Piecewise Linearization of Real Analytic Functions

By

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§ 1. Introduction

Many mathematicians considered the problem of triangulations of algebraic sets, analytic sets, semi-algebraic sets, etc. [2], [4], [7], [8], [9], [10], [14] and [19]. We want to consider more generally a global piecewise linearization of real analytic functions. For this, we need simultaneous triangulations of all levels of the functions.

A function on a polyhedron is called *piecewise linear* ($=PL$) if we have a triangulation of the polyhedron such that the restriction of the function on each simplex is linear. A C^∞ *triangulation* of a real analytic manifold M is a pair of a simplicial complex K and a homeomorphism $g: |K| \to M$, $|K|$ meaning the underlying polyhedron of K, such that the restriction of g on each simplex is a C^∞ diffeomorphism onto the image. The existence of C^∞ triangulation is well-known (e. g. [12]). In this paper manifolds have not boundary unless otherwise specified.

Theorem I. *Let $M \subset R^n$, (K, g) be a real analytic manifold of dimension $\neq 4, 5$ and its C^∞ triangulation respectively, let ε be a positive continuous function on $|K|$, and let f be an analytic function on M. Then there exists a homeomorphism $\tau: |K| \to M$ such that $f \circ \tau$ is PL and that $|\tau(x) - g(x)| < \varepsilon(x)$ for $x \in |K|$.*

Theorem II. *Let M be an analytic manifold of dimension 4, and let f be a proper analytic function on M. Then there exist a PL manifold M' and a homeomorphism $\tau: M' \to M$ such that $f \circ \tau$ is PL.*

We call a set $X \subset R^n$ subanalytic, (see [3]), if for any point $x \in R^n$, there exist an open neighborhood U of x in R^n and a finite number of proper analytic maps f_{ij} from real analytic manifolds to U, $j=1, 2$, such that

$$X \cap U = \bigcup_i (\operatorname{Im} f_{i1} - \operatorname{Im} f_{i2}).$$

A continuous map $f: X_1 \to X_2$ of subanalytic sets, $X_1 \subset R^{n_1}$, $X_2 \subset R^{n_2}$, is called *subanalytic* if the graph is subanalytic in $R^{n_1} \times R^{n_2}$. We remark that a polyhedron X closed in R^n is subanalytic and that a PL function on X is subanalytic. We

define the subanalyticness of a map or a subset of polyhedrons or of analytic manifolds after imbedding the spaces closedly in Euclidean spaces. It does not depend on the choice of imbeddings. Subanalytic versions of the above theorems are the following.

Theorem I′. *Let $M \subset \mathbf{R}^n$ be a PL manifold of dimension $\neq 4, 5$, let ε be a positive continuous function on M, and f be a proper subanalytic function on M. Then there exists a homeomorphism τ of M such that $f \circ \tau$ is PL and that $|\tau(x) - x| < \varepsilon(x)$ for $x \in M$.*

Theorem II′. *Let M be a PL manifold of dimension 4, and f be a proper subanalytic function on M. There exists a PL manifold M' and a homeomorphism $\tau : M' \to M$ such that $f \circ \tau$ is PL.*

As the existence of analytic triangulation follows from the C^∞ triangulation (Proposition 6.11), these are generalizations of Theorems I, II in the case of proper f. The reason why we will treat the subanalytic case too is that it is convenient for the most part to consider our problem in the subanalytic category.

The process of the proof of Theorem I is the following. At first we divide M to semi-analytic X_1, X_2 so that X_1 contains all critical points of f and that f is locally constant on X_1. Secondly f is piecewise linearized on a large domain of X_2, using Lemma 6.15 which is a consequence of the C^∞ triangulation theory of C^∞ manifolds. Thirdly we triangulate X_1 by Proposition 3.1, and then we piecewise linearize f on a neighborhood of X_1 by Proposition 5.1. Lastly, applying Concordance Implies Isotopy Theorem of Kirby-Siebenmann [6], we piecewise linearize f globally. Proposition 3.1 shows a fine triangulation of subanalytic sets. A triangulation of subanalytic sets is known [4]. But if we had only that triangulation, we might not apply Concordance Implies Isotopy Theorem, because the polyhedron treated might be not a *PL* manifold. Proposition 5.1 is a subanalytic generalization of a result in [16] that if two analytic function germs vanish at the origin and have the same sign at each point near the origin, they are topologically equivalent (namely a local homeomorphism transfers one to the other).

The reason why we assume f to be proper in Theorem II is that we can not apply Concordance Implies Isotopy Theorem but apply the Hauptvermutung theorem for 3-manifolds of Moise instead.

Two functions f_1, f_2 on a manifold M is called *topologically R-L equivalent* if there exist homeomorphisms τ_1, τ_2 of M, $\mathbf{R}$ respectively such that $\tau_2 \circ f_1 \circ \tau_1 = f_2$. An application of the theorems is the following.

Corollary III. *Let M be a real compact analytic (or compact PL) manifold of dimension $\neq 5$. Then the topological R-L equivalence classes of all subanalytic functions on M are countable.*

In §9 we show necessary conditions for real (or complex) analytic maps to be piecewise linearizable. In this paper we use confusedly the differential of a function on R^n and the gradient by the usual Riemannian metric.

§2. Subanalytic Sets

Let us show some fundamental properties of subanalytic sets which we will need later. The results below without proof were proved in [3].

(2.1) *The property of being subanalytic is closed under the following operations*:

(2.1.1) *finite union,*

(2.1.2) *finite intersection,*

(2.1.3) *difference of any two.*

(2.2) *Let $A \subset R^n$ be a subanalytic set. Then the interior $\hat{A}$ in R^n, the closure $\bar{A}$ in R^n and every connected component of A are subanalytic. Besides, $\dim A > \dim(\bar{A} - A)$, and the family of all connected components of A is locally finite.*

(2.3) *Any semi-analytic subset of R^n is subanalytic.*

(2.4) *Let $A \subset R^n$ be closed, subanalytic and of dimension r. Then there exist an analytic manifold M and a proper analytic map $f: M \to R^n$ such that $f(M) = A$. Moreover, for these M and f, there exist an r-dimensional analytic manifold N and a proper analytic map $g: N \to M$ such that $f \circ g(N) = f(M)$.*

(2.5) *Let $f: X \to Y$ be a proper subanalytic map of analytic manifolds.*

(2.5.1) *If B is a subanalytic subset of Y, then so is $f^{-1}(B)$ in X.*

(2.5.2) *If A is a subanalytic subset of X, then so is $f(A)$ in Y.*

(2.6) *Let $A \subset R^n$, $f: R^n \to R^m$ be subanalytic. Assume that the restriction of f to $\bar{A}$ is proper (e.g. A is bounded). Then $f(A)$ is subanalytic.*

(2.7) *Let $A \subset R^n$ be subanalytic. Then there exists a subanalytic subset A' of A with $\dim A' < \dim A$ such that $A - A'$ is an analytic manifold.*

Definition 2.8. Let $M, N \subset R^n$ be C^∞ manifolds. Assume that $M \cap N = \emptyset$. Let $y \in N$. We say that (M, N) satisfies the *Whitney condition at* y if the following is satisfied.

(2.8.1) If $\{x_i\}$, $\{y_i\}$ are sequences in M, N respectively, both converging to y, if the sequence of tangent spaces $\{T_{x_i}M\}$ converges to a subspace $T \subset R^n$ (in the Grassmannian of q-dimensional subspaces of R^n where $q = \dim M$), and if the sequence $\{\overrightarrow{x_i y_i}\}$ of lines containing $x_i - y_i$ converges to a line $L \subset R^n$ (in the Grassmannian of 1-dimensional subspaces of R^n), then $L \subset T$.

We say that (M, N) satisfies the *Whitney condition* when it does so at every point in N.

Definition 2.9. A *stratification* of a subset X of $\boldsymbol{R}^n$ is a partition $\{X_i\}$ of X into bounded connected analytic submanifolds of $\boldsymbol{R}^n$ such that

(2.9.1) $\{X_i\}$ is locally finite in $\boldsymbol{R}^n$, and that

(2.9.2) $\bar{X}_i \cap X_j \neq \emptyset$ implies $\bar{X}_i \supset X_j$.

If $\{X_i\}$ satisfies moreover

(2.9.3) for any X_i, X_j, (X_i, X_j) satisfies the Whitney condition,

then we call it a *Whitney stratification*. The stratification is called *subanalytic*, if all strata are subanalytic. In Section 4 we weaken the above conditions on strata, we only assume they are C^∞ manifolds and satisfy (2.9.1).

Definition 2.10. Let $\{A_\nu\}$, $\{B_\mu\}$ be families of subsets of $\boldsymbol{R}^n$. We say that $\{A_\nu\}$ is *compatible with* $\{B_\mu\}$ if for each ν, μ,

$$A_\nu \subset B_\mu \quad \text{or} \quad A_\nu \cap B_\mu = \emptyset .$$

(2.11) *For a locally finite family $\{A_\nu\}$ of subanalytic sets in $\boldsymbol{R}^n$, there exists a Whitney subanalytic stratification of $\boldsymbol{R}^n$ compatible with $\{A_\nu\}$.*

In the case in which $\{A_\nu\}$ consists of one set, it is shown in [3]. The following is the key to the proof.

(2.12) *Let M, $N \subset \boldsymbol{R}^n$ be subanalytic analytic manifolds with $M \cap N = \emptyset$. Then there exists a closed subanalytic subset N' of N such that $\dim N' < \dim N$ and that $(M, N-N')$ satisfies the Whitney condition.*

Proof of (2.11). We may add $\boldsymbol{R}^n$ to $\{A_\nu\}$. We proceed by downward induction. Induction hypothesis: There exists a Whitney subanalytic stratification $\{X_i\}$ of a subanalytic subset Y_{k+1} of $\boldsymbol{R}^n$ compatible with $\{A_\nu\}$ such that $Z_k = \boldsymbol{R}^n - Y_{k+1}$ is closed and of dimension $\leq k$ and that each X_i is of dimension $> k$ (the case $k = n$ is trivial). Put $\{A'_\nu\} = \{A_\nu \cap Z_k\}$. For each A'_ν we have a subanalytic partition B_ν and C_ν of A'_ν by (2.7.12) such that B_ν is an analytic manifold of dimension k, that C_ν is of dimension $< k$ and that (X_i, B_ν) satisfies the Whitney condition for any X_i. Let Z_{k-1} be the union of all $\bar{B}_\nu - B_\nu$, $\bar{C}_\nu$ and connected components of dimension $< k$ of the sets in the form $B_\nu \cap B_{\nu'}$. Then Z_{k-1} is closed, subanalytic and of dimension $< k$ by (2.1.2) and the locally finite assumption of $\{A_\nu\}$. It follows also that the union of $B_\nu - Z_{k-1}$ is $Z_k - Z_{k-1}$ because of $\boldsymbol{R}^n \in \{A_\nu\}$, and it is also a subanalytic analytic manifold. Let $\{X'_j\}$ be all connected components of $Z_k - Z_{k-1}$. Then each X'_j is a subanalytic analytic manifold contained in some B_ν, and moreover $X'_j \cap B_{\nu'} \neq \emptyset$ implies $B_{\nu'} \supset X'_j$. Namely $\{X'_j\}$ is compatible with $\{A_\nu\}$. Hence $\{X_i\} \cup \{X'_j\}$ is a Whitney subana-

lytic stratification of $Y_k = R^n - Z_{k-1}$ compatible with $\{A_\nu\}$ whose strata are of dimension$\geqq k$. Therefore (2.11) follows.

(2.13) *Let $X \subset R^m$ be a closed subanalytic set.*

(2.13.1) *A map $f = (f_1, \cdots, f_n) : X \to R^n$ is subanalytic if and only if so are all f_i.*

(2.13.2) *The set of all subanalytic functions on X is a ring.*

(2.13.3) *Let $f : X \to R^n$, $g : R^n \to R^l$ be subanalytic. Then $g \circ f : X \to R^l$ is subanalytic.*

Proof. (2.13.1). Put
$$Y = \text{graph } f, \ Y_i = \text{graph } f_i, \ i = 1, \cdots, n .$$

Let $\pi_i : R^m \times R^n \to R^m \times R$, $\tau_i : R^m \times R^n \to R^m \times R^n$ be defined by
$$\pi_i(x_1, \cdots, x_m, y_1, \cdots, y_n) = (x_1, \cdots, x_m, y_i) \quad \text{and}$$
$$\tau_i(x_1, \cdots, x_m, y_1, \cdots, y_n) = (x_1, \cdots, x_m, y_2, \cdots, y_i, y_1, y_{i+1}, \cdots, y_n).$$

Then we have
$$\pi_i Y = Y_i, \ i = 1, \cdots, n ,$$
$$Y = \bigcap_{i=1}^{n} \tau_i(Y_i \times R^{n-1}) .$$

Since X is closed in R^m, so is Y in $R^m \times R^n$. Hence, by (2.6) and the above former equality, if Y is subanalytic, then so is Y_i. The converse follows from (2.1.2) and the latter equality.

(2.13.2). Let f_1, f_2 be subanalytic functions on X. Put
$$Z_- = \{(y_1, y_2, y_3) \in R^3 \mid y_3 = y_1 - y_2\} .$$
$$Z_\times = \{(y_1, y_2, y_3) \in R^3 \mid y_3 = y_1 y_2\} ,$$
$$Y_i = \text{graph } f_i, \ i = 1, 2 ,$$
$$Y_- = \text{graph}(f_1 - f_2), \ Y_\times = \text{graph } f_1 f_2 .$$

Let $\pi : R^m \times R^3 \to R^m \times R$, $\tau : R^m \times R^3 \to R^m \times R^3$ be defined by
$$\pi(x_1, \cdots, x_m, y_1, y_2, y_3) = (x_1, \cdots, x_m, y_3) ,$$
$$\tau(x_1, \cdots, x_m, y_1, y_2, y_3) = (x_1, \cdots, x_m, y_2, y_1, y_3) .$$

Then we have
$$Y_- = \pi((R^m \times Z_-) \cap (Y_1 \times R^2) \cap \tau(Y_2 \times R^2)) ,$$
$$Y_\times = \pi((R^m \times Z_\times) \cap (Y_1 \times R^2) \cap \tau(Y_2 \times R^2)) .$$

Hence it follows from the closedness of X, (2.1.2) and (2.6) that Y_-, $Y_\times$ are

subanalytic.

As (2.13.3) follows in the same way as above, we omit its proof.

(2.14) *Let $M \subset R^n$ be a connected subanalytic analytic manifold of dimension r, and $f : R^n \to R^m$ be an analytic map. Then there exists a closed subanalytic set $A \subset M$ of dimension$<r$ such that the differential $d(f|_M)$ of $f|_M$ has constant rank on $M-A$.*

Proof. Let r' be the maximal rank of $d(f|_M)$. Since M is connected, and since f is analytic, the set

$$Y = \{y \in M \,|\, \text{rank } d(f|_M)_y < r'\}$$

is an analytic subset of M of dimension$<r$. Now by (2.2), $\dim(\overline{M}-M)<r$ and $\dim \overline{M} = r$. Hence there exist by (2.4) an analytic manifold N of dimension r and a proper analytic map $g : N \to R^n$ such that $g(N) = \overline{M}$. Put

$$X = \{x \in N \,|\, \text{rank } d(f \circ g)_x < r'\} \,,$$

$$X' = \{x \in N \,|\, \text{rank } dg_x < r\} \,.$$

Then X, X' are analytic subsets of N. Let N_1 be the union of all connected components of N being contained in X, and let $N_2 = N - N_1$. We see $N_1 \subset X'$ by reduction to absurdity. Assume $N_1 - X' \neq \emptyset$. Then $g(N_1 - X')$ has inner points in $\overline{M}$. Hence it follows that $(M-Y) \cap g(N_1 - X') \neq \emptyset$. This means that there exists a point $y \in N_1$ such that $g(y) \in M$, that g is an imbedding on a neighborhood of y and that rank $d(f|_M)_{g(y)} = r'$, and hence rank $d(f \circ g)_y = r'$. This contradicts the definitions of X and N_1. Thus $N_1 \subset X'$. Hence we have $\dim g(N_1) < r$, which together with the inequality $\dim(N_2 \cap X) < r$ implies $\dim g(X) < r$. Put $A = M \cap \overline{g(X)}$, then the lemma follows from (2.2.6).

(2.15) *Let $f : R^n \to R^m$ be analytic, and X, X_i, $i = 1, 2, \cdots$ be locally finite subanalytic sets in R^n with $X \supset X_i$. Then there exists a Whitney subanalytic stratification $\{Y_i\}$ of X compatible with $\{X_i\}$ such that $f|_{Y_i}$ for each i has constant rank.*

Proof. We prove it by induction on $r = \dim X$. The case $r = 0$ is trivial. Hence we assume it for $\dim X < r$. By (2.11) there exists a subanalytic stratification $\{X_i'\}_{i=1,\dots}$ of X compatible with $\{X_i\}$. Let $\dim X_i' = r$ for $i \in \Lambda_1$, $\Lambda_1 \subset N$, and $\dim X_i' < r$ for $i \notin \Lambda_1$. Apply (2.14) to each X_i', $i \in \Lambda_1$. Then we have a subanalytic set Z_i in X_i' of dimension$<r$ such that $f|_{X_i'-Z_i}$ has constant rank. Apply once more (2.11) to $\{X_i'-Z_i, Z_i, X_j' | i \in \Lambda_1, j \notin \Lambda_1\}$, and let $\{Y_i'\}_{i=1\dots}$ be the resulting Whitney subanalytic stratification of X compatible with $\{X_i'-Z_i, Z_i, X_j' | i \in \Lambda_1, j \notin \Lambda_1\}$ with $\dim Y_i' = r$ for $i \in \Lambda_2$, $\Lambda_2 \subset N$, and $\dim Y_i' < r$ for $i \notin \Lambda_2$. It clearly follows that $\{Y_i'\}$ is compatible with $\{X_i\}$. By induction hypothesis there exists a Whitney subanalytic stratification $\{Y_j''\}$ of $\bigcup_{i \notin \Lambda_2} Y_i'$ compatible with

$\{Y'_i\}_{i \notin \Lambda_2}$ such that $f|_{Y''_j}$ for each j has constant rank. Then (Y'_i, Y''_j) for any $i \in \Lambda_2$ and any j satisfies the Whitney condition, because Y''_j is contained in some $Y'_{i'}$, $i' \notin \Lambda_2$, and $(Y'_i, Y'_{i'})$ satisfies the Whitney condition. Hence $\{Y'_i\}_{i \notin \Lambda_2} \cup \{Y''_j\}$ satisfies the required properties.

§3. Triangulation of Subanalytic Sets

The purpose of this section is to prove the following refinement of a result in [4].

Proposition 3.1. *Let K be a triangulation of $\boldsymbol{R}^n$. Let $\{A_i\}$ be a locally finite family of subanalytic subsets of $\boldsymbol{R}^n$. Then there exist a subdivision K' of K and a homeomorphism τ of $\boldsymbol{R}^n$ such that*

(3.1.1)   *τ is subanalytic,*

(3.1.2)   *$\tau(\sigma) = \sigma$ for any $\sigma \in K$,*

(3.1.3)   *for any $\sigma \in K'$, $\tau(\mathring{\sigma})$ is an analytic submanifold of $\boldsymbol{R}^n$ and $\tau|_{\mathring{\sigma}} : \mathring{\sigma} \to \tau(\mathring{\sigma})$ is an analytic diffeomorphism and that*

(3.1.4)   *$\{\tau(\mathring{\sigma}) \,|\, \sigma \in K'\}$ is compatible with $\{A_i\}$.*

A simplex (or a cell) is understood to be a closed one, and $\mathring{\sigma}$ is the interior of a simplex or a cell σ. By $\partial\sigma$ we denote the boundary. In this paper we always consider the usual polyhedron structure on $\boldsymbol{R}^n$.

The condition (3.1.2) was not under consideration in [4], and it will play an important part in our application (see Remark 6.23). We proceed with the proof in a similar way to [10].

Let $A \subset \boldsymbol{R}^n$ be subanalytic, and $c \in \boldsymbol{R}^n$ be a point. A line λ through c is called *non-singular for A at c* if $\lambda \cap A = c$ or $= \emptyset$ in a neighborhood of c. In the other case, namely when $\dim(\lambda \cap A) = 1$ in any neighborhood of c, λ is called *singular for A at c*. If λ is non-singular for A at any point of λ, we call λ *non-singular for A*. In the other case λ is called *singular for A*.

Let $a \in \boldsymbol{R}^n$ be a point, and $S_a^{n-1} \subset \boldsymbol{R}^n$ denote the sphere centered at a with radius 1. Let $q_a : \boldsymbol{R}^n - a \to S_a^{n-1}$, $q'_a : \boldsymbol{R}^n - a \to S_a^{n-1} \times \boldsymbol{R}$ be defined by

$$q_a(x) = a + (x-a)/|x-a|,$$

$$q'_a(x) = (q_a(x), |x-a|).$$

Let $X_1, X_2 \subset \boldsymbol{R}^n$ and $Y \subset S_a^{n-1}$ be subsets with $a \notin X_1, X_2$. Assume the existence of functions ϕ_1, ϕ_2 on Y such that $q'_a(X_i) = \mathrm{graph}\,\phi_i$, $i = 1, 2$. Then we write

$$X_1 < X_2 \ \textit{with respect to } (\boldsymbol{R}^n, a) \quad \text{if} \quad \phi_1 < \phi_2,$$

$$X_1 \leqq X_2 \ \textit{with respect to } (\boldsymbol{R}^n, a) \quad \text{if} \quad \phi_1 \leqq \phi_2.$$

In either case we denote by (X_1, X_2) and $[X_1, X_2]$ the sets

$$\{q_a'^{-1}(y, t) \mid (y, t) \in Y \times \boldsymbol{R}, \ \phi_1(y) < t < \phi_2(y)\} \quad \text{and}$$

$$\{q_a'^{-1}(y, t) \mid \phi_1(y) \leqq t \leqq \phi_2(y)\}$$

respectively.

An analytic submanifold $\Gamma \subset \boldsymbol{R}^n$ is called *topographic with respect to* $(\boldsymbol{R}^n, a)$ if $a \notin \Gamma$, if $q_a(\Gamma)$ is an analytic manifold and if $q_a|_\Gamma : \Gamma \to q_a(\Gamma)$ is an analytic diffeomorphism. Let $p : \boldsymbol{R}^n \times \boldsymbol{R}^m \to \boldsymbol{R}^n$ be the projection. An analytic submanifold $\Gamma \subset \boldsymbol{R}^n \times \boldsymbol{R}^m$ is called *topographic with respect to* $\boldsymbol{R}^m$ if $p(\Gamma)$ is an analytic manifold and if $p|_\Gamma : \Gamma \to p(\Gamma)$ is an analytic diffeomorphism. We remark that an analytic submanifold $\Gamma \subset \boldsymbol{R}^n$ is topographic with respect to $(\boldsymbol{R}^n, a)$ if and only if $a \notin \Gamma$ and if $q_a'(\Gamma) \subset \boldsymbol{R}^n \times \boldsymbol{R}$ is topographic with respect to $\boldsymbol{R}$.

We say that a subset $X \subset \boldsymbol{R}^n - a$ has *property* (P) *with respect to* $(\boldsymbol{R}^n, a)$ if the restriction $q_a|_X$ is an open map. If $X \subset \boldsymbol{R}^n \times \boldsymbol{R}^m$, we define *the property* (P) *of* X *with respect to* $\boldsymbol{R}^m$ in the same way as above. We also remark that $X \subset \boldsymbol{R}^n$ has property (P) with respect to $(\boldsymbol{R}^n, a)$ if and only if $a \notin \Gamma$ and if $q_a'(X) \subset \boldsymbol{R}^n \times \boldsymbol{R}$ has property (P) with respect to $\boldsymbol{R}$.

Lemma 3.2. *Let $A \subset \boldsymbol{R}^n$ be subanalytic and of dimension $< n$. Then the union of all points on lines singular for A is meager in $\boldsymbol{R}^n$ (a countable union of nowhere dense sets).*

Proof (Compare with Lemma 3 in [10]). Let $\{A_i\}$ be a subanalytic stratification of A (2.11). Since $\{A_i\}$ is locally finite, any singular line for A is singular for some A_i. Hence we can assume A to be an analytic submanifold of $\boldsymbol{R}^n$. Then there exist an open set $U \subset \boldsymbol{R}^n$ and an analytic function f on U such that $A \subset U$, $A = f^{-1}(0)$. Put

$$\Theta = \{(x, y) \in U \times \boldsymbol{R}^n \mid y \neq 0, \text{ the line through } x$$

$$\text{and } x + y \text{ is singular for } A \text{ at } x\},$$

$$\pi(x, y) = x + y \quad \text{for} \quad (x, y) \in U \times \boldsymbol{R}^n.$$

Then Θ is a semi-analytic subset of $U \times \boldsymbol{R}^n$ [7], because of

$$\Theta = \left\{(x, y) \in U \times \boldsymbol{R}^n \mid y \neq 0, \ \frac{\partial^i}{\partial t^i} f(x+ty)_{t=0} = 0, \ i=1, 2, \cdots\right\}.$$

If a line λ is singular for A at a point x, f vanishes identically on the connected component of $\lambda \cap U$ containing x. This implies that for any point $x \in U$, there exist neighborhoods V, V' of x, 0 respectively such that $\pi(\Theta \cap (V \times V'))$ is contained in $f^{-1}(0)$. Hence $\pi(\Theta \cap (V \times V'))$ is of dimension $< n$. On the other hand we have

$$(3.2.1) \qquad \Theta \cap (V \times \boldsymbol{R}^n) = \{(x', ty) \mid (x', y) \in \Theta \cap (V \times V'), \ t \neq 0 \in \boldsymbol{R}\}.$$

Choose as V a closed small ball centered at x. Then $\Theta \cap (V \times \mathbf{R}^n)$ is semi-analytic in $\mathbf{R}^n \times \mathbf{R}^n$ and hence subanalytic by (2.3). Apply (2.11) to $\Theta \cap (V \times S_0^{n-1})$, and choose a finite stratification of the set. Then, by (3.2.1) there exists a finite stratification $\{M_i\}$ of $\Theta \cap (V \times \mathbf{R}^n)$ such that for each i, if $(x', y) \in M_i$, $t > 0$, then $(x', ty) \in M_i$. Therefore the rank of the restriction $\pi|_{M_i}$ for each i is smaller than n in a neighborhood of $(x, 0)$ and hence on M_i. This implies that $\pi(\Theta)$ is meager in $\mathbf{R}^n$. As the set in question is contained in $\pi(\Theta) \cup A$, Lemma 3.2 follows.

Lemma 3.2'. *Let A be the same as Lemma 3.2. Let B be the subset of the projective space $P^{n-1}(\mathbf{R})$ consisting of lines L such that $a + L = \{a + x \mid x \in L\}$ are singular for A at a for some $a \in A$. Then B is meager in $P^{n-1}(\mathbf{R})$.*

Proof. We reduce in the same way as the proof of Lemma 3.2 the problem to the case in which A is an analytic submanifold of $\mathbf{R}^n$ and hence in which A is the zero set of an analytic function on an open set. Then the lemma is a special case of Lemma 3, [10].

Lemma 3.3. *Let $A_i \subset \mathbf{R}^n$, $i = 1, 2, \cdots$ be subanalytic sets of dimension $< n$, and $a \in \mathbf{R}^n$ be a point. Then there exist a point $b \in \mathbf{R}^n$ arbitrarily close to a such that any line through b is non-singular simultaneously for all A_i.*

Proof. Trivial from Lemma 3.2.

Lemma 3.3'. *Let A_i be the same as Lemma 3.3, and let $L \in P^{n-1}(\mathbf{R})$. Then there exists $L' \in P^{n-1}(\mathbf{R})$ arbitrarily close to L such that for any $a \in \mathbf{R}^n$, $a + L'$ is non-singular simultaneously for all A_i.*

Proof. It follows from Lemma 3.2'.

Lemma 3.4. *Let $\{A_i\}$ be a finite family of compact subanalytic sets in $\mathbf{R}^n \times \mathbf{R}$. Assume that for any $x \in \mathbf{R}^n$, the line $x \times \mathbf{R}$ is non-singular for $\cup A_i$. Then there exists a subanalytic stratification $\{X_i\}$ of $\cup A_i$ compatible with $\{A_i\}$ such that each X_i is topographic with respect to $\mathbf{R}$.*

Proof. Let r be the dimension of $A = \cup A_i$. We prove the lemma by induction on r. The case $r = 0$ is trivial. Assume the lemma for dimension $< r$. Let $p : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ be the projection. By the non-singularity assumption on A_i we have $\dim p(A) = r$. Apply (2.7) to $\{p(A)\}$ and (2.11) to $\{A_i\}$. Then there exist a subanalytic stratification $\{B_i\}_{i=1,\cdots,s}$ of A compatible with $\{A_i\}$ and a subanalytic partition C_1, C_2 of $p(A)$ such that the dimensions of B_i and C_j are equal to r for $i = 1, \cdots, s'$, $j = 1$ and smaller than r for $i = s'+1, \cdots, s$, $j = 2$ and that C_1 is an analytic manifold. Now apply (2.14) to p and each B_i, $i = 1, \cdots, s'$. Then we have subanalytic subsets $D_i \subset B_i$ of dimension $< r$ closed in B_i such that $d(p|_{B_i})$ has rank r on $B_i - D_i$ by the non-singularity assumption.
    Put

$$M = C_1 - p\left(\bigcup_{i=1}^{s'} D_i \cup \bigcup_{i=s'+1}^{s} B_i\right)$$

$$N = p^{-1}(M) \cap A\,.$$

Then (i) M, N are r-dimensional subanalytic analytic manifolds by (2.5), (ii) $p(N) = M$, (iii) $p|_N$ is an immersion and (iv) $\dim(A-N) = \dim(p(A)-M) < r$. Let $\{X_i\}_{i=1,\cdots,k}$ be the connected components of N. Then each $p(X_i)$ is an analytic manifold, moreover it follows from the compactness of A that $(X_i, p|_{X_i}, p(X_i))$ is an analytic covering. The covering is 1-fold. Indeed, if not so, we have a subset $X_i' \subset X_i$ homeomorphic to S' such that $(X_i', p|_{X_i'}, p(X_i'))$ is a non-onefold covering. We easily see that to be impossible by the inclusion $X_i' \subset p(X_i') \times \boldsymbol{R}$. Thus X_i are topographic with respect to $\boldsymbol{R}$. We remark that $\{X_i\}$ is compatible with $\{B_i\}$ and hence with $\{A_i\}$.

If a connected subset Y of $A-N$ is compatible with $\{A_i'\} = \{\bar{D}_i, \bar{B}_j | i = 1, \cdots, s', j = s'+1, \cdots, s\}$, then we see easily $Y \subset B_j$ for some j, hence Y is compatible with $\{A_i\}$. Hence for the proof we only need a subanalytic stratification $\{X_i\}_{i=k+1,\cdots,k'}$ of $A-N$ compatible with $\{A_i'\} \cup \{A-N\}$ such that each X_i is topographic with respect to $\boldsymbol{R}$. Since $A-N$ is compact and of dimension $<r$, and since $A_i' \subset A-N$, it follows from the induction hypothesis, hence the lemma is proved.

