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On the Stable Hurewicz Image of Some Stunted Projective Spaces, I

Dedicated to Professor N. Shimada on his 60th birthday

By

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§1. Introduction

Let Z be the set of integers, and $Z\{a_1, a_2, \cdots\}$ denote a free abelian group with basis $\{a_1, a_2, \cdots\}$. If $h: G \to H$ is a homomorphism between groups, then we denote its image, its kernel and its cokernel by $\mathrm{Im}\, h$, $\mathrm{Ker}\, h$ and $\mathrm{Coker}\, h$ respectively.

Let HP^n (resp. CP^n) $(0 \leqq n \leqq \infty)$ be the quaternionic (resp. complex) n-dimensional projective space, and $HP^n_k = HP^n/HP^{k-1}$ (resp. $CP^n_k = CP^n/CP^{k-1}$) $(1 \leqq k \leqq n)$ be the stunted projective space. Then, as is well known,

$$\tilde{H}_*(HP^n_k; Z) = Z\{\beta_k, \beta_{k+1}, \cdots, \beta_n\},$$

where $\beta_i \in H_{4i}(HP^n_k; Z)$ $(k \leqq i \leqq n)$ are the standard generators.

Let

$$h_{n,k}: \pi^s_{4n}(HP^\infty_k) \longrightarrow H_{4n}(HP^\infty_k; Z) \tag{1.1}$$

be the stable Hurewicz homomorphism. Then we denote the order of $\mathrm{Coker}\, h_{n,k}$ by $|h_{n,k}|$. Thus, $\mathrm{Im}\, h_{n,k}$ is the subgroup generated by $|h_{n,k}|\beta_n$, equivalently, $|h_{n,k}|$ is equal to the stable order of the attaching map of the top cell of HP^n_k.

D. M. Segal [10] has shown that

$$|h_{n,1}| = (2n)!/a(n) \qquad (n \geqq 1),$$

where $a(n)=1$ if n is even and $a(n)=2$ if n is odd. In this paper we investigate the order $|h_{n,2}|$.

Let $\nu_2(i)$ be the exponent of 2 in the prime power decomposition of an integer i. Then our main result is stated as follows:

Theorem A. *Let* $n \geqq 2$. *Then*

$$\nu_2(|h_{n,2}|) = \nu_2(a(n)((2n)!)/8),$$

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where $a(n)=1$ *if* n *is even and* $a(n)=2$ *if* n *is odd.*

Let $p: HP^n_2 \to HP^n_n = S^{4n}$ be the collapsing map, and $p_*: \pi^s_{4n}(HP^n_2) \to \pi^s_{4n}(S^{4n})$ be the induced homomorphism. Then by definition, $|h_{n,2}|$ is equal to the order of Coker p_*.

Analogously, let $q: CP^{n-1}_{n-k} \to CP^{n-1}_{n-1} = S^{2n-2}$ $(1 \leqq k < n)$ be the collapsing map, and $q_*: \pi^s_{2n-2}(CP^{n-1}_{n-k}) \longrightarrow \pi^s_{2n-2}(S^{2n-2})$ be the induced homomorphism. Then, we denote the order of Coker q_* by $U(n, k)$. For some (n, k), these numbers $U(n, k)$ are investigated by various authors (cf. [3], [8], [9], [4], [13]).

As an application of Theorem A and its proof, we have the following:

Theorem B. *Let* $n \geqq 2$. *Then*

$$\nu_2(U(2n+1, 2n-2)) = \nu_2((2n)!/4).$$

Remark. According to Knapp [4; (7.45)], the odd primary components of $|h_{n,2}|$ and $U(2n+1, 2n-2)$ are already known. $U(2n, 2n-3)$ is also determined by Walker [13].

In our forthcoming paper we shall investigate the analogous problems for the quaternionic quasi-projective space, and apply to the complex projective space.

