideals of X, $I=\mathrm{H}_{*}^{0}(\mathcal{J})$ and $(a\,;\,\bar{\nu}^{1}\,;\,\bar{\nu}^{2})=(a\,;\,\nu_{1},\,\cdots,\,\nu_{a}\,;\,\nu_{a+1},\,\cdots,\,\nu_{a+b})$ its basic sequence. Suppose the minimal free resolution for $\mathrm{H}_{*}^{1}(\mathcal{J})$ over $k(2)$ is of the form*

$$0\longrightarrow k(2)[-\bar{\varepsilon}^{2}]\xrightarrow{G}k(2)[-\bar{\varepsilon}^{1}]\xrightarrow{H}k(2)[-\bar{\varepsilon}^{0}]\xrightarrow{\alpha'}\mathrm{H}_{*}^{1}(\mathcal{J})\longrightarrow 0 \tag{2.4.1}$$

with $\bar{\varepsilon}^{0}=(\varepsilon_{1}^{0},\,\cdots,\,\varepsilon_{p}^{0})$, $\bar{\varepsilon}^{1}=(\varepsilon_{1}^{1},\,\cdots,\,\varepsilon_{q}^{1})$, $\bar{\varepsilon}^{2}=(\varepsilon_{1}^{2},\,\cdots,\,\varepsilon_{r}^{2})$. If the homogeneous coordinates x_{1}, x_{2}, x_{3}, x_{4} are chosen sufficiently general, we have $r=b$ and $\bar{\nu}^{2}=\bar{\varepsilon}^{2}$ up to a permutation. In addition, for the $k(2)$-module N defined by (2.3.6), $CN=\mathrm{Im}^{k(2)}({}^{t}G)$ with a suitable $C\in GL(b,\,k(2))$.

Proof. Let

$$0\longrightarrow k(2)[-\bar{c}^{2}]\longrightarrow k(2)[-\bar{c}^{1}]\longrightarrow N\longrightarrow 0 \tag{2.4.2}$$

be a minimal free resolution of N, where $\bar{c}^{1}=(c_{1}^{1},\,\cdots,\,c_{b'}^{1})$ and $\bar{c}^{2}=(c_{1}^{2},\,\cdots,\,c_{b''}^{2})$. If the variables x_{1}, x_{2}, x_{3}, x_{4} are chosen generally, all entries of U_{4} lie in $(x_{3},\,x_{4})k(2)$ by Lemma 1.6, so that all entries of N are in $(x_{3},\,x_{4})k(2)^{b}$. Consequently the sequence (2.4.2) followed by

$$0\longrightarrow N\longrightarrow k(2)[\bar{\nu}^{2}-2]\longrightarrow\mathrm{Hom}_{k}(\mathrm{H}_{*}^{1}(\mathcal{J}),\ k)\longrightarrow 0$$

gives rise to a minimal free resolution of $\mathrm{Hom}_{k}(\mathrm{H}_{*}^{1}(\mathcal{J}),\ k)$ as a $k(2)$-module. Comparing this resolution with the one obtained by taking the duals of (2.4.1) shows that $r=b$ and $\bar{\nu}^{2}=\bar{\varepsilon}^{2}$ up to a permutation. The last assertion is then obvious. Q.E.D.

With the use of this proposition and Lemma 2.2, we can determine $\bar{\nu}^{2}$ of the basic sequence of a given curve, if the structure of the module $\mathrm{H}_{*}^{1}(\mathcal{J})$ is known well.

Now let us proceed to a description of the free resolution for $\mathrm{H}_{*}^{0}(\mathcal{O}_{X})$. We can treat of this subject minutely only in a special case, and later a restriction will be imposed on the structure of the module $\mathrm{H}_{*}^{1}(\mathcal{J})$. Suppose the homogeneous coordinates are chosen sufficiently general so that Proposition 1.3 should hold with basic sequence $(a\,;\,\bar{\nu}^{1}\,;\,\bar{\nu}^{2})=(a\,;\,\nu_{1},\,\cdots,\,\nu_{a}\,;\,\nu_{a+1},\,\cdots,\,\nu_{a+b})$. Since $\mathrm{H}_{*}^{0}(\mathcal{O}_{X})$ is a Cohen-Macaulay R-module of dimension 2 (See [11; (1.1)]), we may assume x_3, x_4 is a $\mathrm{H}^0_*(\mathcal{O}_X)$-regular sequence. The $k(2)$-module $\mathrm{H}^1_*(\mathcal{J})$ has a minimal free resolution of the form

$$(2.5.1)\quad 0 \longrightarrow k(2)[-\bar{\nu}^2] \xrightarrow{G} k(2)[-\bar{\varepsilon}^1] \xrightarrow{H} k(2)[-\bar{\varepsilon}^0] \xrightarrow{\alpha'} \mathrm{H}^1_*(\mathcal{J}) \longrightarrow 0$$

by the previous proposition, where we assume $\varepsilon^0_1 \leqq \cdots \leqq \varepsilon^0_p$, $\varepsilon^1_1 \leqq \cdots \leqq \varepsilon^1_q$ for convenience sake. Let

$$(2.5.2)\qquad 0 \longrightarrow R/I \xrightarrow{\iota} \mathrm{H}^0_*(\mathcal{O}_X) \xrightarrow{\delta} \mathrm{H}^1_*(\mathcal{J}) \longrightarrow 0$$

be the exact sequence arising from the short exact sequence $0 \to \mathcal{J} \to \mathcal{O}_{\mathbf{P}^3_k} \to \mathcal{O}_X \to 0$. Put $e_i = (0, \cdots, \overset{\underset{\smile}{i}}{1}, \cdots, 0) \in k(2)[-\bar{\varepsilon}^0]$, $\alpha'(e_i) = \bar{e}_i$ and let ϕ_i denote a section of $\mathrm{H}^0(\mathcal{O}_X(\varepsilon^0_i))$ such that $\alpha'(e_i) = \delta(\phi_i)$ for each $1 \leqq i \leqq p$. Since $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_p$ generate the $k(2)$-module $\mathrm{H}^1_*(\mathcal{J})$, we can write

$$(2.5.3)\qquad \begin{cases} (\bar{e}_1, \cdots, \bar{e}_p)x_1 1_p = (\bar{e}_1, \cdots, \bar{e}_p)V_1 \\ (\bar{e}_1, \cdots, \bar{e}_p)x_2 1_p = (\bar{e}_1, \cdots, \bar{e}_p)V_2 \end{cases}$$

where V_i $(i=1, 2)$ are $p \times p$ matrices of homogeneous polynomials of $k(2)$. We have

$$(2.5.4)\qquad R^p = (x_1 1_p - V_1)k(0)^p \oplus (x_2 1_p - V_2)k(1)^p \oplus k(2)^p$$

(cf.(2.3.2)), and, since $(\bar{e}_1, \cdots, \bar{e}_p)(x_i 1_p - V_i) = 0$ $(i=1, 2)$, the kernel of the map

$$\alpha : R[-\bar{\varepsilon}^0] \longrightarrow \mathrm{H}^1_*(\mathcal{J})$$

defined by $\alpha(e_i) = \bar{e}_i$ coincides with

$$(x_1 1_p - V_1)k(0)^p \oplus (x_2 1_p - V_2)k(1)^p \oplus (k(2)^p \cap \mathrm{Ker}(\alpha)).$$

We have $k(2)^p \cap \mathrm{Ker}(\alpha) = \mathrm{Ker}(\alpha') = \mathrm{Im}^{k(2)}(H)$, whence

$$(2.5.5)\qquad \mathrm{Ker}(\alpha) = (x_1 1_p - V_1)k(0)^p \oplus (x_2 1_p - V_2)k(1)^p \oplus \mathrm{Im}^{k(2)}(H).$$

Let A^1 denote the matrix $[x_1 1_p - V_1 \ x_2 1_p - V_2 \ H]$ and A^1_j its j-th column $(1 \leqq j \leqq 2p+q)$. We see $\delta((\phi_1, \cdots, \phi_p)A^1_j) = 0$, so there exists a homogeneous polynomial $s_j \in R$ such that $-\iota(s_j) = (\phi_1, \cdots, \phi_p)A^1_j$ for each j $(1 \leqq j \leqq 2p+q)$. These polynomials are found in N_E (see Proposition 1.3.3)) and we will always take them from N_E in our consideration. We have thus

$$(2.5.6)\qquad \begin{cases} s_j \in N_E \quad \text{for } 1 \leqq j \leqq 2p+q \\ \deg(s_j) = \deg(s_{j+p}) = \varepsilon^0_j + 1 \quad \text{for } 1 \leqq j \leqq p \\ \deg(s_{2p+j}) = \varepsilon^1_j \quad \text{for } 1 \leqq j \leqq q \end{cases}$$

and the columns of

$$\sigma' = [\sigma'^{(0)} \ \sigma'^{(1)} \ \sigma'^{(2)}]$$

$$\begin{cases} \sigma'^{(0)} = \begin{bmatrix} f_0 & s_1, \cdots, s_p \\ 0 & x_1 1_p - V_1 \end{bmatrix} & \sigma'^{(1)} = \begin{bmatrix} f_1, \cdots, f_a & s_{p+1}, \cdots, s_{2p} \\ 0 & x_2 1_p - V_2 \end{bmatrix} \\ \sigma'^{(2)} = \begin{bmatrix} s_{2p+1}, \cdots, s_{2p+q} & f_{a+1}, \cdots, f_{a+b} \\ H & 0 \end{bmatrix} \end{cases}$$

generate $\mathrm{Ker}(\rho)$ over R, where ρ is the surjection of degree zero

$$\rho : R \oplus R[-\bar{\varepsilon}^0] \longrightarrow \mathrm{H}_*^0(\mathcal{O}_X)$$

defined by $\rho(h_0, \cdots, h_p) = h_0 + \sum_{i=1}^{p} h_i \phi_i$.

We will now look into $\mathrm{Im}^R(\sigma')$, first without regard to the degrees of polynomials, and then taking the degrees into account. Set

$$\text{(2.5.7)} \qquad \begin{cases} Q = \mathrm{Im}^R(\sigma') \subset R^{p+1} \\ P = \sigma'^{(0)} k(0)^{p+1} \oplus \sigma'^{(1)} k(1)^{p+a} \subset Q\,. \end{cases}$$

Consider the exact sequence of $k(2)$-modules

$$\text{(2.5.8)} \qquad 0 \longrightarrow Q/P \longrightarrow R^{p+1}/P \longrightarrow R^{p+1}/Q = \mathrm{H}_*^0(\mathcal{O}_X) \longrightarrow 0\,.$$

$R^{p+1}/P \cong N_E \oplus k(2)^p$ by (2.5.4) and Proposition 1.3, so that R^{p+1}/P is a finite $k(2)$-free module. $\mathrm{H}_*^0(\mathcal{O}_X)$ is, on the other hand, $k(2)$-flat, since x_3, x_4 is a $\mathrm{H}_*^0(\mathcal{O}_X)$-regular sequence, therefore we find by (2.5.8) that $\mathrm{H}_*^0(\mathcal{O}_X)$ and Q/P are $k(2)$-free and that

$$\text{(2.5.9)} \qquad Q/P \underset{k(2)}{\otimes} k \longrightarrow R^{p+1}/P \underset{k(2)}{\otimes} k = N_E \underset{k(2)}{\otimes} k \oplus k^p$$

is injective. Recall that $f_{a+j} \in (x_3, x_4)N_E$ for $1 \leqq j \leqq b$ (see Proposition 1.3.1)). The image of ${}^t(f_{a+j}, 0)$ through the map (2.5.9) is zero by this fact, so ${}^t(f_{a+j}, 0)$ is zero in $Q/P \underset{k(2)}{\otimes} k$ for $1 \leqq j \leqq b$. Furthermore we see by (2.5.7) that Q/P is generated over R by the columns of $\sigma'^{(2)}$, and the formula (2.5.5) and Proposition 1.2.3) imply that this Q/P is in fact generated over $k(2)$ by the columns of $\sigma'^{(2)}$. Consequently by Nakayama's lemma the columns of $\begin{bmatrix} s_{2p+1}, \cdots, s_{2p+q} \\ H \end{bmatrix}$ generate Q/P over $k(2)$. Since the sequence (2.5.1) is a minimal free resolution by assumption, any column of H is not a linear combination of other columns over $k(2)$, so that the columns of $\begin{bmatrix} s_{2p+1}, \cdots, s_{2p+q} \\ H \end{bmatrix}$ minimally generate Q/P over $k(2)$. We therefore obtain

$$\text{(2.5.10)} \qquad Q/P \cong \sigma''^{(2)} k(2)^q$$

$$\text{(2.5.11)} \qquad Q = \sigma'^{(0)} k(0)^{p+1} \oplus \sigma'^{(1)} k(1)^{p+a} \oplus \sigma''^{(2)} k(2)^q$$

where $\sigma''^{(2)}=\begin{bmatrix} s_{2p+1}, \cdots, s_{2p+q} \\ H \end{bmatrix}$.

Corollary 2.6. *Let G_j denote the j-th column of G (see (2.4.1)), and put $f'_{a+j}=(s_{2p+1}, \cdots, s_{2p+q})G_j$. Then we have*

$$\bigoplus_{j=1}^{b} f_{a+j}k(2)=\bigoplus_{j=1}^{b} f'_{a+j}k(2)\,.$$

Proof. By the discussion above there exists ${}^t(t^j_1, \cdots, t^j_q)\in k(2)^q$ such that ${}^t(f_{a+j}, 0)=\sigma''^{(2)}\,{}^t(t^j_1, \cdots, t^j_q)$ for each $1\leqq j\leqq b$. From this equation $H\,{}^t(t^j_1, \cdots, t^j_q)=0$, so that ${}^t(t^j_1, \cdots, t^j_q)\in \mathrm{Im}^{k(2)}(G)$ $(1\leqq j\leqq b)$ by (2.5.1). We have therefore $\bigoplus_{j=1}^{b} f_{a+j}k(2)\subset \sum_{j=1}^{b} f'_{a+j}k(2)$. We see, on the other hand, that ${}^t(f'_{a+j}, 0)=\sigma''^{(2)}G_j$ $(1\leqq j\leqq b)$ are linearly independent over $k(2)$ and that $\deg f'_{a+j}=\nu_{a+j}=\deg f_{a+j}$ $(1\leqq j\leqq b)$ by (2.5.11), (2.5.1) and (2.5.6). Consequently the sum $\sum_{j=1}^{b} f'_{a+j}k(2)$ is a direct sum $\bigoplus_{j=1}^{b} f'_{a+j}k(2)$ and coincides with $\bigoplus_{j=1}^{b} f_{a+j}k(2)$. Q.E.D.

In the following we impose a restriction on the structure of the module $\mathrm{H}^1_*(\mathcal{J})$. That is, we will assume from now on that V_1, V_2 defined by (2.5.3) take the simplest form

$$V_i=\begin{bmatrix} v_i & & 0 \\ & \ddots & \\ 0 & & v_i \end{bmatrix}=v_i 1_p \quad \text{with} \quad v_i\in k(2),\ (i=1, 2)\,.$$

This condition is satisfied for example by arithmetically Buchsbaum curves or curves with $b=1$.

Remark 2.7. If $V_i=0$ $(i=1, 2)$ and $\mathrm{H}^1_*(\mathcal{J})$ has a minimal free resolution of the form (2.4.1) for one system of homogeneous coordinates, then from the proof of Lemma 2.2 follows that p, q, r, ε^0_i, ε^1_i, ε^2_i are the unique integers stated in the same lemma.

Proposition 2.8. *Let the notation be as above. Suppose $V_i=v_i 1_p$, $i=1, 2$. Then $q\leqq a$, and there exist integers $i_1, \cdots, i_q$ $(1\leqq i_1<i_2< \cdots <i_q\leqq a)$ satisfying $\varepsilon^1_j+1=\nu_{i_j}$ $(1\leqq j\leqq q)$, and such that*

$$(2.8.1)\qquad Q=\begin{bmatrix} f_0 & s_1, \cdots, s_p \\ 0 & (x_1-v_1)1_p \end{bmatrix}k(0)^{p+1}$$

$$\oplus\begin{bmatrix} f_{i'_1}, \cdots, f_{i'_{a-q}} & s_{p+1}, \cdots, s_{2p} & s_{2p+1}, \cdots, s_{2p+q} \\ 0 & (x_2-v_2)1_p & H \end{bmatrix}k(1)^{a+p}$$

where $\{i'_1, \cdots, i'_{a-q}\}=\{1, \cdots, a\}\setminus\{i_1, \cdots, i_q\}$.