Lemma 3.5. *Let $A \subset \boldsymbol{R}^n \times \boldsymbol{R}$ be a compact subanalytic set. Assume that $0 \in A$ and that for any $x \in \boldsymbol{R}^n$, the line $x \times \boldsymbol{R}$ is non-singular for A. Then there exists a compact subanalytic set $B \subset \boldsymbol{R}^n \times \boldsymbol{R}$ such that $A \subset B$, that $x \times \boldsymbol{R}$ for any $x \in \boldsymbol{R}^n$ is non-singular for B and that $B \cap U$ has property (P) with respect to $\boldsymbol{R}$ for some open subset U of $\boldsymbol{R}^n \times \boldsymbol{R}$ containing 0.*

Proof. We prove the lemma by induction on n. If $n=0$, the lemma is trivial. Hence we assume it for $n-1$. Let $p: \boldsymbol{R}^n \times \boldsymbol{R} \to \boldsymbol{R}^n$ be the projection. By Lemma 3.4, there exists a subanalytic partition A_1, A_2 of A such that A_1 (may be empty) is an analytic manifold of dimension n, that A_2 is compact and of dimension $<n$ and that $p|_{A_1}$ is open. We clearly have $A_2 \neq \varnothing$. We assume moreover $0 \in A_2$, otherwise we have nothing to do. Since $\dim p(A_2) < n$, there exists by Lemma 3.3' a line $L \in P^{n-1}(\boldsymbol{R})$ such that for any $a \in \boldsymbol{R}^n$, $a+L$ is non-singular for $p(A_2)$. Hence, without loss of generality we can assume that for any $x' \in \boldsymbol{R}^{n-1}$ the line $x' \times \boldsymbol{R} \subset \boldsymbol{R}^{n-1} \times \boldsymbol{R} = \boldsymbol{R}^n$ is non-singular for $p(A_2)$. This means that for any $x' \in \boldsymbol{R}^{n-1}$, $x' \times \boldsymbol{R} \times \boldsymbol{R} \cap A_2$ is of dimension 0 and hence consists of finite points.

Let $p_2: \boldsymbol{R}^n \times \boldsymbol{R} \to \boldsymbol{R}^{n-1} \times \boldsymbol{R}$ be the projection defined by

$$p_2(x_1, \cdots, x_n, y) = (x_1, \cdots, x_{n-1}, y)\,.$$

Then it follows that $p_2(A_2)$ is a compact subanalytic set containing 0 and that for any $x' \in \boldsymbol{R}^{n-1}$, the line $x' \times \boldsymbol{R}$ is non-singular for $p_2(A_2)$. So by induction

hypothesis we have a compact subanalytic set $B_2 \subset R^{n-1} \times R$ such that $p_2(A_2) \subset B_2$, that $x' \times R$ for any $x' \in R^{n-1}$ is non-singular for B_2 and that B has property (P) with respect to R in a neighborhood of 0. Hence $B' = p_2^{-1}(B_2) \cap$ (a large closed ball centered at 0) is a compact subanalytic set in $R^n \times R$ such that $A_2 \subset B'$, that $x \times R$ for any $x \in R^n$ is non-singular for B' and that B' has property (P) with respect to R in a neighborhood of 0. Therefore $B = A_1 \cup B' = A \cup B'$ satisfies the required properties in Lemma 3.5.

Using q'_a, we immediately obtain from Lemmas 3.4, 5 the following.

Lemma 3.4'. *Let $\{A_i\}$ be a finite family of compact subanalytic subsets of R^n, and $a \in R^n$ be a point outside of $A = \cup A_i$. Assume that any line through a is non-singular for A. Then there exists a subanalytic stratification $\{X_j\}$ of A compatible with $\{A_i\}$ such that each X_j is topographic with respect to (R^n, a).*

Lemma 3.5'. *Let $A \subset R^n$ be a compact subanalytic set, and let $a \in R^n - A$, $b \in A$. Assume that any line through a is non-singular for A. Then there exists a compact subanalytic set $B \subset R^n$ such that $A \subset B$ and $a \notin B$, that any line through a is non-singular for B and that $B \cap U$ has property (P) with respect to (R^n, a) for some open neighborhood U of b.*

Lemma 3.6. *Under the same assumption as Lemma 3.5', there exist a compact subanalytic set $B \subset R^n$ and a subanalytic stratification $\{B_i\}$ of B such that $A \cap U$ is a union of some B_i's for a neighborhood U of b, that each B_i is topographic with respect to (R^n, a) and that B has property (P) with respect to (R^n, a).*

Proof. Assume $a = (0, \cdots, 0)$, $b = (0, \cdots, 0, 1)$. Let $\delta_n, \delta > 0$ be small numbers. Put

$$h(x) = (x_1^2 + \cdots + x_{n-1}^2)^{1/2} - \delta_n x_n,$$

$$Q = \{x \in R^n \mid h(x) \leq 0, \ |1 - |x|| \leq \delta\}.$$

Then Q is a small neighborhood of b. It is easy to find an analytic diffeomorphism π of $R^n - 0$ such that

$$|\pi(x)| = |x|,$$

$$\pi(x) = x \quad \text{if} \quad h(x) = 0,$$

$$h(\pi(x)) > 0 \quad \text{if} \quad h(x) < 0,$$

$$h(\pi(x)) < 0 \quad \text{if} \quad h(x) > 0,$$

and that π carries each half-line with end point 0 to some such half-line.

Apply Lemma 3.5' to A, let A_1 be the intersection of Q and the resulting subanalytic set, and put

$$A_2 = A_1 \cap \partial Q, \qquad A_3 = \pi(A_1),$$

$$A_4 = A \cap Q, \qquad B = A_1 \cup A_3.$$

Then $A_4 \subset B$, and any line through 0 is non-singular for A_1, A_3 and hence for B. Choose δ_n, δ small enough. Then $B - A_2$ has property (P) with respect to $(\mathbf{R}^n, 0)$, and we have $0 \times \mathbf{R} \cap A_1 = b$. Furthermore, we lessening δ_n so that

$$A_2 \subset \{x \in \mathbf{R}^n \mid h(x) = 0\},$$

B comes to have property (P) with respect to $(\mathbf{R}^n, 0)$. Now apply Lemma 3.4′ to $\{A_i\}_{i=1,\dots,4}$. Then the resulting subanalytic stratification $\{B_i\}$ of B satisfies the required properties.

Let $\sigma \subset \mathbf{R}^n$ be an n-simplex, and $a \in \mathring{\sigma}$. We define $p_{\sigma a} : \mathbf{R}^n - a \to \partial\sigma$ by $p_{\sigma a}(x) =$ *the intersection of $\partial\sigma$ and the half-line through x with end point a.*

For cell complexes K_1, K_2, $K_1 \times K_2$ denotes the product cell complex $\{\sigma_1 \times \sigma_2 \mid \sigma_1 \in K_1, \sigma_2 \in K_2\}$. For a point a, a simplex σ and a complex K, we write as $a * \sigma$ the cone with vertex a and base σ, and as $a * K$ the complex generated by $a * \sigma'$, $\sigma' \in K$, if they are well-defined.

Let σ be a simplex, and $a \in \mathring{\sigma}$. For any homeomorphism ϕ of $\partial\sigma$, we define the conic extension homeomorphism $a * \phi$ of σ by

$$a * \phi(x) = \begin{cases} t(\phi \circ p_{\sigma a}(x) - a) + a & \text{for } x \neq a \\ x & \text{for } x = a \end{cases}$$

where

$$t(p_{\sigma a}(x) - a) = x - a.$$

Then we have

Lemma 3.7. *If ϕ is subanalytic, so is $a * \phi$.*

Proof. Let $\mathbf{R}^n$ be the affine space spanned by σ. Assume $a = 0$. By (2.4) there exist a compact analytic manifold M and an analytic map $f : M \to \mathbf{R}^n \times \mathbf{R}^n$ such that $f(M) = \text{graph } \phi$. Let $F : M \times [0, 1] \to \mathbf{R}^n \times \mathbf{R}^n$ be defined by $F(x, t) = tf(x)$. Then $F(M \times [0, 1]) = \text{graph}(0 * \phi)$, and it is a subanalytic set. Clearly $0 * \phi$ is continuous. Hence $a * \phi$ is subanalytic.

Proposition 3.1′. *Let K be a simplicial complex in $\mathbf{R}^n$. Assume $|K|$ to be closed in $\mathbf{R}^n$. Let $\{A_i\}$ be a locally finite family of subanalytic subsets of $\mathbf{R}^n$ contained in $|K|$. Then there exist a subdivision K' of K and a homeomorphism τ of $|K|$ such that the conditions (3.1.1), $\cdots$, (3.1.4) in Proposition 3.1 are satisfied.*

Proof of Proposition 3.1. Trivial if we assume Proposition 3.1′.

Proof of Proposition 3.1′. Let m be the dimension of K. We will prove the proposition by induction on m. The case $m = 0$ is trivial. Assume the proposition for dimension $\leq m - 1$. Since a simplex is a subanalytic set, we assume $\{A_i\} \supset K^{m-1}$, where K^{m-1} is the $(m-1)$-skeleton of K. Considering $A_i \cap \sigma$ for

each A_i and $\sigma \in K$, we may suppose that each A_i is contained in some $\sigma \in K$.

Moreover we can assume that A_i are closed. Indeed, if A_i is not closed, we replace A_i by a finite system of closed subanalytic sets as follows. Let $A_{i0} = A_i$, and $A_{ij+1} = \overline{A}_{ij} - A_{ij}$, $j = 0, 1, \cdots$. Then, by (2.2), $\dim A_{ij+1} < \dim A_{ij}$ so long as $A_{ij} \neq \varnothing$. Replace A_i by $\{\overline{A}_{ij}\}$.

We may assume also $\dim A_i < m$ for each i. In fact, if $\dim A_i = m$ and A_i is contained in $\sigma \in K$, then we consider $A_i - \mathrm{Int}(A_i)$ in place of A_i. Here $\mathrm{Int}(A_i)$ means the interior of A_i in the affine space spanned by σ. Then a partition of $|K|$ compatible with the new family is automatically compatible with $\{A_i\}$.

(I) Let σ be an m-simplex of K, and Σ denote the affine space spanned by σ. We regard Σ as a Euclidean space too. Then there exist a point a_σ of σ and a finite family C_σ of subanalytic analytic submanifolds of Σ such that

(I.1)   each $\Gamma \in C_\sigma$ is contained in σ, topographic with respect to (Σ, a_σ),

(I.2)   the union S_σ of all $\Gamma \in C_\sigma$ is closed and has property (P) with respect to (Σ, a_σ), and that

(I.3)   any connected subset of σ compatible with C_σ is also compatible with $\{A_i\}$.

Before proving this, we remark that (I.3) is equivalent to the following. Let Λ be the index set of C_σ. Let C_σ' be the family of all connected components of sets in the form $\bigcap_{i \in \Lambda_0} \Gamma_i \cap \bigcap_{i \notin \Lambda_0} \Gamma_i^c$, Λ_0 being subsets of Λ and Γ_i^c denoting the complement in σ. Then C_σ' is compatible with $\{A_i\}$.

Another remark is that if (I.3) is satisfied for any m-simplex of K, then the condition (3.1.4) can be replaced by

(3.1.4)′ $\{\tau(\dot\sigma) \mid \sigma \in K'\}$ is compatible with

$$\bigcup_{\sigma \in K - K^{m-1}} C_\sigma \cup \{A_i \mid A_i \subset |K^{m-1}|\} .$$

Proof of (I). Apply Lemma 3.3 to all A_i contained in σ. Then there exists a point $a_\sigma \in \mathring\sigma$ such that $a_\sigma \notin A_i$ for any i and that any line through a_σ is non-singular simultaneously for all $A_i \subset \sigma$. Now for each $A_i \subset \sigma$ and each $b \in A_i$, apply Lemma 3.6 to A_i, a_σ and b. Then we have a compact subanalytic set $B \subset \Sigma$ and a subanalytic stratification $\{B_j\}$ of B such that A_i is a union of some B_j's in a neighborhood U_{ib} of b, that each B_j is topographic with respect to (Σ, a_σ) and that B has property (P) with respect to (Σ, a_σ).

Since A_i is compact, we can choose finite b's so that U_{ib}'s cover A_i. Gather B_j for those b, and let the family be denoted by $C_{\sigma i}$. Let C_σ be the union of $C_{\sigma i}$ for all $A_i \subset \sigma$. Then C_σ clearly is finite and satisfies the properties (I.1, 2) except that $\Gamma \subset \sigma$. Moreover, if we replace $\{A_i\}$ in (I.3) by $\{A_i \mid A_i \subset \sigma\}$, then (I.3) is satisfied. If $\Gamma \not\subset \sigma$, we consider $\Gamma \cap \mathring\sigma$ in place of Γ. Then the inclusion $\Gamma \subset \sigma$ is satisfied and the other properties remain valid because of $\{A_i\} \supset K^{m-1}$.

In addition to that, (I.3) in its former condition follows from $\Gamma \subset \sigma$. Hence (I) is proved.

(II)   Put $\{B_i\} = \{p_{\sigma a_\sigma}(\Gamma \cap \Gamma') \mid \Gamma,\ \Gamma' \in C_\sigma,\ \sigma \in K - K^{m-1}\} \cup \{A_j \mid A_j \subset |K^{m-1}|\}$. Then $\{B_i\}$ is a locally finite family of subanalytic subsets of $|K^{m-1}|$. Hence there exist by induction hypothesis a subdivision $K^{m-1'}$ of K^{m-1} and a homeomorphism π of $|K^{m-1}|$ such that

(II.1) π is subanalytic,

(II.2) $\pi(\sigma) = \sigma$ for any $\sigma \in K^{m-1}$,

(II.3)   for any $\sigma \in K^{m-1'}$, $\pi(\mathring{\sigma})$ is an analytic submanifold of $\boldsymbol{R}^n$ and $\pi|_{\mathring{\sigma}} : \mathring{\sigma} \to \pi(\mathring{\sigma})$ is an analytic diffeomorphism and that

(II.4) $\{\pi(\mathring{\sigma}) \mid \sigma \in K^{m-1'}\}$ is compatible with $\{B_i\}$.

We remark that $\pi|_{\partial\sigma}$, for any $\sigma \in K$, is a homeomorphism of $\partial\sigma$ because of (II.2). Let $\tilde{\pi}$ be an extension of π to $|K|$ defined by

$$\tilde{\pi} = \begin{cases} a_\sigma * (\pi|_{\partial\sigma}) & \text{on} \quad \sigma \in K - K^{m-1} \\ \pi & \text{on} \quad |K^{m-1}|. \end{cases}$$

Then the following are trivial.

(II.5) $\tilde{\pi}$ is a subanalytic homeomorphism of $|K|$ by Lemma 3.7.

(II.6) $\tilde{\pi}(\sigma) = \sigma$ for any $\sigma \in K$.

(II.7)   Let K_1' be the subdivision of K

$$\bigcup_{\sigma \in K - K^{m-1}} a_\sigma * K_\sigma^{m-1'} \cup K^{m-1'}$$

where

$$K_\sigma^{m-1'} = \{\sigma' \in K^{m-1'} \mid \sigma' \subset \sigma\}.$$

Then for any $\sigma \in K_1'$, $\tilde{\pi}(\mathring{\sigma})$ is an analytic submanifold of $\boldsymbol{R}^n$ and $\tilde{\pi}|_{\mathring{\sigma}} : \mathring{\sigma} \to \pi(\mathring{\sigma})$ is an analytic diffeomorphism.

For each $\sigma \in K - K^{m-1}$, put

$$\sigma_{[b,c]} = \{t(x - a_\sigma) + a_\sigma \mid x \in \partial\sigma,\ b \leq t \leq c\}$$

$$\sigma_1 = \sigma_{[0,\varepsilon]},\ \sigma_2 = \sigma_{[\varepsilon,1]} \quad \text{for} \quad 0 < \varepsilon < 1,$$

and fix $\varepsilon > 0$ so small that $\sigma_1 \cap \Gamma = \varnothing$ for any $\Gamma \in C_\sigma$. After adding some sets to $\{A_i\}$ and C_σ, we assume that

(II.8) $\sigma_1 \cap \sigma_2$ is at once an element of $\{A_i\}$ and a union of elements of C_σ, and $\mathring{\sigma}_1$ does not intersect with any $\Gamma \in C_\sigma$.

(III)   Let σ be an m-simplex of K, and Σ denote the affine space spanned

by σ. Put

$$\mathcal{K}_\sigma = \{\pi(\mathring{\sigma}') \mid \sigma' \in K_\sigma^{m-1'}\}$$

$$\mathcal{L}_\sigma = \{\text{connected components of } \Gamma \cap p_{\sigma a_\sigma}^{-1}(\beta) \mid \Gamma \in \mathcal{C}_\sigma, \ \beta \in \mathcal{K}_\sigma\}.$$

For any $\beta \in \mathcal{K}_\sigma$, denote by $s(\beta)$ the image under π of the set of vertexes of σ' such that $\pi(\mathring{\sigma}') = \beta$. Then the following properties hold.

(III.1)　Each $\gamma \in \mathcal{L}_\sigma$ is a subanalytic analytic submanifold of Σ with $p_{\sigma a_\sigma}(\gamma) \in \mathcal{K}_\sigma$, topographic with respect to (Σ, a_σ).

(III.2)　$\mathcal{L}_\sigma$ is a finite family of disjoint sets.

(III.3)　The union of all $\gamma \in \mathcal{L}_\sigma$ coincides with S_σ, hence closed in Σ and has property (P).

(III.4)　Any connected subset of σ compatible with $\mathcal{L}_\sigma$ is also compatible with $\{A_i\}$.

(III.5)　For any $\gamma \in \mathcal{L}_\sigma$, the restriction of $p_{\sigma a_\sigma}$ to the closure $\bar\gamma$ is a homeomorphism onto $\overline{p_{\sigma a_\sigma}(\gamma)}$, $\bar\gamma$ is a union of elements of $\mathcal{L}_\sigma$, and we have $\bar\gamma \cap p_{\sigma a_\sigma}^{-1}(\beta) \in \mathcal{L}_\sigma$ for any $\beta \in \mathcal{K}_\sigma$ contained in $\overline{p_{\sigma a_\sigma}(\gamma)}$.

Proof. If γ is a connected component of $\Gamma \cap p_{\sigma a_\sigma}^{-1}(\beta)$ for $\Gamma \in \mathcal{C}_\sigma$, $\beta \in \mathcal{K}_\sigma$, then we have by (II.4) $\beta \subset p_{\sigma a_\sigma}(\Gamma)$ which, together with (I.1) and (II.1, 3), implies the first half of (III.1). As to (III.2), the finiteness of $\mathcal{L}_\sigma$ is trivial by (2.2). If γ_i, $i=1, 2$, are respective connected components of $\Gamma_i \cap p_{\sigma a_\sigma}^{-1}(\beta_i)$ with $\gamma_1 \cap \gamma_2 \neq \varnothing$, then $\beta_1 \cap \beta_2 \cap p_{\sigma a_\sigma}(\Gamma_1 \cap \Gamma_2) \neq \varnothing$, hence $\beta_1 = \beta_2 \subset p_{\sigma a_\sigma}(\Gamma_1 \cap \Gamma_2)$ by (II.4), which gives $\gamma_1 = \gamma_2$. The property (III.3) is trivial by (I.2), and (III.4) is an easy consequence of (I.3) since any $\Gamma \in \mathcal{C}_\sigma$ is a union of some elements of $\mathcal{L}_\sigma$.

　　To prove (III.5), let $\gamma \in \mathcal{L}_\sigma$, $\beta = p_{\sigma a_\sigma}(\gamma) \in \mathcal{K}_\sigma$. We have $p_{\sigma a_\sigma}(\bar\gamma) = \bar\beta$ because of $\gamma \subset \sigma_2$, and $\bar\gamma \subset S_\sigma$ by (III.3). For any $x \in \bar\beta - \beta$, the subanalytic set $\bar\gamma \cap p_{\sigma a_\sigma}^{-1}(x)$ has dimension 0 by (I.1) and hence is discrete. This easily implies that $p_{\sigma a_\sigma}|_{\bar\gamma}$: $\bar\gamma \to \bar\beta$ is a homeomorphism (the first part of III.5). Let $\beta_1 \in \mathcal{K}_\sigma$ with $\beta_1 \subset \bar\beta$. Then $\bar\gamma \cap p_{\sigma a_\sigma}^{-1}(\beta_1)$ is homeomorphic to β_1 under $p_{\sigma a_\sigma}$, and it is contained in $S_\sigma \cap p_{\sigma a_\sigma}^{-1}(\beta_1)$, which is a finite union of disjoint elements of $\mathcal{L}_\sigma$, each being homeomorphic to β_1 under $p_{\sigma a_\sigma}$. Hence $\bar\gamma \cap p_{\sigma a_\sigma}^{-1}(\beta_1)$ must be one of those elements. This establishes the second and third parts of (III.5).

　　Considering the barycentric subdivision $\mathrm{Sd}(K^{m-1'})$ of $K^{m-1'}$ in place of $K^{m-1'}$ if necessary, we can assume

(III.6)　Let $\gamma_1, \gamma_2 \in \mathcal{L}_\sigma$ with $p_{\sigma a_\sigma}(\gamma_1) = p_{\sigma a_\sigma}(\gamma_2) = \beta$. Assume $\gamma_1 < \gamma_2$ with respect to (Σ, a_σ). Then $\bar\gamma_1 \leqq \bar\gamma_2$ with respect to (Σ, a_σ) and

$$\bar\gamma_1 \cap p_{\sigma a_\sigma}^{-1}(s(\beta)) \not\equiv \bar\gamma_2 \cap p_{\sigma a_\sigma}^{-1}(s(\beta)).$$

　　The reason is the following. It is clear that $\bar\gamma_1 \leqq \bar\gamma_2$ with respect to (Σ, a_σ).

Let $\gamma_i \subset \Gamma_i$ with $\Gamma_i \in \mathcal{C}_\sigma$, $i=1, 2$, $\beta=\pi(\mathring{\tilde{\sigma}}')$ with $\sigma' \in \mathrm{Sd}(K_\sigma^{m-1'})$ and $\mathring{\tilde{\sigma}}' \subset \mathring{\tilde{\sigma}}'$ with $\mathring{\tilde{\sigma}}' \in K_\sigma^{m-1'}$. Let $\tilde{\gamma}_i$, $i=1, 2$, be the connected components of $\Gamma_i \cap p_{\sigma a_\sigma}^{-1}(\pi(\mathring{\tilde{\sigma}}'))$ containing γ_i. Then, using (II.3) we see in the same way as the above proof that $\tilde{\gamma}_i$ are diffeomorphic to $\pi(\mathring{\tilde{\sigma}}')$ under $p_{\sigma a_\sigma}$ and that $\tilde{\gamma}_1 < \tilde{\gamma}_2$. As $\pi(\mathring{\tilde{\sigma}}')$ contains a point of $s(\beta)$, that implies the required inequality.

Let $\gamma_1, \gamma_2 \in \mathcal{L}_\sigma$. We call γ_1, γ_2 a *consecutive couple of* $\mathcal{L}_\sigma$ if $p_{\sigma a_\sigma}(\gamma_1)=p_{\sigma a_\sigma}(\gamma_2)$, $\gamma_1 < \gamma_2$ with respect to (Σ, a_σ) and if (γ_1, γ_2) does not contain any element of $\mathcal{L}_\sigma$. Then we have

(III.7)   Let γ_1, γ_2 be a consecutive couple of $\mathcal{L}_\sigma$. For each $\beta \in \mathcal{K}_\sigma$ contained in $\overline{p_{\sigma a_\sigma}(\gamma_1)}$, $\tilde{\gamma}_1 \cap p_{\sigma a_\sigma}^{-1}(\beta)$ and $\tilde{\gamma}_2 \cap p_{\sigma a_\sigma}^{-1}(\beta)$ are identical or consecutive.

In fact $\gamma_i' = \tilde{\gamma}_i \cap p_{\sigma a_\sigma}^{-1}(\beta)$, $i=1, 2$, are elements of $\mathcal{L}_\sigma$ (III.5). Hence $\gamma_1' = \gamma_2'$ or $\gamma_1' < \gamma_2'$ with respect to (Σ, a_σ). Assume that an element of $\mathcal{L}_\sigma$ is contained in (γ_1', γ_2'). Then, since S_σ has property (P) (III.3), we have $S_\sigma \cap (\gamma_1', \gamma_2') \neq \varnothing$. As $S_\sigma \cap p_{\sigma a_\sigma}^{-1}(p_{\sigma a_\sigma}(\gamma_1))$ is a union of some elements of $\mathcal{L}_\sigma$, that implies $\gamma_3 \subset (\gamma_1', \gamma_2')$ for some $\gamma_3 \in \mathcal{L}_\sigma$, which is a contradiction.

Denote by $\mathcal{L}_\sigma^\#$ the family of all sets (γ_1, γ_2) where γ_1, γ_2 are consecutive couples of $\mathcal{L}_\sigma$. Put

$$\mathcal{L}_\sigma^* = \mathcal{L}_\sigma \cup \mathcal{L}_\sigma^\# .$$

Then we have

(III.8) $\mathcal{L}_\sigma^*$ is a finite family of disjoint subanalytic analytic submanifolds of Σ, whose union is equal to σ_2 (in II).

Indeed the first half follows from (III.1, 2), and the latter half does from (I.1), (II.8), (III.3) and the inclusion $\{A_i\} \supset K^{m-1}$.

By (III.4)

(III.9) $\mathcal{L}_\sigma^*$ is compatible with $\{A_i\}$.

Finally it follows from (III.2,5,7) that

(III.10)   For any $\gamma \in \mathcal{L}_\sigma^*$, $\bar{\gamma}$ is a union of some elements of $\mathcal{L}_\sigma^*$. If $\gamma=(\gamma_1, \gamma_2) \in \mathcal{L}_\sigma^\#$, any element of $\mathcal{L}_\sigma^*$ contained in $\bar{\gamma}$ is in the form $\bar{\gamma}_1 \cap p_{\sigma a_\sigma}^{-1}(\beta)$, $\bar{\gamma}_2 \cap p_{\sigma a_\sigma}^{-1}(\beta)$ or $(\bar{\gamma}_1 \cap p_{\sigma a_\sigma}^{-1}(\beta), \bar{\gamma}_2 \cap p_{\sigma a_\sigma}^{-1}(\beta))$ with $\beta \in \mathcal{K}_\sigma$, $\beta \subset \overline{p_{\sigma a_\sigma}(\gamma)}$.

(IV)   We use the same notations as (III). For any $\gamma \in \mathcal{L}_\sigma$, we define a simplex $\rho(\gamma) \subset \sigma$ as follows. Put $\beta=p_{\sigma a_\sigma}(\gamma)$. Consider $\tilde{\pi}^{-1}(\bar{\gamma} \cap p_{\sigma a_\sigma}^{-1}(s(\beta)))$. It consists of indepentdent (dim $\gamma+1$)-points by (III.5) (see [13] for the definition of independence), because the image of the set under $p_{\sigma a_\sigma}$ is $\pi^{-1}(s(\beta))$. Let $\rho(\gamma)$ be the simplex spanned by the set, and $\mathring{\rho}(\gamma)$ be the open simplex. Then we have

(IV.1) $p_{\sigma a_\sigma}(\mathring{\rho}(\gamma))=\pi^{-1}(p_{\sigma a_\gamma}(\gamma))$

since

$$\mathring{\psi}^{-1} \circ \tilde{\pi} \circ \psi(x, \, t) = (\pi(x), \, t) \qquad \text{for} \quad (x, \, t) \in \psi^{-1}(\gamma)$$

where the subanalytic homeomorphism $\psi: \partial\sigma \times (0, \, 1] \to \sigma - a_\sigma$ is defined by

$$\psi(x, \, t) = t(x - a_\sigma) + a_\sigma \, .$$

(IV.2) If $\gamma < \gamma'$ with respect to $(\Sigma, \, a_\sigma)$, then $\mathring{\rho}(\gamma) < \mathring{\rho}(\gamma')$ with respect to $(\Sigma, \, a_\sigma)$ by (III.6),

(IV.3) If $\gamma' \subset \bar{\gamma}$ then $\rho(\gamma')$ is a face of $\rho(\gamma)$ by (III.5), and

(IV.4) If $\gamma \in \mathcal{L}_\sigma$ is contained in $\partial\sigma$ or $\partial\sigma_1$, then $\mathring{\rho}(\gamma) = \tilde{\pi}^{-1}(\gamma)$ by III.8), where σ_1 is defined in (II).

Put

$$L_\sigma = \{\rho(\gamma) \mid \gamma \in \mathcal{L}_\sigma\} \, ,$$

$$L_{\partial\sigma_1} = \{\rho(\gamma) \mid \gamma \in \mathcal{L}_\sigma, \, \gamma \subset \partial\sigma_1\} \, ,$$

$$L_\sigma^\sharp = \{[\rho(\gamma), \, \rho(\gamma')] \mid \gamma, \, \gamma' : \text{consecutive couples of } \mathcal{L}_\sigma\} \, .$$

$$L_\sigma^* = L_\sigma \cup a_\sigma * L_{\partial\sigma_1} \cup L_\sigma^\sharp \, .$$

Then, by (III.8, 10) and (IV.1, 2, 3, 4)

(IV.5) L_σ^* is a finite cell complex whose underlying polyhedron is equal to σ.

Let τ_σ be a homeomorphism of σ defined so that

$$\tau_\sigma(\mathring{\rho}(\gamma)) = \gamma \qquad \text{for} \quad \gamma \in \mathcal{L}_\sigma \, ,$$

$$\tau_\sigma((\mathring{\rho}(\gamma_1), \, \mathring{\rho}(\gamma_2)) = (\gamma_1, \, \gamma_2) \qquad \text{for} \quad (\gamma_1, \, \gamma_2) \in \mathcal{L}_\sigma^\sharp \, ,$$

$$p_{\sigma a_\sigma} \circ \tau_\sigma = \pi \circ p_{\sigma a_\sigma} \qquad \text{on} \quad \sigma_2 \, ,$$

$$\tau_\sigma = \tilde{\pi} \qquad \text{on} \quad \sigma_1 \, ,$$

and that for any $(\gamma_1, \, \gamma_2) \in \mathcal{L}_\sigma^\sharp$ and any $x \in \partial\sigma$, τ_σ is linear on the segment $(\mathring{\rho}(\gamma_1),$ $\mathring{\rho}(\gamma_2)) \cap p_{\sigma a_\sigma}^{-1}(x)$. Then τ_σ is uniquely well-defined and equal to π on $\partial\sigma$ by the definition of $\tilde{\pi}$, (III.8, 10) and (IV.4).

Hence we have a homeomorphism τ of $|K|$ whose restriction to each σ coincides with τ_σ. Letting K' be a simplicial subdivision of $\bigcup_{\sigma \in K - K^{n-1}} L_\sigma^* \cup K^{m-1}$, we want to see that τ, K' satisfy (3.1.1, 2, 3, 4). First note that (3.1.1 is equivalent to

(IV.6) $\tau_\sigma \mid_{\sigma'}$ is subanalytic for each $\sigma' \in L_\sigma^*$.