Throughout this paper we use the following notations:

(1.2) For a pointed space X, $\Sigma^n X$ denotes the n fold iterated reduced suspension of X. As we shall work only in the stable category, for a space X, its suspension spectrum is also denoted by the same letter X, and, for spaces X and Y, a map $h: X \to Y$ denotes the degree 0 map between their suspension spectra. Moreover, we denote the stable homotopy class $[h] \in \{X, Y\}$ of a map h simply by the same letter h.

(1.3) In the stable stems $\pi^s_*(S^0)$, we denote a generator of $\mathrm{Im}\, J_{4k-1}$ by j_{4k-1}, where J_l is the stable J-homomorphism $\pi_l(SO) \to \pi^s_l(S^0)$ $(l=4k-1)$. For classes α, β and γ in $\pi^s_*(S^0)$ satisfying $\alpha\beta = \beta\gamma = 0$, we denote the Toda bracket [12] of them by $\langle \alpha, \beta, \gamma \rangle$. We refer to [12] for various properties on the stable homotopy groups of spheres.

(1.4) We denote the Adams e'_R-invariant [1] of a class $\alpha \in \pi^s_*(S^0)$ by $e(\alpha)$, and refer to [1] for its various properties.

(1.5) For the stunted projective spaces, $p_{k,l}: HP^n_k \to HP^n_l$ and $i_{k,l}: HP^k_m \to HP^l_m$ $(1 \leqq m \leqq k \leqq l \leqq n \leqq \infty)$ denote the collapsing map and the inclusion map respectively. Also $\partial_k: HP^n_{k+1} \to \Sigma HP^k$ $(n \geqq k)$ denotes the map which appears in the cofiber sequence as follows:

$$HP^k \xrightarrow{i_{k,n}} HP^n \xrightarrow{p_{1,k+1}} HP^n_{k+1} \xrightarrow{\partial_k} \Sigma HP^k.$$

For the stunted complex projective spaces, we use the similar notations.

This paper is organized as follows:

In Section 2, we prepare a stable map between the quaternionic projective spaces, and state Theorem 3 which is essential in the proof of Theorem 5. In Section 3 we state and prove the main theorem (Theorem 5), and Sections 4 and 6 are devoted to the proofs of Theorems A and B respectively. In Section 5, we prove Theorem 3.

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§ 2. The Stable Map f

In [7], the second named author showed the existence of some stable maps $\Sigma^{16}HP^{\infty}\to HP^{\infty}$. Using its result, we have the following

Theorem 1. ([7; Theorem 1]) *There is a stable map*

$$f:\ \Sigma^8 HP^{\infty}\longrightarrow HP^{\infty}$$

which satisfies

$$f_*(\beta_n)=((2n+4)!/(2n)!)\beta_{n+2} \qquad \textit{for} \quad n\geqq 1\,.$$

Consider the restriction of f to $\Sigma^8 HP^n$ for some n. Then, by the cellular approximation, the range of f can be taken as HP^{n+2}, that is, we have a map $\Sigma^8 HP^n\to HP^{n+2}$. Especially, let

$$f_1:\ S^{12}=\Sigma^8 HP^1\longrightarrow HP^3 \tag{2.1}$$

be the restriction to $\Sigma^8 HP^1$. Then, by Theorem 1, we have

$$f_{1*}(\beta_1)=360\beta_3\,. \tag{2.2}$$

Since HP^3_2 is the mapping cone of $2j_3$ and the order of j_3 is equal to 24, there is a unique map

$$g:\ S^{12}\longrightarrow HP^3_2 \text{ satisfying } g_*(\iota_{12})=12\beta_3\,. \tag{2.3}$$

where $\iota_{12}\in H_{12}(S^{12})$ is a generator.

By an easy computation, we have the following lemma, which is an immediate consequence of [5] or [6] for the 2-localized case. In its equality, $\partial_1:\ HP^3_2\to\Sigma HP^1=S^5$ denotes the map mentioned in (1.5).