Proof. Put

$$\begin{cases} \sigma_1=[\sigma_1^{(0)}\ \sigma_1^{(1)}\ \sigma_1^{(2)}] \\ \sigma_1^{(0)}=\begin{bmatrix} f_0 & s_1, \cdots, s_p \\ 0 & (x_1-v_1)1_p \end{bmatrix} \qquad \sigma_1^{(1)}=\begin{bmatrix} f_1, \cdots, f_a & s_{p+1}, \cdots . s_{2p} \\ 0 & (x_2-v_2)1_p \end{bmatrix} \\ \sigma_1^{(2)}=\begin{bmatrix} s_{2p+1}, \cdots, s_{2p+q} \\ H \end{bmatrix} \end{cases}$$

and denote the columns of σ_1 by u_i $(0\leqq i\leqq a+2p+q)$. Then by 2.5.11)

$$Q=\bigoplus_{i=0}^{p} u_i k(0)\oplus\bigoplus_{i=1}^{p+a} u_{p+i}k(1)\oplus\bigoplus_{i=1}^{q} u_{a+2p+i}k(2)\,. \tag{2.5.11)$'$}$$

We first compute the relations among $u_0, u_1, \cdots, u_{a-2p-q}$ following [1; Theorem 1.6]. Define W_{01}, W_1', W_{21} by

$$(u_{p+1}, \cdots, u_{2p+a})x_1 1_p=-\sigma_1\left(\begin{array}{c|l} W_{01} & \} \ p+1 \\ W_1' & \} \ a+p \\ W_{21} & \} \ q \end{array}\right) \tag{2.8.2}$$

where i) entries of W_{01} are in $k(0)$
ii) entries of W_1' are in $k(1)$
iii) entries of W_{21} are in $k(2)$,

and put

$$W_1=x_1 1_{a+p}+W_1'\,. \tag{2.8.3}$$

Observe that $\sigma_1\left(\begin{array}{c|l} 0 & \} \ 1 \\ -H & \} \ p \\ 0 & \} \ a+p \\ (x_1-v_1)1_q & \} \ q \end{array}\right)$ is a $(p+1)\times q$ matrix of the form $\begin{bmatrix} t_1, \cdots, t_q \\ 0 \end{bmatrix}$

with $t_i\in I$ $(1\leqq i\leqq q)$. With the use of Corollary 2.6 and Proposition 1.3.3) we define W_{02}, W_2, Z_1 by the equation

$$\sigma_1\begin{pmatrix} 0 \\ -H \\ 0 \\ (x_1-v_1)1_q \end{pmatrix}=-\sigma_1\left(\begin{array}{c|l} W_{02} & \} \ 1 \\ 0 & \} \ p \\ W_2 & \} \ a\,, \\ 0 & \} \ p \\ GZ_1 & \} \ q \end{array}\right) \tag{2.8.4}$$

where i) entries of W_{02} are in $k(0)$
ii) entries of W_2 are in $k(1)$
iii) entries of Z_1 are in $k(2)$.

Finally $\sigma_1\begin{pmatrix} 0 \\ -H \\ (x_2-v_2)1_q \end{pmatrix}=\begin{bmatrix} t'_1, \cdots, t'_q \\ 0 \end{bmatrix}$ with $t'_i \in \bigoplus_{i=1}^{a} f_i k(2) \oplus \bigoplus_{j=1}^{b} f_{a+j} k(2)$ by (2.5.6) and Proposition 1.3. therefore we can define again W_1, Z_2 by the equation

$$(2.8.5) \qquad \sigma_1\begin{pmatrix} 0 \\ -H \\ (x_2-v_2)1_q \end{pmatrix}=-\sigma_1\begin{pmatrix} 0 \\ W_4 \\ 0 \\ GZ_2 \end{pmatrix}\begin{matrix} \}\ p+1 \\ \}\ a \\ \}\ p \\ \}\ q \end{matrix}$$

where the entries of W_4 and Z_2 lie in $k(2)$. Now the formulae (2.8.2), (2.8.3), (2.8.4), (2.8.5) and [1; Theorem 1.6] imply

$$(2.8.6) \qquad \mathrm{Ker}(\sigma_1)=\mathrm{Im}^R(\sigma_2)=\sigma_2^{(1)}k(0)^{a+p+q}\oplus\sigma_2^{(2)}k(1)^q$$

where $\sigma_2=[\sigma_2^{(0)}\ \sigma_2^{(1)}]$,

$$\sigma_2^{(0)}=\underbrace{\begin{pmatrix} W_{01} & W_{02} \\ & -H \\ W_1 & W_2 \\ & 0 \\ W_{21} & (x_1-v_1)1_q+GZ_1 \end{pmatrix}}_{a+p \qquad\qquad q}\begin{matrix} \}\ 1 \\ \}\ p \\ \}\ a \\ \}\ p \\ \}\ q \end{matrix}$$

$$\sigma_2^{(1)}=\underbrace{\begin{pmatrix} 0 \\ W_4 \\ -H \\ (x_2-v_2)1_q+GZ_2 \end{pmatrix}}_{q}\begin{matrix} \}\ p+1 \\ \}\ a \\ \}\ p \\ \}\ q \end{matrix}.$$

Set $\sigma_3=\begin{array}{c}\left[\begin{array}{c} -W_4 \\ H \\ -(x_2-v_2)1_q-GZ_2 \\ (x_1-v_1)1_q+GZ_1 \end{array}\right]\begin{array}{l} \}\ a \\ \}\ p \\ \}\ q \\ \}\ q \end{array} \\ \underbrace{\hphantom{-(x_2-v_2)1_q-GZ_2}}_{q} \end{array}$. Then we can prove $\sigma_2\sigma_3=0$ as in the proof of [1; Corollary 3.5] and get a free resolution for $H_*^0(\mathcal{O}_X)$ of length three:

$$
\begin{aligned}
(2.8.7)\qquad 0 &\longrightarrow R[-\bar{\varepsilon}^1-2] \xrightarrow{\sigma_3} R[-\bar{\nu}^1-1]\oplus R[-\bar{\varepsilon}^0-2]\oplus R[-\bar{\varepsilon}^1-1]^2 \\
&\xrightarrow{\sigma_2} R[-a]\oplus R[-\bar{\varepsilon}^0-1]\oplus R[-\bar{\nu}^1]\oplus R[-\bar{\varepsilon}^0-1]\oplus R[-\bar{\varepsilon}^1] \\
&\xrightarrow{\sigma_1} R\oplus R[-\bar{\varepsilon}^0] \xrightarrow{\rho} H_*^0(\mathcal{O}_X) \longrightarrow 0\,.
\end{aligned}
$$

Here all maps are degree zero. $\operatorname{depth}_{\mathfrak{m}} H_*^0(\mathcal{O}_X)=2$ and $\operatorname{Proj.dim}_R H_*^0(\mathcal{O}_X)=2$ by Auslander-Buchsbaum's theorem, so that rank σ_3 (mod $\mathfrak{m}$)$=q$. This implies rank W_4 (mod $\mathfrak{m}$)$=q$ because all the entries of $\sigma_3\binom{1,\ \cdots,\ a}{}$ lie in $\mathfrak{m}$. In other words, we have $q\leqq a$ and there exist $i_1, \cdots i_q$ $(1\leqq i_1<i_2<\cdots<i_q\leqq a)$ such that $\det W_4\binom{i'_1,\ \cdots,\ i'_{a-q}}{}$ is a nonzero constant in k for $i'_1, \cdots, i'_{a-q}$ defined by $\{i'_1, \cdots, i'_{a-q}\}=\{1, \cdots, a\}\setminus\{i_1, \cdots, i_q\}$. Hence we find by (2.8.7) that $\varepsilon_j^1+2=\nu_{i_j}+1$ i.e. $\varepsilon_j^1+1=\nu_{i_j}$ for $1\leqq j\leqq q$.

We next go on to the proof of (2.8.1). Set

$$
(2.8.8)\qquad \begin{cases} \sigma=[\sigma^{(0)}\ \sigma^{(1)}] \qquad \sigma^{(0)}=\sigma_1^{(0)} \\ \sigma^{(1)}=\begin{bmatrix} f_{i'_1}, \cdots, f_{i'_{a-q}} & s_{p+1}, \cdots, s_{2p} & s_{2p+1}, \cdots, s_{a+2p} \\ 0 & (x_2-v_2)1_p & H \end{bmatrix}. \end{cases}
$$

Since $\sigma_1\sigma_2^{(1)}=0$ by (2.8.6), and since $\det W_4\binom{i'_1,\ \cdots,\ i'_{a-q}}{}$ is a nonzero constant in k, each ${}^t(f_{i_j}, \underbrace{0}_{p})$ $(1\leqq j\leqq q)$ is in fact a linear combination of the columns of $\sigma^{(1)}$ over $k(1)$. Consequently $Q=\sigma^{(0)}k(0)^{p+1}+\sigma^{(1)}k(1)^{a+p}$ by (2.5.11). It remains to prove that this sum is direct. Suppose $\sigma^t(g_0, g_1, \cdots, g_{a+2p})=0$ with $g_i\in k(0)$ for $0\leqq i\leqq p$ and $g_j\in k(1)$ for $p+1\leqq j\leqq a+2p$. Then clearly $g_1=g_2=\cdots=g_p=0$. Since the first row of $\sigma\binom{}{1}{}^t(g_1, \cdots, g_{a+2p})=\sigma\binom{}{1}{}^t(0, \cdots, 0, g_{p+1}, \cdots, g_{2p+a})$ is in $\bigoplus_{i=1}^{a} f_i k(1)\oplus\bigoplus_{j=1}^{b} f_{a+j}k(2)\oplus N_I$ by Proposition 1.3, g_0 is also zero. Thus $\sigma^{(1)}\,{}^t(g_{p+1}, \cdots, g_{2p+a})=0$ and this can be rewritten $[\sigma_1^{(1)}\ \sigma_1^{(2)}]^t(t_1, \cdots, t_a, g_{p+a-q+1}, \cdots, g_{a+2p})=0$

where $\qquad \begin{cases} t_{i'_j}=g_{p+j} & \text{for}\quad 1\leqq j\leqq a-q \\ t_j=0 & \text{for}\quad j\in\{i_1, \cdots, i_q\}\,. \end{cases}$

It follows from this that ${}^t(t_1, \cdots, t_a, g_{p+a-q+1}, \cdots, g_{a+2p}) \in \sigma_2^{(1)} k(1)^q$ by (2.8.6), namely ${}^t(t_1, \cdots, t_a, g_{p+a-q+1}, \cdots, g_{a+2p}) = \sigma_2^{(1)}\,{}^t(c_1, \cdots, c_q)$ for some ${}^t(c_1, \cdots, c_q) \in k(1)^q$. This implies $W_4\binom{i'_1, \cdots, i'_{a-q}}{}{}^t(c_1, \cdots, c_q)=0$, therefore ${}^t(c_1, \cdots, c_q)=0$ and $g_{p+1}= \cdots =g_{a+2p}=0$. Q.E.D.

Proposition 2.9. *Under the same notation and assumption as in the previous proposition, let τ denote the matrix of relations among the columns of σ computed by* [1; *Theorem* 1.6]. *Then τ takes the following form*:

$$
(2.9.1)\qquad
\left\{
\begin{aligned}
&\tau=\tau'+\tau''\\
&\tau'=\left[\begin{array}{c} 0 \\ \\ \\ x_1 1_{a+p} \\ \\ \end{array}\right]
-\begin{array}{c}
\left[\begin{array}{ccc}
0 & 0 & 0\\
0 & (x_2-v_2)1_p & H\\
0 & 0 & 0\\
0 & v_1 1_p & 0\\
0 & 0 & v_1 1_q
\end{array}\right]
\begin{array}{l}
\}\,1\\
\}\,p\\
\}\,a-q\\
\}\,p\\
\}\,q
\end{array}\\
\begin{array}{ccc}
\underbrace{\hphantom{000}}_{a-q} & \underbrace{\hphantom{(x_2-v_2)1_p}}_{p} & \underbrace{\hphantom{v_11_q}}_{q}
\end{array}
\end{array}\\
&\tau''=\left[\begin{array}{c}
W_5\\
0\\
W_6\\
\begin{bmatrix} -H \\ (x_2-v_2)1_q \end{bmatrix} Z_3+\begin{bmatrix} 0 \\ G \end{bmatrix} Z_4
\end{array}\right]
\begin{array}{l}
\}\,1\\
\}\,p\\
\}\,a-q\\
\}\,p+q
\end{array}
\end{aligned}
\right.
$$

where W_5, W_6 are matrices with entries in $k(1)$, and Z_3 (resp. Z_4) is a $q\times(a+p)$ (resp. $b\times(a+p)$) matrix with entries in $k(1)$ (resp. $k(2)$).

Proof. Let τ' be the matrix defined by (2.9.1) and t one of its columns. Since $s_j \in N_E$ $(1 \leqq j \leqq 2p+q)$, $\sigma t={}^t(f, 0)$ with $f \in f_0 k(1) \oplus \bigoplus_{i=1}^{a} f_i k(1) \oplus \bigoplus_{j=1}^{b} f_{a+j} k(2)$ by Proposition 1.3. We see therefore by (2.5.11) and Proposition 2.8 that σt is in the module generated by the columns of $\sigma\binom{}{1, \cdots, p}$ over $k(1)$. This enables us to put

$$
\sigma\tau'=-\sigma\binom{}{1, \cdots, p}
\left[\begin{array}{c} W_5 \\ W_6 \\ W_7 \end{array}\right]
\begin{array}{l}
\}\,1\\
\}\,a-q\\
\}\,p+q
\end{array}
$$

where W_5, W_6 and W_7 are matrices with entries in $k(1)$. W_7 must satisfy the equation $[(x_2-v_2)1_p\ H]W_7=0$, so each column of W_7 is in the module of relations among the columns of $[(x_2-v_2)1_p\ H]$. This module of relations are easily computed and are generated by the columns of the matrix

$$\begin{bmatrix} -H & 0 \\ (x_2-v_2)1_q & G \end{bmatrix}$$

therefore W_7 takes the form

$$\begin{bmatrix} -H \\ (x_2-v_2)1_q \end{bmatrix} Z_3 + \begin{bmatrix} 0 \\ G \end{bmatrix} Z_4$$

where Z_3 (resp. Z_4) is a $q\times(a+p)$ (resp. $b\times(a+p)$) matrix with entries in $k(1)$ (resp. $k(2)$). Thus the formula (2.9.1) follows. Q.E.D.

We summarise below some results used in section five, most of which are found in [11; Sections 3 and 4] or elsewhere. For a matrix U of polynomials of R we will denote by $I(U)$ the ideal generated by the $r\times r$ minors of U where r is the rank of U. Let X, $\mathcal{J}$ and I be as in the beginning of this section and let

$$(2.10.1) \qquad 0 \longrightarrow R[-\bar{\gamma}^2] \xrightarrow{\psi_2} R[-\bar{\gamma}^1] \xrightarrow{\psi_1} R \oplus R[-\bar{\gamma}^0] \xrightarrow{\rho} \mathrm{H}^0_*(\mathcal{O}_X) \longrightarrow 0$$

be an arbitrary free resolution for $\mathrm{H}^0_*(\mathcal{O}_X)$, where $\bar{\gamma}^0=(\gamma^0_1, \cdots, \gamma^0_l)$, $\bar{\gamma}^1=(\gamma^1_1, \cdots, \gamma^1_{m'})$ and $\bar{\gamma}^2=(\gamma^2_1, \cdots, \gamma^2_m)$. Let A denote the matrix corresponding to $pr_2\circ\psi_1 : R[-\bar{\gamma}^1] \longrightarrow R[-\bar{\gamma}^0]$ and B the matrix corresponding to ψ_2. Then $m'=m+l+1$ by [3; Corollary 1] and

$$(2.10.2) \qquad 0 \longrightarrow R[-\bar{\gamma}^2] \xrightarrow{B} R[-\bar{\gamma}^1] \xrightarrow{A} R[-\bar{\gamma}^0] \xrightarrow{\alpha} M \longrightarrow 0$$

is a complex which is exact except at $R[-\bar{\gamma}^1]$, where $M=\mathrm{H}^1_*(\mathcal{J})$. Since the degree of the Hilbert polynomial of X is one, we deduce from (2.10.2)

$$(2.10.3) \qquad \sum_{i=1}^{l} \gamma^0_i - \sum_{i=1}^{m+l+1} \gamma^1_i + \sum_{i=1}^{m} \gamma^2_i = 0 .$$

Note that $\operatorname{depth}_{I(A)} R \geqq 4$. Let $\mathcal{E}$ be the locally free sheaf of rank $m+1$ on $\mathbf{P}^3_k$ defined by the following exact sequence:

$$(2.10.4) \qquad 0 \longrightarrow \bigoplus_{i=1}^{l} \mathcal{O}_{\mathbf{P}^3_k}(\gamma^0_i) \xrightarrow{{}^tA} \bigoplus_{i=1}^{m+l+1} \mathcal{O}_{\mathbf{P}^3_k}(\gamma^1_i) \xrightarrow{\phi} \mathcal{E} \longrightarrow 0 .$$