Proof of (IV.6). If $\sigma' \subset \sigma_1$ it follows from (II.5) and from $\tau_\sigma = \tilde{\pi}$ on σ'. Assume $\sigma' = \rho(\gamma)$ with $\gamma \in \mathcal{L}_\sigma$. Then

$$\tau_\sigma \mid_{\sigma'} = (p_{\sigma a_\sigma} \mid_{\bar{\gamma}})^{-1} \circ \pi \circ p_{\sigma a_\sigma} \mid_{\sigma'} \, .$$

Hence (2.13.3) shows that $\tau_\sigma \mid_{\sigma'}$ is subanalytic. Let $\sigma' = [\rho(\gamma), \, \rho(\gamma')]$ with $\gamma, \, \gamma'$

$\in \mathcal{L}_\sigma$. Then, by the above statement and (2.4) we have compact analytic manifolds M, M' and analytic maps $f=(f_1, f_2): M \to \sigma \times \sigma$, $f'=(f_1', f_2'): M' \to \sigma \times \sigma$ such that $f(M)=\text{graph } \tau_\sigma |_{\rho(i)}$, $f(M')=\text{graph } \tau_\sigma |_{\rho(i')}$. Define an analytic map $F: M \times M' \times [0, 1] \to \sigma \times \sigma$ by

$$F(x, x', t)=tf(x)+(1-t)f'(x')$$

$$=(tf_1(x)+(1-t)f_1'(x'), \ tf_2(x)+(1-t)f_2'(x')).$$

Put

$$M''=\{(x, x')\in M \times M' \mid p_{\sigma a_\sigma} \circ f_1(x)=p_{\sigma a_\sigma} \circ f_1'(x')\}.$$

Then M'' is a compact analytic set, and we have

$$F(M'' \times [0, 1])=\text{graph } \tau_\sigma |_{\sigma'}.$$

Hence, by (2.3) and (2.5.2), graph $\tau_\sigma |_{\sigma'}$ is subanalytic which proves (IV.6).

(3.1.2, 3) are trivial, and (3.1.4) follows from (II.8) and (III.9). Hence we complete the proof of Proposition 3.1'.

Remark 3.8. We can refine Propositions 3.1, 1' as follows. Let L be a subcomplex of K. Assume $\{\mathring{\sigma} \mid \sigma \in K-L\}$ is compatible with $\{A_i\}$. Then τ can satisfy moreover, for any $\sigma \in K$ with $\sigma \cap |L|=\varnothing$,

(3.1.5) $\tau=\text{ident}$ on σ, and

(3.1.6) $\sigma \in K'$.

It is clear by the method of construction of τ in the proof of Proposition 3.1'.

The condition (3.1.5) will be important when we will enlarge the domain where f is piecewise linearized in the proof of Theorem I, §7. This is one of the reasons why we refined a result of [4].

We will show a semi-algebraic version of Propositions 3.1, 1'. A subset of R^n is called *semi-algebraic* if it is a finite union of sets in the form

$$\{x \in R^n \mid f_1(x)>0, \cdots, f_k(x)>0, f_{k+1}(x)=0, \cdots, f_l(x)=0\}$$

where f_i are polynomials on R^n. A continuous map between semi-algebraic sets is called *semi-algebraic* if the graph is semi-algebraic. We remark that the semi-algebraic versions of the results in Section 2 hold true, in which the locally finite condition must be replaced by the finite one (see [1], [4], [10]). Since the stereographic projection $R^n \to S^n$ is semi-algebraic, in most problems of semi-algebraic sets we can restrict the sets to being bounded.

Proposition 3.9. *Let K and L be a finite simplicial complex in R^n and a subcomplex respectively. Let $\{A_i\}$ be a finite family of bounded semi-algebraic subsets of R^n contained in $|K|$ such that $\{\mathring{\sigma} \mid \sigma \in K-L\}$ is compatible with $\{A_i\}$. Then there exist a subdivision K' of K and a homeomorphism τ of R^n such that*

(3.9.1) τ is semi-algebraic,

(3.9.2) $\tau(\sigma)=\sigma$ for any $\sigma \in K$,

(3.9.3) for any $\sigma \in K'$, $\tau(\mathring{\sigma})$ is an analytic submanifold of $\mathbf{R}^n$ and $\tau\vert_{\mathring{\sigma}}: \mathring{\sigma} \to \tau(\mathring{\sigma})$ is an analytic diffeomorphism,

(3.9.4) $\{\tau(\mathring{\sigma}) \vert \sigma \in K'\}$ is compatible with $\{A_i\}$,

(3.9.5) $\tau = ident$ on any $\sigma \in K$ with $\sigma \cap \vert L \vert = \emptyset$ and that

(3.9.6) any $\sigma \in K$, with $\sigma \cap \vert L \vert = \emptyset$, is a simplex of K'.

This is a refinement of a theorem of [4] and Theorem 3. [10]. As we can prove it in the same way as Proposition 3.1', we omit the proof.

Remark 3.10. In propositions 3.1 and 3.1' or 3.9, τ is subanalytically or semi-algebraically isotopic to the identity respectively. Namely there exists an isotopy $\tau_t: \vert K \vert \to \vert K \vert$, $0 \leq t \leq 1$, such that $\tau_0 = \tau$, $\tau_1 = ident$ and that the map $T: \vert K \vert \times I \to \vert K \vert \times I$, $I = [0, 1]$, defined by $T(x, t) = (\tau_t(x), t)$ is subanalytic or semi-algebraic respectively. Moreover we can choose the isotopy so that for any $\sigma \in K'$, the restriction of T^{-1} to $\tau(\mathring{\sigma}) \times I$ is an analytic diffeomorphism. This is also clear by the method of construction of τ.

§4. Whitney Stratifications and Vector Fields

In this section, we prepare for the proof of Proposition 5.1. All results are derived from the Thom-Mather theory of Whitney stratification. The result which will be applied is only Lemma 4.14, and the others need for its proof. Manifolds, vector bundles and maps are of class C^∞, and we do not necessarily assume the frontier condition (2.9.2), the connectedness of strata, nor the boundedness for the definition of stratification. Let us recall some definitions (see [1]).

Definition 4.1. Let $X \subset \mathbf{R}^n$ be a submanifold. A *tube at* X is a quadruple $T = (E, \pi, \rho, e)$ where $\pi: E \to X$ is a vector bundle of dimension = codim X, $\rho: E \to \mathbf{R}$ is the quadratic function of a Riemannian metric on E, and $e: E_0 \to \mathbf{R}^n$ is an imbedding, commuting with the zero section $\zeta: X \to E$ so that $e \circ \zeta$ is the inclusion $X \subset \mathbf{R}^n$, E_0 being an open neighborhood of $\zeta(X)$. We set $\vert T \vert = e(E_0)$. By identifying E_0 with $\vert T \vert$, we use π, ρ also as the map $\pi \circ e^{-1}$ and the function $\rho \circ e^{-1}$ on $\vert T \vert$ respectively. For any open subset X' of X, $T_{X'}$ means $(\pi^{-1}(X'), \pi\vert_{\pi^{-1}(X')}, \rho\vert_{\pi^{-1}(X')}, e\vert_{\pi^{-1}(X')})$, and for two tubes at X, $T = (E, \pi, \rho, e)$ and $T' = (E', \pi', \rho', e')$, we write as $T \equiv T'$ if we have a neighborhood U of X in $\mathbf{R}^n$ such that $U \subset \vert T \vert \cap \vert T' \vert$, $\pi\vert_U = \pi'\vert_U$ and $\rho\vert_U = \rho'\vert_U$.

Definition 4.2. Let $\{X_i\}$ be a Whitney stratification of a subset X of $\mathbf{R}^n$. A *tube system for* $\{X_i\}$ consists of one tube $T_i = (E_i, \pi_i, \rho_i, e_i)$ at each X_i. The tube system is called *weakly controlled* if

(4.2.1) $\pi_i \circ \pi_j(x) = \pi_i(x)$, $x \in |T_i| \cap |T_j| \cap \pi_j^{-1}|T_i|$.

We call it *controlled* if furthermore

(4.2.2) $\rho_i \circ \pi_j(x) = \rho_i(x)$, $x \in |T_i| \cap |T_j| \cap \pi_j^{-1}|T_i|$.

Definition 4.3. Let $\{X_i\}$, $\{Y_j\}$ be Whitney stratifications of subsets X, Y of $\mathbf{R}^n$, $\mathbf{R}^m$ respectively. We call a C^∞ map f from $\mathbf{R}^n$ to $\mathbf{R}^m$ a *Thom map with respect to* $\{X_i\}$. $\{Y_j\}$ when for any X_i, $X_{i'}$ the following are satisfied.

(4.3.1) f maps X_i, $X_{i'}$ submersively into some strata (not necessarily onto).

(4.3.2) Let $\{x_k\}$ be a sequence of points in X_i converging to $x \in X_{i'}$ such that $\{\ker d(f|_{X_i})_{x_k}\}_k$ converges, in the appropriate Grassmann bundle, to a plane $L \subset \mathbf{R}^n$. Then $\ker d(f|_{X_{i'}})_x \subset L$.

Definition 4.4. Let $\{X_i\}$, $\{Y_j\}$ be Whitney stratifications of subsets X, Y of $\mathbf{R}^n$, $\mathbf{R}^m$ respectively, and $f: \mathbf{R}^n \to \mathbf{R}^m$ be a Thom map with respect to $\{X_i\}$, $\{Y_j\}$. A tube system $\{T_i\}$ for $\{X_i\}$ is *controlled over* a tube system $\{R_j\}$ for $\{Y_j\}$ if $\{T_i\}$ satisfies (4.2.1) and (4.4.1) below and if (4.2.2) holds for any strata X_i, $X_{i'}$ with $f(X_i) \cup f(X_{i'}) \subset Y_j$ for some j.

(4.4.1) For any X_i, Y_j with $f(X_i) \subset Y_j$

$$f \circ \pi_i(x) = \pi_j \circ f(x), \qquad x \in |T_i| \cap f^{-1}|R_j|.$$

Definition 4.5. Let $M, N \subset \mathbf{R}^n$ be C^∞ manifolds with $M \cap N = \varnothing$. We say that (M, N) satisfies the *Whitney condition* (a) if the following is satisfied.

(4.5.1) Let $\{x_i\}$ be a sequence of points in M converging to $x \in N$. If the sequence of tangent spaces $\{T_{x_i}M\}$ converges to a subspace $L \subset \mathbf{R}^n$, then $L \supset T_x N$.

Remark 4.6. If (M, N) satisfies the Whitney condition, then it does the Whitney condition (a) (see [11]).

Lemma 4.7. *Let $\{X_i\}$, $\{Y_j\}$ be Whitney stratifications of subsets X, Y of $\mathbf{R}^n$, $\mathbf{R}^m$ respectively, and $f: \mathbf{R}^n \to \mathbf{R}^m$ be a Thom map with respect to $\{X_i\}$, $\{Y_j\}$. Then for any weakly controlled tube system $\{R_j\}$ for $\{Y_j\}$ there exists a tube system $\{T_i\}$ for $\{X_i\}$ which is controlled over $\{R_j\}$. Moreover, if $\{T_i^0\}$ is a tube system for $\{X_i \cap U\}$ controlled over $\{R_j\}$, U being an open set in $\mathbf{R}^n$, then $\{T_i\}$ can be chosen so that $T_i|_{U' \cap X_i} \equiv T_i^0|_{U' \cap X_i}$ for any given open set $U' \subset \mathbf{R}^n$ with $\bar{U}' - U \subset \bar{X} - X$.*

Proof. The first half coincides with Theorem 2.6, Chapter II, [1], and the latter half is clear from the construction of $\{T_i\}$ in its proof. So we omit the proof.

Corollary 4.8. *Let $\{X_i\}_{i=1,2,\ldots}$ be a Whitney stratification of a subset X of $\mathbf{R}^n$. Let V be an open neighborhood of X_1 in $\mathbf{R}^n$, and $\pi: V \to X_1$ be a submersion*

with $\pi|_{X_1}=ident$. Then there exist an open neighborhood $V'\subset V$ of X_1 and a controlled tube system $\{T_i=(E_i,\,\pi_i,\,\rho_i,\,e_i)\}$ for $\{X_i'=X_1\cap V'\}$ such that $|T_1|\subset V$ and $\pi_1=\pi$ on $|T_1|$. Moreover if $\{T_i^0=(E_i^0,\,\pi_i^0,\,\rho_i^0,\,e_i^0)\}$ is a controlled tube system for $\{X_i\cap U\}$ such that $\pi_1^0=\pi$ on $|T_1^0|\cap V$, U being an open set in $\mathbf{R}^n$, then $\{T_i\}$ can be chosen so that $T_i|_{V'\cap U'\cap X_i}\equiv T_i^0|_{V'\cap U'\cap X_i}$ for any given open set U' with $\overline{V\cap U'}-U\subset\overline{X}_1-X_1$.

Proof. As $(\mathbf{R}^n-\overline{X},\,X_i)$ satisfies the Whitney condition for any i, we can add $\mathbf{R}^n-\overline{X}$ to $\{X_i\}$. Hence we may assume $\overline{X}=\mathbf{R}^n$. By Remark 4.6 there exists an open neighborhood $V'\subset V$ of X_1 such that for any $x\in X_1$,

(4.8.1) $\pi^{-1}(x)$ and X_i' are transversal, $i=1,\,2,\,\cdots$.

Put $\{Y_j\}=\{Y\}=\{X_1\}$, $\mathbf{R}^m=\mathbf{R}^n$, and let f be any C^∞ extension of $\pi:V'\to Y$ to $\mathbf{R}^n\to\mathbf{R}^n$. Then if tube systems $\{T_i=(E_i,\,\pi_i,\,\rho_i,\,e_i)\}$ for $\{X_i'\}$ and $\{R\}$ for $\{Y\}$ satisfy (4.4.1), then we have $\pi_1=\pi$ on $|T_1|$ because of $\overline{X}=\mathbf{R}^n$, and if $\{T_i\}$ is controlled over $\{R\}$, $\{T_i\}$ itself is controlled. Hence Corollary 4.8 follows from Lemma 4.7 if we see that f is a Thom map with respect to $\{X_i'\}$ and $\{Y\}$.

Let X_i, $X_{i'}$ be any strata. It is clear by (4.8.1) that $\pi|_{X_i'}$, $\pi|_{X_{i'}'}$ are submersions into Y (4.3.1). Let $\{x_j\}$ be a sequence of points in X_i' converging to $x\in X_{i'}'$ such that $\{\ker d(\pi|_{X_i'})_{x_j}\}_j$ converges to a plane $L\subset\mathbf{R}^n$. We want to see $\ker d(\pi|_{X_{i'}'})_x\subset L$. This follows from the following facts.

(4.8.2) We can assume that $\{T_{x_j}X_i'\}_j$ converges to a plane in $\mathbf{R}^n$.

(4.8.3) We have $\ker d(\pi|_{X_i'})_{x_j}=T_{x_j}X_i'\cap T_{x_j}\pi^{-1}(\pi(x_j))$.

(4.8.4) Let $\{K_j\}\{L_j\}$ be sequences of linear subspaces of $\mathbf{R}^n$ converging to K, L respectively such that K, L are transversal. Then, for sufficiently large j, K_j and L_j are transversal, and $\{K_j\cap L_j\}$ converges to $K\cap L$.

Thus we saw that f is a Thom map with respect to $\{X_i'\}$ and $\{Y\}$, which proves the lemma.

We regard $\mathbf{R}^l$, $l=1,\,\cdots,\,n$, as linear subspaces

$$\{(x_1,\,\cdots,\,x_n)\in\mathbf{R}^n\,|\,x_{l+1}=\cdots=x_n=0\}$$

of $\mathbf{R}^n$, and we put

$$x+\mathbf{R}^l=\{x+y\,|\,y\in\mathbf{R}^l\}\qquad\text{for}\quad x\in\mathbf{R}^n.$$

Lemma 4.9. *Let $\{X_i\}$ be a Whitney stratification of a subset X of $\mathbf{R}^n$, and U_i be an open neighborhood of each X_i in $\mathbf{R}^n$. Assume that for each X_i and $x\in\mathbf{R}^n$, $x+\mathbf{R}^l$ and X_i are transversal where $l=\text{codim }X_i$. Then there exists a controlled tube system $\{T_i\}$ for $\{X_i\}$ such that for each i*

(4.9.1) $|T_i|\subset U_i$, *and*

(4.9.2) $\pi_i^{-1}(x) \subset x + \mathbf{R}^{n-m}$ for $x \in X_i \cap (\bigcup_{j \in I_m} U_j - \bigcup_{j \in I_{m-1}} U_j)$,

$$m = 0, \cdots, \dim X_i,$$

where I_j, $j = 0, \cdots, n$, are the index subsets consisting of i with $\dim X_i \leq j$, and $I_{-1} = \varnothing$.

Proof. Considering $\{\bigcup_{\dim X_i = j} X_i\}_{j=0, 1, \cdots}$ in place of $\{X_i\}$, we can assume $\{X_i\} = \{X_i\}_{i=0, \cdots}$ and $\dim X_i = i$. Put $Y_j = \bigcup_{i=0}^{j} X_i$. We prove the lemma inductively as follows; assume that we have already a controlled tube system $\{T_i' = (E_i', \pi_i', \rho_i', e_i')\}$ for $\{X_i \cap V_{j-1}\}$ which satisfy (4.9.1, 2) where V_{j-1} is an open neighborhood of Y_{j-1} contained in $\bigcup_{k=0}^{j-1} U_k$ (the case $j=0$ is trivial). Here we remark that U_j, $U_{j+1}, \cdots$ are not necessary when we see (4.9.2). Later we will replace $U_1, \cdots,$ U_{j-1}, V_{j-1} by smaller ones. Hence the induction hypothesis is settled for any $U_1, \cdots, U_{j-1}$. After defining V_j, we want to extend $\{T_i'\}$ to a controlled tube system for $\{X_i \cap V_j\}$.

Let $T_{j1} = (E_{j1}, \pi_{j1}, \rho_{j1}, e_{j1})$ be a tube at X_j such that

$$\pi_{j1}^{-1}(x) \subset x + \mathbf{R}^{n-j}   \text{for}   x \in X_j.$$

The existence of T_{j1} is clear by the definition of tube. We choose $|T_{j1}|$ so small that

$$\pi_{j1}^{-1}(X_j \cap V_{j-1}') \subset |T_j'| \cap V_{j-1}$$

for some open neighborhood V_{j-1}' of Y_{j-1} with $\overline{V}_{j-1}' - V_{j-1} \subset \overline{Y}_{j-1} - Y_{j-1}$. Let ϕ be a C^∞ function on X_j such that $\phi = 1$ on $X_j - V_{j-1}'$ and $= 0$ on $X_j \cap V_{j-1}''$ for an open neighborhood V_{j-1}'' of Y_{j-1} with $\overline{V}_{j-1}'' - V_{j-1}' \subset \overline{Y}_{j-1} - Y_{j-1}$. Put

$$\pi_{j2}(x) = \begin{cases} \pi_{j1}((1 - \phi \circ \pi_{j1}(x))\pi_j'(x) + \phi \circ \pi_{j1}(x)\pi_{j1}(x)) & \\ & \text{for}   x \in \pi_{j1}^{-1}(X_j \cap V_{j-1}') \\ \pi_{j1}(x) & \text{for}   x \in \pi_{j1}^{-1}(X_j - V_{j-1}'). \end{cases}$$

As $\pi_{j2}|_{X_j} = \text{ident}$, this is well-defined in a neighborhood of X_j, and we find an open neighborhood U_j' such that $U_j' \subset U_j \cap |T_{j1}|$, $\overline{U}_j' \cap \overline{Y}_{j-1} \subset \overline{X}_j - X_j$, and $\pi_{j2} \colon U_j' \to X_j$ is a C^∞ fiber bundle. Let W_i, $i = 0, \cdots, j-1$, be open neighborhoods of X_i such that $\overline{U}_i - W_i \subset \overline{X}_i - X_i$. Then by the definition of π_{j2} we can choose U_j' so small that

(4.9.3) $\pi_{j2}^{-1}(x) \subset x + \mathbf{R}^{n-m}$ for $x \in X_j \cap (W_m - W_0 - \cdots - W_{m-1})$,

$$m = 0, \cdots, j-1.$$

Now replace W_i by U_i, $i = 0, \cdots, j-1$, and U_i by some smaller open neighborhoods

U_i' of X_i such that $\overline{U}_i'-U_i\subset\overline{X}_i-X_i$, and regard the induction hypothesis as considered for U_i'. Then we can assume by (4.9.3)

$$\pi_{j2}^{-1}(x)\subset x+\boldsymbol{R}^{n-m}\qquad\text{for}\quad x\in X_j\cap(U_m-U_0-\cdots-U_{m-1}),\ m=0,\cdots,j-1.$$

It is clear that $\pi_{j2}=\pi_j'$ on $\pi_{j1}^{-1}(X_j\cap V_{j-1}'')$. Therefore, considering $\{T_i'|_{X_i\cap V_{j-1}''}\}$ in place of $\{T_i'\}$, and shrinking $|T_i'|$, $i=0,\cdots,j$, we may suppose

$$|T_j'|=\pi_{j2}^{-1}(X_j\cap V_{j-1}''),$$

$$\pi_{j2}=\pi_j'\quad\text{on}\quad|T_j'|\quad\text{and}$$

$$|T_i'|\subset V_{j-1}'',\ i=0,\cdots,j-1.$$

Then $\{T_i'\}$ continues to be a controlled tube system for $\{X_i\cap V_{j-1}''\}$ which satisfies (4.9.1, 2).

Now apply Corollary 4.8 to $\{X_i\cap U_j'\}_{i=j,j+1,\cdots}$, $\pi=\pi_{j2}\colon U_j'\to X_j$, $U=V_{j-1}''$ and a tube system $\{T_i^0\}=\{T_i'|_{X_i\cap U_j'\cap V_{j-1}''}\}_{i=j,\cdots}$ for $\{X_i\cap U_j'\cap V_{j-1}''\}_{i=j,\cdots}$. Then we have a controlled tube system $\{T_{i3}=(E_{i3},\pi_{i3},\rho_{i3},e_{i3})\}_{i=j,\cdots}$ for $\{X_i\cap U_j''\}_{i=j,\cdots}$, U_j'' being an open neighborhood of X_j contained in U_j' such that $\pi_{i3}=\pi_{j2}$ on $|T_{j3}|\subset U_j'$ and that

$$T_{i3}|_{X_i\cap U_j''\cap V_{j-1}'''}\equiv T_i'|_{X_i\cap U_j''\cap V_{j-1}'''},\ i=j,\ j+1,\ \cdots$$

where V_{j-1}''' is any given open neighborhood of Y_{j-1} so that $\overline{V}_{j-1}'''-V_{j-1}''\subset\overline{Y}_{j-1}-Y_{j-1}$. We remark that the last inclusion implies

$$\overline{U_j'\cap V_{j-1}'''}-V_{j-1}''\subset\overline{X}_j-X_j$$

because of $\overline{U}_j'\cap\overline{Y}_{j-1}\subset\overline{X}_j-X_j$. Shrink V_{j-1}''' and U_j'', and put $V_j=V_{j-1}'''\cup U_j''$. Then, since each T_i', $i=j,\ j+1,\ \cdots$, is defined at $X_i\cap V_{j-1}''$, T_{i3} can be easily extended to a tube at $X_i\cap V_j$. Here we can not necessarily extend so that $|\text{extension}|=|T_i'|\cup|T_{i3}|$. We use the same T_{i3} for the extension. Shrink $|T_i'|$, $i=0,\cdots,j-1$, $|T_{i3}|$, $i=j,\cdots$. Then it is easy to see that $\{T_i'\}_{i=0,\cdots,j-1}\cup\{T_{i3}\}_{i=j,\cdots}$ is a controlled tube system for $\{X_i\cap V_j\}$ which satisfies (4.9.1, 2), since $\pi_{j2}=\pi_{j3}$ satisfies (4.9.2). Hence Lemma 4.9 follows from the induction.

Definition 4.10. Let $\{X_i\}$ be a Whitney stratification of a subset X of $\boldsymbol{R}^n$, and $\{T_i\}$ be a controlled tube system for $\{X_i\}$. A *vector field* ξ on $\{(X_i,T_i)\}$ (or $\{X_i\}$) consists of one C^∞ vector field ξ_i on each X_i. We call ξ *controlled* if for any X_i, $X_{i'}$

(4.10.1) $d(\pi_i|_{X_{i'}})\xi_{i'x}=\xi_{i\pi_i(x)},\qquad x\in X_{i'}\cap U_i$

(4.10.2) $d(\rho_i|_{X_{i'}})\xi_{i'x}=0,\qquad x\in X_{i'}\cap U_i$

where $U_i\subset|T_i|$ is some neighborhood of X_i in $\boldsymbol{R}^n$. We call ξ *continuous* if for any sequence $\{x_j\}$ of points in $X_{i'}$ converging to x of X_i, $\{\xi_{i'x_j}\}$ converges to ξ_{ix} as vectors in $\boldsymbol{R}^n$.

Lemma 4.11. *Let $\{X_i\}$, $\{T_i\}$ be the same as the above definition. Assume the index set contains 1. Let ξ_1' be a C^∞ vector field on X_1. Then there exists a continuous controlled vector field $\xi = \{\xi_i\}$ on $\{(X_i \cap U, T_i|_{x_i \cap U})\}$, U being an open neighborhood of X_1 in $\mathbf{R}^n$, such that $\xi_1 = \xi_1'$.*

Proof. Since $\pi_1 \colon |T_1| \to X_1$, $T_1 = (E_1, \pi_1, \rho_1, e_1)$, is the restriction of a Thom map between $\mathbf{R}^n$ with respect to $\{X_i' = X_i \cap U\}$, $\{X_i\}$ for an open neighborhood U of X_1 (as shown in the proof of Corollary 4.8), the existence of ξ is a consequence of Theorem 3.2, Chapter II, [1], if we do not require ξ to be continuous. For the continuity we need a careful reconsideration of the proof as follows. Using a C^∞ partition of unity, we can reduce the problem to the case in which

$$\{X_i\} = \{X_1, X_2\}, \qquad \bar{X}_2 \cap X_1 \neq \varnothing, \qquad |T_1| = \mathbf{R}^n.$$

$$X_1 = \{(x_1, \cdots, x_n) \in \mathbf{R}^n \mid x_1 = \cdots = x_k = 0\} = 0 \times \mathbf{R}^{n-k},$$

and $\pi_1 \colon \mathbf{R}^n \to X_1$ is the projection, and we only have to define ξ_2 in a neighborhood of $0 \in \mathbf{R}^n$.

Moreover we can assume

(4.11.1) $$\rho_1(x) = x_1^2 + \cdots + x_k^2, \quad x = (x_1, \cdots, x_n).$$

Indeed, by the property of ρ_1, we can suppose

$$\rho_1(x_1, \cdots, x_k, 0, \cdots, 0) = x_1^2 + \cdots + x_k^2.$$

Then it follows that

$$\rho_1(x) = x_1^2 + \cdots + x_k^2 + \sum_{\substack{j l = 1, \cdots, k \\ i = 1, \cdots, n}} a_{ijl}(x) x_i x_j x_l$$

as germs at 0 for some C^∞ function germs a_{ijl}. Let us consider the ring of C^∞ function germs at 0 in $\mathbf{R}^n$, let $\mathfrak{m}$ be the maximal ideal, and let I be the ideal generated by $\dfrac{\partial \rho_1}{\partial x_i}$, $i = 1, \cdots, k$. Then it is easy to see that I is generated by $x_1, \cdots, x_k$. Hence $\rho_1(x) - x_1^2 - \cdots - x_k^2$ is an element of $\mathfrak{m}I^2$. Then, by a refinement of a result of Tougeron and Mather (Lemma 2, [15]), there exists a local diffeomorphism τ at 0 in $\mathbf{R}^n$ such that

$$\rho_1 \circ \tau(x) = x_1^2 + \cdots + x_k^2,$$

$$\tau(x) = (\tau_1(x), \cdots, \tau_k(x), x_{k+1}, \cdots, x_n).$$

As $\pi_1 \circ \tau = \pi_1$, considering π_1, $\rho_1 \circ \tau$ in place of π_1, ρ_1, we may assume (4.11.1).

For any $y \in X_2$, the vector $y - \pi_1(y)$ is vertical to the tangent space $T_y \rho_1^{-1}(\rho_1(y))$ because of (4.11.1). Hence, by the Whitney condition, the tangent spaces $T_y X_2$ and $T_y \rho_1^{-1}(\rho_1(y))$ are transversal if y is near to $X_1 \cap B$, $B = \{x \in \mathbf{R}^n \mid |x| \leq 1\}$. We set

$$V_y = T_y X_2 \cap T_y \rho_1^{-1}(\rho_1(y))$$

Now the following statement is sufficient for the proof.

(4.11.2) Let $\{x^i\}$ be a sequence of points in X_2 converging to $x \in X_1 \cap B$. If the sequence $\{V_{x^i}\}$ converges to a subspace $L \subset \mathbf{R}^n$, then $L \supset T_x X_1 = X_1$.

The reason is the following. Let $y \in X_2$ be a point near to $X_1 \cap B$. (4.11.2) means that we have a basis $(v_1, \cdots, v_{k'})$ of the linear space V_y with $k + k' \geqq n$ such that

$$v_i = \mathbf{x}_{k+i} + \sum_{j=1}^{n} a_{ij} \mathbf{x}_j, \ i = 1, \cdots, n-k, \ a_{ij} \in \mathbf{R},$$

where $\mathbf{x}_j = (0, \cdots, 0, \overset{j}{1}, 0, \cdots, 0)$ and if y converges to a point of X_1 then $|a_{ij}|$ are chosen to converge to 0. Clearly we can assume $a_{ij} = 0$ for $j = k+1, \cdots, n$. If

$$\xi_{1\pi_1(y)} = \sum_{i=1}^{n-k} b_i \mathbf{x}_{k+i}, \qquad b_i \in \mathbf{R},$$

then put

$$\xi_{2y} = \sum_{i=1}^{n-k} b_i v_i.$$

Then ξ_{2y} satisfies (4.10.1, 2). On the other hand, by the transversality of $T_y X_2$ and $T_y \rho_1^{-1}(\rho_1(y))$, $\bigcup_{y \in X_2} V_y$ comes out a C^∞ vector subbundle of the tangent bundle of X_2, and the restriction of $d\pi_1$ to $\bigcup_{y \in X_2} V_y$ is a submersion into the tangent bundle of X_1. Hence ξ_{2y} can be extended to a small neighborhood of y holding (4.10.1, 2). Using once more a partition of unity, we easily obtain ξ_2 on X_2 so that (ξ_1, ξ_2) is a continuous controlled vector field on $\{(X_1, T_1), (X_2, T_2)\}$.