Lemma 2. $$\partial_1 g=8j_7\,.$$

Let M_t be the mod t Moore spectrum $S^0\bigcup_t e^1$, and $i_0:\ S^0\to M_t$ and $p_1:\ M_t\to S^1$ be the inclusion and the projection respectively.

By (2.2) and (2.3), we have $p_{1,2}f_1=30g$, where $p_{1,2}:\ HP^3\to HP^3_2$ is the collapsing map. Hence, there is a map $h(1):\ \Sigma^7 M_{30}\to S^0$ such that the following diagram is commutative up to sign:

(2.4)
$$\begin{array}{ccccccc}
\Sigma^{11}M_{30} & \xrightarrow{p_1} & S^{12} & \xrightarrow{30} & S^{12} & \xrightarrow{i_0} & \Sigma^{12}M_{30} \\
\downarrow h(1) & & \downarrow f_1 & & \downarrow g & & \downarrow h(1) \\
S^4 & \xrightarrow{i_{1,3}} & HP^3 & \xrightarrow{p_{1,2}} & HP^3_2 & \xrightarrow{\partial_1} & S^5 ,
\end{array}$$

where horizontal sequences are cofiberings. Then, by Lemma 2 and (2.4), it follows that

(2.5) $\qquad h(1)$ is an extension of $8j_7$.

Also the order of $h(1)$ is a divisor of 30, because, by (2.5) and [12; Proposition 1.9, (3.10)], $30h(1)\equiv p_1^*\langle 30, 8j_7, 30\rangle=0$. Thus, there is a coextension $A(1): \Sigma^8 M_{30} \to M_{30}$ of $h(1)$. Then $A(1)$ satisfies $p_1 A(1) i_0=8j_7$.

Since the order of j_3 is equal to 24, we have an extension $h(2): \Sigma^3 M_{24}\to S^0$ of j_3. On the other hand, since $\langle 24, 10j_7, 24\rangle=0$ by [12; (3.10)], there is a map $A(2): \Sigma^8 M_{24}\to M_{24}$ which satisfies $p_1 A(2) i_0=10j_7$.

Now, let $M(\varepsilon)=M_{30}$ if $\varepsilon=1$ and $M(\varepsilon)=M_{24}$ if $\varepsilon=2$. Then, we have the following

Theorem 3. *Let $\varepsilon=1$ or 2, and let $k(\varepsilon)=7$ (resp. 3) if $\varepsilon=1$ (resp. 2). Then the following diagram is commutative:*

$$\begin{array}{ccccc}
\Sigma^{12+k(\varepsilon)}M(\varepsilon) & \xrightarrow{h(\varepsilon)} & S^{12} & \xrightarrow{f_1} & HP^3 \\
\downarrow A(\varepsilon) & & & & \uparrow i_{1,3} \\
\Sigma^{4+k(\varepsilon)}M(\varepsilon) & & \xrightarrow{h(\varepsilon)} & & S^4 .
\end{array}$$

We prove this theorem in Section 5, and in Sections 3 and 4, we assume it.

§ 3. The Main Theorem

Let $M(\varepsilon)$, $h(\varepsilon)$ and $A(\varepsilon)$ ($\varepsilon=1, 2$) be the spaces and maps defined in the previous section.

Now, we define a class $\alpha(n)\in\pi^s_{4n-5}(S^0)$ ($n\geqq 2$) as follows:

(3.1)
$$\alpha(n)=\begin{cases} h(1)A(1)^{m-1}i_0 & \text{if } n=2m+1\ (m\geqq 1), \\ h(2)A(2)^{m-1}i_0 & \text{if } n=2m \quad (m\geqq 1), \end{cases}$$

where $i_0: S^r\to\Sigma^r M(\varepsilon)$ are the respective inclusions. Note that $\alpha(2)=j_3$ and $\alpha(3)=8j_7$. Then, from this definition, the following proposition follows immediately, but we shall not use it in this paper:

Proposition 4. *Let $m\geqq 1$.*

(i) *The order of $\alpha(2m+1)$ is equal to 30, and*

$$\alpha(2m+3)\in\langle 8j_7, 30, \alpha(2m+1)\rangle .$$

(ii) *The order of* $\alpha(2m)$ *is equal to* 24, *and*

$$\alpha(2m+2) \in \langle \alpha(2m), 24, 10j_7 \rangle .$$

Consider the following Atiyah-Hirzebruch spectral sequence:

$$E^2_{p,q} = \tilde{H}_p(HP^\infty) \otimes \pi^s_q(S^0) \Longrightarrow \pi^s_{p+q}(HP^\infty), \tag{3.2}$$

and let $d^r: E^r_{p,q} \to E^r_{p-r,q+r-1}$ $(r \geqq 2)$ be the differential in this spectral sequence. If an element $\gamma \in E^2_{p,q}$ persists to $E^r_{p,q}$, then we denote its class in $E^r_{p,q}$ simply by γ.

Let t_n $(n \geqq 2)$ be the following integer:

(3.3) $t_n = (2n)!/60$ if n is odd, and $t_n = (2n)!/24$ if n is even.

Now, we can state the main theorem as follows:

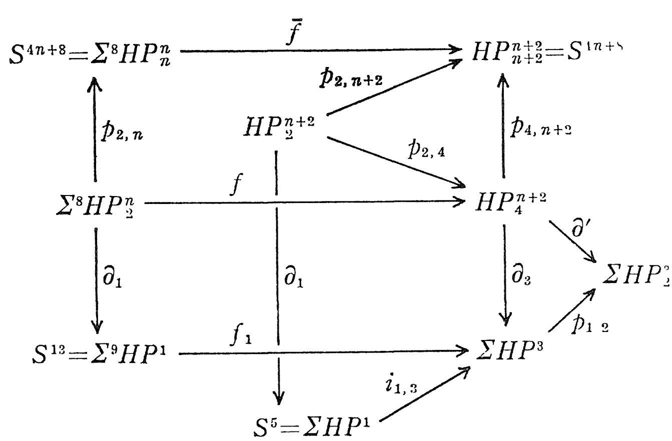
Theorem 5. *For* $n \geqq 2$,

$$t_n \beta_n \in E^{4n-4}_{4n,0} \quad \textit{and} \quad d^{4n-4}(t_n \beta_n) = \beta_1 \otimes \alpha(n) .$$

The rest of this section is devoted to the proof of Theorem 5.

First, we prepare the following diagram $(n \geqq 2)$:

(3.4)



Here, the maps f and $\bar{f}$ are defined from the map of Theorem 1 by restricting to $\Sigma^8 HP^k (k=1, n-1, n)$, f_1 is the map in (2.1), the maps $i_{k,l}$, $p_{k,l}$ and ∂_k are mentioned in (1.5), and $\partial' = p_{1,2}\partial_3$. Then the squares and the triangles in (3.4) are commutative, and the two sequences

$$HP^{n+2}_2 \xrightarrow{p_{2,4}} HP^{n+2}_4 \xrightarrow{\partial'} \Sigma HP^3_2 \quad \text{and} \quad \Sigma HP^1 \xrightarrow{i_{1,3}} \Sigma HP^3 \xrightarrow{p_{1,2}} \Sigma HP^3_2$$

in (3.4) are cofiberings.

We prove Theorem 5 by induction on n.

When $n=2$, the theorem is clear, since HP^2 is the mapping cone of j_3. Also, when $n=3$, the theorem follows from Lemma 2.