We define a map $\psi : \mathcal{E} \rightarrow \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbf{P}^3_k}(\gamma^2_i)$ by putting $\psi(v)={}^tB(u)$ where $u \in \bigoplus_{i=1}^{m+l+1} \mathcal{O}_{\mathbf{P}^3_k}(\gamma^1_i)$ is such that $\phi(u)=v$. From (2.10.3) and (2.10.4) follows $\overset{m+1}{\wedge} \mathcal{E}^{\vee} \cong \mathcal{O}_{\mathbf{P}^3_k}(-\sum_{i=1}^{m} \gamma^2_i)$, therefore using the isomorphism $\mathcal{E}^{\vee} \xrightarrow{\sim} \overset{m}{\wedge} \mathcal{E} \otimes \overset{m+1}{\wedge} \mathcal{E}^{\vee}$, the exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbf{P}^3_k}(-\gamma^2_i) \xrightarrow{\psi^{\vee}} \mathcal{E}^{\vee} \xrightarrow{\overset{m}{\wedge}\psi} \mathcal{O}_{\mathbf{P}^3_k}(\sum_{i=1}^{m} \gamma^2_i) \otimes \overset{m+1}{\wedge} \mathcal{E}^{\vee} \cong \mathcal{O}_{\mathbf{P}^3_k}$$

is obtained. We see by the definitions of $\mathcal{E}$ and ϕ that the image of $\overset{m}{\wedge}\phi$ is the sheaf of ideals in $\mathcal{O}_{\mathbf{P}_k^3}$ defined by all the maximal minors of B, and hence coincides with $\mathcal{J}$ itself by [11; (3.3) and (4.2)]. (Observe that the proposition of [11] we have referred here are valid also for curves in our sense.) Hence we get

$$(2.10.5) \qquad \begin{cases} 0 \longrightarrow \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbf{P}_k^3}(-\gamma_i^2) \xrightarrow{\phi^\vee} \mathcal{E}^\vee \xrightarrow{\overset{m}{\wedge}\phi} \mathcal{J} \longrightarrow 0 \\ \operatorname{Proj} R/I(B) = X \end{cases}$$

$$(2.10.6) \qquad \operatorname{depth}_{I(B)} R = 2\,.$$

Proposition 2.11. *Conversely, suppose a given complex* (2.10.2) *with* $\operatorname{depth}_{I(A)} R \geqq 4$ *satisfy the conditions* (2.10.3) *and* (2.10.6). *Then by the procedure above we obtain the exact sequence* (2.10.5) *and the sheaf of ideals* $\mathcal{J}$ *in* $\mathcal{O}_{\mathbf{P}_k^3}$ *which defines the curve* $\operatorname{Proj} R/I(B)$ *in* $\mathbf{P}_k^3$. *For such curves the following holds*:

$$(2.11.1) \qquad \mathrm{H}_*^1(\mathbf{P}_k^3, \mathcal{J}(\nu)) \cong M$$

$$(2.11.2) \qquad h^0(\mathcal{O}_X(\nu)) = h^0(\mathcal{O}_{\mathbf{P}_k^3}(\nu)) + \sum_{i=1}^{m} h^0(\mathcal{O}_{\mathbf{P}_k^3}(\nu-\gamma_i^2))$$

$$+ \sum_{i=1}^{l} h^0(\mathcal{O}_{\mathbf{P}_k^3}(\nu-\gamma_i^0)) - \sum_{i=1}^{m+l+1} h^0(\mathcal{O}_{P_k^3}(\nu-\gamma_i^1))\,.$$

Proof. Formulae (2.11.1)-2) are deduced from the long exact sequences arising from (2.10.4) and (2.10.5).

§ 3. Structure Theorem for the Ideals Defining Arithmetically Buchsbaum Curves in $\mathbf{P}_k^3$ [1]

Let $X \subset \mathbf{P}_k^3$ be a curve with the property $\mathfrak{m}\mathrm{H}_*^1(\mathcal{J}) = \mathfrak{m}\mathrm{H}_*^1(R/I) = 0$, where $\mathcal{J}$ and I denote the sheaf of ideals of X and $\mathrm{H}_*^0(\mathcal{J})$ respectively. We know that the ring R/I is Buchsbaum for such a curve (see for example [12; Korollar 1.2.3 or Korollar 4.1.3]), and in this case X is called an arithmetically Buchsbaum curve. We will give a structure theorem for these curves in the language of our Proposition 1.3. For an arithmetically Buchsbaum curve X we set $i(X) = \dim_{R/\mathfrak{m}} \mathrm{H}_*^1(\mathcal{J})$ (see [5; p. 11]), and we denote by $\sharp A$ the number of elements of a finite set A. In the following we abbreviate 'arithmetically Buchsbaum' to 'a. B.'.

Theorem 3.1. *Let* X *be an a. B. curve in* $\mathbf{P}_k^3$ *and* $(a\,;\, \nu_1, \cdots, \nu_a\,;\, \nu_{a+1}, \cdots, \nu_{a+b})$ *its basic sequence. Then we have*

1) $i(X) = b$

2) *For each integer* ν

$$\sharp\{i \mid \nu_i=\nu \ \ 1\leqq i\leqq a\} \geqq 2\cdot\sharp\{j \mid \nu_{a+j}=\nu \ \ 1\leqq j\leqq b\}\,,$$

and hence $a\geqq 2b$.

Proof. Suppose the homogeneous coordinates x_1, x_2, x_3, x_4 are sufficiently generally chosen. Since $\mathfrak{m}\mathrm{H}^1_*(\mathcal{J})=0$, $\mathrm{H}^1_*(\mathcal{J})$ has the following minimal free resolution as a $k(2)$-module:

$$(3.1.3)\qquad 0 \longrightarrow k(2)[-\bar{\varepsilon}^0-2] \xrightarrow{G} k(2)[-\bar{\varepsilon}^0-1]^2 \xrightarrow{H} k(2)[-\bar{\varepsilon}^0] \longrightarrow \mathrm{H}^1_*(\mathcal{J}) \longrightarrow 0$$

where $\bar{\varepsilon}^0=(\varepsilon_1^0, \cdots, \varepsilon_{i(X)}^0)\,(\varepsilon_1^0\leqq\varepsilon_2^0\leqq\cdots\leqq\varepsilon_{i(X)}^0)$, $H=[x_3 1_{i(X)}\ \ x_4 1_{i(X)}]$ and $G=\begin{bmatrix} -x_4 1_{i(X)} \\ x_3 1_{i(X)} \end{bmatrix}$. Hence $i(X)=b$ and

$$(3.1.4)\qquad \varepsilon_j^0+2=\nu_{a+j} \qquad \text{for} \quad 1\leqq j\leqq b$$

by Proposition 2.4. We see, on the other hand, there exist integers $1\leqq i_1<\cdots<i_{2b}\leqq a$ satisfying $((\varepsilon_1^0+1)+1, (\varepsilon_1^0+1)+1, (\varepsilon_2^0+1)+1, (\varepsilon_2^0+1)+1, \cdots, (\varepsilon_b^0+1)+1, (\varepsilon_b^0+1)+1)=(\varepsilon_1^0+2, \varepsilon_1^0+2, \varepsilon_2^0+2, \varepsilon_2^0+2, \cdots, \varepsilon_b^0+2, \varepsilon_b^0+2)=(\nu_{i_1}, \nu_{i_2}, \cdots, \nu_{i_{2b}})$ by Proposition 2.8. Consequently $(\nu_{a-1}, \nu_{a+1}, \nu_{a+2}, \nu_{a+2}, \cdots, \nu_{a+b}, \nu_{a+b})=(\nu_{i_1}, \nu_{i_2}, \cdots, \nu_{i_{2b}})$, which proves 2). Q.E.D.

Theorem 3.2. *Let* (μ_{ij}) $\nu_0=a$, $\nu_i(1\leqq i\leqq a+b)$ *and* $\lambda_2=\begin{pmatrix} U_{01} & U_{02} & 0 \\ U_1 & U_2 & U_4 \\ U_{21} & U_3 & U_5 \end{pmatrix}$ *be as in Proposition* 1.7, *and suppose that* $\lambda_2\lambda_3=0$ *with* $\lambda_3=\begin{pmatrix} -U_4 \\ -U_5 \\ U_3 \end{pmatrix}$ *and that the entries of* U_4 *are in* $\mathfrak{m}$. *Suppose, in addition, that* (2.1.1) *is satisfied and that* $\mathrm{ht}\, I\geqq 2$ *for* $I=f_0k(0)\oplus\bigoplus_{i=1}^{a} f_ik(1)\oplus\bigoplus_{j=1}^{b} f_{a+j}k(2)$ *where* f_i $(0\leqq i\leqq a+b)$ *are defined by the formula in Remark* 1.8. *Let* X *denote the curve* Proj R/I *and* $\mathcal{J}$ *its sheaf of ideals. Then,* X *is an a.B. curve whose basic sequence is* $(a\,;\, \nu_1, \cdots, \nu_a\,;\, \nu_{a+1}, \cdots, \nu_{a+b})$ *if and only if*

1) *The entries of* U_3 *and* U_5 *as well as those of* U_4 *lie in* $\mathfrak{m}$.
2) $\mathrm{Im}^{k(2)}({}^tU_4)=(x_3, x_4)k(2)^b$.

Proof. Observe first that the argument concerning (2.3.1), (2.3.5), (2.3.7) or Proposition 2.4 is valid in the present case where $(a\,;\,\bar{\nu}^1\,;\,\bar{\nu}^2)=(a\,;\,\nu_1, \cdots, \nu_a\,;\,\nu_{a+1}, \cdots, \nu_{a+b})$ is not necessarily known in advance to be the basic sequence of X. We have thus

$$(3.2.3)\qquad \mathrm{Hom}_k(\mathrm{H}^1_*(\mathcal{J}), k)\cong\begin{cases} R[\bar{\nu}^2-2]/\mathrm{Im}^R({}^t\lambda_3) & \text{as } R\text{-modules} \\ k(2)[\bar{\nu}^2-2]/N & \text{as } k(2)\text{-modules} \end{cases}$$

where $N=\sum_{i\geqq 0}\mathrm{Im}^{k(2)}(({}^t\mathring{U}_5)^i\, {}^tU_4)$, and $N\subset(x_3, x_4)k(2)[\bar{\nu}^2-2]$, because all the entries of U_4 are in $\mathfrak{m}$ by hypothesis. Suppose X is a.B.. Then $\mathrm{H}^1_*(\mathcal{J})$ has a free resolution of the form (3.1.3), so that $i(X)=b$ and $N=\mathrm{Im}^{k(2)}({}^tG)=(x_3, x_4)k(2)^b$ by Proposition 2.4. Furthermore, since $\mathfrak{m}\mathrm{H}_*^1(\mathcal{J})=0$ and $\mathrm{H}_*^1(\mathcal{J})$ is minimally generated over R by b elements, the R-module $R^b/\mathrm{Im}^R({}^t\lambda_3)\cong\mathrm{Hom}_k(\mathrm{H}_*^1(\mathcal{J}), k)\cong\mathrm{Ext}_R^4(\mathrm{H}_*^1(\mathcal{J}), R)$ (see (2.3.1) and [8]) has its minimal free resolution

$$\longrightarrow R^{4b} \xrightarrow{[x_1 1_b\ x_2 1_b\ x_3 1_b\ x_4 1_b]} R^b \longrightarrow R^b/\mathrm{Im}^R({}^t\lambda_3) \longrightarrow 0$$

by taking the dual of the minimal free resolution for $\mathrm{H}_*^1(\mathcal{J})$ over R. We have therefore $\mathrm{Im}^R({}^t\lambda_3)=\mathfrak{m}R^b$ and find that the entries of ${}^tU_3$ and ${}^tU_5$ are in $\mathfrak{m}$. Since $(x_3, x_4)k(2)^b=N=\sum_{i\geq 0}\mathrm{Im}^{k(2)}(({}^t\mathring{U}_5)^i\,{}^tU_4)$ and since the entries of $({}^t\mathring{U}_5)^i\,{}^tU_4$ lie in $(x_3, x_4)^2k(2)$ for $i\geqq 1$, we have $\mathrm{Im}^{k(2)}({}^tU_4)=(x_3, x_4)k(2)^b$.

Conversely, suppose the conditions 1) and 2) are satisfied. In this case (3.2.3) becomes

$$(3.2.4)\quad \mathrm{Hom}_k(\mathrm{H}_*^1(\mathcal{J}), k)\cong\begin{cases} R[\bar{\nu}^2-2]/\mathfrak{m}R[\bar{\nu}^2-2] & \text{as } R\text{-modules} \\ k(2)[\bar{\nu}^2-2]/(x_3, x_4)k(2)[\bar{\nu}^2-2] & \text{as } k(2)\text{-modules}, \end{cases}$$

since $N=(x_3, x_4)k(2)[\bar{\nu}^2-2]$ by 2) and $\mathrm{Im}^R({}^t\lambda_3)=\mathfrak{m}R^b$ by 1) and 2). This implies $\mathfrak{m}\mathrm{H}_*^1(\mathcal{J})=0$, hence X is an a. B. curve with $i(X)=b$. It remains to prove that the basic sequence of X is in fact $(a\,;\bar{\nu}^1\,;\bar{\nu}^2)$ if X is a. B.. Let $(a\,;\bar{\nu}'^1\,;\bar{\nu}'^2)=(a\,;\nu_1', \cdots, \nu_a'\,;\nu_{a+1}', \cdots, \nu_{a+b'}')$ be the basic sequence of X. We have $i(X)=b$ and (3.2.4) implies that ε_j^0 in (3.1.3) coincides with $\nu_{a+j}-2$ for $1\leqq j\leqq b$. Hence $b=b'$ and $\bar{\nu}'^2=\bar{\nu}^2$ by Proposition 2.4. Counting $\dim_k I_\nu(\nu\geqq 0)$ then shows $\bar{\nu}'^1=\bar{\nu}^1$.

Q.E.D.

Corollary 3.3. *Let the notation be as in the previous theorem and suppose X is a. B.. Then there exists a matrix $L\in GL(a+b, k(2))$ of homogeneous polynomials such that for $(f_1', \cdots, f_{a+b}')=(f_1, \cdots, f_{a+b})L$ the following holds.*

1) *f_i' $(1\leqq i\leqq a+b)$ are homogeneous polynomials and*

$$\deg f_1'\leqq\cdots\leqq\deg f_{a-2b}',\ \Delta(f_{a-2b+1}', \cdots, f_a')$$
$$=(\nu_{a+1}, \cdots, \nu_{a+b}, \nu_{a+1}, \cdots, \nu_{a+b}).$$

2) $I=f_0k(0)\oplus\bigoplus_{i=1}^{a}f_i'k(1)\oplus\bigoplus_{j=1}^{b}f_{a+j}'k(2)$

3) *The matrix of relations among $f_0, f_1', \cdots, f_{a+b}'$ computed by* [1 ; *Theorem* 1.6] *takes the form*

$$\lambda_2'=\left(\begin{array}{ccc} U_{01}' & U_{02}' & 0 \\ & & 0 \\ U_1' & U_2' & x_3 1_b \\ & & x_4 1_b \\ U_{21}' & U_3' & x_2 1_b \end{array}\right)\begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} a-2b \\ \\ \\ \\ \end{array}$$

where the entries of λ_2' *of course satisfy the conditions of Proposition* 1.3.