We want to prove (4.11.2). Without loss of generality, we may suppose that sequence $\{\overrightarrow{x^i \pi_1(x^i)}\}$ of lines containing 0 and $x^i - \pi_1(x^i)$ converges to a line $L_1 \subset \mathbf{R}^n$ and that the sequence of tangent spaces $\{T_{x^i} X_2\}$ converges to a subspace $L_2 \subset \mathbf{R}^n$. Then, by the Whitney condition and Remark 4.6, we have

$$L_1 \subset L_2, \qquad X_1 \subset L_2.$$

Here we must remark that the Whitney condition is invariant under a diffeomorphism of $\mathbf{R}^n$. Let L_{1x^i} be a unit vector contained in $\overrightarrow{x^i \pi_1(x^i)}$ and L_{2x^i} be a unit vector in $T_{x^i} X_2$ such that $\{L_{1x^i}\}$ and $\{L_{2x^i}\}$ converge to L_1. Then $\{L_{1x^i} - L_{2x^i}\}$ converges to 0, and it follows from (4.11.1) that L_{1x^i} is vertical to V_{x^i} and to $T_{x^i} \rho_1^{-1}(\rho_1(x^i))$. Hence if $L_{2x^i} = a_i L_{1x^i} + L_{3x^i}$, $a_i \in \mathbf{R}$, $L_{3x^i} \in T_{x^i} \rho_1^{-1}(\rho_1(x^i))$, is the orthogonal decomposition, then $\{a_i\}$ and $\{L_{3x^i}\}$ converge to 1 and 0 respectively.

For (4.11.2), we only need to see that for a unit vector $v \in X_1$, there exist $v^i \in V_{x^i}$, $i = 1, \cdots$, such that $\{v^i\}$ converges to v. By the inclusion $X_1 \subset L_2$, we have $v'^i \in T_{x^i} X_2$, $i = 1, \cdots$, such that $\{v'^i\}$ converges to v. Since $T_{x^i} X_2$ is the direct sum of $\mathbf{R} L_{2x^i}$ and V_{x^i}, there exist uniquely $v^i \in V_{x^i}$ and $b_i \in \mathbf{R}$ such that $v'^i = v^i + b_i L_{2x^i}$. The sequence $\{b_i\}$ converges to 0. In fact

$$v'^i = (v^i + b_i L_{3xi}) + a_i b_i L_{1xi}$$

is the orthogonal decomposition of v'^i to the $T_{x^i}\rho_1^{-1}(\rho_1(x^i))$-factor and its normal factor. Since $T_{x^i}\rho_1^{-1}(\rho_1(x^i)) \supset X_1$ as linear spaces, the convergence of $\{v'^i\}$ to v implies that of $\{v^i + b_i L_{3xi}\}$ to v, which shows that $\{a_i b_i\}$ converges to 0. Therefore $b_i \to 0$ as $i \to \infty$. Hence we prove the convergence of $\{v^i\}$ to v and hence the lemma.

Let $\{X_i\}$ be a Whitney stratification of a subset X of $\mathbf{R}^n$, and $\xi = \{\xi_i\}$ be a vector field on $\{X_i\}$. Integrating each vector field ξ_i on X_i we obtain C^∞ flows $\Theta_i : D_i \to X_i$ where Θ_i are given by $\Theta_i(x, 0) = x$, $\dfrac{\partial}{\partial t}\Theta_i(x, t) = \xi_{\Theta_i(x, t)}$ and $D_i \subset X_i \times \mathbf{R}$ are the maximal domains containing $X_i \times 0$. Let D be the union of D_i, and define $\Theta : D \to X$ by $\Theta|_{D_i} = \Theta_i$.

Definition 4.12. We call $\Theta : D \to X$ the *flow of* ξ. ξ is called *locally integrable* if D is open in $X \times \mathbf{R}$ and if Θ is continuous (see Lemma 4.4, Chapter II, [1]).

Lemma 4.13 (Corollary 4.7, Chapter II, [1]). *Let* $\{X_i\}$ *be a Whitney stratification of a subset* X *of* $\mathbf{R}^n$, $\{T_i\}$ *be a controlled tube system for* $\{X_i\}$, *and* ξ *be a controlled vector field on* $\{(X_i, T_i)\}$. *If* X *is locally closed,* ξ *is locally integrable.*

Lemma 4.14. *Let* $\{X_i\}$ *be a Whitney stratification of a locally closed subset* X *of* $\mathbf{R}^n$, *let* f_1, f_2 *be* C^∞ *functions on* $\mathbf{R}^n$, *let* $\xi = \{\xi_i\}$ *be a vector field on* $\{X_i\}$, *and let* $\{V_j\}_{j=0,\cdots,n-1}$, $\{V_j^1\}_{j=0,\cdots,n-1}$ *be open coverings of* X *in* $\mathbf{R}^n$ *with* $\overline{V}_j^1 - V_j \subset \overline{X} - X$ *for each* j. *Assume for each* i, j

(4.14.1) $\xi_i(f_1|_{X_i}), \xi_i(f_2|_{X_i}) > 0$,

(4.14.2) $x + \mathbf{R}^l$ *and* X_i *are transversal for any* $x \in \mathbf{R}^n$ *where* $l = \operatorname{codim} X_i$, *and*

(4.14.3) $\xi_{ix} \in \mathbf{R}^{n-j}$ *for* $x \in V_j \cap X_i$.

Then there exists a locally integrable vector field $\xi^1 = \{\xi_i^1\}$ *on* $\{X_i\}$ *which satisfies* (4.14.1) *and* (4.14.3) *for* $x \in V_j^1 \cap X_i$.

Proof. By the same reason as the proof of Lemma 4.9 we assume $\{X_i\} = \{X_i\}_{i=1,2,\cdots,n}$ and $\dim X_i = i$. Here we remark that X_0 does not exist because of (4.14.1). By (4.14.2, 3), the property $V_j \cap X_i \neq \emptyset$ implies $j < i$, hence $V_j \cap X_i = \emptyset$ for $j \geq i$, and for each i $\{V_j\}_{j=0,\cdots,i-1}$ is a covering of X_i. Let $\{V_j^k\}_{j=0,\cdots,n-1}$, $k = 2, \cdots, n$, be open coverings of X such that $\overline{V}_j^n - V_j \subset \overline{X} - X$ and $\overline{V}_j^{k-1} - V_j^k \subset \overline{X} - X$, $k = 2, \cdots, n$, $j = 0, \cdots, n-1$. Then we have $\overline{V}_j^k \cap X_i = \emptyset$, $k = 1, \cdots, n$, if $j \geq i$. Let U_i be an open neighborhood of each X_i such that

(4.14.4) $U_i \subset \bigcup\limits_{j=0}^{i-1} V_j^1$, $\overline{U}_i \cap \overline{V}_j^n \subset \overline{X}_i - X_i$ if $j \geq i$.

Then, by Lemma 4.9 we have a controlled tube system $\{T_i = (E_i, \pi_i, \rho_i, e_i)\}$ for $\{X_i\}$ such that for each i

$$(4.14.5) \qquad\qquad\qquad |T_i| \subset U_i$$

$$(4.14.6) \qquad \pi_i^{-1}(x) \subset x + \boldsymbol{R}^{n-k} \qquad \text{for} \quad x \in X_i \cap (U_k - U_1 - \cdots - U_{k-1})$$

$$k = 1, \cdots, i\,.$$

We will show the existence of a controlled vector field ξ^1 on $\{(X_i, T_i)\}$ which satisfies (4.14.1) and (4.14.3) for $x \in V_j^1 \cap X_i$. We prove it by induction. Assume we have already constructed a vector field $\xi^{m+1} = \{\xi_i^{m+1}\}$ on $\{(X_i, T_i)\}$ satisfying (4.14.1) and (4.14.3) for $x \in V_j^{m+1} \cap X_i$ such that $\{\xi_i^{m+1}\}_{i \geq m-1}$ is controlled on $\{(X_i, T_i)\}_{i \geq m+1}$. In the case $m = n-1$, it is trivial, so we want to find ξ^m. Now Lemma 4.11 implies the existences of an open neighborhood W of X_m and a continuous controlled vector field $\xi' = \{\xi_i'\}$ on $\{(X_i \cap W, T_i|_{X_i \cap W})\}$ such that $\xi_m' = \xi_m^{m+1}$. Choose W so small that

$$X_i \cap W = \varnothing, \qquad i = 1, \cdots, m-1, \qquad W \subset |T_m|\,,$$

$$W \cap \pi_m^{-1}(V_j^m \cap X_m) \subset V_j^{m+1}, \quad j = 0, \cdots, m-1, \quad \text{and}$$

$$\xi_i'(f_j|_{X_i \cap W}) > 0 \qquad \text{for} \quad i = m, \cdots, j = 1, 2\,.$$

The last inequality follows from the continuity of ξ'.

It is easy to find a positive C^∞ function θ on X_m such that $W' = \{x \in W \,|\, \theta \circ \pi_m(x) \cdot \rho_m(x) < 1\}$ satisfies $\overline{W}' - W \subset \overline{X}_m - X_m$. Let ϕ be a C^∞ function on $\boldsymbol{R}$ such that $\phi = 0$ on $[1, \infty)$, $= 1$ on $(-\infty, 1/2]$ and $0 < \phi < 1$ on $(1/2, 1)$. Put

$$\Psi(x) = \begin{cases} \phi(\theta \circ \pi_m(x) \cdot \rho_m(x)) & \text{for} \quad x \in W \\ 0 & \text{for} \quad x \in \boldsymbol{R}^n - W - (\overline{X}_m - X_m)\,. \end{cases}$$

Then Ψ is a C^∞ function on $\boldsymbol{R}^n - (\overline{X}_m - X_m)$ such that $\Psi = 1$ in a neighborhood of X_m, $= 0$ outside W, $0 \leq \Psi \leq 1$ and

$$(4.14.7) \qquad \Psi(x) = \Psi \circ \pi_i(x) \qquad \text{for} \quad x \in |T_i|, \ i \geq m+1$$

by (4.2.1, 2). Here we have to shrink $|T_i|$, $i \geq m+1$, in order to assure that if $x \in W' \cap |T_i|$, then $\pi_i(x) \in |T_m|$ and conversely that if $x \in |T_i|$, $\pi_i(x) \in W'$, then $x \in |T_m|$. Put

$$\tilde{\xi}_i^m = \begin{cases} \Psi \xi_i' + (1 - \Psi) \xi_i^{m+1} & \text{for} \quad i \geq m \\ \xi_i^{m+1} & \text{for} \quad i < m\,. \end{cases}$$

Then $\xi^m = \{\xi_i^m\}$ is a vector field on $\{(X_i, T_i)\}$ satisfying (4.14.1) and (4.14.3) for $x \in V_j^m \cap X_i$ and $\{\xi_i^m\}_{i \geq m}$ is controlled on $\{(X_i, T_i)\}_{i \geq m}$ as follows.

Indeed (4.14.1) is clear. For (4.14.3) it is sufficient to see

$$(4.14.8) \qquad \xi_{ix}' \in \boldsymbol{R}^{n-j} \qquad \text{for} \quad x \in V_j^m \cap X_i \cap W$$

because of the assumption on ξ^{m+1}. By the property $X_i \cap W = \emptyset$, $i=1, \cdots, m-1$, X_i in (4.14.8) is of dimension $\geq m$. Since ξ' is controlled, we see

(4.14.9) $\xi'_{ix} \in R^{n-\min(j,k)}$ for $x \in W$ with $\pi_m(x) \in X_m \cap$

$$(\overline{U}_k - \overline{U}_1 - \cdots - \overline{U}_{k-1}) \cap V_j^{m+1},\ 1 \leq k \leq m,\ 0 \leq j \leq m-1$$

by (4.10.1), (4.14.6), the assumption on $\xi'_m = \xi_m^{m+1}$ and by easy basic calculations. For any $x \in W$ with $\pi_m(x) \in X_m \cap (\overline{U}_k - \overline{U}_1 - \cdots - \overline{U}_{k-1}) \cap V_j^{m+1}$, let j_x denote the maximum of such j. Since $\overline{U}_k \cap V_{j_x}^{m+1} \subset \overline{X}_k - X_k$ for $k \leq j_x$ (4.14.4), and since $1 \leq k \leq m$, the property $X_m \cap \overline{U}_k \cap V_{j_x}^{m+1} \neq \emptyset$ means that $k > j_x$. Hence we have by (4.14.9)

$$\xi'_{ix} \in R^{n-j}   for   x \in W \text{ with } \pi_m(x) \in V_j^{m+1} \cap X_m.$$

Therefore, shrinking $|T_m|$ and hence W, W', we obtain (4.14.8).

It rests to see that the vector field $\{\xi_i^m\}_{i \geq m}$ on $\{(X_i, T_i)\}_{i \geq m}$ is controlled. For any X_i, $X_{i'}$, $i' > i \geq m$ and $x \in X_{i'}$ near to X_i, we have if $x \in W$,

$$d(\pi_i|_{X_{i'}})\xi_{i'x}^m = \Psi(x) d(\pi_i|_{X_{i'}})\xi'_{i'x} + (1-\Psi(x))d(\pi_i|_{X_{i'}})\xi_{i'x}^{m+1}$$

$$= \Psi \circ \pi_i(x) d(\pi_i|_{X_{i'}})\xi'_{i'x} + (1-\Psi \circ \pi_i(x))d(\pi_i|_{X_{i'}})\xi_{i'x}^{m+1},\ \text{by (4.14.7)},$$

$$= \Psi \circ \pi_i(x)\xi'_{i\pi_i(x)} + (1-\Psi \circ \pi_i(x))\xi_{i\pi_i(x)}^{m+1},\ \text{by the controlledness}$$

$$\text{of } \xi' \text{ and } \{\xi_i^{m+1}\}_{i \geq m+1},$$

$$= \xi_{i\pi_i(x)}^m,$$

and if $x \notin W$, by (4.14.7) and the controlledness of $\{\xi_i^{m+1}\}_{i \geq m+1}$

$$d(\pi_i|_{X_{i'}})\xi_{i'x}^m = d(\pi_i|_{X_{i'}})\xi_{i'x}^{m+1} = \xi_{i\pi_i(x)}^{m+1} = \xi_{i\pi_i(x)}^m.$$

Hence (4.10.1) holds true. (4.10.2) is shown in the same way by the controlledness of ξ' and $\{\xi_i^{m+1}\}_{i \geq m+1}$. We omit the proof. Hence, by induction we have a controlled vector field ξ^1 on $\{(X_i, T_i)\}$ which satisfies (4.14.1) and (4.14.3) for $x \in V_j^1 \cap X_i$, which, together with Lemma 4.13, proves Lemma 4.14.

§ 5. Topological Equivalence of Subanalytic Functions

In this section we prove the following.

Proposition 5.1. *Let X be a closed subanalytic subset of R^n. Let f_1, f_2 be subanalytic functions on X such that for each point $x \in X$, both $f_1(x)$ and $f_2(x)$ have the same sign. Put $Z = f_1^{-1}(0)$. Then there exist neighborhoods W_1, W_2 of Z in X and a homeomorphism $\tau: W_1 \to W_2$ such that $f_2 \circ \tau = f_1$ on W_1 and $\tau|_Z = ident.$*

The case of analytic function germs is Theorem 4.3 in [16]. There I gave a sketch of proof, whose idea is the main tool also in the proof below. Two

functions f_1, f_2 on a manifold M is called *topologically equivalent* if there exists a homeomorphism τ of M such that $f_1 \circ \tau = f_2$. Another version of Proposition 5.1 is Corollary 4.4, [16], which should be amended to the following.

Proposition 5.2. *Let f_1, f_2 be analytic functions on a compact analytic manifold M. Put*

$$S = \{x \in M \mid a_1 df_{1x} + a_2 df_{2x} = 0 \text{ for some } a_1, a_2 \geq 0 \text{ with } a_1 + a_2 > 0\}.$$

Assume $f_1^{-1} f_1(S) = f_2^{-1} f_2(S)$ and $f_1 = f_2$ on $f_1^{-1} f_1(S)$. Then f_1 and f_2 are topologically equivalent, and the homeomorphism of equivalence can be the identity on $f_1^{-1} f_1(S)$ and analytic outside it.

The proof of Proposition 5.1 consists of 5 steps, (5.3), $\cdots$, (5.7).

(5.3)　Let $\{X_i\}_{i=1,\dots}$ be a subanalytic stratification of X compatible with Z (2.11). We can reduce the problem to the case in which (5.3.1, $\cdots$, 5) below hold true.

(5.3.1)　f_1, f_2 are restrictions to X of analytic functions on $\mathbf{R}^n$.

(5.3.2)　$\{X_i\}$ is a Whitney stratification of X.

(5.3.3)　There exist compact analytic manifolds M_i, $i=1, \cdots$, subanalytic open subsets M_i' of M_i and analytic maps $\phi_i : M_i \to \mathbf{R}^n$ such that $\phi_i|_{M_i'}$ are diffeomorphisms onto X_i.

Put

$$\mathbf{R}^k = \{(x_1, \cdots, x_n) \in \mathbf{R}^n \mid x_{k+1} = \cdots = x_n = 0\}, \qquad 1 \leq k \leq n,$$

and

$$x + \mathbf{R}^k = \{x + y \mid y \in \mathbf{R}^k\} \qquad \text{for} \quad x \in \mathbf{R}^n.$$

(5.3.4)　For each i, k and $x \in \mathbf{R}^n$, X_i and $x + \mathbf{R}^k$ are transversal if $\dim X_i + k \geq n$, and $X_i \cap (x + \mathbf{R}^k)$ is empty or of dimension 0 if $\dim X_i + k < n$.

Let $N \subset X$ be a subanalytic analytic submanifold in $\mathbf{R}^n$, and let $1 \leq k \leq n$ with $\dim N + k > n$. Assume that for any $x \in \mathbf{R}^n$, N and $x + \mathbf{R}^k$ are transversal. Put

$$S(N, k) = \{x \in N \mid a_1 d(f_1|_{N \cap x + \mathbf{R}^k})_x + a_2 d(f_2|_{N \cap x + \mathbf{R}^k})_x = 0$$

$$\text{for some} \quad a_1, a_2 \geq 0 \quad \text{with} \quad a_1 + a_2 > 0\}.$$

(5.3.5)　For each $X_i \subset Z$, $X_{i'} \not\subset Z$ and $1 \leq k \leq n$ with $\dim X_i + k \geq n$ and $\bar{X}_{i'} \supset X_i$, $S(X_{i'}, k)$ is empty in a neighborhood of X_i.

The reduction to (5.3.1) is trivial if we consider the graph of (f_1, f_2) in place of X. Hence we assume (5.3.1). Before beginning the proof, we remark some facts.

Let $N \subset \mathbf{R}^n$ be a subanalytic analytic manifold, M be a compact analytic manifold, M' be a subanalytic open subset of M, and $\phi : M \to \mathbf{R}^n$ be an analytic

map such that $\phi|_{M'}$ is a diffeomorphism onto N and that $\phi(M)\subset X$. Let $1\leqq k$ $\leqq n$ with dim $N+k>n$. Assume that for any $x\in\boldsymbol{R}^n$, N and $x+\boldsymbol{R}^k$ are transversal. Then

(5.3.6) $S(N, k)$ is closed in N and subanalytic.

Proof of (5.3.6). The closedness is clear. For the subanalyticness, consider

$$V=\{v\in T_xM\,|\,x\in M,\ d\phi_x v\in\boldsymbol{R}^k\}\ .$$

Then V is an analytic subset of the tangent bundle TM of M since $d\phi:TM\to TR^n$ is analytic. Put

$$V'=\{v\in V\,|\,|v|=1\}$$

for some analytic Riemannian metric on TM,

$$V''=\{v\in T_xM\,|\,x\in M,\ d(f_1\circ\phi)_x v\cdot d(f_2\circ\phi)_x v>0\}\ .$$

Then V' is an analytic subset of TM, and V'' is a well-defined semi-analytic subset of TM by (5.3.1). It is easy to see that

$$N-S(N,\ k)=p\circ d\phi(V'\cap V''\cap TM')$$

where $p:T\boldsymbol{R}^n\to\boldsymbol{R}^n$ be the projection to the base space. As the closure of $V'\cap V''\cap TM'$ in TM is compact, and as $p\circ d\phi:TM\to\boldsymbol{R}^n$ is analytic, it follows from (2.3) and (2.6) that $p\circ d\phi(V'\cap V''\cap TM')$ is subanalytic. Hence (2.1.3) implies that $S(N, k)$ is subanalytic.

(5.3.7) Let $N\subset X-Z$ be a subanalytic analytic submanifold in $\boldsymbol{R}^n$. Assume $S(N, n)$ is subanalytic. Then $S(N, n)$ is empty in a neighborhood of Z.

Proof of (5.3.7). Assume $\overline{S(N, n)}\cap Z\neq\varnothing$. Apply Proposition 3.1 to $S(N, n)$. Then we have a 1-dimensional subanalytic analytic submanifold $N'\subset S(N, n)$ so that $\overline{N'}\cap Z\neq\varnothing$. Since $n+\dim N'>n$, $S(N', n)$ is well-defined. This is the reason why we assumed $k=n$. Now it is easy to see that

$$S(N',\ n)=S(N,\ n)\cap N'=N'\ .$$

Applying (2.4) to N', we obtain a compact 1-dimensional analytic manifold M and an analytic map $\phi:M\to\boldsymbol{R}^n$ such that $\phi(M)=\overline{N'}$. We remark that $f_1\circ\phi(x)$, $f_2\circ\phi(x)$ have the same sign for each point $x\in M$, that

$$f_1\circ\phi(x_0)=f_2\circ\phi(x_0)=0$$

for some $x_0\in M$ and that

$$a_1 d(f_1\circ\phi)_x+a_2 d(f_2\circ\phi)_x=0$$

for each $x\in M$ and some $a_1, a_2\geqq 0$ with $a_1+a_2>0$. The third property implies that if one of $f_i\circ\phi$ is monotone increasing on a subset of M diffeomorphic to an interval, then the other is monotone decreasing. This contradicts the first

and second properties. Hence $\overline{S(N, n)} \cap Z = \emptyset$, which proves (5.3.7).

We will prove (5.3) by induction. Let $0 \leq l \leq n$ be an integer. If $l = n$, the next statement coincides with (5.3).

(5.3)$_l$ We can reduce the problem to the case in which (5.3.3) for codim X_l $\leq l$, (5.3.2)$_l$ below and (5.3.4, 5) for $k \leq l$ hold true.

(5.3.2)$_l$ $\{X_i | \text{codim } X_i \leq l\}$ is a Whitney stratification of the union of such X_i's.

Proof of (5.3)$_l$ If $l > 0$, assume (5.3)$_{l-1}$. Let $\{X_i'\}$ be a subanalytic stratification of X compatible with Z such that (5.3.3) for codim $X_i' < l$, (5.3.2)$_{l-1}$ and (5.3.4, 5) for $k < l$ hold true. Here X_i in (5.3.3, 4, 5) and (5.3.2)$_{l-1}$ are replaced by X_i'. Let I_1, I_2 be the index subsets of all i with codim $X_i' = l$, $\geq l$ respectively, and Y be the union of X_i', $i \in I_2$.

At first we want to change the x_l-axis fixing the $x_1, \cdots, x_{l-1}$-axes so that $Y \cap (x + \boldsymbol{R}^l)$ is empty or of dimension 0 for any $x \in \boldsymbol{R}^n$. This condition is equivalent to that $X_i' \cap (x + \boldsymbol{R}^l)$ is a finite set for any $i \in I_2$ and $x \in \boldsymbol{R}^n$, since a bounded subanalytic set of dimension 0 consists of finite points. Let

$$r_{l-1} : \boldsymbol{R}^n \to \boldsymbol{R}^{l-1\perp} = \{x \in \boldsymbol{R}^n \mid x = (0, \cdots, 0, x_l, \cdots, x_n)\}$$

be the projection. Then, since $r_{l-1}(X_i')$ is a subanalytic set of dimension $\leq n - l$ for any $i \in I_2$, we can apply Lemma 3.2' to $r_{l-1}(X_i')$. Hence there exists a 1-dimensional linear subspace L of $\boldsymbol{R}^{l-1\perp}$ such that for any $a \in \boldsymbol{R}^{l-1\perp}$, $a + L$ is nonsingular for each $r_{l-1}(X_i')$, $i \in I_2$. This means that $(a + L) \cap r_{l-1}(X_i')$ is a finite set for each $i \in I_2$. Changing the x_l-axis, we assume that L is the x_l-axis. Let $x \in \boldsymbol{R}^n$, $i \in I_2$. Put $a = r_{l-1}(x)$. Let $(a + L) \cap r_{l-1}(X_i')$ consists of $a_1, \cdots, a_{l'} \in \boldsymbol{R}^{l-1\perp} \subset \boldsymbol{R}^n$, and let $X_i' \cap (a_j + \boldsymbol{R}^{l-1})$ consist of at most l'' points for any j (5.3)$_{l-1}$. Then we have

$$\#(X_i' \cap (x + \boldsymbol{R}^l)) = \#(X_i' \cap (x + \boldsymbol{R}^{l-1} + L))$$

$$\leq \max_j \#(X_i' \cap (a_j + \boldsymbol{R}^{l-1})) \cdot \#((a + L) \cap r_{l-1}(X_i')) = l' l''.$$

Hence $X_i' \cap (x + \boldsymbol{R}^l)$ is a finite set.

Let $r_l : \boldsymbol{R}^n \to \boldsymbol{R}^{l\perp} = \{x = (0, \cdots, 0, x_{l+1}, \cdots, x_n)\}$ be the projection. Since $X_i' \cap (x + \boldsymbol{R}^l)$ is a finite set for any $i \in I_1$ and $x \in \boldsymbol{R}^n$, $d(r_l|_{X_i'})$ has the maximal rank $n - l$. By (2.4) there exist compact analytic manifolds M_i, $i \in I_1$, of dimension $n - l$ and analytic maps $\phi_i : M_i \to \boldsymbol{R}^n$ such that $\phi_i(M_i) = \overline{X}_i'$. Let X_i^2 be the images under ϕ_i of the sets of critical points of $r_l \circ \phi_i$, and $X_i^3 \subset X_i'$ be subanalytic subsets closed in X_i' of codimension $> l$ in $\boldsymbol{R}^n$ such that $(X_j', X_i' - X_i^3)$ satisfies the Whitney condition for any $j \notin I_2$ (2.12). Here we put $X_i^3 = \emptyset$ if $l = 0$. Then X_i^2 are of codimension $> l$, and $X_i' - X_i^2$ and $x + \boldsymbol{R}^l$ are transversal for any $x \in \boldsymbol{R}^n$ since $r_l : X_i' - X_i^2 \to \boldsymbol{R}^{l\perp}$ are immersions.

For each $i \in I_1$ with $X'_i \subset Z$, put

$$X_i^4 = X'_i \cap \bigcup_{j \notin I_2} \bar{S}(X'_j, l).$$

Here $\bar{S}$ means the closure of S in $\mathbf{R}^n$, and S is well-defined because the transversality of X'_j and $x + \mathbf{R}^{l-1}$ implies that of X'_j and $x + \mathbf{R}^l$. By (5.3.6), $S(X'_j, l)$ is closed in X'_j and subanalytic. Hence (2.1.2) imply that X_i^4 is subanalytic.

Moreover X_i^4 is of codimension $> l$ in $\mathbf{R}^n$. We will prove this by reduction to absurdity. Assume codim $X_i^4 = l$. Then (2.2) implies that $\bigcup_{j \notin I_2} S(X'_j, l)$ is codimension $< l$ as germ at X'_i. Apply Proposition 3.1 to $\{S(X'_j, l)\}_{j \notin I_2}$, X_i^4. There exist then a triangulation K of $\mathbf{R}^n$ and a subanalytic homeomorphism τ of $\mathbf{R}^n$ such that two properties corresponding to (3.1.3, 4) are satisfied. Hence we have a simplex $\sigma \in K$ of codimension $< l$ so that $\tau(\mathring{\sigma})$ is a subanalytic analytic manifold contained in $S(X'_j, l)$ for some $j \notin I_2$ and that an open face $\mathring{\sigma}'$ of σ of codimension l is analytically imbedded by τ into X_i^4. By (2.12) there is an inner point x of $\tau(\sigma')$ such that $(\tau(\mathring{\sigma}), \tau(\mathring{\sigma}') \cap$ a neighborhood of $x)$ satisfies the Whitney condition. Choosing moreover $x \notin X_i^2$, we assume $\tau(\mathring{\sigma}')$ and $x + \mathbf{R}^l$ are transversal at x. Then it follows from the definition of the Whitney condition that $\tau(\mathring{\sigma})$ and $x + \mathbf{R}^l$ are transversal in a neighborhood of x, and hence $\tau(\mathring{\sigma}) \cap (x + \mathbf{R}^l)$ is a subanalytic analytic submanifold of $\mathbf{R}^n$ in a neighborhood of x whose closure contains x. We see easily that

$$S(\tau(\mathring{\sigma}), l) = S(X'_j, l) \cap \tau(\mathring{\sigma}) = \tau(\mathring{\sigma})$$

in a neighborhood of x. Hence it follows that

$$a_1 d(f_1|_{\tau(\mathring{\sigma}) \cap (x+R^l)})_y + a_2 d(f_2|_{\tau(\mathring{\sigma}) \cap (x+R^l)})_y = 0$$

in a neighborhood of x for each $y \in \tau(\mathring{\sigma}) \cap (x + \mathbf{R}^l)$ and some $a_1, a_2 \geq 0$ with $a_1 + a_2 > 0$. These imply that

$$S(\tau(\mathring{\sigma}) \cap (x + \mathbf{R}^l), n) = \tau(\mathring{\sigma}) \cap (x + \mathbf{R}^l) \text{ in a neighborhood of } x,$$

$$\overline{\tau(\mathring{\sigma}) \cap (x + \mathbf{R}^l)} \ni x.$$

These contradict (5.3.7), hence codim $X_i^4 > l$.

If $X'_i \not\subset Z$, put $X_i^4 = \emptyset$. Let $\{X_i\}^l$ be a subanalytic stratification of Y compatible with $\{X'_i, X'_i \cap X_i^2, X_i^3, X_i^4\}_{i \in I_1} \cup \{Z\}$. By Proposition 3.1 we may assume that each X_i is simply connected. Put

$$\{X_i\} = \{X'_i\}_{i \notin I_2} \cup \{X_i\}^l.$$

Then $\{X_i\}$ is a subanalytic stratification of X compatible with Z and satisfying (5.3.3) for codim $X_i \leq l$, (5.3.2)$_l$ and (5.3.4, 5) for $k \leq l$.

Indeed, for any X_i with codim $X_i = l$ we have $X_i \subset X'_j - X_j^2 - X_j^3 - X_j^4$ for some $j \in I_1$. It is easy to see that for any connected component A of $\phi_j^{-1}(X'_j - X_j^2 - X_j^3 - X_j^4)$

$$\phi_j : A \longrightarrow X_j' - X_j^2 - X_j^3 - X_j^4$$

is an analytic covering because of the compactness of M_j. Hence if we let M_i' be any connected component of $\phi_j^{-1} X_i$, then M_i' is a subanalytic open subset of M_i by (2.2), (2.5.1), and $\phi_j : M_i' \to X_i$ is an analytic diffeomorphism since X_i is simply connected. Hence (5.3.3) holds true for codim $X_i = l$ and obviously for codim $X_i < l$. Now, since each X_i of codimension l is contained in $X_j' - X_j^3$ for some $j \in I_1$, (5.3.2)$_l$ is clear.