By the definition of d^r in the spectral sequence (3.2), the assertion that Theorem 5 holds for n is equivalent to that there is a map

$$X(n): S^{4n} \longrightarrow HP_2^n \tag*{$(3.5)_n$}$$

which satisfies that

$$h_{n,2}(X(n))=t_n\beta_n \tag*{$(3.6)_n$}$$

and

$$\partial_1 X(n)=\alpha(n)\,, \tag*{$(3.7)_n$}$$

where $h_{n,2}$ is the stable Hurewicz homomorphism (see (1.1)). Therefore, by induction, we may assume the existence of such a map $X(n)$ in $(3.5)_n$ for $n\geqq 2$ that satisfies the properties in $(3.6)_n$ and $(3.7)_n$, and under these assumptions it is enough to prove that the assertion also holds for $n+2$, that is, there exists a map $X(n+2)$ in $(3.5)_{n+2}$ satisfying $(3.6)_{n+2}$ and $(3.7)_{n+2}$.

By Theorem 3 and the definition of $\alpha(n)$ in (3.1), we have

$$f_1\alpha(n)=i_{1,3}\alpha(n+2)\,. \tag{3.8}$$

Then, using $(3.7)_n$ and the commutativity of (3.4), we have $\partial' fX(n)=0$. Hence, there is a map $X'\colon S^{4n+8}\to HP_2^{n+2}$ such that $p_{2,4}X'=fX(n)$. Then, by (3.8) and (3.4), we have $i_{1,3}\partial_1 X'=\partial_3 fX(n)=f_1\alpha(n)=i_{1,3}\alpha(n+2)$. Thus $\partial_1 X'-\alpha(n+2)\in \operatorname{Ker} i_{1,3*}$, and so there is a map $y'\colon S^{4n+8}\to HP_2^3$ which satisfies that

$$\partial_1(X'+i_{3,n+2}y')=\alpha(n+2)\,. \tag{3.9}$$

Now we define a map $X(n+2)$ by

$$X(n+2)=X'+i_{3,n+2}y'\,. \tag*{$(3.5)_{n+2}$}$$

Then, (3.9) yields that

$$\partial_1 X(n+2)=\alpha(n+2)\,. \tag*{$(3.7)_{n+2}$}$$

On the other hand, $p_{2,n+2}i_{3,n+2}y'=0$, since $i_{3,n+2}y'$ is a torsion element in $\pi^s_{4n+8}(HP_2^{n+2})$. Hence, we have

$$p_{2,n+2}X(n+2)=p_{2,n+2}X'=\bar{f}p_{2,n}X(n)\,.$$

This implies that $h_{n+2,2}(X(n+2))=\bar{f}_* h_{n,2}(X(n))=f_* h_{n,2}(X(n))$. But, by Theorem 1 and the assumption $(3.6)_n$, we have $f_* h_{n,2}(X(n))=((2n+4)!/(2n)!)t_n\beta_{n+2}=t_{n+2}\beta_{n+2}$. Thus

$$h_{n+2,2}(X(n+2))=t_{n+2}\beta_{n+2}\,, \tag*{$(3.6)_{n+2}$}$$

and we have completed the proof of Theorem 5.

§4. Proof of Theorem A

We use the following notations: Let k, l be any integers. Then $k|l$ means that k is a divisor of l, $\nu_2(k)$ is the exponent of 2 in the prime power decomposition of k, and

$$a(k)=1 \text{ if } k \text{ is even} \quad\text{and}\quad a(k)=2 \text{ if } k \text{ is odd}. \tag{4.1}$$

Recall that $|h_{n,2}|$ is the order of the cokernel of the stable Hurewicz homomorphism $h_{n,2}$ (see (1.1)). Then, Theorem 5 yields that

$$|h_{n,2}|\,|\,t_n\,. \tag{4.2}$$

Hence, we have

$$\nu_2(|h_{n,2}|)\leqq\nu_2(t_n)=\nu_2(a(n)((2n)\,!)/8)\,. \tag{4.3}$$

Thus, in order to complete the proof of Theorem A, it is sufficient to show the following proposition.