Proof. We see by Theorem 3.2 that there exists a matrix

$$L_1=\begin{bmatrix} L_{11} & 0 \\ L_{21} & 1_b \end{bmatrix} \in GL(a+b,\ k(2)) \text{ such that } L_1\begin{bmatrix} U_4 \\ U_5 \end{bmatrix}=\begin{bmatrix} 0 \\ x_3 1_b \\ x_4 1_b \\ x_2 1_b \end{bmatrix}.$$

We set $L=L_1^{-1}$. Since $\Delta(\lambda_2)$ satisfies the condition [1; (3.4)], each f_i' $(1\leqq i\leqq a+b)$ becomes homogeneous and

$$(3.3.4) \qquad (f_1', \cdots, f_{a+b}')\begin{bmatrix} 0 \\ x_3 1_b \\ x_1 1_b \\ x_2 1_b \end{bmatrix}=0 .$$

Hence we obtain 1) by changing the order if necessary. In fact we can take L_1 so that $\Delta(f_1, \cdots, f_{a+b})=\Delta(f_1', \cdots, f_{a+b}')$ should be satisfied up to a permutation. Note that the formula (3.3.4) can be rewritten as

$$(3.3.4)' \qquad (f_{a+1}', \cdots, f_{a+b}')x_2 1_b=-(f_1', \cdots, f_a')\begin{bmatrix} 0 \\ x_3 1_b \\ x_1 1_b \end{bmatrix}$$

To prove the assertion 2) we first show that I is contained in the set $I' := f_0 k(0)+\sum_{i=1}^{a} f_i' k(1)+\sum_{j=1}^{b} f_{a+j}' k(2)$. Let $f=\sum_{i=0}^{a+b} f_i g_i$ $({}^t(g_0, \cdots, g_{a+b})\in k(0)\oplus k(1)^a \oplus k(2)^b)$ be an element of I. Then

$$\begin{aligned} f&=(f_0, \cdots, f_{a+b})\cdot{}^t(g_0, \cdots, g_{a+b}) \\ &=f_0 g_0+(f_1', \cdots, f_{a+b}')L_1{}^t(g_1, \cdots, g_{a+b}) \\ &=f_0 g_0+(f_1', \cdots, f_a')L_{11}{}^t(g_1, \cdots, g_a) \\ &\quad +(f_{a+1}', \cdots, f_{a+b}')\{L_{21}{}^t(g_1, \cdots, g_a)+{}^t(g_{a+1}, \cdots, g_{a+b})\} . \end{aligned}$$

Observe that $L_{21}{}^t(g_1, \cdots, g_a)$ is in $k(1)^b$, hence

$$(3.3.5) \qquad \begin{aligned} L_{21}{}^t(g_1, \cdots, g_a)&=x_2 1_b{}^t(h_{a+1}, \cdots, h_{a+b}) \\ &\quad +{}^t(r_{a+1}, \cdots, r_{a+b}) \end{aligned}$$

with $h_{a+j}\in k(1)$ and $r_{a+j}\in k(2)$ for $1\leqq j\leqq b$. Using (3.3.4)$'$ and (3.3.5) we obtain

$$\begin{aligned} f&=f_0 g_0 \\ &\quad +(f_1', \cdots, f_a')\left\{L_{11}{}^t(g_1, \cdots, g_a)-\begin{bmatrix} 0 \\ x_3 1_b \\ x_4 1_b \end{bmatrix}{}^t(h_{a+1}, \cdots, h_{a+b})\right\} \\ &\quad +(f_{a+1}', \cdots, f_{a+b}')\{{}^t(g_{a+1}, \cdots, g_{a+b})+{}^t(r_{a+1}, \cdots, r_{a+b})\} . \end{aligned}$$

This implies $f \in I'$ and $I \subset I'$, therefore $I = I'$ by obvious inclusion $I' \subset I$. Comparing $\dim_k I_\nu$ with $\dim_k I'_\nu$ for $\nu \geqq 0$ then shows I is the direct sum $f_0 k(0) \oplus \bigoplus_{i=1}^{a} f'_i k(1) \oplus \bigoplus_{j=1}^{b} f'_{a+j} k(2)$. Let $\lambda'_2 = \begin{bmatrix} U'_{01} & U'_{02} & 0 \\ U'_1 & U'_2 & U'_4 \\ U'_{21} & U'_3 & U'_5 \end{bmatrix}$ be the matrix of relations among $f_0, f'_1, \cdots, f'_{a+b}$ computed by [1; Theorem 1.6], then we find by (3.3.4) that $\begin{bmatrix} U'_4 \\ U'_5 \end{bmatrix}$ must be $\begin{bmatrix} 0 \\ x_3 1_b \\ x_4 1_b \\ x_2 1_b \end{bmatrix}$. Q.E.D.

Let $X \subset \mathbf{P}^3_k$ be an a.B. curve with basic sequence $(a\,;\ \nu_1, \cdots, \nu_a\,;\ \nu_{a+1}, \cdots, \nu_{a+b})$. In view of the corollary above, it seems convenient to use $(f_0, f'_1, \cdots, f'_{a+b})$ instead of $(f_0, f_1, \cdots, f_{a+b})$. And we will always assume from now on that $(f_0, f_1, \cdots, f_{a+b})$ itself satisfies the conditions 1), 2) and 3) of Corollary 3.3 when we deal with a.B. curves. We set $\Delta(f_1, \cdots, f_{a-2b}) = (m_1, \cdots, m_{a-2b}) = \bar{m}$, $(\nu_{a+1}, \cdots, \nu_{a+b}) = (n_1, \cdots, n_b) = \bar{n}$ and call $(a\,;\ \bar{m}\,;\ \bar{n}) = (a\,;\ m_1, \cdots, m_{a-2b}\,;\ n_1, \cdots, n_b)$ the short basic sequence of X. Note that $\Delta(f_0, f_1, \cdots, f_{a+b}) = (a, m_1, \cdots, m_{a-2b}, n_1, \cdots, n_b, n_1, \cdots, n_b, n_1, \cdots, n_b)$. $a \leqq m_1 \leqq \cdots \leqq m_{a-2b}$ and $a \leqq n_1 \leqq \cdots \leqq n_b$.

We have (3.1.3) and (3.1.4) for X, therefore by the results of Section two the R-module $\mathrm{H}^0_*(\mathcal{O}_X)$ has a free resolution of the following form (see Proposition 2.8 and (2.9.1)):

$$\begin{aligned} & 0 \longrightarrow R[-\bar{m}-1] \oplus R[-\bar{n}]^3 \\ (3.4.1) \qquad & \xrightarrow{\tau} R[-a] \oplus R[-\bar{n}+1] \oplus R[-\bar{m}] \oplus R[-\bar{n}+1]^3 \\ & \xrightarrow{\sigma} R \oplus R[-\bar{n}+2] \xrightarrow{\rho} \mathrm{H}^0_*(\mathcal{O}_X) \longrightarrow 0 \end{aligned}$$

where

$$\sigma = \begin{bmatrix} f_0 & s_1, \cdots, s_b & f_{i'_1}, \cdots, f_{i'_{a-2b}} & s_{b+1}, \cdots, s_{2b} & s_{2b+1}, \cdots, s_{4b} \\ 0 & x_1 1_b & 0 & x_2 1_b & x_3 1_b \ \ x_4 1_b \end{bmatrix}$$

$$\tau = \begin{pmatrix} & 0 & & \\ 0 & -x_2 1_b & -x_3 1_b & -x_4 1_b \\ & & & \\ & x_1 1_{a+b} & & \end{pmatrix} + \underbrace{\left(\begin{matrix} W_5 \\ 0 \\ W_6 \\ \begin{pmatrix} -x_3 1_b & -x_4 1_b \\ \multicolumn{2}{c}{x_2 1_{2b}} \end{pmatrix} Z_3 + \begin{pmatrix} 0 \\ -x_4 1_b \\ x_3 1_b \end{pmatrix} Z_4 \end{matrix}\right)}_{a+b} \begin{matrix} \}\ 1 \\ \}\ b \\ \}\ a-2b \\ \}\ 3b \end{matrix}$$

And

$$\mathrm{H}^1_*(\mathcal{J})\cong R[-\bar{n}+2]/\mathfrak{m}R[-\bar{n}+2]\,. \tag{3.4.2}$$

It should be noted that $\varDelta(f_{\iota'_1},\cdots,f_{\iota'_{a-2b}})=(m_1,\cdots,m_{a-2b})$ by Proposition 2.8.

§ 4. Examples of Integral Arithmetically Buchsbaum Curves with Basic Sequence $(a\,;\underbrace{n,\cdots,n}_{a}\,;\underbrace{n,\cdots,n}_{b})\,(n\geqq a\geqq 2b)$

Let us begin by giving a solution to the equation $\lambda_2\lambda_3=0$ in the case where $U_4=\begin{pmatrix}0\\ x_3 1_b\\ x_4 1_b\end{pmatrix}$ and $U_5=x_2 1_b$. Let a, b, $a\leqq m_1\leqq m_2\leqq\cdots\leqq m_{a-2b}$, $a\leqq n_1\leqq n_2\leqq\cdots\leqq n_b$ be integers and set $\bar{m}=(m_1,\cdots,m_{a-2b})$, $\bar{n}=(n_1,\cdots,n_b)$, $(\nu_1,\cdots,\nu_{a+b})=(\bar{m},\bar{n},\bar{n},\bar{n})$. Define $\varLambda_2:=(\mu_{ij})\,(0\leqq i\leqq a+b,\ 1\leqq j\leqq a+2b)$ as in [1; (3.4)]. Set

$$\lambda_2=\begin{pmatrix} U_{01} & U_{02} & 0\\ & & 0\\ U_1 & U_2 & x_3 1_b\\ & & x_4 1_b\\ U_{21} & U_3 & x_2 1_b\end{pmatrix},\qquad \lambda_3=\begin{pmatrix}0\\ -x_3 1_b\\ -x_4 1_b\\ -x_2 1_b\\ U_3\end{pmatrix} \tag{4.1.1}$$

where U_3 is a matrix with entries in $\mathfrak{m}$, $\varDelta(\lambda_2)=\varLambda_2$, and λ_2 satisfies the conditions of Proposition 1.3.5). Put

$$\begin{bmatrix}U_{01}\\ U_1\end{bmatrix}=\begin{bmatrix}0\\ x_1 1_a\end{bmatrix}+\sum_{i\geqq 0} x_2^i V^{(i)}\,,\qquad \mathring{U}_3=x_1 1_b-U_3 \tag{4.1.2}$$

where $V^{(i)}$ are matrices of homogeneous polynomials in $k(2)$. Then $\lambda_2\lambda_3=0$ is equivalent to

$$\begin{cases}\begin{bmatrix}0\\ 0\\ x_3 1_b\\ x_4 1_b\end{bmatrix}\hat{U}_3+V^{(0)}\begin{pmatrix}0\\ x_3 1_b\\ x_4 1_b\end{pmatrix}=0\,, \qquad U_{21}\begin{pmatrix}0\\ x_3 1_b\\ x_4 1_b\end{pmatrix}=0\\[2ex] \begin{bmatrix}U_{02}\\ U_2\end{bmatrix}=-\sum_{r\geqq 1} x_2^{r-1}V^{(r)}U_4\,.\end{cases} \tag{4.1.3}$$

(see [1; Remark 4.1].) The solution of this equation is given by the following formula:

$$(4.1.4)\quad \begin{cases} \begin{bmatrix} V^{(0)} \\ U_{21} \end{bmatrix} = -\begin{array}{l} \left[\begin{array}{ccc} 0 & \multicolumn{2}{c}{0} \\ 0 & \mathring{U}_3 & 0 \\ & 0 & \mathring{U}_3 \\ 0 & \multicolumn{2}{c}{0} \end{array}\right] \begin{array}{l} \}\, a-2b+1 \\ \\ \\ \}\, b \end{array} \\ \underbrace{\hphantom{000}}_{a-2b} \end{array} + \left[\begin{array}{cc} \underbrace{W_8}_{a-2b} & \underbrace{Z_5[-x_4 1_b \ x_3 1_b]}_{2b} \end{array}\right], \\ \begin{bmatrix} U_{02} \\ U_2 \end{bmatrix} = -\sum_{r\geqq 1} x_2^{r-1} V^{(r)} U_4 . \end{cases}$$

where W_8, Z_5, $V^{(r)}$ $(r\geqq 1)$ and $\mathring{U}_3$ are arbitrary matrices of homogeneous polynomials of $k(2)$ (of $(x_3, x_4)k(2)$ for $\mathring{U}_3$) whose degrees are determined by $(a\,;\,\bar{m}\,;\,\bar{n})$.

Since the degree of each entry of W_8, Z_5, U_3 and $V^{(r)}$ $(r\geqq 1)$ is fixed for the given $(a\,;\,\bar{m}\,;\,\bar{n})$, all the entries of these matrices are parameterized by a finite dimensional affine space. Let $S(a\,;\,\bar{m}\,;\,\bar{n})$ denote this affine space. That is

$$(4.1.5)\qquad S(a\,;\,\bar{m}\,;\,\bar{n}) = \operatorname{Spec} k[\xi_i\,;\ 1\leqq i\leqq \rho]$$

where the set of parameters $\{\xi_i \mid 1\leqq i\leqq \rho\}$ corresponds to all the coefficients of the entries of W_8, Z_5, U_3 and $V^{(r)}$ $(r\geqq 1)$ as homogeneous polynomials in x_1, x_2, x_3, x_4 of the fixed degrees determined by $(a\,;\,\bar{m},\,\bar{n})$. Let $\tilde{\tilde{W}}_8$, $\tilde{\tilde{Z}}_5$, $\tilde{\tilde{\mathring{U}}}_3$, $\tilde{\tilde{V}}^{(r)}$ $(r\geqq 1)$ denote the corresponding family of matrices over $S(a\,;\,\bar{m},\,\bar{n})$, and, using these instead of W_8, Z_5, $\mathring{U}_3$, $V^{(r)}$ $(r\geqq 1)$, define $\tilde{\tilde{\lambda}}_2$, $\tilde{\tilde{\lambda}}_3$ by the formulae (4.1.4), (4.1.2) and (4.1.1). Denote the ring $k[\xi_i\,;\ 1\leqq i\leqq \rho]$ by $k[S]$. Since $\tilde{\tilde{\lambda}}_2\tilde{\tilde{\lambda}}_3 = 0$,

$$F_\iota = (-1)^\iota \det \tilde{\tilde{\lambda}}_2 \binom{\qquad i \qquad}{a+b+1,\ \cdots,\ a+2b} \Big/ \det \tilde{\tilde{\mathring{U}}}_3$$

is indeed a homogeneous polynomial in x_1, x_2, x_3, x_4 with coefficients in $k[S]$ by Proposition 1.7. Put

$$\tilde{\tilde{I}} = (F_0, F_1, \cdots, F_{a+b}) R \underset{k}{\otimes} k[S]$$

$$\mathfrak{X}' = \operatorname{Proj}_{k[S]} R \underset{k}{\otimes} k[S] / \tilde{\tilde{I}}$$

$$\pi : \mathfrak{X}' \longrightarrow S(a\,;\,\bar{m},\,\bar{n}) \text{ the natural projection.}$$

The set

$$\mathring{S}(a\,;\,\bar{m},\,\bar{n}) := \{s\in S(a\,;\,\bar{m}\,;\,\bar{n}) \mid ht\, \tilde{\tilde{I}} \otimes k(s) \geqq 2\}$$

is Zarisky open and by Remarks 1.8-1.9 the Hilbert polynomial of $\pi^{-1}(s)$ is independent of $s\in \mathring{S}(a\,;\,\bar{m},\,\bar{n})$, so that the family $\mathfrak{X}'|_{\mathring{S}(a;\bar{m},\bar{n})} \xrightarrow{\pi} \mathring{S}(a\,;\,\bar{m}\,;\,\bar{n})$ is a flat family of curves (see [10; Chap. III Theorem 9.9]). We denote $\mathfrak{X}'|_{\mathring{S}(a,\bar{m},\bar{n})}$ simply by $\mathfrak{X}$. In this way

(4.1.6)
$$\begin{tikzcd}
\mathfrak{X} \arrow[r, hookrightarrow] \arrow[dr, "\pi : \text{flat}"'] & \mathbf{P}^3_{\mathring{S}(a;\bar{m};\bar{n})} \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathring{S}(a;\,\bar{m};\,\bar{n})
\end{tikzcd}$$

Lemma 4.2. *$\mathring{S}(a;\bar{m};\bar{n})$ is not empty for an arbitrary $(a;\bar{m};\bar{n})$ such that $a\geqq 2b$, $a\leqq m_1\leqq\cdots\leqq m_{a-2b}$, $a\leqq n_1\leqq\cdots\leqq n_b$.*

Proof. Let P be a matrix representing the permutation of $\nu_1,\cdots,\nu_a$ such that for $(\nu'_1,\cdots,\nu'_a)=(\nu_1,\cdots,\nu_a)P$ the inequality $\nu'_1\leqq\nu'_2\leqq\cdots\leqq\nu'_a$ holds. We set

$$\begin{cases}
\begin{bmatrix} U_{01} \\ U_1 \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 1_a \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix} \begin{pmatrix} x_2^{\nu'_1+1-a} & & 0 \\ 0 & x_2^{\nu'_2+1-\nu'_1} & \\ & \ddots & x_2^{\nu'_a+1-\nu'_{a-1}} \\ 0 & \cdots\cdots\cdots\cdots & 0 \end{pmatrix} {}^tP \\
U_{21}=0, \quad U_3=x_1 1_b\,,
\end{cases}$$

and define λ_2, λ_3 by (4.1.1). Then $\varDelta(\lambda_2)=\Lambda_2$, $\lambda_2\lambda_3=0$ and the ideal defined by λ_2 satisfies $ht\,I\geqq 2$. Q.E.D.

We will assume in the rest of this section that k is an algebraically closed field of characteristic zero and under this assumption prove the existence of an integral a. B. curve with basic sequence $(a;\underbrace{n,\cdots,n}_{a};\underbrace{n,\cdots,n}_{b})$ where $n\geqq a\geqq 2b$ and $b\geqq 1$. We fix such a, b, n and put $\bar{m}=(\underbrace{n,\cdots,n}_{a-2b})$, $\bar{n}=(\underbrace{n,\cdots,n}_{b})$. In this case the matrix Λ_2 takes the following form:

$$\Lambda_2=\begin{pmatrix} n-a+1 & n-a+1 & \cdots\cdots & n-a+1 \\ 1 & 1 & \cdots\cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots\cdots & 1 \end{pmatrix}$$

Put $n-a+1=e$ and let t_i $(1\leqq i\leqq 6)$, u, v, w be parameters.