Consider (5.3.4) for $k \leq l$. We have five cases; (i) codim $X_i < l$ and $k < l$, (ii) codim $X_i < l$ and $k = l$, (iii) codim $X_i = l$ and $k = l$, (iv) codim $X_i \geq l$ and $k < l$ and (v) codim $X_i > l$ and $k = l$. In the case of (i), (5.3.4) is contained in the assumption. If (ii), we have dim $X_i + k > n$. Hence, by the assumption, X_i and $x + R^{k-l}$ are transversal for any $x \in R^n$, which implies also the transversality of X_i and $x + R^k$. (5.3.4) in the case of (iii) follows from the fact already seen that $X_j' - X_j^2$ and $x + R^l$ are transversal for any $x \in R^n$, $j \in I_1$. In the case of (iv) we have $X_i', j \in I_2$, so that $X_i \subset X_j'$. Hence it is sufficient to see that $X_j' \cap (x + R^k)$ is empty or of dimension 0. But it is a part of the assumption because of $k < l$, dim $X_i' + k < n$. We have chosen the x_l-axis so that $Y \cap (x + R^l)$ is empty or of dimension 0 for any $x \in R^n$. That proves the case of (v).

For (5.3.5), let $X_i \subset Z$, $X_{i'} \not\subset Z$, $1 \leq k \leq l$ such that dim $X_i + k \geq n$, $\bar{X}_{i'} \supset X_i$. If $S(X_{i'}, k)$ is empty in a neighborhood of X_i, then so is $S(X_{i'}, k-1)$ because of the inclusion

$$S(X_i', k) \supset S(X_i', k+1).$$

Hence we only have to prove the case dim $X_i + k = n$. There are then two cases $k < l$ or $k = l$. If $k < l$, then $X_i = X_j'$, $X_{i'} = X_{j'}'$ for some j, j'. Hence, by the assumption, $S(X_{i'}, k)$ is empty in a neighborhood of X_i. If $k = l$, the same statement follows from the facts that X_i is contained in $X_j' - X_j^2$ for some j and that $X_{i'}$ coincides with some $X_{j'}'$.

Thus we have proved (5.3)$_0$ and (5.3)$_l$, $l > 0$, assuming (5.3)$_{l-1}$. Hence (5.3)$_n$ = (5.3) follows.

(5.4) Let $\{X_i\}$ be the stratification which appeared in (5.3), X_i be a stratum with $X_i \not\subset Z$, and l be an integer bigger than codim X_i. Let Y_{il} denote the union of X_j's such that $X_j \subset Z \cap \bar{X}_i$ and codim $X_j = l$. Then there exists an analytic vector field ξ_{il} on the intersection U_{il} of X_i and an open neighborhood of Y_{il} in $X_i \cup Y_{il}$ such that

(5.4.1) for any $x \in U_{il}$, ξ_{ilx} is contained in R^l, and
(5.4.2) $\xi_{il} f_k$, $k = 1, 2$, are positive on U_{il}.

Here $\xi_{il} f_k$ are well-defined because of (5.3.1).

Proof. Put
$$U_{il} = X_i - S(X_i, l),$$

$$v_{kilx}=d(f_k\,|\,_{X_i\cap(x+R^l)})_x,\ \ k=1,\,2,\ \ x\in U_{il},$$

$$\tilde{\xi}_{ilx}=|v_{2ilx}|v_{1ilx}+|v_{1ilx}|v_{2ilx},\ \ x\in U_{il}.$$

Then, by (5.3.4), v_{kil} are well-defined analytic vector fields on U_{il}. On the other hand the definition of $S(X_i,\,l)$ shows that $|v_{kil}|$ do not vanish and that v_{1ilx} and v_{2ilx}, $x\in U_{il}$, do not point the opposite directions each other. Hence $\tilde{\xi}_{il}$ also is a well-defined non-singular analytic vector field. By (5.3.5), U_{il} is the intersection of X_i and a neighborhood of Y_{il} in $X_i\cup Y_{il}$. Clearly (5.4.1, 2) follow from the above definition of $\tilde{\xi}_{il}$. Hence (5.4) is proved.

(5.5) Under the same notations as (5.4), let U'_{il} and U''_{il} be open subsets of X_i having the same property as U_{il}, namely being the intersections of X_i and neighborhoods of Y_{il} in $X_i\cup Y_{il}$, and moreover satisfying

$$\bar{U}'_{il}\cap X_i\subset U''_{il},\ \bar{U}''_{il}\cap X_i\subset U_{il}.$$

Put $U_i=\bigcup\limits_{l>\mathrm{codim}\,X_i} U'_{il}$. Then there exists a C^∞ vector field $\tilde{\xi}_i$ on X_i such that

(5.5.1)   for any $x\in U'_{il}$, ξ_{ix} is contained in R^l, and

(5.5.2) $\xi_i f_k$, $k=1,\,2$, are positive on U_i.

Proof. Let g_l be C^∞ functions on X_i such that $g_l=1$ on U'_{il}, $=0$ on $X_i-U''_{il}$ and $0<g<1$ on $U''_{il}-\bar{U}'_{il}$. Then, putting $g_l\tilde{\xi}_{ilx}=0$ on X_i-U_{il}, we extend $g_l\tilde{\xi}_{il}$ to X_i as C^∞ vector fields. Put

$$\xi_{ix}=\sum_{l>\mathrm{codim}\,X_i}\ \prod_{l'<l}(1-g_{l'})g_l\tilde{\xi}_{ilx}.$$

Then ξ_i is a C^∞ vector field on X_i. We will see that ξ_i satisfies (5.5.1, 2).

For (5.5.1), let x be a point of U'_{il_1}. Since $1-g_{l_1}(x)=0$, we have

$$\prod_{l'<l_2}(1-g_{l'})g_{l_2}\tilde{\xi}_{il_2 r}=0$$

for any $l_2>l_1$. Hence

$$\xi_{ix}=\sum_{l\leq l_1}\prod_{l'<l}(1-g_{l'})g_l\tilde{\xi}_{ilx}.$$

By (5.4.1), $g_l\tilde{\xi}_{il}$ is contained in R^{l_1} for any $l\leq l_1$. Hence (5.5.1) follows.

For (5.5.2), consider a point $x\in U_i\cap \bar{U}'_{il_1}$. We can assume $x\in \bar{U}'_{il}$ for any $l<l_1$. Since $g_l\tilde{\xi}_{il}f_k(x)$ and $1-g_l(x)$ are non-negative for any l, k, it is sufficient to see

(5.5.3) $$\prod_{l<l_1}(1-g_l)g_{l_1}\tilde{\xi}_{il_1}f_k(x),$$

$k=1,\,2$, are positive. By the above assumption on x and the property of g_l, we have

$$(1-g_l(x))>0\ \ \ \ \mathrm{for}\ \ l<l_1,$$

and
$$g_{l_1}(x)>0\,.$$

These inequalities together with (5.4.2) imply that (5.5.3) are positive. Hence (5.5) is proved.

(5.6) For any X_i of codimension l contained in Z, let V_i be an open neighborhood of X_i in X such that $V_i\cap X_j\subset U'_{jl}$ for any X_j with $X_j\subset Z$ and $\bar X_j\supset X_i$. We denote by V the union of all V_i. Then V is an open neighborhood of Z in X. Considering $X\cap V$, $\{X_i\cap V\}$ in place of X, $\{X_i\}$ respectively, we reduce Proposition 5.1 to the next proposition.

Proposition 5.1'. *Let X be a locally closed subset of R^n, $\{X_i\}$ be a Whitney stratification of X, f_1 and f_2 be the restrictions to X of C^∞ functions $\tilde f_1$ and $\tilde f_2$ on R^n respectively, $\{\xi_i\}$ be a vector field on $\{X_i\}$, and V_i be an open neighborhood of each X_i in X such that*

(5.6.1) $f_1^{-1}(0)=f_2^{-1}(0)$,

(5.6.2) $Z=f_1^{-1}(0)$ *is closed in* R^n,

(5.6.3) $\{X_i\}$ *is compatible with* Z,

(5.6.4) *for each i, $\bar X_i\cap Z\neq\varnothing$,*

(5.6.5) *for each i, k and $x\in R^n$, X_i and $x+R^k$ are transversal if $k\geqq$ codim X_i,*

(5.6.6) *for each i, j and $x\in X_i\cap V_j$ with $X_i\not\subset Z$, $X_j\subset Z$ and $\bar X_i\supset X_j$, ξ_{ix} is contained in R^l where $l=$ codim X_j, and*

(5.6.7) *$\xi_i(f_k|_{X_i})$, for any i, k with $X_i\not\subset Z$, is positive.*

Then there exist neighborhoods W_1, W_2 of Z in X and a homeomorphism $\tau:$ $W_1\to W_2$ such that $f_2\circ\tau=f_1$ on W_1, $\tau|_Z=$ ident, and $\tau(W_1\cap X_i)=W_2\cap X_i$ for each i.

(5.7) *Proof of Proposition 5.1'.* For the sake of simplicity we assume f_1, $f_2\geqq0$. The general case requires no more than complicated notations. For each $X_i\subset Z$, there obviously exists an open neighborhood V'_i of X_i in X such that $\bar V'_i-V_i\subset\bar X_i-X_i\subset Z$. Let J be the index subset consisting of i with $X_i\not\subset Z$. Apply Lemma 4.14 to $\{X_i\}_{i\in J}$, $X^*=X-Z$, f_1, f_2, $\xi=\{\xi_i\}_{i\in J}$, $\{\underset{\underset{i\notin J}{\dim X_i=j}}{\bigcup}V_i-X_i\}$,

and $\{\underset{\underset{i\notin J}{\dim X_i=j}}{\bigcup}V'_i-X_i\}_j$. Then we can modify ξ so that it is locally integrable. Here we have to replace X by $X\cap\underset{i\notin J}{\bigcup}V'_i$ and the belonging $x\in X_i\cap V_j$ in (5.6.6) by $x\in X_i\cap V'_j$. But the replacement does not influence the conclusion of Proposition 5.1'. So we add the assumption that ξ is locally integrable to the assumptions of Proposition 5.1'. Put

$$c_i=\inf_{x\in\bar X_i-X}\{\tilde f_1(x),\tilde f_2(x)\}\,,\ i\in J\,.$$

If $\overline{X}_i - X = \varnothing$ (we see later this is not the case), then we put $c_i = \infty$. Then c_i are positive by (5.6.2), the local closedness of X and by the boundedness of X_i. Let $\Theta : D \to X^* = X - Z$, $D \subset X^* \times \boldsymbol{R}$, be the flow of ξ (4.12). Then D is open, Θ is continuous, and for any $x \in X^*$ we have $t_x^- < 0 < t_x^+$ (may be infinity) such that

$$D \cap (x \times \boldsymbol{R}) = x \times (t_x^-, t_x^+) .$$

It follows from (5.6.7) that $f_j \circ \Theta |_{x_0, (t_{x_0}^-, t_{x_0}^+)}$ is strictly increasing for each $x_0 \in X_i$, $i \in J$, and each j.

Moreover if $f_j(x_0) < c_i$, then $f_j \circ \Theta(x_0, t)$ converges to 0, a number larger or equal to c_i as $t \to t_{x_0}^- + 0$, $t_{x_0}^+ - 0$ respectively. In fact, if not so, we have by (5.6.7) and by the boundedness of X_i a sequence $\{t_k\}$ of numbers in $(t_{x_0}^-, t_{x_0}^+)$ such that $\{\Theta(x_0, t_k)\}$ converges to a point y of $(\overline{X}_i - X_i) - Z - (\overline{X}_i - X) \subset X^* - X_i$. Since D is open, we have a neighborhood O_y of y in X and $\varepsilon > 0$ such that $D \supset O_y \times [-\varepsilon, \varepsilon]$. This means that for sufficiently large k, $\Theta(x_0, t_k) \in O_y$ and hence $(x_0, [t_k - \varepsilon, t_k + \varepsilon]) \subset D$. Hence $t_k \to \infty$ or $-\infty$ as $k \to \infty$. By the continuity of Θ, $\{\Theta(x_0, t_k \pm \varepsilon)\}_k$ converges to $\Theta(y, \pm \varepsilon)$. Hence $\{f_j \circ \Theta(x_0, t_k \pm \varepsilon)\}_k$ converges to $f_j \circ \Theta(y, \pm \varepsilon) \neq f_j(y)$. This contradicts the fact that $f_j \circ \Theta |_{x_0 < (t_{x_0}^-, t_{x_0}^+)}$ is increasing. Since f_j is bounded on X_i, we have seen also $\overline{X}_i - X \neq \varnothing$ and hence $c_i \neq \infty$.

The above property of $f_j \circ \Theta(x_0, t)$ shows that for each $x_0 \in X_i$, $i \in J$, and each j with $f_j(x_0) < c_i$, $f_j \circ \Theta$ maps diffeomorphically $(x_0 \times (t_{x_0}^-, t_{x_0}^+)) \cap (f_j \circ \Theta)^{-1}((0, c_i))$ onto $(0, c_i)$. Hence we uniquely obtain a C^∞ diffeomorphism τ_{x_0} from $\Theta((x_0 \times (t_{x_0}^-, t_{x_0}^+)) \cap (f_1 \circ \Theta)^{-1}((0, c_i)))$ to $\Theta((x_0 \times (t_{x_0}^-, t_{x_0}^+)) \cap (f_2 \circ \Theta)^{-1}((0, c_i)))$ such that $f_2 \circ \tau_{x_0} = f_1$ on the domain of definition of τ_{x_0}. We remark that $\tau_{x_0} \equiv \tau_{x_0'}$ if both x_0 and x_0' are contained in one integral curve of ξ_i. Put

$$W_{ji} = \{x \in X_i \mid f_i(x) < c_i\} \quad \text{for} \quad j = 1, 2 \text{ and } i \in J,$$

$$W_j = \bigcup_{i \in J} W_{ji} \cup Z .$$

Then W_1 is the union of Z and the domains of definition of τ_{x_0} for all $x_0 \in X_i$, $i \in J$, with $f_1(x_0) < c_i$, W_2 is the union of Z and all the images of τ_{x_0}'s, and they are both neighborhoods of Z in X. We define a map $\tau : W_1 \to W_2$ by

$$\tau(x) = \begin{cases} \tau_x(x) & \text{for} \quad x \in W_1 - Z \\ x & \text{for} \quad x \in Z . \end{cases}$$

Then clearly τ is well-defined and one-to-one, we have $f_2 \circ \tau = f_1$ on W_1 and $\tau = \text{ident}$ on Z, and for any i $\tau |_{W_1 \cap X_i}$ is a C^∞ diffeomorphism onto $W_2 \cap X_i$.

We only need to see that τ is a homeomorphism. Let $\{x_k\}$ be a sequence of points in $W_1 \cap X_i$ converging to $y_1 \in W_1 \cap X_{i'}$. We want to see that $\{\tau(x_k)\}$ converges to $\tau(y_1)$. If $i = i'$ or $i \notin J$, it is trivial by the continuity of $\tau |_{W_1 \cap X_i}$ and by the definition of $\tau |_Z$. Hence we assume $i \neq i'$ and $i \in J$. The case $i' \notin J$ clearly follows from the next statement.

(5.7.1) For each $j \notin J$ and for any $\varepsilon > 0$ there exists a neighborhood Q_j of X_j in X such that

(5.7.1)′ $\qquad$ $\mathrm{dist}\,(x,\, \Theta(x,\, t)) \leqq \varepsilon \qquad$ for $\quad x \in Q_j - Z,\, t \in (t_x^-,\, 0)$.

We prove (5.7.1) by induction on codimension of X_j. (5.7.1) is trivial if $\mathrm{codim}\, X_j = 0$. Hence, assume (5.7.1) for $\mathrm{codim}\, X_j < l$. Let $\varepsilon > 0$, and for each $j' \notin J$ with $\mathrm{codim}\, X_{j'} < l$, let $Q_{j'}$ be a neighborhood of $X_{j'}$ in X such that

(5.7.2) $\qquad$ $\mathrm{dist}\,(x,\, \Theta(x,\, t)) \leqq \varepsilon/3 \qquad$ for $\quad x \in Q_{j'} - Z,\, t \in (t_x^-,\, 0)$.

Put $Q = \bigcup\limits_{\mathrm{codim}\, X_{j'} < l} Q_{j'}$. Let X_j, $j \notin J$, be of codimension l. By (5.6.5, 6) we have an open neighborhood Q_j' of X_j in $\boldsymbol{R}^n$ and a C^∞ fiber bundle $\pi_j : Q_j' \to X_j$ such that

$$\pi_j|_{X_j} = \mathrm{ident},$$

(5.7.3) $\qquad$ $\pi_j^{-1}(y) \subset (y + \boldsymbol{R}^l) \cap B_{\varepsilon/3}(y) \qquad$ for $\quad y \in X_j$, and

$$\xi_{ix} \in \boldsymbol{R}^l \qquad \text{for} \quad x \in X_i \cap Q_j',\, i \in J,$$

where

$$B_{\varepsilon/3}(y) = \{y' \in \boldsymbol{R}^n \mid |y - y'| \leqq \varepsilon/3\}.$$

We remark that

(5.7.4) if $x_0 \in X^*$ and $t_1 < t_2 \in \boldsymbol{R}$ satisfy $\Theta(x_0,\, (t_1,\, t_2)) \subset Q_j'$, then $\pi_j \circ \Theta(x_0,\, (t_1,\, t_2))$ $=$ a point.

If necessary, shrink Q so small that for any $y \in X_j$, $\lambda(y) = \overline{\pi_j^{-1}(y)} - Q_j' - Q$ is not empty. Put

$$d(y) = \inf_{x \in \lambda(y)} f_1(x),$$

$$Q_j = \{x \in X \cap Q_j' \mid f_1(x) < d \circ \pi_j(x)\}.$$

Then we see easily $d(y) > 0$ and that Q_j is a neighborhood of X_j in X.

Moreover Q_j satisfies (5.7.1)′. In fact, let $x \in Q_j - Z$. Then there are two cases,

(i) $\qquad$ $\Theta(x,\, (t_x^-,\, 0)) \subset Q_j' \quad$ and

(ii) $\qquad$ $\Theta(x,\, (t_x^-,\, 0)) \not\subset Q_j'$.

If (i), we have by (5.7.3, 4)

$$\mathrm{dist}\,(x,\, \Theta(x,\, t)) \leqq 2\varepsilon/3 \qquad \text{for any} \quad t \in (t_x^-,\, 0),$$

which proves (5.7.1). Now consider the case of (ii). Let t_0 be the inferior of t with $\Theta(x,\, (t,\, 0)) \subset Q_j'$. Then we see by (5.7.3, 4)

$$\Theta(x,\, (t_0,\, 0)) \subset \pi_j^{-1}(\pi_j(x)) \quad \text{and}$$

$$\Theta(x,\, t_0) \in \overline{\pi_j^{-1}(\pi_j(x))} - Q_j' \quad \text{hence}$$

(5.7.5) $\mathrm{dist}(x, \Theta(x, t)) \leqq 2\varepsilon/3$ for $t \in [t_0, 0)$.

Moreover

$$\Theta(x, t_0) \in Q.$$

Indeed, if not so, then

$$\Theta(x, t_0) \in \overline{\pi_j^{-1}(\pi_j(x))} - Q_j' - Q.$$

Hence, by the definition of $d(y)$

$$d \circ \pi_j(x) \leqq f_1 \circ \Theta(x, t_0).$$

But, since $f_1(x) < d \circ \pi_j(x)$ and since f_1 is strictly increasing on $\Theta(x, (t_x^-, 0))$, we have

$$f_1 \circ \Theta(x, t_0) < d \circ \pi_j(x),$$

which is a contradiction. Hence $\Theta(x, t_0) \in Q$. This implies by (5.7.2)

$$\mathrm{dist}(\Theta(x, t_0), \Theta(x, t)) \leqq \varepsilon/3   \text{ for } t \in (t_x^-, t_0),$$

which together with (5.7.5) proves that Q_j satisfies (5.7.1)'.

Now the case $i' \in J$ remains in the proof of the convergence of $\{\tau(x_k)\}$ to $\tau(y_1)$. Let $t_1 \in (t_{y_1}^-, t_{y_1}^+)$ satisfy $\Theta(y_1, t_1) = \tau(y_1)$, and let ε be a small positive number. Then, by the local integrability of ξ there exist neighborhoods $R' \subset R$ of y_1 in X and $\delta > 0$ such that

$$R \times [t_1 - \delta, t_1 + \delta] \subset D,$$

(5.7.6) $f_2 \circ \Theta(x, t_1 - \delta) < f_2 \circ \Theta(y_1, t_1) = f_2 \circ \tau(y_1)$

$$= f_1(y_1) < f_2 \circ \Theta(x, t_1 + \delta) \text{ for any } x \in R',$$

(5.7.7) $|\Theta(x, t) - \tau(y_1)| < \varepsilon$ for $(x, t) \in R \times [t_1 - \delta, t_1 + \delta]$.

By the definition of τ there exists a number δ_k for each k such that

(5.7.8) $\Theta(x_k, t_1 + \delta_k) = \tau(x_k)$.

Since $\{x_k\}$ converges to y_1, and since $f_2 \circ \tau(x_k) = f_1(x_k)$, (5.7.6, 8) show that for sufficiently large k

$$f_2 \circ \Theta(x_k, t_1 - \delta) < f_2 \circ \Theta(x_k, t_1 + \delta_k) < f_2 \circ \Theta(x_k, t_1 + \delta),$$

which implies that $|\delta_k| < \delta$. Therefore it follows from (5.7.7, 8) that

$$|\tau(x_k) - \tau(y_1)| < \varepsilon.$$

Namely $\{\tau(x_k)\}$ converges to $\tau(y_1)$. As the continuity of τ^{-1} is shown in the same way, we complete the proof of Proposition 5.1'.

Remark 5.8. In Proposition 5.1, given a locally finite family $\{Y_j\}$ of sub-analytic subsets of $\boldsymbol{R}^n$, we can choose $\tau: W_1 \to W_2$ so that $\tau(W_1 \cap Y_j) = W_2 \cap Y_j$ for any j. In fact, using (2.11), we can construct the Whitney stratification

$\{X_i\}$ of X in (5.3) so that it is compatible with $\{Y_j\}$. Then τ defined in (5.7) automatically satisfies $\tau(W_1 \cap Y_j) = W_2 \cap Y_j$ for any j.

Remark 5.9. We can refine Proposition 5.1 as follows. For any small neighborhood W_3 of Z in X there exists moreover a homeomorphism $\tilde{\tau}$ of X such that $\tilde{\tau}|_{W_1} = \tau$ and $\tilde{\tau}|_{X-W_3} = \mathrm{ident}$, here W_1 and W_2 are chosen to be contained in W_3.

Proof. (Continued from (5.7).) Let Λ be the quotient topological space of $W_1 - Z$ under the equivalence relation $x \sim \Theta(x, t)$ for any t, and let $\theta_1: W_1 - Z \to \Lambda$ be the canonical surjection, namely Λ is an orbit space. Then for any finite subset J^0 of J, we have a positive number c^0 by the local integrability of ξ such that $c^0 < c_i$ for any $i \in J^0$ and that the restriction of θ_1 to $\bigcup_{i \in J^0} X_i \cap f_1^{-1}(c^0)$ is homeomorphic to the image $\theta_1(W_1 \cap \bigcup_{i \in J^0} X_i)$. Repeat this argument for a sequence $\{J^j\}_j$ of finite subsets of J such that $\{\bigcup_{i \in J^j} X_i\}_j$ is a locally finite open covering of X^*. Then let $\{c^j\}$ be the consequent numbers, and let $c(y)$ be a positive continuous function on Λ such that $c \circ \theta_1(x) < c^j$ for $x \in X_i$, $i \in J^j$. Here the existence of such c follows easily from the remarks that $\{\theta_1(W_1 \cap \bigcup_{i \in J^j} X_i)\}_j$ is a locally finite open covering of Λ and that Λ is a normal space. Then it follows that the restriction of θ_1 to $\{x \in W_1 - Z \mid f_1(x) = c \circ \theta_1(x)\}$ is a homeomorphism onto Λ.

Choose $c(y)$ so small that W_3' contains

$$W_3' = Z \cup \{x \in W_1 - Z \mid f_1(x) \leq c \circ \theta_1(x)\},$$

and let $c'(y)$ be a positive continuous function on Λ such that $c'(y) < c(y)$ and that $\tau(W_1')$ is contained in the interior of W_3' in X where

$$W_1' = Z \cup \{x \in W_1 - Z \mid f_1(x) \leq c' \circ \theta_1(x)\}.$$

Put $\qquad W_2' = \tau(W_1'), \quad \theta = (\theta_1, f_1): W_3' - Z \longrightarrow \Lambda \times (0, \infty).$

Then θ is a homeomorphism onto the image, and we have

$$\theta(W_3' - Z) = \{(y, t) \in \Lambda \times (0, \infty) \mid t \leq c(y)\},$$

$$\theta(W_1' - Z) = \{t \leq c'(y)\},$$

$$\theta(W_2' - Z) \subset \mathrm{Int}\, \theta(W_3' - Z).$$

Since each integral curve of ξ is invariant under τ, the homeomorphism $\tau': \theta(W_1' - Z) \to \theta(W_2' - Z)$ defined by $\tau' \circ \theta = \theta \circ \tau$ is in the form $\tau'(y, t) = (y, \tau''(y, t))$ for some continuous function τ''. Let $y \in \Lambda$ be fixed. Then $\tau''(y, t)$ is a homeomorphism from $y \times \{t \in (0, \infty) \mid t \leq c'(y)\}$ to $\theta(W_2' - Z) \cap y \times \mathbf{R}$. Hence we can extend $\tau''(y, t)$ to a homeomorphism of $y \times \{t \leq c(y)\}$ so that it is linear on the

complement of $y \times \{t \leq c'(y)\}$. Let the extension be denoted by $\tilde{\tau}''_y$. Then clearly

$$\tilde{\tau}'(y, t) = (y, \tilde{\tau}''_y(t)) \qquad \text{for} \quad (y, t) \in \theta(W'_3 - Z)$$

is a homeomorphism of $\theta(W'_3 - Z)$ such that

$$\tilde{\tau}' = \begin{cases} \tau' \text{ on } \theta(W'_1 - Z) \\ \text{ident on } \{(y, t) \in \Lambda \times (0, \infty) \,|\, t = c(y)\} \ . \end{cases}$$

Hence the extension $\tilde{\tau}$ of τ induced by $\tilde{\tau}'$ is a homeomorphism of W'_3 which is the identity on the boundary of W'_3 in X. Therefore we can extend $\tilde{\tau}$ moreover to X so that it is the identity outside W'_3. After replacing W_1, W_2 by W'_1, W'_2 respectively, we obtain the remark.

Remark 5.10.   In Proposition 5.1, τ is the identity on $\{x \in W_1 \,|\, f_1(x) = f_2(x)\}$. In Remark 5.9, assume $f_1 = f_2$ on a neighborhood of a closed subset C of X. Then $\tilde{\tau}$ can be chosen to be the identity on a neighborhood of C. These are clear by the method of construction τ and $\tilde{\tau}$.

Remark 5.11.   In Proposition 5.1 and Remarks 5.9, 10, assume moreover that X is an analytic manifold and that f_1, f_2 are analytic. Then τ and $\tilde{\tau}$ can be chosen to be analytic and C^∞ differentiable on $X - Z$ respectively.

Proof.   Come back to (5.3). The stratification $\{X_i\}$ can be chosen so that $\{X_i - Z\}$ is the family of all connected components of $X - Z$. Then $\Theta : D \to X^*$ in (5.7) turns out to be of class C^∞. Here the strata may fail to satisfy the boundedness condition. But the condition is used only in the construction of W_i in (5.7), namely of c_i. In case in which X is an analytic manifold and f_1 and f_2 are analytic, it is clear that there exists a continuous function $c(y)$ on Λ in the proof of Remark 5.9, since Θ is differentiable. Replace W_i by W'_i of the proof of Remark 5.9. Then the C^∞ differentiability of τ is trivial by the definition of τ. About $\tilde{\tau}$, it is not of class C^∞ at

$$\theta^{-1}\{(y, t) \in \Lambda \times (0, \infty) \,|\, t = c(y) \text{ or } = c'(y)\} \ .$$

To make $\tilde{\tau}$ differentiable, at first choose $c(y)$, $c'(y)$ to be differentiable. It has a meaning since Λ naturally has a C^∞ manifold structure induced by X. Nextly we modify $\tilde{\tau}$ to be differentiable. It is easy because of the differentiability of Θ and since we only need to consider $\tilde{\tau}'$ on $\{(y, t) \in \Lambda \times (0, \infty) \,|\, t \leq c(y)\}$. We omit the details.

For an analytic modification of τ, we apply Theorem 8.4 in [16] to $f_2 \circ \tau_1$, f_1 on $W_1 - Z$ where $\tau_1 : W_1 - Z \to W_2 - Z$ is an analytic approximation diffeomorphism of $\tau|_{W_1 - Z}$ in the Whitney topology [5]. Then there exists an analytic diffeomorphism $\tau_2 : W_1 - Z \to W_1 - Z$ such that $f_1 = f_2 \circ \tau_1 \circ \tau_2$. Moreover $\tau_1 \circ \tau_2$ can be chosen to be arbitrarily close to $\tau|_{W_1 - Z}$ in the Whitney topology by Remark 8.5 in [16]. Hence, if we extend $\tau_1 \circ \tau_2$ to W_1 by putting $\tau_1 \circ \tau_2 = \text{ident}$ on Z,

then the extension is a homeomorphism from W_1 to W_2 which satisfies the requirement of Remark 5.11.

Problem. In Proposition 5.1, can τ be subanalytic?

Proof of Proposition 5.2. Let $M \subset \mathbf{R}^n$, and let b be a number contained in $f_1(S)$. Put $S' = f_1^{-1} f_1(S)$. By the definition, S is identical with $S(M, n)$ defined in (5.3). Clearly $S(M, n)$ is closed, and (5.3.7) implies that $S(M, n) - f_1^{-1}(b)$ is empty in a neighborhood of $f_1^{-1}(b)$. Hence it follows that $f_1(S)$ is a discrete set and hence that S' is an analytic set.

Let A be a connected component of $M - S'$. Then we have $a_1, a_2 \in \overline{A} - A$ such that $f_1(A) = (f_1(a_1), f_1(a_2))$ since M is compact. We, consequently, have $f_1(A) = f_2(A)$. Hence, by the proofs of Proposition 5.1 and Remark 5.11, we have neighborhoods W_1, W_2 and U of S' in M with $W_1, W_2 \subset U$, a C^∞ vector field ξ on $U_1 = U - S'$ and a homeomorphism $\tau: W_1 \to W_2$ such that $f_2 \circ \tau = f_1$ on W_1, $\tau = $ident on S' and $\xi(f_i|_U) > 0$, $i = 1, 2$ and that τ is defined separately on each integral curve of ξ, namely, $\tau(\text{curve} \cap W_1) = \text{curve} \cap W_2$.