Proposition 6. $\nu_2(a(n)((2n)\,!)/8)\leqq\nu_2(|h_{n,2}|)$ *for* $n\geqq 2$.

We shall prove this proposition by using standard arguments of K-theory and Chern character. For this, we prepare some notations.

Let K, KO and KSp denote the complex, real and symplectic K-theory respectively, and $K^*(X)$ and $KO^*(X)$ denote the K- and KO-cohomology respectively. Let ξ be the canonical quaternionic line bundle over HP^n $(1\leqq n\leqq\infty)$, and $\tilde{\xi}=\xi-1 \in\widetilde{KSp}(HP^n)=\widetilde{KO}^4(HP^n)$. Then, as is well known, $p_{1,k}^*:\widetilde{KO}^*(HP_k^n)\to\widetilde{KO}^*(HP^n)$ is monomorphic and $\widetilde{KO}^*(HP_k^n)$ is a free module over $\pi_*(KO)$ with basis $\{\tilde{\xi}(s): k\leqq s\leqq n\}$ whose element $\xi(s)$ satisfies $p_{1,k}^*(\xi(s))=\tilde{\xi}^s$.

Let $c:\widetilde{KO}^*(X)\to\tilde{K}^*(X)$ be the complexification, and $ch:\tilde{K}^*(X)\to\tilde{H}^*(X;Q)$ be the Chern character. The composition $ch\circ c$ is called the Pontrjagin character and we denote it by $ph:\widetilde{KO}^*(X)\to\tilde{H}^*(X;Q)$.

Let $x\in H^4(HP^n)$ be the Euler class of ξ. Then $H^*(HP^n)=Z[x]/(x^{n+1})$. We use the same letter x^i to denote the element of $H^{4i}(HP_k^n)$. Then it is well known that

$$ph(\xi(s))=(e^{\sqrt{x}}+e^{-\sqrt{x}}-2)^s=(\sum_{j\geqq 1}(2/(2j)\,!)x^j)^s\,. \tag{4.4}$$

Let $N(m,s)$ be the coefficient of x^m in (4.4), that is,

$$\sum_{m\geqq 1}N(m,s)x^m=(\sum_{j\geqq 1}(2/(2j)\,!)x^j)^s\,. \tag{4.5}$$

Then we have

Lemma 7. *For* $k\leqq s$,

$$|h_{n,k}|N(n,s)\in a(n-s)Z\,.$$

Proof. Let $\alpha\in\pi^s_{4n}(HP^\infty_k)$ be a class such that $h_{n,k}(\alpha)=|h_{n,k}|\beta_n$. We can consider α to be a map $S^{4n}\to HP^n_k$ by the cellular approximation. Then, by definition, we have

$$\text{(4.6)}\qquad \alpha^*(x^n)=|h_{n,k}|\iota_{4n} \text{ for a generator } \iota_{4n}\in H^{4n}(S^{4n}; Z).$$

By (4.4), (4.5) and (4.6), we have

$$\text{(4.7)}\qquad ph(\alpha^*\xi(s))=\alpha^*(N(n, s)x^n)=|h_{n,k}|N(n, s)\iota_{4n}.$$

On the other hand, as is well known, both $\tilde{K}^{4s}(S^{4n})$ and $\widetilde{KO}^{4s}(S^{4n})$ are isomorphic to Z, and the complexification $c: \widetilde{KO}^{4s}(S^{4n})\to\tilde{K}^{4s}(S^{4n})$ sends the generator of $\widetilde{KO}^{4s}(S^{4n})$ to the $a(n-s)$ times of the one of $\tilde{K}^{4s}(S^{4n})$. Moreover the Chern character gives an isomorphism $\tilde{K}^{4s}(S^{4n})\to\tilde{H}^*(S^{4n}; Z)$. Since $\alpha^*\xi(s)\in\widetilde{KO}^{4s}(S^{4n})$, it follows that

$$\text{(4.8)}\qquad ph(\alpha^*\xi(s))\in a(n-s)H^{4n}(S^{4n}; Z),$$

and the result follows from (4.7) and (4.8). q.e.d.