(4.3.1) When $b\geqq 2$ we set

$$\begin{bmatrix} U_{01} \\ U_1 \end{bmatrix}=$$

$$\left(\begin{array}{lll|l}
wx_4^e & & t_1x_2^e\ t_2x_2^e\ t_3x_2^e & \\
x_1\quad wx_4 & & t_4x_2\ t_5x_2\ t_6x_2 & \\
-ux_2\quad x_1\quad \cdot & 0 & & a-2b+1 \\
\qquad -ux_2\cdot\quad \cdot & & & \\
\qquad\quad \cdot\quad \cdot\quad wx_4 & & & \\
\qquad\qquad -ux_2\quad x_1 & -w'x_4 & w'x_3 & \\
& ux_2\quad x_1\ -wx_4-vx_3 & 0\quad wx_3 & \\
& \quad ux_2\quad x_1\ -wx_4-vx_3 & \quad\cdot\quad\cdot & \\
0 & \qquad\cdot\quad\cdot\quad\cdot & \qquad\cdot\quad\cdot & b \\
& \qquad\quad\cdot\quad\cdot\quad\cdot & \qquad\quad\cdot\quad wx_3 & \\
& \qquad\qquad ux_2\quad x_1 & -x_2\qquad\qquad 0 & \\
& \qquad\qquad\quad ux_2 & x_1\ -vx_3 & \\
& & ux_2\quad x_1\ -vx_3 & \\
0 & 0 & \quad ux_2\quad x_1 & b \\
& & \qquad\cdot\quad\cdot\ -vx_3 & \\
& & \qquad\qquad ux_2\quad x_1 & \\
\hline
a-2b & b & b &
\end{array}\right)$$

where $w'=\begin{cases} w & \text{if} \quad a-2b\neq 0 \\ wx_2^{e-1} & \text{if} \quad a-2b=0\,. \end{cases}$

$$\begin{bmatrix} U_{02} \\ U_2 \end{bmatrix}=-V^{(1)}U_4-x_2^{e-1}V^{(e)}U_4$$

$$=-\left(\begin{array}{lll}
0 & & t_1x_2^{e-1}\ \ t_2x_2^{e-1}\ \ t_3x_2^{e-1} \\
0 & & t_4\qquad t_5\qquad t_6 \\
-u\quad\cdot & 0 & \\
\quad -u\quad 0 & & \\
\qquad\ \cdot\quad\cdot & & \\
\qquad\quad -u\quad 0 & & \\
& u\quad 0 & \\
0 & \quad u\quad 0 & 0 \\
& \qquad\cdot\quad\cdot & \\
& \qquad\quad\cdot\quad\cdot & \\
& \qquad\qquad u\quad 0\quad -1 & \\
& & u\quad 0 \\
0 & 0 & \quad u\quad 0 \\
& & \qquad\cdot\quad\cdot \\
& & \qquad\quad\cdot\quad\cdot \\
& & \qquad\qquad u\quad 0
\end{array}\right)
\left(\begin{array}{c}
0 \\
\begin{matrix} x_3 & & 0 \\ & \ddots & \\ 0 & & x_3 \end{matrix} \\
\begin{matrix} x_4 & & 0 \\ & \ddots & \\ 0 & & x_4 \end{matrix}
\end{array}\right)$$

$$U_3=\begin{pmatrix} x_1 & -vx_3 & & & 0 \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ 0 & & & \cdot & -vx_3 \\ & & & & x_1 \end{pmatrix}, \qquad U_{21}=0.$$

and define λ_2, λ_3 by (4.1.1).

(4.3.2) When $b=1$ and $a\geqq 3$ we set

$$\begin{bmatrix} U_{01} \\ U_1 \end{bmatrix}=\left(\begin{array}{ccccccc} wx_4^e & & & & t_1x_2^e & t_2x_2^e & t_3x_2^e \\ x_1 & wx_4 & & & t_4x_2 & t_5x_2 & t_6x_2 \\ -ux_2 & x_1 & \cdot & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & wx_4 & & \\ & & & -ux_2 & x_1 & -x_2 & 0 \\ & & 0 & & -ux_2 & x_1-wx_4 & wx_3 \\ & & & & & ux_2 & x_1 \end{array}\right) \qquad \begin{array}{l} U_{21}=0 \\ U_3=x_1 \end{array}$$

$$\begin{bmatrix} U_{02} \\ U_2 \end{bmatrix}=\left(\begin{array}{ccccccc} 0 & & & & t_1x_2^{e-1} & t_2x_2^{e-1} & t_3x_2^{e-1} \\ 0 & & & & t_4 & t_5 & t_6 \\ -u & 0 & & & & & \\ & \cdot & \cdot & & & & \\ & & \cdot & \cdot & & & \\ & & & -u & 0 & -1 & 0 \\ & & 0 & & -u & 0 & 0 \\ & & & & & u & 0 \end{array}\right)\cdot\begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_3 \\ x_4 \end{pmatrix}$$

and define λ_2, λ_3 by (4.1.1).

It is easy to check $\lambda_2\lambda_3=0$ in both cases. Denote the ring $k[t_i(1\leqq i\leqq 6), u, v, w]$ by B and Spec B by T. We see B is a factor ring of $k[S]$, so that T is a closed subscheme of $S(a\,;\,\bar{m}\,;\,\bar{n})$. Put $\mathring{T}=T\cap\mathring{S}(a\,;\,\bar{m}\,;\,\bar{n})$. The family of curves $\mathfrak{X}_{\mathring{T}}\xrightarrow{\pi}\mathring{T}$ induced from $\mathfrak{X}\xrightarrow{\pi}\mathring{S}(a\,;\,\bar{m}\,;\,\bar{n})$ by the embedding $\mathring{T}\hookrightarrow\mathring{S}(a\,;\,\bar{m}\,;\,\bar{n})$ coincides with the family of curves obtained by the family of ideals determined by λ_2 set in (4.3.1) or (4.3.2). For this family we have the following theorem.

Theorem 4.4. *Suppose k is an algebraically closed field of characteristic zero. In the notation above, there exists a Zarisky open set $D\subset\mathring{T}$ such that for every point $s\in D$, $\pi^{-1}(s)$ is an integral a. B. curve with short basic sequence* $(a\,;\,\underbrace{n,\cdots,n}_{a-2b}\,;\,\underbrace{n,\cdots,n}_{b})$.

The proof is divided into several lemmas. We give its full detail only in the case $b\geqq 2$, leaving the proof for the case $b=1$ to the interested reader. Put $L=\operatorname{Proj} R/(x_1, x_2)R$ and $H=\operatorname{Proj} R/x_2R$. We will denote $(-1)^i\det\begin{pmatrix} U_{01} & U_{02} \\ U_1 & U_2 \\ 0 & U_3 \end{pmatrix}\binom{i}{}/\det U_3$ at a point $s\in\mathring{T}$ simply by f_i, without indicating the point s explicitly.

Lemma 4.5. *There exists a Zarisky open set $D_1 \subset \mathring{T}$ such that $f_{a+b}(0, 0, x_3, x_4) \neq 0$ for every point of D_1.*

Proof. It is enough to prove the existence of a point $s \in T$ such that $f_{a+b}(0, 0, x_3, x_4) \neq 0$ at s. For this purpose we set $u=0$ and $t_i=0$ for $1 \leqq i \leqq 6$. Then

$$f_{a+b}(x_1, 0, x_3, x_4) = v^{b-1} x_3^{b-1} x_4 x_1^b w^{a-2b+1} x_4^{a-2b+e} (w x_4 + v x_3)^{b-1} / x_1^b$$

$$= v^{b-1} w^{a-2b+1} (w x_4 + v x_3)^{b-1} x_3^{b-1} x_4^{a-2b+e+1}$$

up to a sign. Hence $f_{a+b}(0, 0, x_3, x_4) \neq 0$ if $vw \neq 0$. Q.E.D.

Lemma 4.6. *Any irreducible component of $\pi^{-1}(s)$ is not contained in H and $\pi^{-1}(s) \cap H = \pi^{-1}(s) \cap L$ for every $s \in D_1$.*

Proof. If $x_2=0$, then $f_0(x_1, 0, x_3, x_4) = x_1^a$ for a point $s \in D_1$, so that all points of $\pi^{-1}(s) \cap H$ must be contained in L. But $L \cap \pi^{-1}(s)$ has its dimension less than one for $s \in D_1$ by the previous lemma. Consequently any irreducible component of $\pi^{-1}(s)$ cannot lie in H for $s \in D_1$. Q.E.D.

Lemma 4.7. *There exists a Zarisky open set $D_2 \subset \mathring{T}$ such that for every $s \in D_2$,* Proj R/f_0R *is an irreducible surface with singularity L.*

Proof. Fix $(u, v, w) \in k^3 (u \neq 0, v \neq 0)$ arbitrarily (abuse of notation). Then

$$f_0 = \det U_1 \tag{4.7.1}$$

$$= \pm t_6 u^{a-1} x_2^a \pm t_5 u^{a-2} x_1 x_2^{a-1} \pm t_4 u^{a-3} x_2^{a-2} (x_1^2 + uv x_2 x_3) + g_0$$

where g_0 is the determinant of U_1 in the case $t_4=t_5=t_6=0$. Note that $g_0 \in (x_1, x_2)^2 R$ and hence $f_0 \in (x_1, x_2)^2 R$ for all points of $\mathring{T}$. (4.7.1) can be taken for a linear system on $\mathbf{P}_k^3$ generated by $u^{a-1} x_2^a$, $u^{a-2} x_1 x_2^{a-1}$, $u^{a-3} x_2^{a-2}(x_1^2 + uv x_2 x_3)$ and by g_0. Let Θ_1 denote this linear system and $\Phi_1 : \mathbf{P}_k^3 \to \mathbf{P}_k^3$ the rational map associated with Θ_1.

Claim. i) *Θ_1 has no fixed components and its base locus is L.*
ii) $\dim \Phi_1(\mathbf{P}_k^3) \geqq 2$.

Proof of i). The equations $u^{a-1} x_2^a = u^{a-2} x_1 x_2^{a-1} = u^{a-3} x_2^{a-2} (x_1^2 + uv x_2 x_3) = g_0 = 0$ imply $x_1 = x_2 = 0$.

Proof of ii). Put $z_\imath = x_\iota / x_2$ for $\imath = 1, 3, 4$, and consider the map Φ_1 restricted on $\mathbf{P}_k^3 \backslash H = \operatorname{Spec} k[z_1, z_3, z_4] =: \mathbf{A}_k^3$.

$$\Phi_1 : \mathbf{A}_k^3 \longrightarrow \mathbf{P}_k^3$$

is given by

$$\Phi_1(z_1, z_3, z_4) = (1 : z_1/u : (z_1^2 + uv z_3)/u^2 : g_0(z_1, 1, z_3, z_4)/u^{a-1})$$

from which follows immediately $\dim \Phi_1(\mathbf{P}_k^3)\geqq 2$. Q.E.D.

We can therefore conclude that $\operatorname{Proj} R/f_0R$ is an irreducible surface with singularity L by Bertini's theorem (see [15; Theorem 4.21] for example).

Q.E.D.

Lemma 4.8. *There exists a Zarisky open set $D_3\subset \mathring{T}$ such that for $s\in D_3$, $f_0(u\alpha, 1, u(\beta-\alpha^2)/v, z_4)$ is a nonconstant polynomial in the variable z_4, where α, β are general elements of k.*

Proof. Put $t_i=0$ $(1\leqq i\leqq 6)$. Then we find by a direct computation that

$$\begin{aligned} f_0(x_1, x_2, 0, x_4) &= \det U_1 \\ &= x_1^a+x_1^{a-1}\{(a-2b)uwx_2x_4+(b-1)uwx_2x_4+ux_2^2\} -f_0^{(1)} \\ &= x_1^a+x_1^{a-1}\{(a-b-1)uwx_2x_4+ux_2^2\} +f_0^{(1)} \end{aligned}$$

where $f_0^{(1)}$ denote the sum of terms of degree less than $a-2$ with respect to x_1. Since $a-b-1\geqq 2b-b-1\geqq b-1\geqq 1$ by hypothesis, $f_0(u\alpha, 1, 0, z_4)$ is a nonconstant polynomial of $k[z_4]$ of degree at least one for a general $\alpha\in k$. Hence the assertion follows. Q.E.D.

Let $\mathring{T}_1$ denote the subspace of $\mathring{T}$ defined by the equation $t_1=t_2=t_3=0$ and $\mathring{T}_2$ the subspace of $\mathring{T}$ defined by $u=0$. By the proofs of the previous lemmas we see $D_1\cap D_2\cap D_3\cap \mathring{T}_1\backslash \mathring{T}_2\neq \phi$, so take and fix a point $s'\in D_1\cap D_2\cap D_3\cap \mathring{T}_1\backslash \mathring{T}_2$. Let Y denote the hypersurface of $\mathbf{P}_k^3$ defined by the equation $f_0=0$ where f_0 is the polynomial determined by λ_2 corresponding to the point s'. We then consider f_1 corresponding to the point $s'+(t_1, t_2, t_3, 0, \cdots, 0)\in \mathring{T}$ with parameters t_1, t_2, t_3. Note that f_0 is independent of t_1, t_2, t_3.

Lemma 4.9. *Under the notation above the affine curve* $\operatorname{Proj} R/(f_0, f_1)R\backslash H\subset Y\backslash H$ *is irreducible and nonsingular for a general* $(t_1, t_2, t_3)\in k^3$.

Proof. By a direct computation

$$\begin{aligned} (4.9.1) \qquad f_1 &= \det\begin{bmatrix} U_{01} \\ U_1 \end{bmatrix}\binom{1}{} \\ &= \pm t_3u^{a-1}x_2^{e+a-1}\pm t_2u^{a-2}x_1x_2^{e+a-2} \\ &\quad \pm t_1u^{a-3}x_2^{e+a-3}(x_1^2+uvx_2x_3)+g_1 \end{aligned}$$

where g_1 denotes $\det \begin{bmatrix} U_{01} \\ U_1 \end{bmatrix}\binom{1}{}$ in the case $t_1=t_2=t_3=0$. We take 4.9.1) for a linear system on Y and denote it by Θ_2. Let $\hat{Y}\xrightarrow{\omega} Y$ be a desingularization of Y, $\hat{\Theta}_2=\omega^*\Theta_2$ and $\Phi_2: \hat{Y}\longrightarrow \mathbf{P}_k^3$ the rational map associated with $\hat{\Theta}_2$.

Claim. i) *All fixed components and base points are contained in $\omega^{-1}(L)$.*
ii) $\dim \Phi_2(\hat{Y})\geqq 2$.

Proof of i). Since $u\neq 0$, $u^{a-1}x_2^{e+a-1}=u^{a-2}x_1x_2^{e+a-2}=u^{a-3}x_2^{e+a-3}(x_1^2+uvx_2x_3)=f_2$ $=0$ on Y implies $x_2=0$ and $f_0=0$, so that $x_1=x_2=0$, which proves i).

Proof of ii). As before we consider the map Φ_2 restricted on $Y':=\hat{Y}\backslash\omega^{-1}(H)\cong Y\backslash H\subset\mathbf{A}_k^3$. Φ_2: $Y'\to\mathbf{P}_k^3$ is given by

$$\Phi_2(z_1, z_3, z_4)=(1: z_1/u: (z_1^2+uvz_3)/u^2: g_1(z_1, 1, z_3, z_4)/u^{a-1}).$$

Let α, β be general elements of k and consider the equations

$$\text{(4.9.2)}\qquad \begin{cases} z_1/u=\alpha \\ (z_1^2+uvz_3)/u^2=\beta \\ f_0(z_1, 1, z_3, z_4)=0\,. \end{cases}$$

This is equivalent to

$$\text{(4.9.3)}\qquad \begin{cases} z_1=u\alpha,\ z_3=u(\beta-\alpha^2)/v \\ f_0(u\alpha, 1, u(\beta-\alpha^2)/v, z_4)=0\,. \end{cases}$$

Since $f_0(u\alpha, 1, u(\beta-\alpha^2)/v, z_4)$ is a nonconstant polynomial in z_4 by Lemma 4.8, (4.9.3) certainly has a solution. This means that $\dim\Phi_2(Y')=2$. Q.E.D.

We can therefore conclude by Bertini's theorem that general members of the variable part of the linear system $\hat{\Theta}_2$ on $\hat{Y}$ are irreducible and nonsingular, from which our assertion follows. Q.E.D.

In the situation of Lemma 4.9, put $X=\pi^{-1}(s)$ for $s=(s', t_1, t_2, t_3)\in\mathring{T}$ where $(t_1, t_2, t_3)\in k^3$ is general. Since $u\neq 0$ and $\det\lambda_2\begin{pmatrix}0, & 1 & & \\ a, & a+1, & \cdots, & a+b\end{pmatrix}=(-1)^{a-2b-1}u^{a-1}x_2^{a+b-1}$, the relation $(f_0, f_1, \cdots, f_{a+b})\lambda_2=0$ implies $f_i\in(f_0, f_1)\mathcal{O}_{\mathbf{A}_k^3}$ on $\mathbf{A}_k^3=\mathbf{P}_k^3\backslash H$ for $i\geqq 2$, so that $X\backslash H=\operatorname{Proj} R/(f_0, f_1)R\backslash H$. Now we have

i) No irreducible component of X is contained in H (Lemma 4.6).

ii) $X\backslash L=X\backslash H$ is a nonsingular irreducible curve (Lemma 4.9).