Let ϕ be a C^∞ function on M such that $\phi = 0$ on a neighborhood of S', $= 1$ on a neighborhood of $M - U$ and $0 \leq \phi \leq 1$. Put

$$\xi' = (1 - \phi)\xi + \phi |df_1| df_2 + \phi |df_2| df_1.$$

Then ξ' is a C^∞ vector field on $M - S'$ satisfying $\xi'(f_i|_{M-S'}) > 0$, $i = 1, 2$, because of $(|df_1| df_2 + |df_2| df_1) f_i > 0$ on $M - S'$ and $\xi' = \xi$ on (a neighborhood of $S') - S'$. In the same way as the construction of τ, we obtain a one-to-one mapping τ': $M \to M$ using the integral curves of ξ' such that $f_2 \circ \tau' = f_1$ on M. Then we automatically have $\tau' = \tau$ on a neighborhood of S', and the differentiability of $\tau'|_{M-S'}$ is clear. Hence τ' is a homeomorphism of M. An analytic modification of $\tau'|_{M-S'}$ proceeds just in the same way as the proof of Remark 5.11, and $\tau'|_{S'}$ is the identity. Hence Proposition 5.2 is proved.

We will apply Proposition 5.1 and Remarks 5.9, 10 later in the following form.

Corollary 5.12. *Let $Z \subset X \subset \mathbf{R}^n$ be polyhedrons, and let f be a subanalytic function on X. Assume that X and Z are closed in $\mathbf{R}^n$ and that for any $x \in Z$, $f^{-1} f(x) \equiv Z$ as germs at x. Then there exists a homeomorphism τ of X such that $f \circ \tau$ is PL on a neighborhood of Z and that τ is the identity on Z and outside a given neighborhood of Z. Moreover if there is a closed subset C of X such that f is PL on a neighborhood of C, then τ is chosen to be the identity on a neighborhood of C.*

Proof. Let $L \subset K$ be simplicial complexes such that $|K| = X$ and $|L| = Z$. By subdividing K and by the assumption we can assume that L is full in K (see [13] for the definition), hence that for any $\sigma \in K$, $f(\sigma \cap L)$ consists of at most one member and that $f^{-1} f(\sigma \cap L) \cap \sigma$ is a simplex of L. The last condi-

tion means that if we define a linear function f_1 on σ by putting $f_1=f$ on the vertexes, then $f-f(\sigma\cap L)$ and $f_1-f(\sigma\cap L)$ satisfy the conditions of Proposition 5.1 on σ. Let f_1 be defined globally on X in this way separately on each simplex. Then f_1 is a well-defined subanalytic function on X. Apply Proposition 5.1 and Remark 5.9 to f and f_1 on a small neighborhood of each connected component of Z. Then we have a homeomorphism τ of X such that $f\circ\tau=f_1$ on a neighborhood of Z and that τ is the identity outside a given neighborhood of Z. Since f_1 is PL, so is $f\circ\tau$ on a neighborhood of Z. Thus the first half of the corollary is proved.

For the latter half, subdivide K so that f is linear on any simplex which intersects with C. Let Z' denote the union of such simplexes. Subdividing K once more, we can assume moreover that f is linear on each simplex which intersects with Z'. Then we automatically have $f=f_1$ on a neighborhood of Z'. Hence, by Remark 5.10, τ can be the identity on a neighborhood of Z' and hence of C. Therefore Corollary 5.12 is proved.

Remark 5.13. Let ε be a positive continuous function on X in Remarks 5.9, 10 and Corollary 5.12. Then $\tilde{\tau}$ in Remarks 5.9, 10 and τ in Corollary 5.12 can be chosen so that

$$|\tilde{\tau}(x)-x|<\varepsilon(x) \quad \text{and} \quad |\tau(x)-x|<\varepsilon(x) \qquad \text{for} \quad x\in X.$$

It is clear by (5.7.1) and by the method of construction of $\tilde{\tau}$ and τ.

§6. Analytic Triangulations

At this stage, let us consider the problem of piecewise linearization of an analytic function $f: R^n \to R$. For any point $x\in R^n$, the pair $(R^n, f^{-1}f(x))$ can be triangulated by Proposition 3.1. Hence Corollary 5.12 shows the existence of a homeomorphism τ of R^n such that $f\circ\tau$ is PL on a neighborhood of $f^{-1}f(x)$. In the extension of this argument to the global R^n there are two difficulties. At first the domains where f is piecewise linearized are too narrow to cover R^n. The other difficulty is that we can not say "τ is subanalytic". If τ were subanalytic, then we would repeat the argument above even if the domains of piecewise linearization of f intersect.

We need another method of piecewise linearization. Fortunately we can obtain by Lemma 6.15 below a large domain of piecewise linearization where f is C^∞ regular and τ is subanalytic. We will show also an analytic triangulation of an analytic manifold (Proposition 6.11). The results in this section are based on [12].

Let K always denote a simplicial complex, $M\subset R^n$ an analytic manifold and $r=1, \cdots, \infty$ or ω. For any subcomplex L of K, let $N(L, K)$ denote the simplicial neighborhood of L in K, namely the subcomplex generated by all $\sigma\in K$ with $\sigma\cap|L|\neq\varnothing$.

Definition 6.0. A *linear isomorphism* $g: K \to L$ of simplicial complexes is a homeomorphism $g: |K| \to |L|$ carrying each simplex of K linearly onto one of L.

Definition 6.1. A C^r *map* $g: K \to M$ means a map $g: |K| \to M$ such that the restrictions $g|_\sigma$, $\sigma \in K$, are of class C^r. We say that g is *analytic* on a subpolyhedron P of $|K|$ if we have a subcomplex K_1 of K with $P = |K_1|$ such that $g|_{K_1}$ is of class C^ω. Let K' be another simplicial complex with $|K'| = |K|$, and let $g': K' \to M$ be a C^r map. We write $g = g'$ on P if we have subcomplexes K_1 of K and K'_1 of K' such that $P = |K_1| = |K'_1|$, $K_1 = K'_1$ and $g|_{K_1} = g'|_{K_1}$.

Definition 6.2. For any $b \in |K|$, let $\mathrm{st}(b, K)$ denote the union of simplexes of K which contain b. Let $g: K \to M \subset R^n$ be a C^r map, and let $b \in |K|$. We define $dg_b^*: \mathrm{st}(b, K) \to R^n$ by

$$dg_b^*(x) = d(g|_\sigma)_b(x - b),$$

where σ is a simplex containing $b + \varepsilon(x - b)$ for small $\varepsilon > 0$. Here K is regarded as to be contained in a Euclidean space, and $x - b$ is regarded as a tangent vector at b. The definition does not depend on the choice of imbedding of K in a Euclidean space.

Definition 6.3. Let $g: K \to M \subset R^n$ be a C^r map. We call g an *imbedding* if g and dg_b^* are homeomorphic onto the images for any $b \in |K|$. If g is also a homeomorphism onto M, it is called a C^r *triangulation* of M.

Definition 6.4. Let $g: K \to R^n$ be a C^r map. Fix an imbedding of K in a Euclidean space. Let $\bar{\partial}$ be a positive continuous function on $|K|$. A map $g': |K| \to R^n$ is called a $\bar{\partial}$-*approximation* of g if

(6.4.1) for some subdivision K' of K, $g': K' \to R^n$ is a C^r map,

(6.4.2) $|g(b) - g'(b)| < \bar{\partial}(b)$ for any $b \in |K|$ and if

(6.4.3) $|dg_b^*(x) - dg_b'^*(x)| \leq \bar{\partial}(b) |x - b|$ for any $b \in |K|$, $x \in \mathrm{st}(b, K')$.

We will use the above definitions in the cell complex case too.

Lemma 6.5 (Theorem 8.8, [12], see also the exercise (c) following it). *Let* $g: K \to R^n$ *be a* C^r *imbedding. Then there exists a positive continuous function* $\bar{\partial}$ *on* $|K|$ *such that any* $\bar{\partial}$-*approximation of* g *is an imbedding.*

Definition 6.6. Let $g: K \to R^n$ be a C^r map. The *secant map* induced by g, $g_K: K \to R^n$, is defined by $g_K = g$ on the vertex set of K so that it is linear on each simplex of K.

Lemma 6.7 (Theorem 9.6, [12]). *Let* $g: K \to R^n$ *be a* C^r *map,* K *finite. Given a constant function* $\delta > 0$ *on* $|K|$, *there is a subdivision* K' *of* K *such that the secant map* $g_{K'}$ *is a* $\bar{\partial}$-*approximation of* g.

Lemma 6.8. *Let* $g: K \to \boldsymbol{R}^n$ *be a* C^r *map. Let* K_1 *be a finite subcomplex of* K. *Given* $\varepsilon > 0$, *there is* $\delta > 0$ *such that any* δ-*approximation* $h: K'_1 \to \boldsymbol{R}^n$ *of* $g|_{K_1}$ *can be extended to an* ε-*approximation* $\tilde{h}: K' \to \boldsymbol{R}^n$ *of* g *for some subdivision* K' *of* K *with* $K'|_{|K_1|} = K'_1$. *Here we can choose* $\tilde{h} = g$ *on* $|K| - |\mathring{N}(K_1, K)|$ *where* $\mathring{N} = N(K_1, K) - \{\sigma \in K \mid \sigma \cap |K_1| = \varnothing\}$.

Proof. If $r \leq \infty$, the lemma coincides with Lemma 9.8, [12], so assume $r = \omega$. In this case we have to modify the proof in [12].

Special case. Assume K is the complex generated by one simplex σ, and $K_1 = K - \{\sigma\}$.

Proof of special case. Let a be the barycenter of σ. Put

$$\sigma'_{[b,c]} = \{t(x - a) + a \mid x \in \sigma', \ b \leq t \leq c\} \qquad \text{for} \quad \sigma' \in K'_1 \text{ and } b, c \in \boldsymbol{R}.$$

$$\sigma'_1 = \sigma'_{[0, 1/2]}, \ \sigma'_2 = \sigma'_{[1/2 \ 1/2]}, \ \sigma'_3 = \sigma'_{[1/2, 1]} \quad \text{and} \quad \sigma_3 = \bigcup_{\sigma' \in K'_1} \sigma'_3.$$

Let K' be the cell complex consisting of a, σ'_1, σ'_2, σ'_3 for all $\sigma' \in K'_1$ and K'_1. Let $\rho: K'_1 \cup \{\sigma'_2, \sigma'_3 \mid \sigma' \in K'_1\} \to \partial\sigma$ be the map defined by $\rho(t(x - a) + a) = x$, $x \in \partial\sigma$. Then ρ is of class C^ω. Hence $h \circ \rho$ can be an arbitrarily close approximation of $g \circ \rho$. Hence if we put

$$\alpha(t(x - a) + a) = 2(t - 1/2) \qquad \text{for} \quad 1/2 \leq t \leq 1, \ x \in \partial\sigma \quad \text{and}$$

$$\tilde{h} = \begin{cases} g + (h \circ \rho - g \circ \rho) \cdot \alpha & \text{on} \quad \sigma_3 \\ g & \text{on} \quad \sigma - \sigma_3, \end{cases}$$

then $\tilde{h}$ can be an ε-approximation of g such that $\tilde{h}|_{\partial\sigma} = h$. Since there is a simplicial subdivision of K' fixed on K'_1 (see e.g. [13]), the special case is proved.

The general case proceeds in the same way as [12]. We prove it by induction. Assume a construction of a map $\tilde{h}_k: |K_1| \cup (|K| - |\mathring{N}(K_1, K)|) \cup |K^k| \to \boldsymbol{R}^n$ which is at once an approximation of the restriction of g to the domain and an extension of h such that $\tilde{h}_k = g$ on $|K| - |\mathring{N}(K_1, K)|$, where K^k is the k-skeleton of K, (if $k = 0$, it is trivial since $|K^0| \subset |K_1| \cup (|K| - |\mathring{N}(K_1, K)|)$.) We want to construct $\tilde{h}_{k+1}$ on $|K_1| \cup (|K| - (\mathring{N}(K_1, K))|) \cup |K^{k+1}|$. For any $\sigma \in (K^{k+1} - K^k) \cap (\mathring{N}(K_1, K) - K_1)$, $\tilde{h}_k$ is already defined on $\partial\sigma$. Hence, by the special case, we can extend $\tilde{h}_k|_{\partial\sigma}$ onto σ so that the extension is an approximation of $g|_\sigma$. Therefore $\tilde{h}_k$ has an extension $\tilde{h}_{k+1}$ to $|K_1| \cup (|K| - |\mathring{N}(K_1, K)|) \cup |K^{k+1}|$ which is an approximation of the restriction of g on the domain. Clearly $\tilde{h}_{k+1} = g$ on $|K| - |\mathring{N}(K_1, K)|$. Thus we construct inductively $\tilde{h} = \tilde{h}_m$ which satisfies the requirement in Lemma 6.8, where m is the dimension of $N(K_1, K)$. Here the grades of approximation of $\tilde{h}_k$, $k = 0, \cdots, m$, to the restriction of g to the domains are decided by downward induction so that $\tilde{h}$ is an ε-approximation of g. We omit the details (see [12]).

Lemma 6.9. *Let $g: K \to M$ be a C^r map, $r \neq \omega$. Let K_1 be a finite sub-complex such that $g(|N(K_1, K)|)$ is contained in a coordinate neighborhood of M. Let $\varepsilon > 0$. Then there exists an ε-approximation $g': K' \to M$ of g such that $g' = g$ on $|K| - |\mathring{N}(K_1, K)|$ and that $g'|_{|K_1|}$ is analytic. Moreover if $g|_{|K_2|}$ is analytic for some subcomplex K_2 of K, so is $g'|_{|K_1 \cup K_2|}$.*

Proof. Since we have an analytic coordinate neighborhood of M containing $g(|N(K_1, K)|)$, and since g' is required to equal g on $|K| - |\mathring{N}(K_1, K)|$, the problem is reduced to the case $M = \mathbf{R}^n$. In this case, the first half of the lemma follows from Lemmas 6.7, 8, namely $g'|_{|K_1|}$ is defined to be the secant map $g_{K_1'}$ for some subdivision K_1' of K_1, and we let g' be its extension to $|K|$ constructed in Lemma 6.8.

For the latter half, we have to check up the above extension of $g_{K_1'}$. Return to the proof of Lemma 6.8. In the special case, there are two cases $K_2 = K$ or $K_2 \subset K_1$. If $K_2 = K$, $g': K' \to \mathbf{R}^n$ is analytic because so are $\rho: K_1' \cup \{\sigma_2', \sigma_3' | \sigma' \in K_1'\} \to \mathbf{R}^n$ and $\alpha: K_1' \cup \{\sigma_2', \sigma_3' | \sigma' \in K_1'\} \to \mathbf{R}$. If $K_2 \subset K_1$, the latter half is trivial. Since the general case is treated by stages, and since each stage is equivalent to the special case, the latter half also in the general case is clear.

Lemma 6.10. *Let $g: K \to M$ be a C^r map, $r \neq \omega$. Let ε be a positive continuous function on $|K|$. Then there exists an analytic ε-approximation $g': K' \to M$ of g. Moreover, let $K_1 \subset K_2$ be subcomplexes of K such that $N(K_1, K) \subset K_2$. If g is analytic on $|K_2|$, then we can choose $g' = g$ on $|K_1|$.*

Proof. Let K_3 be a subcomplex of K such that $|K_3| \cap |K_1| = \emptyset$ and $|K_3| \cup |K_2| = |K|$, for example $K_3 = K - \mathring{N}(K_1, K)$. Subdivide K so that for any $\sigma \in K_3$, $g(|N(\sigma, K)|)$ is contained in some coordinate neighborhood of M and that the restriction of the subdivision to $|K_1|$ remains K_1. We use the same notation K for the subdivision. Let us order all simplexes of K_3 as $\sigma_1, \sigma_2, \cdots$. We will construct g' inductively.

Put $g^{-1} = g^0 = g$. Let k be a non-negative integer. Assume an $\varepsilon/2^k$-approximation $g^k: K(k) \to M$ of $g^{k-1}: K(k-1) \to M$ such that $g^k = g^{k-1}$ on $|K| - |\mathring{N}(\sigma_k, K)|$ and that g^k is analytic on $|K_2| \cup \bigcup_{i=1}^{k} \sigma_i$, where $K(k)$ is a subdivision of $K(k-1)$ such that $K(-1) = K(0) = K$. Replacing ε by smaller one, we can assume that the ε_k-neighborhood of $g(|N(\sigma_k, K)|)$ in M for any k is contained in some coordinate neighborhood of M where $\varepsilon_k = \sup_{x \in |N(\sigma_k, K)|} \varepsilon(x)$. Hence $g^k(|N(\sigma_{k+1}, K)|)$ is contained in a coordinate neighborhood of M. Apply Lemma 6.9 to g^k and the subcomplex of $K(k)$ whose underlying polyhedron is σ_{k+1}. Then we obtain a subdivision $K(k+1)$ of $K(k)$ and an $\varepsilon/2^{k+1}$-approximation $g^{k+1}: K(k+1) \to M$ of g^k such that $g^{k+1} = g^k$ on $|K| - |\mathring{N}(\sigma_{k+1}, K)|$ and that g^{k+1} is analytic on $|K_2| \cup \bigcup_{i=1}^{k+1} \sigma_i$. Since $\{|\mathring{N}(\sigma_m, K)|\}_{m=1,2,\cdots}$ is locally finite in $|K|$, and since $K(k+1)|_{|K|-|\mathring{N}(\sigma_{k+1}, K)|} =$

$K(k)|_{|K|-|\mathring{N}(\sigma_{k+1}, K)|}$, the limit of $g^k : K(k) \to M$, as $k \to \infty$, is an ε-approximation of g. Let the limit be $g' : K' \to M$. As $|K|$ is covered by $|K_2|$ and $\sigma_1, \cdots$, it follows from the definition of g' that g' is analytic on $|K|$. It is clear that $g' = g$ on $|K_1|$ also by the definition. Hence we proved the lemma.

Proposition 6.11. *M has a C^ω triangulation.*

Proof. By Theorem 10.6, [12] we have a C^∞ triangulation $g : K \to M$. There exists a positive continuous function ε on $|K|$ such that any ε-approximation of g is an imbedding (Lemma 6.5). Moreover we easily choose ε so that such an ε-approximation is surjective on M, namely, a C^∞ triangulation of M. Hence Proposition 6.11 follows from Lemma 6.10.

Remark 6.12. Let M may have boundary. Let $g_1 : K_1 \to M$ and $g_2 : K_2 \to M$ be C^1 triangulations. Then there are arbitrarily close approximations g_1', g_2' of g_1, g_2 respectively such that $g_2'^{-1} \circ g_1'$ is a linear isomorphism (Theorem 10.5, [12]). Hence, if $g : K \to M$ is a C^1 triangulation, then $|K|$ is automatically a *PL* manifold since we have at least one C^1 triangulation $g_1 : K_1 \to M$ such that $|K_1|$ is a *PL* manifold (Cairns-Whitehead).

Lemma 6.13. *Let $g_1 : K_1 \to M$ and $g_2 : K_2 \to M$ be C^r imbeddings whose images are closed in M. Let $\varepsilon > 0$ be a continuous function on the disjoint union of $|K_1|$ and $|K_2|$. Let L_1, L_2 be subcomplexes of K_1, K_2 respectively such that*

$$g_1(|N(L_1, K_1)|) \cap g_2(|K_2|) = \varnothing \quad and$$

$$g_2(|N(L_2, K_2)|) \cap g_1(|K_1|) = \varnothing .$$

Then there exist a complex K, linear isomorphisms i_1 and i_2 from K_1' and K_2', subdivisions of K_1 and K_2 respectively, to subcomplexes of K and a C^r imbedding $g : K \to M$ such that $g_1' = g \circ i_1$, $g_2' = g \circ i_2$ are ε-approximations of g_1, g_2 respectively, that $i_1(|K_1'|) \cup i_2(|K_2'|) = |K|$ and that $g_1' = g_1$ on $|L_1|$, $g_2' = g_2$ on $|L_2|$.
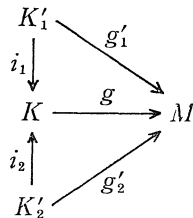
Proof. If $r \neq \omega$, the lemma coincides with Theorem 10.4, [12] except the last requirements, $g_1' = g_1$ on $|L_1|$, $g_2' = g_2$ on $|L_2|$. But these conditions are easy to see by the method of construction of g_1', g_2' in [12]. It follows moreover that $g_1' = g_1$ on $|N(L_1, K_1)|$, $g_2' = g_2$ on $|N(L_2, K_2)|$. We omit making sure of it.

If $r = \omega$, regard g_1 and g_2 as C^∞ imbeddings, and let K, i_1, i_2, K_1', K_2', g_1' and

g_2' be the results in the C^∞ case. We assume however that g_1', g_2' are $\varepsilon/2$-approximations of g_1, g_2 respectively and that $g_1'=g_1$ on $|N(L_1, K_1)|$, $g_2'=g_2$ on $|N(L_2, K_2)|$. We want to modify g to be analytic. We remark that g is analytic on $i_1|N(L_1, K_1)|\cup i_2|N(L_2, K_2)|$. Let $\delta: |K|\to R$ be a positive continuous function such that for any δ-approximation $h: K'\to R^n$ of the zero map $0: K\to R^n$, $h\circ i_1$ and $h\circ i_2$ are $\varepsilon/2$-approximations of the zero maps $0: K_1'\to R^n$ and $0: K_2'\to R^n$ respectively. Apply Lemma 6.10 to $g: K\to M$ and δ. Let $g': K'\to M$ be the resulting analytic δ-approximation of g such that $g=g'$ on $i_1(|L_1|)\cup i_2(|L_2|)$. Let K_1'', K_2'' be the subdivisions of K_1', K_2' respectively such that i_1 on K_1'' and i_2 on K_2'' are linear isomorphisms onto the images. Then K', $i_1: K_1''\to K'$, $i_2: K_2''\to K'$ and $g': K'\to M$ satisfy the requirements of Lemma 6.13. Hence the lemma is proved.

Proposition 6.14. *Even if M has boundary, it has a C^ω triangulation.*

Proof. Let $h: L\to\partial M$, $g_1: K_1\to M-\partial M$ be C^ω triangulations (Lemma 6.11), and $\phi: \partial M\times[0, 1]\to M$ be an analytic collar such that $\phi(x, 0)=x$. Let K_2 be a simplicial subdivision of the cell complex $L\times[0, 1]$. Then $g_2=\phi\circ(h, \text{ident}): K_2\to M$ is a C^ω imbedding whose image is a closed collar of M.

Subdivide K_1 finely enough, and assume that if we put

$$K_3=\{\sigma\in K_1|g_1(\sigma)\cap\phi(\partial M\times[0, 1/3])=\varnothing\},$$

we have

$$g_1(|K_3|)\supset M-\phi(\partial M\times[0, 2/3]).$$

Put $g_3=g_1|_{K_3}$. Then $g_3: K_3\to M$ is a C^ω imbedding whose image is closed in M. Now we remark that Lemma 6.13 holds true even if M has boundary in the case in which $g_1^{-1}(\partial M)\subset|L_1|$ and $g_2^{-1}(\partial M)\subset|L_2|$. Hence, by Lemma 6.13, we have a complex K, linear isomorphisms i_1 and i_2 from K_2' and K_3', subdivisions of K_2 and K_3 respectively, to subcomplexes of K and a C^ω imbedding $g: K\to M$ such that $g_2'=g\circ i_1$, $g_3'=g\circ i_2$ are close approximations of g_2, g_3 respectively and that $g_2'=g_2$ on a polyhedral neighborhood of $|L|\times0$ in $|K_2|$. If the above approximations are sufficiently close, we have

$$g_2'(|K_2|)\supset\phi(\partial M\times[0, 3/4])$$

because of $g_2'=g_2$ near $|L|\times0$, and

$$g_3'(|K_3|)\supset M-\phi(\partial M\times[0, 3/4]),$$

namely g is surjective. Hence g can be a C^ω triangulation of M.

Lemma 6.15. *Let M_1 be an analytic manifold of dimension$=\dim M-1$ possibly with boundary, let $\phi: M_1\times[0, 1]\to M$ be an analytic imbedding whose image is closed in M, let $h: L\to M_1$ be a C^ω triangulation, let K_1 be a simplicial subdivision of the cell complex $L\times[0, 1]$, and let K_2 be a subcomplex of K_1 whose image*

under (h, *ident*) *does not intersect with* $\partial(M_1 \times [0, 1])$. *Put*

$$g_2 = \phi \circ (h, \ ident)|_{K_2}.$$

Then there exists a C^ω *triangulation* $g : K \to M$ *and a subcomplex* K_3 *of* K *such that*

$$g(|K_3|) = g_2(|K_2|)$$

and that $g_2^{-1} \circ g : K_3 \to K_2$ *is a linear isomorphism.*

The proof proceeds just in the same way as Proposition 6.14, so we omit it.

We will apply Lemma 6.15 to the proof of Theorem I later in the following situation. The analytic function f is C^∞ regular on $\phi(M_1 \times [0, 1])$, and for any connected component C of M_1 there is a constant c such that $f \circ \phi(x, t) = t + c$, $(x, t) \in C \times [0, 1]$. Then $f \circ g$ is subanalytic on $|K|$ and PL on $|K_3|$, namely f can be piecewise linearized on an arbitrarily large subset of $\operatorname{Int} \phi(M_1 \times [0, 1])$ closed in M.

Corollary 6.16. *Let* $g : K \to M$ *be a* C^ω *triangulation. Given a locally finite family* $\{X_i\}$ *of subanalytic subsets of* M, *there exist a subdivision* K' *of* K *and a subanalytic homeomorphism* $\tau : |K| \to M$ *such that for any* $\sigma \in K$, $\tau(\sigma) = g(\sigma)$, *that for any* $\sigma \in K'$, $\tau(\mathring{\sigma})$ *is an analytic submanifold of* M *and* $\tau|_{\mathring{\sigma}} : \mathring{\sigma} \to \tau(\mathring{\sigma})$ *is an analytic diffeomorphism and that* $\{\tau(\mathring{\sigma}) \,|\, \sigma \in K'\}$ *is compatible with* $\{X_i\}$.

Proof. Assume K is contained in $\mathbf{R}^m$ so that $|K|$ is closed in $\mathbf{R}^m$. Since g is of class C^ω, by (2.5.1) $\{g^{-1}(X_i)\}$ is a locally finite family of subanalytic subsets of $\mathbf{R}^m$. Hence the corollary follows from Proposition 3.1′.

Let us consider the semi-algebraic case of the above results. The proofs of the results below proceed in the same way as the C^ω case, so we omit the details of proofs.

Definition 6.17. If M is semi-algebraic in $\mathbf{R}^n$, we call it a *Nash manifold*. An analytic map between Nash manifolds is called a *Nash map* if the graph is semi-algebraic. A C^ω map $g : K \to M$, K being finite, is called *of class Nash* if the graph is semi-algebraic.

If K is finite, Lemma 6.8 holds true in the case of Nash map too since ρ and α in the proof of Lemma 6.8 are of class Nash. In Lemmas 6.9, 10, if M is a Nash manifold, if K is finite and if $g|_{|K_2|}$ is of class Nash, then $g'|_{|K_1 \cup K_2|}$ in Lemma 6.9 and g' in Lemma 6.10 can be of class Nash. We remark here that any secant map on a finite complex is of class Nash. We obtain also the Nash case of Lemma 6.13. As the replacement is clear, we omit the details. By these facts, we have the following Nash case of Propositions 6.11, 14.

Proposition 6.18. *If* M *is a compact Nash manifold possibly with boundary, then it has a Nash triangulation, namely, there exist a finite complex* K *and a*

triangulation $g: K \to M$ *of class Nash.*

For the case of non-compact Nash manifolds, we need the following

Lemma 6.19 (Theorem 1, [17]). *If M is a non-compact Nash manifold, there exists a compact Nash manifold M' with boundary such that M is Nash diffeomorphic to $M' - \partial M'$.*

Lemma 6.20. *Let M be a non-compact Nash manifold. Then there exist a complex K in $\mathbf{R}^m$ and a C^ω triangulation $g: K \to M$ such that $|K|$ and g are semi-algebraic.*

Proof. Apply Proposition 6.18 to M' in Lemma 6.19. Let $g': K' \to M'$, $K' \subset \mathbf{R}^m$, be a Nash triangulation. Then $|K'|$ is a *PL* manifold with boundary (Remark 6.12). Hence we have $g'(\partial |K'|) = \partial M'$, so a triangulation K of $|K'| - \partial |K'|$ compatible with K' and $g = g'|_{|K|}$ satisfy the requirement of Lemma 6.20.

Corollary 6.21. *Let M be a compact Nash manifold possibly with boundary, and let $g: K \to M$ be a Nash triangulation. Given a finite family $\{X_i\}$ of semi-algebraic subsets of M, there exist a subdivision K' of K and a semi-algebraic homeomorphism $\tau: |K| \to M$ such that $\tau(\sigma) = g(\sigma)$ for any $\sigma \in K$, that for any $\sigma \in K'$, $\tau(\mathring{\sigma})$ is a Nash submanifold of M and $\tau|_{\mathring{\sigma}}: \mathring{\sigma} \to \tau(\mathring{\sigma})$ is a Nash diffeomorphism and that $\{\tau(\mathring{\sigma}) | \sigma \in K'\}$ is compatible with $\{X_i\}$.*

Proof. Clear by Proposition 3.9.

Corollary 6.22. *Let M be a non-compact Nash manifold. Given a finite family $\{X_i\}$ of semi-algebraic subsets of M, there exist a complex K in $\mathbf{R}^m$, a C^ω triangulation $g: K \to M$ and a homeomorphism $\tau: |K| \to M$ such that $|K|$ and τ are semi-algebraic, that for any $\sigma \in K$, $\tau(\mathring{\sigma})$ is a Nash submanifold of M and $\tau|_{\mathring{\sigma}}: \mathring{\sigma} \to \tau(\mathring{\sigma})$ is a Nash diffeomorphism and that $\{\tau(\mathring{\sigma}) | \sigma \in K\}$ is compatible with $\{X_i\}$.*

Proof. We only need to remark that for a compact Nash manifold M' with boundary, any semi-algebraic subset of $M' - \partial M'$ is also semi-algebraic in M'.

Corollaries 6.21, 22 are generalizations of Theorem 3, [10] which treated only the case $M = \mathbf{R}^n$.

Remark 6.23. $|K|$ in Corollaries 6.16, 21, 22 are *PL* manifolds by Remark 6.12. If it were not so, we could not apply Concordance Implies Isotopy Theorem to the proof of Theorem I. This is one of the reasons why we refined a result of [4] in Proposition 3.1. It is difficult to obtain the corollaries by the result of [4].

§7. Proof of Theorem I

The last but not least tool of the proof is Concordance Implies Isotopy Theorem of Kirby-Siebenmann [6].

Let P, Q be polyhedrons. A map $f: P \to \boldsymbol{R}^n$ is called a *PL map* if there is a simplicial complex K with $|K|=P$ such that f is linear on each simplex of K. A map $f: P \to Q$ is called a *PL map* if Q is PL imbedded in some $\boldsymbol{R}^m$ so that $f: P \to \boldsymbol{R}^m$ is a PL map. A homotopy ϕ_t, $0 \leq t \leq 1$, between topological spaces is called an *isotopy* if for each t, ϕ_t is a homeomorphism.