Proof of Proposition 6. By an elementary arithmetic, we have that $N(n, 2)=8(4^{n-1}-1)/(2n)!$. Hence, by Lemma 7, $8(4^{n-1}-1)|h_{n,2}|/(a(n)((2n)!))$ is an integer. Since $a(n)((2n)!)/8$ is an integer for $n\geqq 2$, we have the desired result. q.e.d.

§5. Proof of Theorem 3

Proof of Theorem 3 *for* $\varepsilon=1$. Using the commutativity of the left and the right squares of (2.4) and by that $A(1)$ is a coextension of $h(1)$, we have

$$f_1h(1)=f_1p_1A(1)=i_{1,3}h(1)A(1),$$

and the theorem holds in this case.

Proof of Theorem 3 *for* $\varepsilon=2$. Recall that $M(2)=M_{24}$ is the mod 24 Moore spectrum, $h(2)$ is any extension of j_3 and $A(2)$ is a map which satisfies $p_1A(2)i_0=10j_7$. In the following diagram, we explain notations which we shall use in the proof:

$$\text{(5.1)}\qquad \begin{array}{ccc} S^{16} & & HP^3_2 \xleftarrow{\;i_{2,3}\;} S^8=HP^2_2 \\ \uparrow p_1 & & \uparrow p_{1,2} \\ \Sigma^{15}M_{24} & \xrightarrow{\;f_1h(2)\;} & HP^3 \\ \uparrow i_0 & & \uparrow i_{1,3} \\ S^{15} & & S^4. \end{array}$$

Lemma 8. *The composition* $p_{1,2}f_1h(2): \Sigma^{15}M_{24}\to HP^3_2$ *is null homotopic.*

Proof. We remark that HP^3_2 is a mapping cone of $2j_3$ and that $f_{1*}(\beta_1)=$

$360\beta_3$ by (2.2). Then, using [12; Proposition 1.8] and [1; Theorem 11.1], we have

$$p_{1,2}f_1h(2)i_0 \in (i_{2,3})_*\langle 2j_3,\ 360,\ j_3\rangle = 0\,.$$

Thus there is a class $\alpha\in\pi_{16}^s(HP_2^3)$ such that

$$p_{1,2}f_1h(2)=\alpha p_1\,. \tag{5.2}$$

Now, we show that

$$\alpha=0\,. \tag{5.3}$$

Then, by (5.2), we have the lemma.

Let $\eta\in\pi_1^s(S^0)$ be a generator. Then there is an extension $\bar{\eta}: HP_2^3\to S^7$ of $\eta: S^8\to S^7$, and we have

$$\bar{\eta}p_{1,2}f_1h(2)=0\,, \tag{5.4}$$

because $\bar{\eta}p_{1,2}f_1\in\pi_5^s(S^0)=0$. Therefore, $\bar{\eta}\alpha p_1=0$. Since $\pi_4^s(S^0)=0$, α factors a map $\alpha_1: S^{16}\to S^8=HP_2^2$, and we have

$$\eta\alpha_1 p_1=0\,. \tag{5.5}$$

But $\eta^*: \pi_8^s(S^0)\to\pi_9^s(S^0)$ is monomorphic by the table of [12; Chapter XIV], and $p_1^*: \{S^{16}, S^7\}\to\{\Sigma^{15}M_{21}, S^7\}$ is monomorphic by that $2\pi_9^s(S^0)=0$. Thus $\alpha_1=0$ by (5.5), and we have (5.3). q. e. d.

By the above lemma, there is a map $\varphi: \Sigma^{15}M_{21}\to S^4$ which satisfies

$$i_{1,3}\varphi=f_1h(2)\,. \tag{5.6}$$

Lemma 9. $e(\varphi i_0)=1/24$,

where $e(\varphi i_0)$ is the Adams e_R'-invariant of φi_0.