Consequently X is an integral curve which is nonsingular except at the points of $X\cap L$. Finally X is in fact a. B. with short basic sequence $(a; \underbrace{n, \cdots, n}_{a-2b}; \underbrace{n, \cdots, n}_{b})$ by Theorem 3.2, and the proof of Theorem 4.4 is completed in the case $b\geqq 2$. Q.E.D.

Remark 4.10. Since $\mathring{T}$ is a thin subspace of $\mathring{S}(a; \bar{m}, \bar{n})$, it may well be hoped that the curves $\pi^{-1}(s)$ for $s\in\mathring{S}(a; \bar{m}, \bar{n})$ general are nonsingular and irreducible, however we have not confirmed it as yet.

Example 4.11. The monomial curve $\operatorname{Proj} k[s^{4n}, s^{2n+1}t^{2n-1}, s^{2n-1}t^{2n+1}, t^{4n}]$ is a. B. by [5] and its short basic sequence is $(2; —; 2n+1)$. It therefore coincides with $\pi^{-1}(s)$ for a certain point $s\in\mathring{S}(2; —; 2n+1)$.

§ 5. Some Irreducible Components of Hilb ($\mathbf{P}^3_k$) Whose General Points Correspond to Arithmetically Buchsbaum Curves

We denote the universal flat family of subschemes of $\mathbf{P}^3_k$ over Hilb($\mathbf{P}^3_k$) by $\mathcal{Z}$:

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook] \arrow[dr, "\tilde{\omega}:\ \text{flat}"'] & \mathbf{P}^3_k \times_k \mathrm{Hilb}(\mathbf{P}^3_k) \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Hilb}(\mathbf{P}^3_k) \end{tikzcd} \tag{5.1}$$

Definition 5.2. Let $(a\,;\ \bar{m}\,;\ \bar{n})=(a\,;\ m_1, \cdots, m_{a-2b}\,;\ n_1, \cdots.\ n_b)$ be a sequence of integers such that $a \leqq m_1 \leqq \cdots \leqq m_{a-2b}$, $a \leqq n_1 \leqq \cdots \leqq n_b$, where $a \geqq 2b$. We say that $(a\,;\ \bar{m}\,;\ \bar{n})$ represents an irreducible component of Hilb($\mathbf{P}^3_k$) if and only if there exists an irreducible component H of Hilb($\mathbf{P}^3_k$) such that $\tilde{\omega}^{-1}(h)$ is an a. B. curve with short basic sequence $(a\,;\ \bar{m}\,;\ \bar{n})$ for every general $h \in H$.

Remark 5.3. Since all a. B. curve with short basic sequence $(a\,;\ \bar{m}\,;\ \bar{n})$ are parametrized by an irreducible variety $\mathring{S}(a\,;\ \bar{m}\,;\ \bar{n})$ (see (4.1.6)) for a given $(a\,;\ \bar{m}\,;\ \bar{n})$, $(a\,;\ \bar{m}\,;\ \bar{n})$ can represent only one irreducible component of Hilb($\mathbf{P}^3_k$).

Our main concern in this section is to find, as far as possible by the methods developped so far, the conditions in order that $(a\,;\ \bar{m}\,;\ \bar{n})$ should actually represent an irreducible component of Hilb($\mathbf{P}^3_k$). Let us seek for a necessary condition first. In the following lemmas, we let $(a\,;\ \bar{m}\,;\ \bar{n})=(a\,;\ m_1, \cdots, m_{a-2b}\,;\ n_1, \cdots, n_b)$ be the short basic sequence of an a. B. curve X.

Lemma 5.4. *If* $n_{j+1}=n_j+1$ *for some* $1 \leqq j \leqq b-1$, *then* $(a\,;\ \bar{m}\,;\ \bar{n})$ *does not represent any irreducible components of* Hilb($\mathbf{P}^3_k$).

Proof. $\mathrm{H}^0_*(\mathcal{O}_X)$ has the free resolution (3.4). Let t be a parameter and set

$$Q_i=\begin{array}{cl} \begin{array}{cccccc} & & \,j & & & \end{array} & \\ \left(\begin{array}{cccccc} x_i & & \vdots & & & \\ & \ddots & \vdots & & 0 & \\ & & x_i & & & \\ & & t & x_i & \cdots & \cdots \\ & 0 & & & \ddots & \\ & & & & & x_i \end{array}\right) & \begin{array}{l} \\ \\ \\ \cdots j+1 \\ \\ \\ \end{array} \end{array} \qquad \text{for} \quad i=3,\ 4$$

$$\tilde{\sigma}^{(1)}=[\underbrace{0}_{1}\ x_1 1_b\ \underbrace{0}_{a-2b}\ x_2 1_b\ Q_3\ Q_4]$$

$$\tilde{\tau}=\begin{pmatrix} & 0 & & \\ 0 & -x_2 1_b & -Q_3 & -Q_4 \\ & & & \\ & x_1 1_{a+b} & & \\ & & & \end{pmatrix}+\begin{pmatrix} W_5 \\ 0 \\ W_6 \\ \begin{pmatrix} -Q_3 & -Q_4 \\ \multicolumn{2}{c}{x_2 1_{2b}} \end{pmatrix} Z_3+\begin{pmatrix} 0 \\ -Q_4 \\ Q_3 \end{pmatrix} Z_1 \end{pmatrix}.$$

We have $\varDelta(\tilde{\sigma}^{(1)})=\varDelta(\sigma^{(1)})$, $\varDelta(\tilde{\tau})=\varDelta(\tau)$ and check easily $\tilde{\sigma}^{(1)}\tilde{\tau}=0$, so that a flat family of curves $\tilde{X}:=\mathrm{Proj}\, R\otimes_k k[t]/I(\tilde{\tau})\xrightarrow{p}\mathrm{Spec}\, k[t]$ is obtained by Proposition 2.11. $p^{-1}(0)=X$ and for each $\eta\in k$ we know by the same proposition $\mathrm{H}^1_*(\mathcal{J}_\eta)\cong\mathrm{Coker}\, \tilde{\sigma}^{(1)}(\eta)$, where $\mathcal{J}_\eta$ denotes the sheaf of ideals of the curve $p^{-1}(\eta)$ and $\tilde{\sigma}^{(1)}(\eta)$ the matrix obtained by putting $t=\eta$ in $\tilde{\sigma}^{(1)}$. Hence, if $\eta\neq 0$, $\mathrm{H}^1_*(\mathcal{J}_\eta)$ cannot be annihilated by $\mathfrak{m}$. We see, in addition, the basic sequence of $p^{-1}(\eta)$ is in fact different from that of X if $\eta\neq 0$, since the $k(2)$-module structures of $\mathrm{H}^1_*(\mathcal{J}_\eta)$ and $\mathrm{H}^1_*(\mathcal{J}_0)$ are different (see Remark 2.7). This implies there exists a curve whose basic sequence is different from that of X and which is not a. B. in an arbitrary small neighborhood of the point of $\mathrm{Hilb}(\mathbf{P}^3_k)$ corresponding to X. Consequently $(a\,;\,\bar{m}\,;\,\bar{n})$ does not represent any irreducible components of $\mathrm{Hilb}(\mathbf{P}^3_k)$. Q.E.D.

Lemma 5.5. *Suppose for some* $1\leqq j\leqq b$

1) $a=n_j-2$ *and* $\#\{i\,|\,m_i=a\}+1>3\#\{i\,|\,n_i=a\}$ *or*

2) $a\neq n_j-2$, $\{i\,|\,m_i=n_j-2\}\neq\phi$ *and*

$$\#\{i\,|\,m_i=n_j-2\}>3\#\{i\,|\,n_i=n_j-2\}+\#\{i\,|\,m_i=n_j-3\}\,.$$

Then $(a\,;\,\bar{m}\,;\,\bar{n}$ *does not represent any irreducible components of* $\mathrm{Hilb}(\mathbf{P}^3_k)$.

Proof. Consider first the case 2). Suppose

$$\begin{cases} m_i=n_j-3 & \text{for}\quad \alpha_0+1\leqq i\leqq\alpha_1 \\ m_i=n_j-2 & \text{for}\quad \alpha_1+1\leqq i\leqq\alpha_2 \\ n_i=n_j-2 & \text{for}\quad \beta_0+1\leqq i\leqq\beta_1 \end{cases}$$

and $\alpha_2-\alpha_1>3(\beta_1-\beta_0)+(\alpha_1-\alpha_0)$. $\mathrm{H}^0_*(\mathcal{O}_X)$ has the free resolution (3.4). We have

$$\varDelta\left(W_0\left(1,\,\cdots,\,\alpha_1,\,\alpha_2+1,\,\cdots,\,a-2b\right)\right)$$

$$=[\underbrace{\;*\;}_{\alpha_0}\;\underbrace{\;0\;}_{\alpha_1-\alpha_0}\;\underbrace{\;*\;}_{a-2b-\alpha_1}\;\varDelta_1\;\varDelta_1\;\varDelta_1\;]\}\,\alpha_2-\alpha_1$$

where $\varDelta_1=[\underbrace{\;*\;}_{\beta_0}\;\underbrace{\;0\;}_{\beta_1-\beta_0}\;\underbrace{\;*\;}_{b-\beta_1}]$, and the entries of $*$ are all positive or negative integers. rank $W_6\left(1, \cdots, \alpha_1, \alpha_2+1, \cdots, a-2b\right)(\operatorname{mod} \mathfrak{m})$ is therefore less than $\alpha_2-\alpha_1$ by hypothesis, and there exists a nonzero vector $\bar{r}=(\underbrace{0, \cdots, 0}_{\alpha_1}, r_{\alpha_1+1}, \cdots, r_{\alpha_2}, \underbrace{0, \cdots, 0}_{a-2b-\alpha_2})\in k^{a-2b}$ such that

$$\bar{r}W_6\equiv 0\,(\operatorname{mod} \mathfrak{m})\,. \tag{5.5.3}$$

Let t be a parameter and set

$$\tilde{\sigma}^{(1)}=[\,0\ \ x_1 1_b\ t\bar{r}\cdots j\text{-th row}\ x_2 1_b\ x_3 1_b\ x_1 1_b]\,.$$

Then $\Delta(\tilde{\sigma}^{(1)})=\Delta(\sigma^{(1)})$ and we see by (5.5.3)

$$\tilde{\sigma}^{(1)}\tau=tP$$

where P is a matrix of homogeneous polynomials in $\mathfrak{m}$. This P can be written

$$P=\sigma^{(1)}Q=\tilde{\sigma}^{(1)}Q$$

where $Q=\underbrace{\begin{pmatrix}0\\ Q_1\\ 0\\ Q_2\\ Q_5\\ Q_6\end{pmatrix}}_{a+b}\begin{matrix}\}\ 1\\ \}\ b\\ \}\ a-2b\\ \}\ b\\ \}\ b\\ \}\ b\end{matrix}$ for suitable matrices Q_i $(i=1, 2, 5, 6)$ of homogeneous polynomials such that $\Delta(Q)=\Delta(\tau)$. Set $\tilde{\tau}=\tau-tQ$. Then $\Delta(\tilde{\tau})=\Delta(\tau)$ and $\tilde{\sigma}^{(1)}\tilde{\tau}=0$. In this way we get a flat family of curves $X=\operatorname{Proj}_{k[t]} R\otimes_k k[t]/I(\tilde{\tau}) \xrightarrow{p} \operatorname{Spec} k[t]$ such that $p^{-1}(0)=X$ by Proposition 2.11. As in the previous lemma $\mathrm{H}^1_*(\tilde{\mathcal{J}}_\eta)\cong \operatorname{Coker} \tilde{\sigma}^{(1)}(\eta)$ for $\eta\in k$, so that, if $\eta\neq 0$, $p^{-1}(\eta)$ is an a.B. curve with $i(p^{-1}(\eta))<i(X)=b$. This implies that, in an arbitrary small neighborhood of the point of $\operatorname{Hilb}(\mathbf{P}^3_k)$ corresponding to X, there is an a.B. curve with short basic sequence different from $(a\,;\,\bar{m}\,;\,\bar{n})$, and hence $(a\,;\,\bar{m}\,;\,\bar{n})$ does not represent any irreducible components of $\operatorname{Hilb}(\mathbf{P}^3_k)$. The proof for the case 1) is similar.

Q.E.D.

Lemma 5.6. *Suppose $n_{i+1}=n_i$ or $n_{i+1}-n_i\geqq 2$ for every $1\leqq i\leqq b-1$. If*

$$\#\{i\,|\,m_i=n_j+1\}>\#\{i\,|\,m_i=n_j+2\}+3\#\{i\,|\,n_i=n_j+2\}$$

for some $1\leqq j\leqq b$, then $(a\,;\,\bar{m}\,;\,\bar{n})$ does not represent any irreducible components of $\operatorname{Hilb}(\mathbf{P}^3_k)$.

Proof. Let

$\lambda_2 = \begin{bmatrix} U_{01} & U_{02} & 0 \\ & & 0 \\ U_1 & U_2 & x_3 1_b \\ & & x_4 1_b \\ U_{21} & U_3 & x_2 1_b \end{bmatrix}$ be the matrix of relations among $f_0, f_1, \cdots, f_{a+b}$ (see the end of Section three). As is described in the beginning of Section four the equation $\lambda_2 \lambda_3 = 0$ is equivalent to (4.1.3). By hypothesis we may assume

$$\begin{cases} m_i = n_j + 1 & \text{for } \alpha_0 + 1 \leqq i \leqq \alpha_1 \\ m_i = n_j + 2 & \text{for } \alpha_1 + 1 \leqq i \leqq \alpha_2 \\ n_i = n_j + 2 & \text{for } \beta_0 + 1 \leqq i \leqq \beta_1 \end{cases}$$

with $\alpha_1 - \alpha_0 > (\alpha_2 - \alpha_1) + 3(\beta_1 - \beta_0)$. Since $\Delta(f_0, f_1, \cdots, f_{a+b}) = (a, \bar{m}, \bar{n}, \bar{n}, \bar{n})$, each entry of $\Delta(U_{01})$ is positive and

$$\Delta\left(\begin{bmatrix} U_1 \\ U_{21} \end{bmatrix}\begin{pmatrix} 1, \cdots, \alpha_0, \alpha_1 + 1, \cdots, a \end{pmatrix}\right) = \overbrace{\begin{bmatrix} * \\ 0 \\ * \\ \Delta_2 \\ \Delta_2 \\ \Delta_2 \end{bmatrix}}^{\alpha_1 - \alpha_0} \begin{matrix} \}\ \alpha_1 \\ \}\ \alpha_2 - \alpha_1 \\ \}\ a - 2b - \alpha_2 \\ \\ \\ \\ \end{matrix}$$

where $\Delta_2 = \begin{bmatrix} * \\ 0 \\ * \end{bmatrix} \begin{matrix} \}\ \beta_0 \\ \}\ \beta_1 - \beta_0 \\ \}\ b - \beta_1 \end{matrix}$ and the entries of $*$ are either positive or negative.

In view of this,

$$\operatorname{rank}\begin{bmatrix} V^{(0)} \\ U_{21} \end{bmatrix}\begin{pmatrix} 1, \cdots, \alpha_0, \alpha_1 + 1, \cdots, a \end{pmatrix} (\operatorname{mod} \mathfrak{m})$$

is less than $\alpha_1 - \alpha_0$ by hypothesis, so that there exists a nonzero vector $\bar{c} = {}^t(\underbrace{0, \cdots, 0}_{\alpha_0}, c_{\alpha_0+1}, \cdots, c_{\alpha_1}, \underbrace{0, \cdots, 0}_{a-2b-\alpha_1}) \in k^{a-2b}$ such that $\begin{bmatrix} V^{(0)} \\ U_{21} \end{bmatrix} \bar{c} \equiv 0 (\operatorname{mod} (x_3, x_4) k(2))$.