Lemma 7.1 ([6]). *Let M_1, M_2 be metrized PL manifolds, $\phi: M_1 \times [0, 1] \to M_2$ be a homeomorphism, $C \subset M_1$ be a closed subset, and ε be a positive continuous function on $M_1 \times [0, 1]$. Assume that $\dim M_2 \neq 4, 5$ and that ϕ is PL on a neighborhood of $M_1 \times 0 \cup C \times [0, 1]$. Then there exists an isotopy $\phi_t: M_1 \times [0, 1] \to M_2$, $0 \leq t \leq 1$, of ϕ such that*

(7.1.1) *ϕ_1 is a PL homeomorphism,*

(7.1.2) *$\phi_t = \phi$, $0 \leq t \leq 1$, on a neighborhood of $M_1 \times 0 \cup C \times [0, 1]$, and that*

(7.1.3) *$\mathrm{dist}(\phi_t(x), \phi(x)) < \varepsilon(x)$ for all $x \in M_1 \times [0, 1]$, $0 \leq t \leq 1$.*

Now we have finished preparing for the proof of Theorem I. We begin to prove it. The proof consists of seven steps $(7.2), \cdots, (7.8)$.

(7.2) We assume M to be non-compact, because the compact case is easier to prove. There exist compact analytic submanifolds with boundary M_1, M_2, $\cdots$ and without boundary $N_0 = \varnothing$, N_1, N_2, $\cdots$ of M such that

$$\bigcup_{i=1}^{\infty} M_i = M, \qquad \partial M_i = N_{i-1} \cup N_i,$$

$$M_i \cap M_{i+1} = N_i, \ M_i \cap M_j = \varnothing \quad \text{for any} \ \ i, j \ \text{with} \ |i-j| \geq 2.$$

They are constructed for example as follows. Assume M is closed in $\boldsymbol{R}^n$. Put $\varphi(x) = |x|^2$, $x \in M$. Then φ is a positive proper analytic function on M. Let $a_0 = -1 < a_1 < a_2 < \cdots$ be a sequence of C^∞ regular values of φ tending to infinity such that $\varphi(M) \ni a_i$. Then

$$M_i = \varphi^{-1}([a_{i-1}, a_i]) \quad \text{and} \quad N_i = \varphi^{-1}(a_i), \ i=1, 2, \cdots$$

satisfy the conditions above.

Put $N = \bigcup_{i=1}^{\infty} N_i$. Then N is an analytic manifold closed in M. We remark the fact that the set of critical values of an analytic function on a compact manifold (may having boundary) is a finite set. Let S_1, S_2, $\cdots$ be a sequence of finite subsets of $\boldsymbol{R}$ such that for each i, S_i contains all critical values of $f|_{M_i}$, $f|_{N_{i-1}}$ and $f|_{N_i}$ and that $S_i \supset S_{i-1} \cup S_{i+1}$ for even i. We put $S_0 = \varnothing$ and $M_0 = \varnothing$ for convenience. Put

$$X = \bigcup_{i=1}^{\infty} (M_i \cap f^{-1}(S_i)) \,,$$

$$Y_1 = \bigcup_{i=1}^{\infty} N_i \cap f^{-1}((S_i - S_{i+1}) \cup (S_{i+1} - S_i))$$

$$Y_2 = X \cap N - Y_1 = \bigcup_{i=1}^{\infty} N_i \cap f^{-1}(S_i \cap S_{i+1}) \,.$$

Then X, Y_1 and Y_2 are a closed semi-analytic set and analytic sets respectively. We remark that X and Y_2 contains all critical points of f and $f|_N$ respectively.

(7.3) Applying Lemma 6.15, we want to piecewise linearize f on a large domain containing Y_1. There exist an open neighborhood U of $Y_1 \cup (M - X)$ in M and a C^∞ vector field ξ on U such that $\xi f > 0$ on U and that $\xi|_{N \cap U}$ is a vector field on $N \cap U$. Indeed, put $\xi_1 = df$, and let ξ_2 be an extension of $d(f|_N)$ to a C^∞ vector field on M. Then we have an open neighborhood U_1 of $N - Y_2$ in M such that $\xi_1 f > 0$ on $U_1 \cup (M - X)$ and $\xi_2 f > 0$ on U_1. Let U_2 be another neighborhood of $N - Y_2$ with $\bar{U}_2 - U_1 \subset Y_2$, and let ρ be a C^∞ function on $M - Y_2$ such that $0 \le \rho \le 1$, $\rho = 0$ on U_2 and $= 1$ outside U_1. Then $\xi = \rho \xi_1 + (1 - \rho) \xi_2$ and $U = U_1 \cup (M - X)$ satisfy the required conditions.

Multiplying ξ by $1/\xi f$, we assume moreover $\xi f = 1$ on U. Let $\Theta : D \to U$ be the C^∞ flow of ξ, namely, Θ is defined by

$$\Theta(x, 0) = x, \quad \frac{\partial}{\partial t} \Theta(x, t) = \xi_{\Theta(x, t)}$$

and $D \subset U \times \boldsymbol{R}$ is the maximal open set containing $U \times 0$. Then we have

(7.3.1) $\quad f \circ \Theta(x, t) = f(x) + t \quad$ for $(x, t) \in D$ because of $\xi f = 1$,

(7.3.2) $\quad \Theta(x, t) \in N \quad$ for $x \in N$, $(x, t) \in D$.

(7.3.3) $\quad \Theta(x, t) \in M_i \quad$ for $x \in M_i$, $(x, t) \in D$, $i = 1, \cdots$.

Let i be a positive odd integer. Let S_i^0 be the open ∂_i-neighborhood of S_i in $\boldsymbol{R}$ for some small $\partial_i > 0$ such that

(7.3.4) $\quad \bar{S}_i^0 \cap (S_{i+1} \cup S_{i-1}) = S_i \,.$

Put

$$S_i^+ = S_i + \partial_i = \{t + \partial_i \,|\, t \in S_i\} \quad \text{and}$$

$$S_i^- = S_i - \partial_i = \{t - \partial_i \,|\, t \in S_i\} \,.$$

Then $f^{-1}(S_i^-) \cap M_i$ and $f^{-1}(S_i^-) \cap M_i$ are compact analytic manifolds with boundary in $N_{i-1} \cup N_i$. Under Θ, $f^{-1}(\boldsymbol{R} - S_i^0) \cap M_i$ is C^∞ diffeomorphic to $(f^{-1}(S_i^+) \cap M_i) \times [0, 1]$. We will enlarge $f^{-1}(\boldsymbol{R} - S_i^0)$ a little, and it is on this set that we will piecewise linearize f in (7.3).

Put

$$L_i = f^{-1}(S_i^+) \cap \varphi^{-1}([a_{i-1} - \delta_i', \ a_i + \delta_i']),$$

$$L_i' = f^{-1}(S_i^+) \cap \varphi^{-1}([a_{i-1} - \delta_i'/2, \ a_i + \delta_i'/2])$$

for small $\delta_i' > 0$ where φ is defined in (7.2). Choose δ_i' small enough. Then L_i, L_i' are compact analytic manifolds with boundary such that $L_i - \partial L_i \supset L_i'$, $L_i' - \partial L_i' \supset f^{-1}(S_i^+) \cap M_i$ and that $(x, t) \in D$ for $x \in L_i$, $0 \leq t \leq \min\{s \in S_i^- \mid s > f(x)\} - f(x)$. It means that we have a C^∞ imbedding $\phi_i : L_i \times [0, 1] \to U$ such that $\phi_i(x, 0) = x$, $f \circ \phi_i(x, 1) \in S_i^-$ for $x \in L_i$, that for each $x \in L_i$, $\phi_i(x \times [0, 1])$ is contained in the integral curve of ξ through x and that if $\Theta(x, t_0) = \phi_i(x, 1)$, $t_0 \in \mathbf{R}$ then $\phi_i(x, t) = \Theta(x, t_0 t)$, $0 \leq t \leq 1$. The last property together with (7.3.1) tells us that $f \circ \phi \mid_{x \times [0,1]}$ is linear for each fixed x.

From the other properties and (7.3.3, 4) it follows that for each connected component C of L_i, $C \times \mathbf{R} \cap D \subset C \times (s_1 - f(C), \ s_2 - f(C))$ where (s_1, s_2) is the connected component of $\mathbf{R} - S_i$ containing $f(C)$. We see easily from (7.3.2) that the set $\Theta(C \times \{s_1 - f(C), s_2 - f(C)\} \cap D)$ has a positive distance from $N_{i-1} \cup N_{i+1}$ if it is not empty. Hence, shrinking U near $f^{-1}(\{s_1, s_2\}) \cap (N_{i-1} \cup N_{i+1})$ if necessary, we can assume $C \times \{s_1 - f(C), s_2 - f(C)\} \cap D = \emptyset$. Then we have

(7.3.5) $$C \times \mathbf{R} \cap D = C \times (s_1 - f(C), \ s_2 - f(C)),$$

and hence $\Theta \mid_{L \times \mathbf{R} \cap D} : L \times \mathbf{R} \cap D \to M$ is a C^∞ imbedding. Choose δ_j' for all odd j so small that

(7.3.6) $$\Theta(L_{j'} \times \mathbf{R} \cap D) \cap \Theta(L_{j''} \times \mathbf{R} \cap D) = \emptyset \quad \text{for odd} \quad j' \neq j''.$$

Before applying Lemma 6.15, we need to modify ξ to be analytic. Remember $M \subset \mathbf{R}^n$. Let us regard ξ_x as a tangent vector of $\mathbf{R}^n$ at x, namely ξ is regarded as a C^∞ map from U to $\mathbf{R}^n$. Approximate ξ by an analytic map ξ' in the Whitney topology [5]. Let p_x, $x \in M$, be the orthogonal projection of the tangent space $T_x \mathbf{R}^n$ to $T_x M$. Then $\{p_x \circ \xi_x' \mid x \in M\}$ is an analytic vector field on M close to ξ. We use the same notation ξ for the approximation. Choose the approximation so close that $\xi f > 0$ on U. Assume $\xi f = 1$ on U by the same reason as before. We then define the flow of new ξ too and use the same notation $\Theta : D \to U$ for it. Consider the conditions (7.3.1, $\cdots$, 6). Clearly (7.3.1) remains true, and (7.3.6) can do so for some smaller δ_i', but (7.3.2, 3) fail in general, (the fact is, we can continue them by using the analytic sheaf theory in the same way as Chapter II, [16]). We assume in place of (7.3.2, 3) that

(7.3.7) $$\Theta((L_j' - \partial L_j') \times \mathbf{R} \cap D) \supset M_j - f^{-1}(S_j) \quad \text{for any odd } j, \text{ and}$$

(7.3.8) $$\Theta(L_i \times \mathbf{R} \cap D) \subset M_i \cup (M_{i-1} - \partial M_{i-1}) \cup (M_{i+1} - \partial M_{i+1})$$

which follows from (7.3.6, 7) for $j = i-2$, i and $i+2$. The assumptions are clearly possible if the approximation is close enough. By (7.3.8) and by the same reason

as before, (7.3.5) can remain true. We remark that Θ and ϕ_i are analytic and that the image of ϕ_i is closed in M.

Put

$$L=\bigcup_{i:\text{ odd}}L_i, \qquad L'=\bigcup_{i:\text{ odd}}L'_i,$$

and let $\phi: L\times[0, 1]\to M$ be defined by $\phi|_{L_i\times[0,1]}=\phi_i$. By Proposition 6.14, we have a C^ω triangulation $h: W\to L$. Subdivide W finely enough. Then we have a subcomplex W'' of W whose underlying polyhedron is a PL manifold with boundary, is contained in Int $|W|$ and contains $h^{-1}(L')$ in its interior (see Chapter 2, 3, [13]). Put $L''=h(|W''|)$. Let $\tilde{\phi}_i: L_i\times[-\delta''_i, 1+\delta''_i]\to M$ be a well-defined analytic extension of ϕ_i which is an imbedding for small $\delta''_i>0$. Apply Lemma 6.15 to $\{\tilde{\phi}_i\}_{i:\text{ odd}}$ and $h: W\to L$. Then we have a C^ω triangulation $g: K\to M$ and a subcomplex K_1 of K such that

$$g(|K_1|)=\phi(L''\times[0, 1]), \quad \text{and that}$$

(7.3.9) $\qquad (h, \text{ident})^{-1}\circ\phi^{-1}\circ g: |K_1| \longrightarrow |W''|\times[0, 1]$

is a PL homeomorphism. Hence we can assume moreover that a subcomplex K_2 of K_1 has the underlying polyhedron which is carried by g onto $\phi(L''\times 0)$. Then

(7.3.10) $\qquad (h, \text{ident})^{-1}\circ\phi^{-1}\circ g: |K_2| \longrightarrow |W''|\times 0$

is a PL homeomorphism. Now, by the definition of ϕ, $f\circ\phi\circ(h, \text{ident})$ is a PL function on $|W|\times[0, 1]$. Hence (7.3.9) tells us that

(7.3.11) $\qquad f\circ g$ is PL on $|K_1|$.

We can assume that $g: K\to M$ given in Theorem I coincides with the above $g: K\to M$. The reason is the following. Let $g^*: K^*\to M$ be another C^∞ triangulation. Then by Remark 6.12 there are C^∞ approximations $g': K'\to M$, $g^{*\prime}: K^{*\prime}\to M$ of g, g^* respectively such that $g'^{-1}\circ g^{*\prime}$ is a linear isomorphism. Hence $g\circ g'^{-1}\circ g^{*\prime}: K^{*\prime}\to M$ is an approximation of g^*. As $g^{*\prime-1}\circ g'(|K_1|)$ is the underlying polyhedron of a subcomplex $K_1^{*\prime}$ of $K^{*\prime}$, and since $f\circ g$ is PL on $|K_1|$, $f\circ g\circ g'^{-1}\circ g^{*\prime}$ is PL on $|K_1^{*\prime}|$. Hence we can replace $g: K\to M$, K_1 by $g\circ g'^{-1}\circ g^{*\prime}: K^{*\prime}\to M$, $K_1^{*\prime}$ respectively. Therefore the assumption above is admitted.

(7.4) For any subset A of M, let A^* denote the inverse image of A under $g: |K|\to M$. Put

$$D^*=\{(x, t)\in U^*\times\boldsymbol{R}\,|\,(g(x), t)\in D\} .$$

Let $\Theta^*: D^*\to U^*$, $\phi^*: L^*\times[0, 1]\to U^*$, $h^*: W\to L^*$, $f^*: M^*\to\boldsymbol{R}$ and $\phi_i^*: L_i^*\times[0, 1]\to M^*$, i: odd, be defined by $\Theta^*=g^{-1}\circ\Theta\circ(g, \text{ident})$, $\phi^*=g^{-1}\circ\phi\circ(g, \text{ident})$, $h^*=g^{-1}\circ h$, $f^*=f\circ g$ and $\phi_i^*=g^{-1}\circ\phi_i\circ(g, \text{ident})$. From now on we consider f^* on M^* in place of f on M and ident: $K\to M^*$ in place of $g: K\to M$ in Theorem

I. We summarize what we obtained.

(7.4.1) M^*, L''^* are a PL manifold and one with boundary respectively by Remark 6.12 and (7.3.10). L'^*, X^*, Y_1^*, and Y_2^* are closed subanalytic subsets of M^*. We have

$$|K|=M^*, \quad |K_2|=L''^*=\phi^*(L''^*\times 0), \quad |K_1|=\phi^*(L''^*\times[0, 1]).$$

Put

$$|K_1|_i=\phi_i^*(L_i''^*\times[0, 1]), \quad i=1, 3, \cdots.$$

(7.4.2) Θ^*, ϕ^*, h^* are subanalytic maps. f^* on X^* is locally constant.

(7.4.3) $f^*\circ\Theta^*(x, t)=f^*(x)+t$ for $(x, t)\in D^*$ by (7.3.1).

(7.4.4) For any connected component C of L_i^*, i: odd,

$$C\times\boldsymbol{R}\cap D^*=C\times(s_1-f^*(C), s_2-f^*(C))$$

where (s_1, s_2) is the connected component of $\boldsymbol{R}-S_i$ containing $f^*(C)$ by (7.3.5).

(7.4.5) $$\phi^*|_{L''^*\times[0,1]}: L''^*\times[0, 1] \longrightarrow |K_1|$$

is a PL homeomorphism by (7.3.9) and because $h^*|_{|W''|}: |W''|\to L''^*$ is a PL homeomorphism by (7.3.10).

(7.4.6) The definition of ϕ shows that

$$\phi^*(x, 0)=x \quad \text{for} \quad x\in L''^*,$$

$$f^*\circ\phi_i^*(x, 1)\in S_i^- \quad \text{for each odd } i \text{ and} \quad x\in L_i''^*,$$

$$\phi^*(x\times[0, 1])\subset\Theta^*(x\times\boldsymbol{R}\cap D) \quad \text{for each} \quad x\in L''^*, \text{ and}$$

$f^*\circ\phi^*|_{x\times[0,1]}$ is linear for each $x\in L''^*$.

(7.4.7) $f^*|_{|K_1|}$ is PL by (7.3.11).

(7.4.8) $\Theta^*((L_i'^*-\partial L_i'^*)\times\boldsymbol{R}\cap D^*)\supset M_i^*-f^{*-1}(S_i)$, i: odd, by (7.3.7).

(7.4.9) $\Theta^*(L_i^*\times\boldsymbol{R}\cap D^*)\subset M_i^*\cup(M_{i-1}^*-\partial M_{i-1}^*)\cup(M_{i+1}^*-\partial M_{i+1}^*)$, i: odd,

$$\text{by (7.3.8).}$$

(7.4.10) Θ^* is one-to-one on $L''^*\times\boldsymbol{R}\cap D$ by (7.3.5, 6) and PL on $L''^*\times\boldsymbol{R}\cap\Theta^{*-1}$ ($|K_1|$) by (7.4.5, 6).

The statement that Θ^* is PL and one-to-one on $L''^*\times\boldsymbol{R}\cap\Theta^{*-1}(|K_1|)$ is equivalent to

(7.4.10)' Θ^* is a PL homeomorphism from $\Theta^*(C, c)\times[-c, s_2-s_1-2\delta_i-c]$ to the connected component of $|K_1|$ containing C for any connected component C of $L_i''^*$ and any $0\leq c\leq s_2-s_1-2\delta_i$ where (s_1, s_2) is defined in (7.4.4).

(7.5) Here we triangulate X^*. For each odd $i>0$, $Z_i=|K_1|_i\cap f^{*-1}((S_{i+1}\cup S_{i-1})-S_i)$ is a polyhedron by (7.4.7) and contains $X^*\cap|K_1|_i$ by the definition

of X, S_i, S_{i+1} and S_{i-1}. Put $Z = \bigcup\limits_{i:\text{ odd}} Z_i$. Apply Proposition 3.1' to K and X^* $\cup Z$. Then we have a subdivision K' of K and a subanalytic homeomorphism τ_1 of M^* such that $\{\tau_1(\mathring{\sigma}) \mid \sigma \in K'\}$ is compatible with $X^* \cup Z$. Since X^* and Z are closed in M, it follows that $\tau_1^{-1}(X^* \cup Z)$ is a subpolyhedron of M^*. Moreover Remark 3.8 tells us that τ_1 can be the identity outside an arbitrarily small neighborhood of $\overline{X^* - Z} = \overline{X^* - |K_1|}$ since Z is a subpolyhedron and since $X^* \cup Z$ coincides with Z outside $\overline{X^* - Z}$. Hence the properties in (7.4) remain true when we shrink $|K_2|$ and $|K_1|$ a little and when we shift the problem by τ_1^{-1}, namely, when we consider $\tau_1^{-1}(X^*)$, $f^* \circ \tau_1$, etc. in place of X^*, f^*, etc., so we can add the following to the properties in (7.4).

(7.5.1) $X^* \cup Z$ is a subpolyhedron of M^*.

Since τ in Theorem I is required to be a C^0 ε-approximation of g ($=$ident), for the admission of replacement of X^*, f^*, etc. by $\tau_1^{-1}(X^*)$, $f^* \circ \tau_1$, etc., τ_1 must be arbitrarily close to the identity in the C^0 fine topology. To be exact, let $M^* \subset \boldsymbol{R}^{n*}$, and ε' be a positive continuous function on M^*. Then τ_1 must be chosen so that $|\tau_1(x) - x| < \varepsilon'(x)$. It is possible by (3.1.2) if we subdivide K so that the diameter of each $\sigma \in K$ is smaller than any $\varepsilon'(x)$, $x \in \sigma$.

(7.6) We want to piecewise linearize f^* on a neighborhood of X^*. If $f^{-1} f(x) \equiv X^*$ as germs at x for any $x \in X^*$, then it would follow directly from Corollary 5.12 and (7.5.1). But the equality above is not correct in general, so we remove subsets from M^* and X^*. Let $P_2 \subset L''^* - \partial L''^*$ be a PL manifold with boundary closed in L''^* and containing L'^* in its interior. Put $P_1 = \phi^*(P_2 \times [0, 1])$. Then P_1 is a PL manifold with boundary by (7.4.5), f^* is PL on a neighborhood of $P_1 \cap X^*$ by (7.4.7), and it also follows from the definitions of S_i, L_i' that $f^{-1}|_{X^* - \text{Int } P_1}(f(x)) \equiv X^* - \text{Int } P_1$ as germs at x for any $x \in X^* - \text{Int } P_1$. Apply Corollary 5.12 to $M^* - \text{Int } P_1$, $X^* - \text{Int } P_1$ and f^*. Then we have a homeomorphism τ_2 of $M^* - \text{Int } P_1$ such that $f^* \circ \tau_2$ is PL on a neighborhood of $X^* - \text{Int } P_1$ in $M^* - \text{Int } P_1$ and that τ_2 is the identity on $\partial P_1 \cup (X^* - \text{Int } P_1)$ and outside a small neighborhood of $X^* - \text{Int } P_1$. Extend τ_2 to M^* by putting $\tau_2 = $ ident on P_1. Consider $f^* \circ \tau_2$, P_1, etc. in place of f^*, L''^*, etc. respectively, and use the former notations for them. Then we have

(7.6.1) f^* is PL on a neighborhood of X^*.

The replacement here is admitted because τ_2 can be arbitrarily close to the identity in the C^0 fine topology by Remark 5.13. We remark that the first half of (7.4.2) fails now and that Θ^*, ϕ^*, h^* are only a continuous flow, continuous maps respectively. The other properties in (7.4) and (7.5.1) remain true.

(7.7) We will piecewise linearize f^* on $\Theta^*((L^{*''} - \partial L''^*) \times \boldsymbol{R} \cap D^*)$. Let C be a connected component of $L''^* - \partial L''^*$ contained in some $L_i''^*$. Put $Q = \Theta^*(C \times \boldsymbol{R} \cap D^*)$. Let (s_1, s_2) be the connected component of $\boldsymbol{R} - S_i$ containing $f^*(C)$.

Then $C \times \boldsymbol{R} \cap D^* = C \times (s_1 - f^*(C), s_2 - f^*(C))$ by (7.4.4). Hence we have

$$f^*(Q) = (s_1, s_2), \quad f^* \circ \psi_i^*(C \times 0) = s_1 + \delta_i, \quad f^* \circ \psi_i^*(C \times 1) = s_2 - \delta_i$$

by (7.4.3) and by the definitions of L_i'' and ψ_i. Let δ_i''' be a positive small number such that f^* is PL on $f^{*-1}([s_1, s_1 + \delta_i''] \cup [s_2 - \delta_i''', s_2]) \cap (M_i^* \cup M_{i-1}^* \cup M_{i+1}^*)$ and that

$$((s_1, s_1 + \delta_i + \delta_i'''] \cup [s_2 - \delta_i - \delta_i''', s_2)) \cap (S_{i+1} \cup S_{i-1}) = \varnothing .$$

We want to find a homeomorphism τ_{3C} of $\bar{Q}$ such that $\tau_{3C} = \text{ident on } (\bar{Q} - Q) \cup \Theta^*(\bar{C} \times [\delta_i''', s_2 - s_1 - 2\delta_i - \delta_i''']) \cup (\text{a neighborhood of } X^* \cap \bar{Q} \text{ in } \bar{Q})$ and that $f^* \circ \tau_{3C}$ is PL on Q. In (7.8) we will shorten δ_i''' moreover. Put $Q_1 = f^{*-1}([s_1 + \delta_i''', s_1 + \delta_i + \delta_i''']) \cap Q$. Let $\chi : C \times [0, 1] \to Q_1$ be a homeomorphism defined by $\chi(x, s) = \Theta^*(x, \delta_i''' - s\delta_i)$. Then we have

$$\chi(C \times 1) = f^{*-1}(s_1 + \delta_i''') \cap Q_1 ,$$

$$\chi(C \times 0) = f^{*-1}(s_1 + \delta_i + \delta_i''') \cap Q_1 ,$$

$$\chi(C \times [0, \delta_i''' / \delta_i]) = |K_1| \cap Q_1 ,$$

$f^* \circ \chi |_{x \times [0, 1]}$ is linear, and the restriction of χ to $C \times [0, \delta_i''' / \delta_i]$ is a PL homeomorphism to $|K_1| \cap Q_1$ by (7.4.3, 10). Let $\tilde{\chi} : \bar{C} \times [0, 1] \to \bar{Q}_1$ be the continuous extension of χ, whose existence is trivial.

Recall the following well-known facts. Let $A \subset B$ be compact polyhedrons, and let ζ be a PL function on B such that $\zeta^{-1}(0) = A$. Then there exists a positive number c such that for any $0 < c' < c$, $\zeta^{-1}([-c', c'])$ is a regular neighborhood of A in B. See [13] for the definition of regular neighborhood from which the above fact follows easily. If B, moreover, is a PL manifold (with boundary), then any regular neighborhood is a PL manifold with boundary (Corollary 3.30, [13]).

Consider a compact polyhedron pair of a small closed neighborhood of $\bar{Q}$ in M^* which is a PL manifold with boundary and the intersection of the neighborhood and $f^{*-1}(s_1) \cup f^{*-1}(s_1 + \delta_i)$. Then, by the above facts, the intersection of the neighborhood and $f^{*-1}((s_1, s_1 + \delta_i''] \cup (s_1 + \delta_i, s_1 + \delta_i + \delta_i''])$ is a PL manifold with boundary for small δ_i'''. Therefore, since

$$Q_1 = Q - f^{*-1}((s_1, s_1 + \delta_i''] \cup (s_1 + \delta_i + \delta_i''', s_2)) ,$$

we assume that Q_1 is a PL manifold with boundary. Hence we can apply Lemma 7.1 to $\chi : C \times [0, 1] \to Q_1$.

Let, accordingly, $\chi_t : C \times [0, 1] \to Q_1$, $0 \leq t \leq 1$, be an isotopy of χ such that

(7.7.1) χ_1 is a PL homeomorphism,

(7.7.2) $\chi_t = \chi$, $0 \leq t \leq 1$, on $C \times 0$ and that

(7.7.3) for all t, χ_t are close approximations of χ in the C^0 fine topology. Hence the isotopy can be extended to an isotopy $\tilde{\chi}_t : \bar{C} \times [0, 1] \to \bar{Q}_1$, $0 \leq t \leq 1$, by putting

$$\tilde{\chi}_t(x, s) = \tilde{\chi}(x, s) \quad \text{for} \quad x \in \bar{C} - C, \ 0 \leq s \leq 1.$$

Now we define τ_{3C} on $Q_2 = f^{*-1}([s_1 + \delta_i'''/2, \ s_1 + \delta_i + \delta_i''']) \cap Q$. Put

$$\tau_{3C}(x) = \begin{cases} \chi \circ \chi_1^{-1}(x) & \text{on} \quad Q_1 - f^{*-1}(s_1 + \delta_i'''), \\ \Theta^*(\chi \circ \chi_t^{-1} \circ \Theta^*(x, \ s_1 + \delta_i''' - f^*(x)), \ f^*(x) - s_1 - \delta_i''') & \text{on} \quad Q_2 - \text{Int } Q_1 \end{cases}$$

where

$$t = 2(f^*(x) - s_1 - \delta_i'''/2)/\delta_i'''.$$

We remark that $Q_2 - \text{Int } Q_1 \ni x \to \Theta^*(x, \ s_1 + \delta_i''' - f^*(x)) \in M^*$ is a projection onto $\chi(C \times 1) = f^{*-1}(s_1 + \delta_i''') \cap Q$, that $\chi_t|_{C \times 1}$, $0 \leq t \leq 1$, are homeomorphisms onto $\chi(C \times 1)$ and that $f^* \circ \tau_{3C}(x) = f^*(x)$ for $x \in Q_2 - \text{Int } Q_1$. Hence $\tau_{3C}|_{Q_2 - \text{Int } Q_1}$ is a homeomorphism of $Q_2 - \text{Int } Q_1$, coincides with $\chi \circ \chi_1^{-1}$ on $\chi(C \times 1)$, equals the identity on $Q_2 - \text{Int } Q_2 - Q_1 = f^{*-1}(s_1 + \delta_i'''/2) \cap Q$ and is a close approximation of the identity in the C^0 fine topology by (7.7.3). Consequently τ_{3C} is a homeomorphism of Q_2 close to the identity. It trivially follows also that $f^* \circ \tau_{3C}$ is PL on $Q_2 - \text{Int } Q_1$. Look into τ_{3C} on Q_1 in detail. $f^* \circ \tau_{3C}|_{Q_1}$ is PL since so are $f^* \circ \chi$ and χ_1^{-1}. From (7.4.10)' and from the equality $\Theta^*((\Theta^*(C \times \delta_i''')) \times [-\delta_i, 0]) = Q_1$ it follows that $\chi^{-1} \circ \Theta^*|_{\Theta^*(C \times \delta_i''') \times [-\delta_i, 0]}$ is a PL homeomorphism onto $C \times [0, 1]$. Hence $\tau_{3C}^{-1} \circ \Theta^*$ is PL on $\Theta^*(C \times \delta_i''') \times [-\delta_i, 0]$. We remark that $\tau_{3C}^{-1} \circ \Theta^* = \tau_{3C}^{-1} \circ \Theta^* \circ (\tau_{3C}, \text{ident})$ on $\Theta^*(C \times \delta_i''') \times [-\delta_i, 0]$.

Define τ_{3C} also on $f^{*-1}([s_2 - \delta_i - \delta_i''', \ s_2 - \delta_i'''/2]) \cap Q$ in the same way as above, and extend it to $\bar{Q}$ by putting $\tau_{3C} = \text{ident}$ on $(f^{*-1}([s_1, \ s_1 + \delta_i'''/2] \cup (s_1 + \delta_i + \delta_i''', \ s_2 - \delta_i - \delta_i''') \cup (s_2 - \delta_i'''/2, \ s_2]) \cap Q) \cup (\bar{Q} - Q)$. Then $f^* \circ \tau_{3C}$ is PL on Q, $\tau_{3C}^{-1} \circ \Theta^* \circ (\tau_{3C}, \text{ident})$ is PL on $\Theta^*(C \times \delta_i''') \times [-\delta_i, \ s_2 - s_1 - \delta_i - 2\delta_i''']$ by (7.4.10)', and τ_{3C} is a close approximation of the identity in the C^0 (fine) topology. We remark that if we put $\tau_{3C} = \text{ident}$ outside $\bar{Q}$ then we can extend τ_{3C} globally to M^*.