Let t be a parameter and put

$$\tilde{U}_4=\left(\begin{array}{c} j \\ \vdots \\ t\bar{c} \\ x_3 1_b \\ x_4 1_b \end{array}\right)\begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} a-2b \\ \\ \\ \end{array},$$

then $\Delta(\tilde{U}_4)=\Delta\left(\left(\begin{array}{c} 0 \\ x_3 1_b \\ x_4 1_b \end{array}\right)\right)$. $n_{i+1}=n_i$ or $n_{i+1}-n_i \geqq 2$ for $1 \leqq i \leqq b-1$ by assumption, so that any entry of $\Delta(\mathring{U}_3)$ is not zero. We therefore get

$$\text{(5.6.1)} \qquad \left\{\begin{array}{l} \begin{bmatrix} 0 \\ \tilde{U}_4 \end{bmatrix}\mathring{U}_3+V^{(0)}\tilde{U}_4=tP_1 \\ U_{21}\tilde{U}_4=tP_2 \end{array}\right.$$

where P_1 and P_2 are matrices of homogeneous polynomials of $(x_3, x_4)k(2)$. These P_1 and P_2 can be written

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}=\begin{array}{r} a+1 \{ \\ b \{ \end{array}\left(\begin{array}{cc} 0 & Q_7 \\ 0 & Q_8 \\ \underbrace{}_{a-2b} & \underbrace{}_{2b} \end{array}\right)\tilde{U}_4$$

with matrices Q_7, Q_8 of homogeneous polynomials of $k(2)$ such that $\Delta\left(\begin{bmatrix} 0 & Q_7 \\ 0 & Q_8 \end{bmatrix}\right)=\Delta\left(\begin{bmatrix} V^{(0)} \\ U_{21} \end{bmatrix}\right)$. We set

$$\left\{\begin{array}{l} \tilde{V}^{(0)}=V^{(0)}-tQ_7, \qquad \tilde{U}_{21}=U_{21}-tQ_8 \\ \begin{bmatrix} \tilde{U}_{01} \\ \tilde{U}_1 \end{bmatrix}=\begin{bmatrix} 0 \\ x_1 1_a \end{bmatrix}+\tilde{V}^{(0)}+\sum_{r\_1} x_2^r V^{(r)} \end{array}\right.$$

and set

$$\tilde{\lambda}_2=\begin{bmatrix} \tilde{U}_{01} & U_{02} & 0 \\ \tilde{U}_1 & U_2 & \tilde{U}_4 \\ \tilde{U}_{21} & U_3 & x_2 1_b \end{bmatrix} \qquad \tilde{\lambda}_3=\begin{bmatrix} -\tilde{U}_4 \\ -x_2 1_b \\ U_3 \end{bmatrix}.$$

For these matrices the equations

$$\left\{\begin{array}{l} \begin{bmatrix} 0 \\ \tilde{U}_4 \end{bmatrix}\mathring{U}_3+\tilde{V}^{(0)}\tilde{U}_4=0 \qquad \tilde{U}_{21}\tilde{U}_4=0 \\ \begin{bmatrix} U_{02} \\ U_2 \end{bmatrix}=-\sum_{r \geqq 1} x_2^{r-1}V^{(r)}\tilde{U}_4 \end{array}\right.$$

hold by (5.6.1) and (4.1.3), hence $\tilde{\lambda}_2\tilde{\lambda}_3=0$ (see [1; Remark 4.1]). Now we get a flat family of curves

$$\tilde{X}=\mathrm{Proj}_{k[t]}\,R\otimes_k k[t]/(\tilde{f}_0,\ \tilde{f}_1,\ \cdots,\ \tilde{f}_{a+b})R \xrightarrow{p} \mathrm{Spec}\ k[t]$$

where $\tilde{f}_i=(-1)^i \det \tilde{\lambda}_2\begin{pmatrix} i \\ a+b+1,\ \cdots,\ a+2b\end{pmatrix}/\det U_3$ for $0\leqq i\leqq a+b$ (see Remark 1.7). As in Section two $\mathrm{Im}^R({}^t\tilde{\lambda}_3(\eta))\supset {}^tU_3k(0)\oplus x_2 1_b k(1)\oplus \mathrm{Im}^{k(2)}({}^t\tilde{U}_4(\eta))$ (c. f. (2.3.5)) for every $\eta\in k$ where ${}^t\tilde{\lambda}_3(\eta)$ and ${}^t\tilde{U}_4(\eta)$ denote the matrices obtained by putting $t=\eta$ in ${}^t\tilde{\lambda}_3$, ${}^t\tilde{U}_4$ respectively. This implies $\mathrm{Hom}_k(\mathrm{H}^1_*(\tilde{\mathcal{J}}_\eta),\ k)\cong R^b/\mathrm{Im}^R({}^t\tilde{\lambda}_3(\eta))$ is annihilated by $\mathfrak{m}$ for all $\eta\in k$, but since $\bar{c}\neq 0$, $i(p^{-1}(\eta))=\dim_k(\mathrm{H}^1_*(\tilde{\mathcal{J}}_\eta),\ k)<i(X)=b$ for $\eta\neq 0$. Consequently an arbitrary neighborhood of the point of Hilb($\mathbf{P}^3_k$) corresponding to X contains an a. B. curve whose short basic sequence is different from $(a\,;\ \bar{m}\,;\ \bar{n})$, whence our assertion follows. Q.E.D.

We summarize the results obtained so far in a theorem.

Theorem 5.7. *In order that a sequence of integers* $(a\,;\ \bar{m}\,;\ \bar{n})=(a\,;\ m_1,\ \cdots,\ m_{a-2b},\ n_1,\ \cdots,\ n_b)$ *with* $a\geqq 2b$, $a\leqq m_1\leqq \cdots \leqq m_{a-2b}$, $a\leqq n_1\leqq \cdots \leqq n_b$ *should represent an irreducible component of* Hilb($\mathbf{P}^3_k$), *it must satisfy the following conditions.*

1) $n_{i+1}=n_i$ *or* $n_{i+1}-n_i\geqq 2$ *for every* $1\leqq i\leqq b-1$.

2) *If* $a=n_j-2$ *for some* j, *then*

$$\#\{i\,|\,m_i=a\}+1\leqq 3\#\{i\,|\,n_i=a\}.$$

3) *For each* $1\leqq j\leqq b$ *such that* $a\neq n_j-2$, *we have*

$$\#\{i\,|\,m_i=n_j-2\}\leqq 3\#\{i\,|\,n_i=n_j-2\}+\#\{i\,|\,m_i=n_j-3\}$$

4) $\#\{i\,|\,m_i=n_j+1\}\leqq \#\{i\,|\,m_i=n_j+2\}+3\#\{i\,|\,n_i=n_j+2\}$ *for every* $1\leqq j\leqq b$.

Our next problem is whether $(a\,;\ \bar{m}\,;\ \bar{n})$ satisfying the conditions of this theorem actually represents an irreducible component of Hilb($\mathbf{P}^3_k$) or not. In any case an a. B. curve with short basic sequence $(a\,;\ \bar{m}\,;\ \bar{n})$ exists by Lemma 4.2, though it may not be even reduced. And if $n_{i+1}=n_i$ or $n_{i+1}-n_i\geqq 3$ for every $1\leqq i\leqq b-1$, we can prove that the conditions of Theorem 5.7 are indeed sufficient for $(a\,;\ \bar{m}\,;\ \bar{n})$ to represent an irreducible component of Hilb($\mathbf{P}^3_k$). In other cases we do not have any answers yet.

To describe the answer in the case mentioned just now we need some lemmas. They may be found somewhere in the literature available, nevertheless we give the proofs for the convenience of the reader.

Lemma 5.8. *Let* $(A,\ \mathfrak{n})$ *be a local ring with residue field* k *such that* $k\hookrightarrow A$, *and let* $\tilde{f}_0,\ \tilde{f}_1,\ \cdots,\ \tilde{f}_{a+b}\in A(0)$ *be homogeneous polynomials with coefficients in* A *of degrees* $a,\ \nu_1,\ \cdots,\ \nu_{a+b}$ *respectively. Suppose* $f_i:=\tilde{f}_i(\mathrm{mod}\ \mathfrak{n})\in k(0)=R$ $(0\leqq i\leqq a+b)$ *satisfy the condition*

(5.8.1) $$R=\{f_0k(0)\oplus\bigoplus_{i=1}^{a}f_ik(1)\oplus\bigoplus_{j=1}^{b}f_{a+j}k(2)\}\oplus L$$

(*direct sum as k-vector spaces*) *where* $L=\bigoplus_{\nu\geqq 0}L_\nu$ $(L_\nu\subset R_\nu)$ *is a graded k-vector subspace of* R. *Then we have*

(5.8.2) $$A(0)=\{\tilde{f}_0A(0)\oplus\bigoplus_{i=1}^{a}\tilde{f}_iA(1)\oplus\bigoplus_{j=1}^{b}\tilde{f}_{a+j}A(2)\}\oplus L\otimes_k A$$

(*direct sum as A-modules*).

Proof. We take a tensor product of (5.8.1) and A over k to obtain

$$A(0)=M^{(1)}\oplus M^{(2)}\qquad\text{(direct sum as } A\text{-modules)}$$

where $M^{(1)}=f_0A(0)\oplus\bigoplus_{i=1}^{a}f_iA(1)\oplus\bigoplus_{j=1}^{b}f_{a+j}A(2)$ and $M^{(2)}=L\otimes_kA$. Let $\theta: M^{(1)}\oplus M^{(2)}=A(0)\to A(0)$ be the map defined by $\theta(\sum_{i=0}^{a+b}\tilde{g}_if_i, r)=\sum_{i=0}^{a+b}\tilde{g}_i\tilde{f}_i+r$ with $(\tilde{g}_0, \tilde{g}_1, \cdots, \tilde{g}_a, \tilde{g}_{a+1}, \cdots, \tilde{g}_{a+b})\in A(0)\oplus A(1)^a\oplus A(2)^b$. Since $\tilde{f}_i$ $(0\leqq i\leqq a+b)$ are homogeneous, θ gives a map $\theta_\nu: A(0)_\nu\to A(0)_\nu$ from a finite free A-module into itself for each $\nu\in\boldsymbol{Z}_0$. θ_ν becomes an identity when considered $(\operatorname{mod}\mathfrak{n})$, therefore θ_ν itself is an isomorphism for every ν. Q.E.D.

Lemma 5.9. *In the situation of the previous lemma, suppose, in addition, that* $\tilde{I}:=\tilde{f}_0A(0)\oplus\bigoplus_{i=1}^{a}\tilde{f}_iA(1)\oplus\bigoplus_{j=1}^{b}\tilde{f}_{a+j}A(2)$ *is an ideal. Then there exists a homogeneous polynomial* $\hat{f}_i$ *for each* $0\leqq i\leqq a+b$ *such that*

1) $\hat{f}_i-\tilde{f}_i\in\tilde{I}$, $\hat{f}_i-f_i\in L\otimes_kA$ *and* $\deg\hat{f}_i=\deg\tilde{f}_i$

2) $$\begin{cases}\tilde{I}=\hat{f}_0A(0)\oplus\bigoplus_{i=1}^{a}\hat{f}_iA(1)\oplus\bigoplus_{j=1}^{b}\hat{f}_{a+j}A(2)\\ A(0)=\tilde{I}\oplus L\otimes_kA.\end{cases}$$

Proof. We can write $\tilde{f}_i-f_i=\tilde{f}_i'+\tilde{f}_i''$ with $\tilde{f}_i'\in\tilde{I}\cap\mathfrak{n}A(0)$ and $\tilde{f}_i''\in L\otimes_kA\cdot\mathfrak{n}A(0)$ by (5.8.2). Set $\hat{f}_i=f_i+\tilde{f}_i''$. Then $\hat{f}_i$ $(0\leqq i\leqq a+b)$ satisfy 1) and $\hat{f}_i=\tilde{f}_i-\tilde{f}_i'\in\tilde{I}$. Since $\hat{f}_i(\operatorname{mod}\mathfrak{n})=f_i+\tilde{f}_i''(\operatorname{mod}\mathfrak{n})=f_i$, (5.8.2) holdes for $\hat{f}_0, \cdots, \hat{f}_{a+b}$ and 2) follows easily. Q.E.D.

Lemma 5.10. *Let* $(A, \mathfrak{n})$ *be a local integral domain with* $A/\mathfrak{n}=k\hookrightarrow A$, o *the closed point of* Spec A. *Let* $p: \tilde{X}\subset\mathbf{P}_A^3\to$ Spec A *be a flat family of curves and* $\tilde{\mathfrak{J}}$ *the sheaf of ideals of* X. *Suppose the ideal* $\mathrm{H}_*^0(\tilde{\mathfrak{J}}_o)\subset R$ *is generated by homogeneous polynomials* f_i $(1\leqq i\leqq l)$ *with* $\deg f_i=d_i$, *where* $\tilde{\mathfrak{J}}_y$ *denotes the sheaf of ideals of the curve* $p^{-1}(y)$ *for* $y\in$ Spec A. *If* $\mathrm{H}^1(\mathbf{P}_k^3, \tilde{\mathfrak{J}}_o(d_i))=0$ *for all* $1\leqq i\leqq l$, *then* $\mathrm{H}^0(\mathbf{P}_A^3, \tilde{\mathfrak{J}}(\nu))$ *is a free A-module for every* $\nu\geqq 0$, *and* $A(0)/\mathrm{H}_*^0(\mathbf{P}_A^3, \tilde{\mathfrak{J}})$ *is a flat A-module.*

Proof. As A is local, we have $R^ip_*(\tilde{\mathfrak{J}}(\nu))=\mathrm{H}^i(\mathbf{P}_A^3, \tilde{\mathfrak{J}}(\nu))$ for all $\nu\in\boldsymbol{Z}$ and $i\geqq 0$. Consider the natural maps

$$\phi^{i}_{\nu}(y): \mathrm{H}^{i}(\mathbf{P}^{3}_{A}, \tilde{\mathcal{J}}(\nu))\otimes k(y) \longrightarrow \mathrm{H}^{i}(\mathbf{P}^{3}_{k(y)}, \tilde{\mathcal{J}}_{y}(\nu))$$

(see [10; Chap. III. Theorem 12.11]). The assumption $\mathrm{H}^1(\mathbf{P}^3_k, \tilde{\mathcal{J}}_o(d_i))=0$ implies $\mathrm{H}^1(\mathbf{P}^3_A, \tilde{\mathcal{J}}(d_i))=0$ and the surjectivity of $\phi^0_{d_i}(0)$ (loc. cit.), so that there exists $\tilde{f}_i\in \mathrm{H}^0(\mathbf{P}^3_A, \tilde{\mathcal{J}}(d_i))$ such that $\tilde{f}_i(\mathrm{mod}\,\mathfrak{n})=f_i$ for each $1\leqq i\leqq l$. Since $(\tilde{f}_1, \cdots, \tilde{f}_l)A(0)\subset \mathrm{H}^0_*(\mathbf{P}^3_A, \tilde{\mathcal{J}})$, $\phi^0_\nu(0)$ turns out to be surjective for all $\nu\in \boldsymbol{Z}$. $\phi^{0-1}_{\nu}(0)=\phi^{-1}_{\nu}(0)$ is trivially surjective hence by the same theorem (loc. cit.) $\mathrm{H}^0(\mathbf{P}^3_A, \tilde{\mathcal{J}}(\nu))$ is A-free and $\phi^0_\nu(y)$ is an isomorphism for all $y\in\mathrm{Spec}\,A$ and $\nu\in\boldsymbol{Z}$. From the commutative diagram

$$\begin{tikzcd}
 & \mathrm{H}^0(\mathbf{P}^3_A, \tilde{\mathcal{J}}(\nu))\otimes_A k(o) \arrow[r, "\zeta"] \arrow[d, phantom, "\|\wr"] & A(0)_\nu\otimes_A k(o) \arrow[d, phantom, "\|\wr"] \\
0 \arrow[r] & \mathrm{H}^0(\mathbf{P}^3_{k(o)}, \tilde{\mathcal{J}}_o(\nu)) \arrow[r] & \mathrm{H}^0(\mathbf{P}^3_{k(o)}, \mathcal{O}_{\mathbf{P}^3_{k(o)}}(\nu))
\end{tikzcd}$$

follows the injectivity of ζ, and we find $\mathrm{Tor}^A_1((A(0)/\mathrm{H}^0_*(\mathbf{P}^3_A, \tilde{\mathcal{J}}))_\nu, k)=0$. $(A(0)/\mathrm{H}^0_*(\mathbf{P}^3_A, \tilde{\mathcal{J}}))_\nu$ is therefore A-free for every $\nu\geqq 0$ and $A(0)/\mathrm{H}^0_*(\mathbf{P}^3_A, \tilde{\mathcal{J}})$ is A-flat.

Q.E.D.

Theorem 5.11. *Let* $(a\,;\ \bar{m}\,;\ \bar{n})=(a\,;\ m_1, \cdots, m_{a-2b}\,;\ n_1, \cdots, n_b)$ *be a sequence of integers such that* $a\leqq m_1\leqq\cdots\leqq m_{a-2b}$, $a\leqq n_1\leqq\cdots\leqq n_b$ *where* $a\geqq 2b$. *Suppose*

1) $n_{i+1}=n_i$ *or* $n_{i+1}-n_i\geqq 3$ *for every* $1\leqq i\leqq b-1$.

2) $a\neq n_j-2$ *and* $\#\{i\,|\,m_i=n_j-2\}\leqq\#\{i\,|\,m_i=n_j-3\}$ *for each* $1\leqq j\leqq b$.

3) $\#\{i\,|\,m_i=n_j+1\}\leqq\#\{i\,|\,m_i=n_j+2\}$ *for each* $1\leqq j\leqq b$.

Then $(a\,;\ \bar{m}\,;\ \bar{n})$ *represents an irreducible component of* $\mathrm{Hilb}(\mathbf{P}^3_k)$.

Remark 5.12. In the case where $n_{i+1}=n_i$ or $n_{i+1}-n_i\geqq 3$ for every $1\leqq i\leqq b-1$, the conditions of Theorem 5.7 reduce to those of the present theorem.

Proof of Theorem 5.11. Let X be an a. B. curve with short basic sequence $(a\,;\ \bar{m}\,;\ \bar{n})$, $\mathcal{J}$ its sheaf of ideals and $I=\mathrm{H}^0_*(\mathcal{J})\subset R$. We can write

$$(5.11.4)\qquad \begin{cases} I=f_0k(0)\oplus\bigoplus\limits_{i=1}^{a} f_ik(1)\oplus\bigoplus\limits_{j=1}^{b} f_{a+j}k(2) \\ R=I\oplus N_I \end{cases}$$

by Proposition 1.3.3), and the matrix of relations is of the form

$$\lambda_2=\begin{pmatrix} U_{01} & U_{02} & 0 \\ & & 0 \\ U_1 & U_2 & x_3 1_b \\ & & x_4 1_b \\ U_{21} & U_3 & x_2 1_b \end{pmatrix}$$

(see the end of Section three). Suppose $\{n_1, \cdots, n_b\}=\{n'_1, \cdots, n'_v\}$ with $n'_1<n'_2<\cdots<n'_v$ and

$$\begin{cases} m_i=n'_u-3 & \text{for} \quad \alpha_0^u+1\leqq i\leqq\alpha_1^u \\ m_i=n'_u-2 & \text{for} \quad \alpha_1^u+1\leqq i\leqq\alpha_2^u\leqq a-2b \end{cases} \tag{5.11.5}$$

$$\begin{cases} m_i=n'_u+1 & \text{for} \quad \beta_0^u+1\leqq i\leqq\beta_1^u \\ m_i=n'_u+2 & \text{for} \quad \beta_1^u+1\leqq i\leqq\beta_2^u\leqq a-2b \end{cases} \tag{5.11.6}$$

where $1\leqq u\leqq v$. We see by the condition 1) and [1; (3.4)]

$$\varDelta\left(\begin{bmatrix} U_{01} \\ U_1 \\ U_{21} \end{bmatrix}\begin{pmatrix} \\ 1, \cdots, \alpha_0^u, \alpha_1^u+1, \cdots, a \end{pmatrix}\right)=\underbrace{\begin{pmatrix} * \\ * \\ 0 \\ * \end{pmatrix}}_{\alpha_1^u-\alpha_0^u}\begin{matrix} \}\,1 \\ \}\,\alpha_1^u \\ \}\,\alpha_2^u-\alpha_1^u \\ \}\,a+b-\alpha_2^u \end{matrix} \tag{5.11.7}$$

$$\varDelta\left(\begin{bmatrix} U_{01} \\ U_1 \\ U_{21} \end{bmatrix}\begin{pmatrix} \\ 1, \cdots, \beta_0^u, \beta_1^u+1, \cdots, a \end{pmatrix}\right)=\underbrace{\begin{pmatrix} * \\ * \\ 0 \\ * \end{pmatrix}}_{\beta_1^u-\beta_0^u}\begin{matrix} \}\,1 \\ \}\,\beta_1^u \\ \}\,\beta_2^u-\beta_1^u \\ \}\,a+b-\beta_2^u \end{matrix} \tag{5.11.8}$$

where the entries of $*$ are either positive or negative integers. Note that $\alpha_1^u-\alpha_0^v\geqq\alpha_2^u-\alpha_1^u$ and $\beta_1^u-\beta_0^u\leqq\beta_2^u-\beta_1^u$ by the conditions 2) and 3) respectively. Since $\lambda_3=\begin{pmatrix} 0 \\ -x_3 1_b \\ -x_4 1_b \\ -x_2 1_b \\ U_3 \end{pmatrix}$ and $\lambda_3\begin{pmatrix} a-2b+1, \cdots, a+2b \\ \\ \end{pmatrix}=0$, the relation $\lambda_2\lambda_3=0$ still holds if the entries of $\lambda_2\begin{pmatrix} \\ a-2b+1, \cdots, a+2b \end{pmatrix}$ are varied freely. We may therefore assume from the first that

$$\operatorname{rank}_k\begin{bmatrix} U_{01} \\ U_1 \\ U_{21} \end{bmatrix}\begin{pmatrix} \\ 1, \cdots, \alpha_0^u, \alpha_1^u+1, \cdots, a \end{pmatrix}(\mathrm{mod}\ \mathfrak{m})=\alpha_2^u-\alpha_1^u \tag{5.11.9}$$

$$\operatorname{rank}_k\begin{bmatrix} U_{01} \\ U_1 \\ U_{21} \end{bmatrix}\begin{pmatrix} \\ 1, \cdots, \beta_0^u, \beta_1^u+1, \cdots, a \end{pmatrix}(\mathrm{mod}\ \mathfrak{m})=\beta_1^u-\beta_0^u \tag{5.11.10}$$

for every $1\leqq u\leqq v$.

All the columns of $\begin{bmatrix} U_{01} \\ U_1 \\ U_{21} \end{bmatrix}$ are relations among $f_0, f_1, \cdots, f_{a+b}$ by its definition, so that (5.11.9) implies that, for each $1 \leqq u \leqq v$, we have $f_i \in (f_0, \cdots, f_{\alpha_1^u}, f_{\alpha_2^u+1}, \cdots, f_{a+b})R$ for $\alpha_1^u+1 \leqq i \leqq \alpha_2^u$. It follows from this that I is generated by f_0, $f_{a-2b+1}, \cdots, f_{a+b}$ and by all f_i such that $1 \leqq i \leqq a-2b$, $i \notin \bigcup_{u=1}^{v} \{w \mid \alpha_1^u+1 \leqq w \leqq \alpha_2^u\}$. Write these generators, say $g_1, \cdots, g_l$. Then we see by 1), 2) and (5.11.5)

$$\deg g_i \neq n'_u - 2 \qquad \text{for any} \quad 1 \leqq i \leqq l \quad \text{and} \quad 1 \leqq u \leqq v\,. \tag{5.11.11}$$

Let H be an arbitrary irreducible component of red(Hilb$(\mathbf{P}_k^3)$) containing the k-rational point o corresponding to X, and let $(A, \mathfrak{n})$ be the local ring at this point. Denote by $p: \tilde{X} \rightarrow \operatorname{Spec} A$ the family induced from the universal family (5.1) through the natural inclusion $\operatorname{Spec} A \hookrightarrow H \hookrightarrow \operatorname{Hilb}(\mathbf{P}_k^3)$, and denote the sheaf of ideals of $\tilde{X}$ (resp. $p^{-1}(h)$, $h \in \operatorname{Spec} A$) by $\tilde{\mathcal{J}}$ (resp. $\tilde{\mathcal{J}}_h$). Since $\mathrm{H}^1(\mathbf{P}_k^3, \tilde{\mathcal{J}}_o(\deg g_i)) = \mathrm{H}^1(\mathbf{P}_k^3, \mathcal{J}(\deg g_i)) = 0$ by (3.4.2) and (5.11.11), we find by Lemma 5.10 that $A(0)/\mathrm{H}_*^0(\mathbf{P}_A^3, \tilde{\mathcal{J}})$ is A-flat and that there exist homogeneous polynomials $\tilde{f}_i \in \mathrm{H}_*^0(\mathbf{P}_A^3, \tilde{\mathcal{J}})$ $(0 \leqq i \leqq a+b)$ such that $\tilde{f}_i (\operatorname{mod} \mathfrak{m}) = f_i$. This, combined with Lemma 5.8, implies

$$\tilde{I} = \tilde{f}_0 A(0) \oplus \bigoplus_{i=1}^{a} \tilde{f}_i A(1) \oplus \bigoplus_{j=1}^{b} \tilde{f}_{a+j} A(2) \tag{5.11.12}$$

where $\tilde{I} = \mathrm{H}_*^0(\mathbf{P}_A^3, \tilde{\mathcal{J}})$, and we may assume by Lemma 5.9 $\tilde{f}_i - f_i \in N_I \otimes_k A$ (see (5.11.4)). Denote the quotient field of A by K. We will then consider the curve $\tilde{X}_K = \operatorname{Proj}_K(K(0)/\tilde{I} \otimes_A K)$, where

$$\tilde{I} \otimes_A K = \tilde{f}_0 K(0) \oplus \bigoplus_{i=1}^{a} \tilde{f}_i K(1) \oplus \bigoplus_{j=1}^{b} \tilde{f}_{a+j} K(2)\,.$$

Denote by $\tilde{\lambda}_2 = \begin{bmatrix} \tilde{U}_{01} & \tilde{U}_{02} & 0 \\ \tilde{U}_1 & \tilde{U}_2 & \tilde{U}_4 \\ \tilde{U}_{21} & \tilde{U}_3 & \tilde{U}_5 \end{bmatrix}$ the matrix of relations among $\tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_{a+b}$ computed by [1; Theorem 1.6] and $\tilde{\lambda}_3 = \begin{bmatrix} -\tilde{U}_4 \\ -\tilde{U}_5 \\ \tilde{U}_3 \end{bmatrix}$ as usual.

Claim. $\tilde{\lambda}_3 \equiv 0 (\operatorname{mod} \mathfrak{m})$.

Proof of Claim. Since $\Delta(\tilde{\lambda}_3) = \Delta(\lambda_3)$, $\Delta(\tilde{U}_3) = \Delta(\tilde{U}_5)$ and $\Delta\left(\tilde{U}_4\begin{pmatrix}1, \cdots, a-2b\end{pmatrix}\right)$ are matrices of nonzero integers by the condition 1). On the other hand

$$\Delta\left(\tilde{U}_4\begin{pmatrix} a-2b+1, \cdots, a \\ 1, \cdots, j-1, j+1, \cdots, b \end{pmatrix}\right) = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \\ \\ * \end{bmatrix} \begin{matrix} \}\ \beta_0^u \\ \\ \left.\vphantom{\begin{matrix}0\\ \vdots \\ 0\end{matrix}}\right\} \beta_1^u - \beta_0^u \quad \text{if} \quad n_j = n'_u \\ \\ \end{matrix} \tag{5.11.13}$$

for $1 \leqq j \leqq b$,

where $*$ consists of nonzero integers. The entries of $\tilde{\lambda}_2$ are in fact in $A(0)$ by (5.11.12) and $\tilde{\lambda}_2 (\operatorname{mod} \mathfrak{n}) = \lambda_2$, so that

$$\text{(5.11.14)} \qquad \operatorname{rank}_k \begin{bmatrix} \tilde{U}_{01} \\ \tilde{U}_1 \\ \tilde{U}_{21} \end{bmatrix} \begin{pmatrix} \\ 1, \cdots, \beta_0^u, \beta_1^u+1, \cdots, a \end{pmatrix} (\operatorname{mod} \mathfrak{m}) = \beta_1^u - \beta_0^u$$

for $1 \leqq u \leqq v$.

From the relation $\tilde{\lambda}_2 \tilde{\lambda}_3 = 0$,

$$\text{(5.11.15)} \qquad \tilde{\lambda}_2 \begin{pmatrix} 0, \cdots, \beta_1^u, \beta_2^u+1, \cdots, a+b \\ \\ \end{pmatrix} \tilde{\lambda}_3 \begin{pmatrix} \\ 1, \cdots, j-1, j+1, \cdots, b \end{pmatrix} = 0$$

for j, u such that $n_j = n'_u$.

In the equation above, we see by (5.11.13) and (5.11.8) that the left hand side is a vector of homogeneous polynomials of degree zero, hence

$$\begin{bmatrix} \tilde{U}_{01} \\ \tilde{U}_1 \\ \tilde{U}_{21} \end{bmatrix} \begin{pmatrix} 0, \cdots, \beta_1^u, \beta_2^u+1, \cdots, a+b \\ 1, \cdots, \beta_0^u, \beta_1^u+1, \cdots, a \end{pmatrix} \tilde{U}_4 \begin{pmatrix} 1, \cdots, \beta_0^u, \beta_1^u+1, \cdots, a \\ 1, \cdots, j-1, j+1, \cdots, b \end{pmatrix} = 0 .$$

From this and (5.11.14) follows immediately $\tilde{U}_4 \begin{pmatrix} \\ 1, \cdots, j-1, j+1, \cdots, b \end{pmatrix} \equiv 0$ $(\operatorname{mod} \mathfrak{m})$. This holds for all $1 \leqq j \leqq b$ and we obtain $\tilde{\lambda}_3 \equiv 0 (\operatorname{mod} \mathfrak{m})$. Q.E.D.

Now we go back to the proof of the theorem. Since X is a. B., we have $\operatorname{Im}^R({}^t\lambda_3) = \mathfrak{m} R^b$. We deduce therefore from the Claim combined with the fact $\tilde{\lambda}_3$ $(\operatorname{mod} \mathfrak{n}) = \lambda_3$ that $\operatorname{Im}^{K(0)}({}^t\tilde{\lambda}_3) = \mathfrak{m} K(0)^b$, and consequently X_K is an a. B. curve over the field K with short basic sequence $(a\,;\ \bar{m}\ ;\ \bar{n})$ (see Theorem 3.2, with k being replaced by K). We may thus assume by Corollary 3.3 that $\tilde{\lambda}_2$ is of the form

$$\begin{bmatrix} \tilde{U}_{01} & \tilde{U}_{02} & 0 \\ & & 0 \\ \tilde{U}_1 & \tilde{U}_2 & x_3 1_b \\ & & x_4 1_b \\ \tilde{U}_{21} & \tilde{U}_3 & x_2 1_b \end{bmatrix} \quad \text{and} \quad \tilde{f}_i = (-1)^i \det \tilde{\lambda}_2 \begin{pmatrix} i \\ a+b+1, \cdots, a+2b \end{pmatrix} / \det \tilde{U}_3$$

for $0 \leqq i \leqq a+b$. Comparing the matrix $\tilde{\lambda}_2$ with $\tilde{\tilde{\lambda}}_2$ we obtain a morphism

$$\Omega : \operatorname{Spec} K \longrightarrow \mathring{S}(a\,;\ \bar{m}\ ;\ \bar{n})$$

such that $X_K = \pi^{-1}(\Omega(\eta))$ where η is the unique point of Spec K (see (4.1.6)). Let

$$\chi : \mathring{S}(a\,;\ \bar{m}\ ;\ \bar{n}) \longrightarrow \operatorname{Hilb}(\mathbf{P}_k^3)$$

be the unique natural morphism such that $\chi^*(\mathcal{Z}) = \mathfrak{X}$ (see (5.1)). Then the diagram

$$\begin{tikzcd}
\mathrm{Spec}\, K \arrow[r, "\Omega"] \arrow[d, hook] & \mathring{S}(a\,;\ \bar{m}\,;\ \bar{n}) \arrow[d, "\chi"] \\
H \arrow[r, hook] & \mathrm{Hilb}(\mathbf{P}^3_k)
\end{tikzcd}$$

commutes by the universality of the family $\mathscr{Z}$, therefore the generic point of H is in the image of χ. This means that $(a\,;\ \bar{m}\,;\ \bar{n})$ represents an irreducible component of $\mathrm{Hilb}(\mathbf{P}^3_k)$. Q.E.D.

Corollary 5.12. *Let a, b and n be integers such that $n \geqq a \geqq 2b$. Then* $(a\,;\ \underbrace{n, \cdots, n}_{a-2b}\,;\ \underbrace{n, \cdots, n}_{b})$ *represents an irreducible component of* $\mathrm{Hilb}(\mathbf{P}^3_k)$ *if and only if $a \neq n-2$. Furthermore, if $a=n-2$, the points of* $\mathrm{Hilb}(\mathbf{P}^3_k)$ *corresponding to a.B. curves with short basic sequence* $(n-2\,;\ \underbrace{n, \cdots, n}_{n-2-2b}\,;\ \underbrace{n, \cdots, n}_{b})$ *are contained in the irreducible component represented by the short basic sequence* $(n-1\,;\ n-1,\ n-1,\ n-1,\ \underbrace{n, \cdots, n}_{n-2b-2}\,;\ \underbrace{n, \cdots, n}_{b-1})$.

Proof. The first half is clear by Theorems 5.7 and 5.11. We see by Theorem 5.11 $(n-1\,;\ n-1,\ n-1,\ n-1,\ \underbrace{n, \cdots, n}_{n-2b-2}\,;\ \underbrace{n, \cdots, n}_{b-1})$ indeed represents an irreducible component of $\mathrm{Hilb}(\mathbf{P}^3_k)$. The detail of the proof of the latter half is left to the reader.

Remark 5.13. In the case where $n_{i+1}=n_i+2$ for some $1 \leqq i \leqq b$, the methods we have developped so far may not be applicable. The crucial point is that we cannot tell in advance whether or not an arbitrary flat deformation of X in $\mathbf{P}^3_k$ comes from a flat deformation either of the ring $R/\mathrm{H}^0_*(\mathscr{J})$ or of the module $\mathrm{H}^0_*(\mathcal{O}_X)$, if $n_{i+1}=n_i+2$ for some i.

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