Repeat this argument for each connected component of $L''^* - \partial L''^*$. Then we have a homeomorphism τ_3 of M^* close to the identity and equal to it outside $\Theta^*((L''^* - \partial L''^*) \times \mathbf{R} \cap D^*)$ such that $f^* \circ \tau_3$ is PL on $\Theta^*((L''^* - \partial L''^*) \times \mathbf{R} \cap D^*) \cup$ (a neighborhood of X^*). Replace f^*, Θ^*, L''^*, $|K_1|$, etc. by $f^* \circ \tau_3$, $\tau_3^{-1} \circ \Theta^* \circ (\tau_3, \text{ident})$, $\bigcup_C \Theta^*(\bar{C} \times \delta_i''')$, $\bigcup_C \Theta^*(\bar{C} \times [\delta_i''', \ s_2 - s_1 - 2\delta_i - \delta_i'''])$, etc. respectively, and use the former notations for them. Then we can assume

(7.7.4) f^* is PL on $\Theta^*((L''^* - \partial L''^*) \times \mathbf{R} \cap D^*)$, and

(7.7.5) Θ^* is PL on $C \times [\delta_i''' - \delta_i, \ s_2 - s_1 - \delta_i - \delta_i''']$ for any connected component C of $L''^* - \partial L''^*$ where δ_i, δ_i''', s_1 and s_2 are given as before.

Clearly (7.7.5) is equivalent to

(7.7.6) Θ^* is PL on $\Theta^*(C \times c) \times [\delta_i''' - \delta_i - c, \ s_2 - s_1 - \delta_i - \delta_i''' - c]$ for any $c \in [\delta_i''' - \delta_i, \ s_2 - s_1 - \delta_i - \delta_i''']$.

(7.8) Finally we piecewise linearize f^* globally. For it we only need to do so on $M^* - X^* - \Theta^*(L'^* \times \mathbf{R} \cap D^*)$ because of (7.6.1) and (7.7.4). Let U be a connected component of $M^* - X^* - \Theta^*(L'^* \times \mathbf{R} \cap D^*)$. Then it is contained in

some M_j^*, j: even, because for each odd i, $M_i^*-X^*$ is contained in $\Theta^*((L'^*-\partial L'^*)\times\boldsymbol{R}\cap D^*)$ by (7.4.8). Put $f^*(U)=(s_1, s_2)$. Let $x\in U$. Then

$$\Theta^*(x\times\boldsymbol{R}\cap D^*)\subset M_j^*$$

by (7.4.8), and s_1, $s_2\in S_j$. Hence, by (7.4.3),

$$x\times\boldsymbol{R}\cap D^*\supset x\times(s_1-f^*(x),\ s_2-f^*(x)),$$

which is equivalent to

$$\Theta^*(x\times\boldsymbol{R}\cap D^*)\cap U=(\text{some connected component of } \Theta^*(x\times\boldsymbol{R}\cap D^*)-X^*)$$

$$=\Theta^*(x\times(s_1-f^*(x),\ s_2-f^*(x))).$$

Hence we obtain the following. Let $s\in(s_1, s_2)$ and put $C=f^{*-1}(s)\cap U$. Then C is a topological manifold, $C\times(s_1-s, s_2-s)\subset D^*$, and $\Theta^*|_{C\times(s_1-s,s_2-s)}: C\times(s_1-s, s_2-s)\to U$ is a homeomorphism. Let C_1 be a closed subset of C such that $C_1\cup(\bar{C}-C)$ is a neighborhood of $\bar{C}-C$ in $\bar{C}$ and that

(7.8.1) $C_1\subset\Theta^*((L''^*-\partial L''^*)\times\boldsymbol{R}\cap D^*)$.

Then, by (7.6.1) and (7.7.4) we have $\lambda_j>0$ such that

(7.8.2) $f^*|_U$ is PL on a neighborhood of $\Theta^*(C_1\times(s_1-s, s_2-s))\cup(U-f^*((s_1+\lambda_j, s_2-\lambda_j))$.

Put $s=s_2-\lambda_j$, and shorten δ_{j-1}''' and δ_{j+1}''' defined in (7.7) so that $\lambda_j>\delta_{j-1}'''$, δ_{j+1}'''. Then C is a PL manifold, and it follows from (7.7.6), (7.8.1) and from the inclusion $S_j\supset S_{j-1}\cup S_{j+1}$ that

(7.8.3) Θ^* is PL on a neighborhood of $C_1\times[-s_2+s_1+2\lambda_j, 0]$ in $C\times[-s_2+s_1+2\lambda_j, 0]$.

Put
$$U_1=\Theta^*(C\times[-s_2+s_1+2\lambda_j,\ 0])=U\cap f^{*-1}([s_1+\lambda_j,\ s_2-\lambda_j]).$$

Consider a compact polyhedron pair of a small closed neighborhood V of $\bar{U}\cap f^{*-1}(\{s_1, s_2\})$ in M^* which is a PL manifold with boundary and $V\cap f^{*-1}(\{s_1, s_2\})$. Then, by the facts about regular neighborhood stated in (7.7) and by (7.6.1), $V-f^{*-1}((s_1+\lambda_j, s_2-\lambda_j))$ is a PL manifold with boundary for small λ_j. Moreover we have a PL homeomorphism $\mu: (V\cap f^{*-1}(s_2-\lambda_j))\times[0, 1)\to V\cap f^{*-1}([s_2-\lambda_j, s_2))$ for small λ_j such that

$$\mu(x, 0)=x \quad \text{and} \quad f^*\circ\mu(x, t)=s_2-\lambda_j+t\lambda_j$$

$$\text{for}\quad x\in V\cap f^{*-1}(s_2-\lambda_j),\ t\in[0, 1),$$

which follows easily from the proof of the fact that $V\cap f^{*-1}([s_2-\lambda_j, s_2+\lambda_j])$ is a regular neighborhood of $V\cap f^{*-1}(s_2)$ in V. These imply that

(7.8.4)   we can assume U_1 and C are PL manifolds with boundary and without boundary respectively

and that we can modify $\Theta^*|_{C \times [0, \lambda_j)}$ fixing it on $C \times 0$ so that Θ^* is PL there. If we replace C by $\Theta^*(C \times \lambda_j')$ for sufficiently small $\lambda_j' > 0$, the last statement means that we can assume

(7.8.5) Θ^* is PL on a neighborhood of $C \times 0$ in $C \times [-s_2 + s_1 + 2\lambda_j, 0]$.

According to (7.8.3, 4, 5) we can apply Lemma 7.1. To be exact, let a homeomorphism $\omega : C \times [0, 1] \to U_1$ be defined by

$$\omega(x, u) = \Theta^*(x, u(-s_2 + s_1 + 2\lambda_j)).$$

Then ω is PL on a neighborhood of $C \times 0 \cup C_1 \times [0, 1]$. Hence, by Lemma 7.1 we have an isotopy $\omega_t : C \times [0, 1] \to U_1$, $0 \leq t \leq 1$, of ω such that

(7.8.6) ω_1 is PL,

(7.8.7) $\omega_t = \omega$, $0 \leq t \leq 1$, on a neighborhood of $C \times 0 \cup C_1 \times [0, 1]$ and that

(7.8.8)   for all t, ω_t are close approximations of ω.

We define a homeomorphism τ_{1C} of $U_2 = f^{*-1}([s_1 + \lambda_j/2, s_2 - \lambda_j] \cap U$ in the same way as τ_{3C}, namely, put

$$\tau_{4C}(x) = \begin{cases} \omega \circ \omega_1^{-1}(x) & \text{for} \quad x \in U_1 - f^{*-1}(s_1 + \lambda_j) \\ \Theta^*(\omega \circ \omega_t^{-1} \circ \Theta^*(x, s_1 + \lambda_j - f^*(x)), f^*(x) - s_1 - \lambda_j) \\ & \qquad \qquad \text{for} \quad x \in U_2 - \text{Int } U_1 \end{cases}$$

where

$$t = 2(f^*(x) - s_1 - \lambda_j/2)/\lambda_j.$$

Then, by the same reason as (7.7),

(7.8.9) τ_{4C} is well-defined, the identity on $U_2 - \text{Int } U_2$, and close to the identity on U_2,

(7.8.10) $f^* \circ \tau_{4C} = f^*$ on $U_2 - U_1$, and $f^* \circ \tau_{4C}$ is PL on U_1. Hence

(7.8.11) $f^* \circ \tau_{4C}$ is PL on U_2.
Moreover it follows from (7.8.7) that

(7.8.12) τ_{4C} is the identity on $\Theta^*(C_1 \times [-s_2 + s_1 + 3\lambda_j/2, 0])$.

Extend τ_{4C} to U by putting $\tau_{1C} = \text{ident}$ on $U - U_2$. Then, by (7.8.2, 9, 11, 12)

(7.8.13) τ_{4C} is a homeomorphism of U, the identity outside a compact subset of U and close to the identity, and $f^* \circ \tau_{4C}$ is PL on U.

Repeating this argument for each connected component of $M^* - X^* - \Theta^*(L'^* \times \mathbf{R} \cap D^*)$, we obtain a homeomorphism τ_4 of M^* close to the identity and equal to it on a neighborhood of $X^* \cup \Theta^*(L'^* \times \mathbf{R} \cap D^*)$ such that $f^* \circ \tau_4$ is PL on M^*

$-X^*-\Theta^*(L'^*\times \boldsymbol{R}\cap D^*)$ (7.8.13). Since f^* is PL on a neighborhood of $X^{*'}\cup$ $\Theta^*(L'^*\times \boldsymbol{R}\cap D^*)$ (7.6.1), (7.7.4), it follows that $f^*\circ\tau_4$ is globally PL. Hence we complete the proof of Theorem I.

§ 8. Proofs of the Other Results

As Theorem II is a special case of Theorem II′ (see Proposition 6.11), we do not need to prove Theorem II.

(8.1) *Proof of Theorem I′.* Let M be closed in $\boldsymbol{R}^n$, and let $\tilde{M}\subset\boldsymbol{R}^n\times\boldsymbol{R}$ be the graph of f. Then $\tilde{M}$ is a closed subanalytic set in $\boldsymbol{R}^n\times\boldsymbol{R}$. Let $\tilde{f}: \tilde{M}\to\boldsymbol{R}$ be the restriction of the projection $\boldsymbol{R}^n\times\boldsymbol{R}\to\boldsymbol{R}$. We remark thar $\tilde{f}$ is proper, since f is so. Recall the proof of Proposition 5.1. (5.3) tells us the following. There exist a Whitney subanalytic stratification $\{\tilde{M}_i\}_{i=1,\dots}$ of $\tilde{M}$, compact analytic manifolds N_i, $i=1, 2, \cdots$, subanalytic open subsets $N_i'\subset N_i$, $i=1, \cdots$, and analytic maps $\phi_i: N_i\to\tilde{M}$, $i=1, \cdots$, such that

(8.1.1)   for each i, $\phi_i|_{N_i'}$ is a diffeomorphism onto $\tilde{M}_i$.

Put for each i

$$\tilde{f}_i=\tilde{f}|_{\tilde{M}_i}, \quad S_i=\{x\in\tilde{M}_i\,|\,d\tilde{f}_{ix}=0\}\ .$$

Then S_i coincides with $S(\tilde{M}_i, n+1)$ for $f_1=f_2=\tilde{f}$ in (5.3). Hence, by (5.3.6), S_i is closed in $\tilde{M}_i$ and subanalytic. Here the condition (8.1.1) is important. It follows from (2.5) that $\tilde{f}_i(S_i)$ is a subanalytic set in $\boldsymbol{R}$, which clearly is bounded and of dimension 0, since $\tilde{M}_i$ is bounded. Let R denote the union of all $\tilde{f}_i(S_i)$, and put

$$\tilde{S}=\tilde{f}^{-1}(R), \quad S=f^{-1}(R)\ .$$

Then the properness of $\tilde{f}$ assures that R is a discrete set, hence S and $\tilde{S}$ are closed subanalytic sets in $\boldsymbol{R}^n$, $\boldsymbol{R}^{n+1}$ respectively by (2.5). Adding $\boldsymbol{Z}$ to R if necessary, we assume R to be unbounded from above and below..

Consider the vector field $\xi=\{\xi_i=d\tilde{f}_i\}$ on $\{\tilde{M}_i-\tilde{S}\}$ (see Definition 4.10). Clearly $\{\tilde{M}_i-\tilde{S}\}$ is a Whitney stratification of a locally closed subsets $\tilde{M}-\tilde{S}$ of $\boldsymbol{R}^{n+1}$, and we have $\xi_i(\tilde{f}|_{\tilde{M}_i-\tilde{S}})>0$, $i=1, 2, \cdots$. Put $f_1=f_2=\tilde{f}$, $V_0=V_0'=\boldsymbol{R}^{n+1}$, $V_j=V_j'=\emptyset$ for $1\le j\le n$. Then ξ, $\tilde{M}-\tilde{S}$, f_1, f_2, V_j and V_j', $0\le j\le n$, satisfy the conditions in Lemma 4.14. Hence we have a locally integrable vector field $\xi'=\{\xi_i'\}$ on $\{\tilde{M}_i-\tilde{S}\}$ such that $\xi_i'(\tilde{f}|_{\tilde{M}_i-\tilde{S}})>0$, $i=1, 2, \cdots$. After multiplying ξ_i' by $1/\xi_i'\tilde{f}$, we assume moreover

(8.1.2) $\xi_i'(\tilde{f}|_{\tilde{M}_i-\tilde{S}})=1$, $i=1, 2, \cdots$.

Let $\tilde{\Theta}: \tilde{D}\to\tilde{M}-\tilde{S}$ be the flow of ξ' (see Definition 4.12). Then, since ξ' is locally integrable, $\tilde{D}$ is open in $(\tilde{M}-\tilde{S})\times\boldsymbol{R}$, and Θ is continuous. (8.1.2) clearly implies

(8.1.3) $$\tilde{f}(\tilde{\Theta}(x, t)) = \tilde{f}(x) + t \quad \text{for} \quad (x, t) \in \tilde{D}.$$

Let $p: \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ be the projection. Put

$$D = \{(p(x), t) \,|\, (x, t) \in \tilde{D}\},$$

$$\Theta(x, t) = p \circ \tilde{\Theta}(x, f(x), t) \quad \text{for} \quad (x, t) \in D.$$

Then D is open in $(M-S) \times \mathbf{R}$, and $\Theta: D \to M-S$ is a continuous flow such that by (8.1.3)

(8.1.4) $$f(\Theta(x, t)) = f(x) + t \quad \text{for} \quad (x, t) \in D.$$

Let U be a connected component of $M-S$. Put $f(U) = (s_1, s_2)$. Then $s_1, s_2 \in R$. Let $s \in (s_1, s_2)$, and put $C = f^{-1}(s) \cap U$. Then, since f is proper, U is bounded, C is compact, hence we have $C \times \mathbf{R} \cap D = C \times (s_1-s, s_2-s)$, and $\Theta|_{C \times (s_1-s, s_2-s)}: C \times (s_1-s, s_2-s) \to U$ is a homeomorphism.

In the same way as (7.5, 6) we can reduce the problem to the case in which S is a subpolyhedron of M and f is PL on a neighborhood of S. Here the importance is that S is closed in M and that for any $x \in S$, $f^{-1}f(x) \equiv S$ as germs at x. We do not repeat the proof of reduction. For each above U, there exists $\lambda > 0$ such that f is PL on a neighborhood of $\bar{U} - f^{-1}((s_1+\lambda, s_2-\lambda))$, since $\bar{U}$ is compact. Choose λ so small that $U \cap f^{-1}([s_1+\lambda, s_2-\lambda])$ is a PL manifold (see (7.7)), and put $s = s_2 - \lambda$. Then $U \cap f^{-1}([s_1+\lambda, s_2-\lambda])$ and C are compact PL manifolds with boundary and without boundary respectively. It is easier than (7.7) to find a homeomorphism τ_U of $U \cap f^{-1}([s_1+\lambda/2, s_2-\lambda])$ such that $f \circ \tau_U$ is PL there and that $\tau_U = \text{ident}$ on $U \cap f^{-1}(\{s_1+\lambda/2, s_2-\lambda\})$. We omit the details. Extend τ_U to M by putting $\tau_U = \text{ident}$ outside $U \cap f^{-1}([s_1+\lambda/2, s_2-\lambda])$. Then $f \circ \tau_U$ is a subanalytic function on M and PL on a neighborhood of $\bar{U}$. Repeating this argument for each component of $M-S$, we obtain a homeomorphism τ of M such that $f \circ \tau$ is PL on M. Here τ can be chosen to be arbitrarily close to the identity by the same reason as the proof of Theorem I. Hence Theorem I′ is proved.

Remark 8.2. The reason why I assumed the properness of f in Theorem I′ is that I could not modify (7.3) for the case of subanalytic f, nor I could proceed with the proof without the modification in the non-proper case.

(8.3) *Proof of Theorem* II′. When we proceed with the proof in the same way as (8.1), there is no problem except in the last step, so we assume the following.

(8.3.1) $R \subset \mathbf{R}$ is a discrete set unbounded from above and below.

(8.3.2) $S = f^{-1}(R)$ is a subpolyhedron of M such that f is PL on a neighborhood of S.

(8.3.3) $\Theta: D \to M-S$, $D \subset (M-S) \times \mathbf{R}$, is a continuous flow such that

$$f \circ \Theta(x, t) = f(x) + t \qquad \text{for} \quad (x, t) \in D$$

and that for any connected component U of $M-S$ with $f(U) = (s_1, s_2)$ and for any $s \in (s_1, s_2)$ we have $C \times \mathbf{R} \cap D = C \times (s_1-s, s_2-s)$, and $\Theta|_{C \times (s_1-s, s_2-s)} : C \times (s_1-s, s_2-s) \to U$ is a homeomorphism where $C = U \cap f^{-1}(s)$.

(8.3.4) λ is a small positive number depending on U such that f is PL on a neighborhood of $\bar{U} - f^{-1}((s_1+\lambda, s_2-\lambda))$ and that $U \cap f^{-1}([s_1+\lambda, s_2-\lambda])$ is a PL manifold.

For the above U, let $s = s_1 + \lambda$, $C' = U \cap f^{-1}(s_2 - \lambda)$. Then, by (8.3.4), $M - U \cap f^{-1}((s_1+\lambda, s_2-\lambda))$ is a PL manifold with boundary $C \cup C'$. As we can not apply Lemma 7.1 in the same way as (7.7), we need the following result of Moise (see p. 15, [6]). Let $\psi : C \to C'$ be a homeomorphism. Then, there exists an isotopy ψ_t, $0 \leq t \leq 1$, of ψ such that ψ_1 is PL. Consider the case in which $\psi(x) = \Theta(x, s_2-s_1-2\lambda)$, $x \in C$. By (8.3.3) ψ is a homeomorphism onto C'. Let ψ_t be the above isotopy of ψ.

Let M_U be the quotient space of $M - U \cap f^{-1}((s_1+\lambda, s_2-\lambda))$ by the identification of $x \in C$ with $\psi_1(x) \in C'$. Then, since ψ_1 is a PL homeomorphism, M_U has naturally a PL manifold structure. We want to find a homeomorphism $\tau_U : M_U \to M$ such that

(8.3.5) $\tau_U = \text{ident}$ on a neighborhood of $M-U$ and that

(8.3.6) $f \circ \tau_U$ is PL on $\tau_U^{-1}(U)$.

If it is possible, then repeating the construction of M_U and τ_U for each connected component U of $M-S$, we obtain a PL manifold M' and a homeomorphism $\tau : M' \to M$ such that $f \circ \tau$ is PL.

We define at first a map $\tilde{\tau}_U : M - U \cap f^{-1}((s_1+\lambda, s_2-\lambda)) \to M$ by

$$\tilde{\tau}_U(x) = \begin{cases} x & \text{for} \quad x \in M - U \cap f^{-1}([s_1+\lambda/2, s_2-\lambda)) \\ \Theta(\psi_t \circ \Theta(x, s_1+\lambda-f(x)), -2(s_2-s_1-3\lambda/2)(s_1+\lambda-f(x))/\lambda) \\ \qquad \text{for} \quad x \in U \cap f^{-1}([s_1+\lambda/2, s_1+\lambda]) \end{cases}$$

where

$$t = 2(f(x) - s_1 - \lambda/2)/\lambda .$$

Then it is well-defined and continuous because of

$$(x, s_1+\lambda-f(x)) \in D \quad \text{and} \quad \Theta(x, s_1+\lambda-f(x)) \in C$$

$$\text{for} \quad x \in U \cap f^{-1}([s_1+\lambda/2, s_1+\lambda]) ,$$

$$\tilde{\tau}_U(x) = x \quad \text{for} \quad x \in U \cap f^{-1}(s_1+\lambda/2) .$$

We also have, by (8.3.3),

$$f \circ \tilde{\tau}_U(x) = \begin{cases} f(x) \quad \text{on} \quad M - U \cap f^{-1}([s_1 + \lambda/2, \ s_2 - \lambda]) \\ s_2 - \lambda - 2(s_2 - s_1 - 3\lambda/2)(s_1 + \lambda - f(x))/\lambda \quad \text{on} \\ \qquad\qquad U \cap f^{-1}([s_1 + \lambda/2, \ s_1 + \lambda]) . \end{cases}$$

Hence, by (8.3.4),

(8.3.7) $f \circ \tilde{\tau}_U$ is PL on $\bar{U} - f^{-1}((s_1 + \lambda, \ s_2 - \lambda))$.

On the other hand

$$\tilde{\tau}_U(x) = \Theta(\psi_1 \circ \Theta(x, \ 0), \ 0) = \psi_1(x) \qquad \text{for} \quad x \in U \cap f^{-1}(s_1 + \lambda) .$$

It is easy also to see that $\tilde{\tau}_U$ is surjective, and the restriction to $M - U \cap f^{-1}$ $([s_1 + \lambda, \ s_2 - \lambda])$ is one-to-one. Hence $\tilde{\tau}_U$ induces a homeomorphism $\tau_U : M_U \to M$. (8.3.5) is then clear by the definition of $\tilde{\tau}_U$, and (8.3.6) follows from (8.3.7). Hence Theorem II′ is proved.

Remark 8.4. I do not know if M' in Theorems II, II′ can be a manifold of C^∞ triangulation of M.

(8.5) *Proof of Corollary III.* For each positive integer m, let Γ_m denote the set of all simplicial complexes consisting of m simplexes. We identify two complexes K, L of Γ_m if they are linearly isomorphic (see Definition 6.0). Then Γ_m is a finite set.

For any complex $K \in \Gamma_m$, let $\varDelta_K$ denote the set of all simplicial functions on $|K|$, namely functions on $|K|$ whose restriction on each simplex of K is linear. We give a equivalence relation to $\varDelta_K$ as follows. For f, $g \in \varDelta_K$, we define $f \sim g$ if there is a homeomorphism τ of $\boldsymbol{R}$ such that $f = \tau \circ g$. Let $\tilde{\varDelta}_K$ denote the quotient set of $\varDelta_K$ under the equivalence relation. Then it is easy to see that $\tilde{\varDelta}_K$ is a finite set.

Let $\varDelta_m$ and $\tilde{\varDelta}_m$ denote the unions of $\varDelta_K$ and $\tilde{\varDelta}_K$, $K \in \Gamma_m$, respectively, and $p_m : \varDelta_m \to \tilde{\varDelta}_m$ be the projection. Put $\varDelta = \bigcup_{m=1}^{\infty} \varDelta_m$, $\tilde{\varDelta} = \bigcup_{m=1}^{\infty} \tilde{\varDelta}_m$, and let $p : \varDelta \to \tilde{\varDelta}$ be defined by $p = p_m$ on $\varDelta_m$. Then $\tilde{\varDelta}_m$ are finite sets, since $\tilde{\varDelta}_K$, Γ_m are finite sets, consequently $\tilde{\varDelta}$ is countable.

Let $\varDelta$ denote the set of all subanalytic funcitons on M. For any $f \in \varDelta$, we have a simplicial complex K and a homeomorphism $\tau : |K| \to M$ such that $f \circ \tau \in \varDelta_K$ by Theorems I′, II′. By putting $q(f) = p(f \circ \tau) \in \tilde{\varDelta}$, we define a map $q : \varDelta \to \tilde{\varDelta}$. If f_1, $f_2 \in \varDelta$ are carried by q to one element of $\tilde{\varDelta}$, f_1 and f_2 clearly are topologically R-L equivalent. Hence there are only countable topological R-L equivalence classes of $\varDelta$. That proves Corollary III.

Remark 8.6. Corollary III does not hold true if M is non-compact. In fact, the set of topological R-L equivalence classes of all subanalytic (or analytic) functions on noncompact M is a continuum. As the proof is easy, we omit it.

§ 9. Piecewise Linearization of Analytic Maps

Let $f: M_1^n \to M_2^m$ be a real (or complex) analytic map between connected real (or complex) analytic manifolds respectively. We call f *piecewise linearizable* if there are C^∞ triangulations (K_1, g_1), (K_2, g_2) of M_1, M_2 respectively and homeomorphisms $\tau_1: |K_1| \to M_1$, $\tau_2: M_2 \to |K_2|$ such that $\tau_2 \circ f \circ \tau_1: |K_1| \to |K_2|$ is a *PL* map (see the beginning of §7). Let $g: X_1 \to X_2$ be a subanalytic map between subanalytic sets. For each $x \in X_1$, let $d_g(x)$ denote the dimension of the germ of $g^{-1}g(x)$ at x. Let l be the maximal rank of df on M_1.

Proposition 9.1 (Real case). *If f is piecewise linearizable, there exists an analytic subset X of M_1 of codimension at least two such that*

$$d_f(x) \begin{cases} = n - l & \text{for} \quad x \in M_1 - X \\ \leq n - l & \text{for} \quad x \in X. \end{cases}$$

Proof. At first we will prove $d_f \leq n - l$ on M_1. Let (K_1, g_1), (K_2, g_2) be C^∞ triangulations of M_1, M_2 respectively, and $\tau_1: |K_1| \to M_1$, $\tau_2: M_2 \to |K_2|$ be homeomorphisms such that $\tau_2 \circ f \circ \tau_1$ is a *PL* map. Imbed K_2 in some $\boldsymbol{R}^{m'}$ so that $g = \tau_2 \circ f \circ \tau_1: |K_1| \to \boldsymbol{R}^{m'}$ is a *PL* map, and assume that for each simplex σ of K, $g|_\sigma$ is linear. Let $\sigma \in K$, Σ be the affine space spanned by σ, and $G: \Sigma \to \boldsymbol{R}^{m'}$ be the linear extension of $g|_\sigma$. Then, for any $x \in \sigma$, $G^{-1}g(x)$ is an affine subspace of Σ, and we have $(g|_\sigma)^{-1}g(x) = G^{-1}g(x) \cap \sigma$. Hence

(9.1.1) $d_{(g|_\sigma)}$ is equal to dim σ − dim $g(\sigma)$ on $\overset{\circ}{\sigma}$ and hence constant there,

(9.1.2) for any $x \in \sigma$, $d_{(g|_\sigma)}(x) \leq d_{(g|_\sigma)}(\overset{\circ}{\sigma})$.

Let $x \in |K_1|$. By the definition of d_g, $d_g(x)$ is the maximum of $d_{(g|_\sigma)}(x)$ for $\sigma \in K$ with $x \in \sigma$. Here we can restrict σ to n-simplexes, since $|K|$ is a *PL* manifold. Hence, there is at least one $\sigma \in K$ of dimension n such that $x \in \sigma$ and that $d_g(x) = d_{(g|_\sigma)}(x)$. Hence, by (9.1.2) we obtain

(9.1.3) $d_g(x) \leq d_{(g|_\sigma)}(\overset{\circ}{\sigma}) = d_g(\overset{\circ}{\sigma})$.

On the other hand, since the set of points $x \in M_1$ where df has rank $< l$ is an analytic subset of M_1 of codimension ≥ 1, we have $d_g = n - l$ on an open dense subset of $|K_1|$. Hence it follows from (9.1.3) that $d_f \leq n - l$ on M_1.

By Proposition 9.3, [16], for any $x_0 \in M_1$, f takes the following form in some analytic local coordinate systems of M_1, M_2 around x_0, $f(x_0)$ respectively except on an analytic subset X of M_1 of codimension ≥ 2.

$$f(x) = (f_1(x), \cdots, f_m(x)), \ x = (x_1, \cdots, x_n), \ x_0 = 0,$$

$$f_1(x) = x_1, \cdots, f_p(x) = x_p,$$

$$f_{p+1} = \pm x_{p+1}^{s_1}$$

$$f_{p+2} = x_{p+2} x_{p+1}^{s_2} + f_{11} x_{p-1}^{s_1},$$
$$\vdots$$
$$f_l = x_l x_{p+1}^{s_{l-p}} + f_{l-p-1\,l-p-1} x_{l-1} x_{p+1}^{s_{l-p-1}} + \cdots + f_{l-p-1\ldots} x_{l-1}^{l-1},$$

$$f_{l+1} = f_{l-p\,l-p} x_l x_{p+1}^{s_{l-p}} + \cdots + f_{l-p1} x_{p+1}^{s_1},$$
$$\vdots$$
$$f_m = f_{m-p-1\,l-p} x_l x_{p+1}^{s_{l-p}} + \cdots + f_{m-p-11} x_{p+1}^{s_1},$$

where $1 \leqq s_1 \leqq \cdots \leqq s_{l-p}$ are integers, f_{ij} are analytic functions in $x_l, \cdots, x_{p+j}$.

Here $f^{-1}f(0) = \{x_1 = \cdots x_{p+1} = 0\}$. Hence $d_f(0) = n-p-1$. Since $l \geqq p+1$ and $d_f \leqq n-l$, l must be equal to $p+1$. Hence we have $d_f(0) = n-l$, which proves Proposition 9.1.

Proposition 9.2 (Complex case). *If f is piecewise linearizable, then $d_f = 2(n-l)$ globally.*

Proof. Proposition 9.1 tells us that $d_f \leqq 2(n-l)$ and that $d_f = 2(n-l)$ on an open dense subset of M_1. As the problem is local, we assume $M_2 = C^m$, and we write $f = (f_1, \cdots, f_m)$. For any $x \in M_1$, let $\mathcal{O}_x$ be the ring of analytic function germs on M_1 at x, and $\mathfrak{p}_x$ be the ideal of $\mathcal{O}_x$ generated by $f_1 - f_1(x), \cdots, f_m - f_m(x)$. By Proposition 1, Chapter 1, [18], the function $M_1 \equiv x \mapsto$ coheight $\mathfrak{p}_x$ is upper semi-continuous. On the other hand, by Hilbert zero point theorem we have

$$d_f(x) = 2 \text{ coheight } \mathfrak{p}_x, \quad x \in M_1,$$

Hence d_f is an upper semi-continuous function on M_1. That proves Proposition 9.2.

Example 9.3. A simple example of map not piecewise linearizable is

$$f : (x, y) \longrightarrow (xy, x).$$

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