Proof. Recall that $ph: \widetilde{KO}(\ )\to\widetilde{H}^*(\ ; Q)$ is the Pontrjagin character and $\tilde{\xi}\in\widetilde{KO}(\Sigma^4HP^3)$ is the element corresponding to $\xi-1\in\widetilde{KSp}(HP^3)$ under the Bott isomorphism (see Section 4). We denote the standard generator of $\widetilde{KO}(S^{8i})$ by g_{8i}. As is well-known (cf. [1], [13]), Adams e_R'-invariant is a functional Pontrjagin character. In our case we have

$$e(\varphi i_0)=(ph_{20})_{\varphi i_0}(g_8)\,, \tag{5.7}$$

where ph_{20} is the 20-dimensional component of ph and $(ph_{20})_{\varphi i_0}: \widetilde{KO}(S^8)\to\widetilde{H}^*(S^{20}; Q)/\mathrm{Im}\, ph_{20}\cong Q/Z$ is the functional Pontrjagin character of φi_0. We put $\psi=\varphi i_0$. By (5.6) we have the following commutative diagram:

$$\begin{array}{ccc} S^{16} & \xrightarrow{\ f_1\ } & \Sigma^4HP^3 \\ \uparrow{\scriptstyle j_3} & \psi & \uparrow{\scriptstyle i_{1,3}} \\ S^{19} & \longrightarrow & S^8\,. \end{array}$$

Then the following diagram is commutative:

$$\begin{array}{ccc} \widetilde{KO}(\Sigma^4 HP^3) & \xrightarrow{i^*_{1,3}} & \widetilde{KO}(S^8) \\ \downarrow f^*_1 & & \downarrow (ph_{20})_\varphi \\ \widetilde{KO}(S^{16}) & \xrightarrow{(ph_{20})_{j_3}} & H^{20}(S^{20}; Q)/\mathrm{Im}\, ph_{20}. \end{array} \tag{5.8}$$

It is clear that $i^*_{1,3}\tilde{\xi}=g_8$. On the other hand,

$$f^*_1\tilde{\xi}=g_{16}. \tag{5.9}$$

In fact, $f^*_1(x^3)=360\iota_{16}$ by (2.2), and so $ph(f^*_1\tilde{\xi})=f^*_1((2/(6!))x^3)=\iota_{16}$, where $x\in H^4(HP^3; Z)\subset H^4(HP^3; Q)$ is the Euler class of ξ. Since $ph: \widetilde{KO}(S^{16})\to H^{16}(S^{16}; Z)$ is isomorphic, we have (5.9).

Now, by the naturality of the functional operation, (5.8) and the above equalities, we have

$$\begin{aligned}(ph_{20})_\varphi(g_8)&=(ph_{20})_\varphi(i^*_{1,3}\tilde{\xi})=(ph_{20})_{j_3}(f^*_1\tilde{\xi})\\ &=(ph_{20})_{j_3}(g_{16}).\end{aligned} \tag{5.10}$$

But, $(ph_{20})_{j_3}(g_{16})=e(j_3)=1/24$. Thus (5.7) and (5.10) give the desired result. q. e. d.

Now, $\pi^s_{11}(S^0)=\mathrm{Im}\, J_{11}$ and the order of $\pi^s_{11}(S^0)$ is equal to 504 (cf. [1; Example 7.17]). Hence by the above lemma and [1], we have

$$\varphi i_0=21j_{11}. \tag{5.11}$$

On the other hand, using [12; Proposition 1.7] and [1; Theorem 11.1], we have

$$h(2)A(2)i_0\in\langle j_3, 24, 10j_7\rangle=21j_{11}. \tag{5.12}$$

But $i^*_0: \{\Sigma^{11}M_{24}, S^0\}\to\pi^s_{11}(S^0)$ is monomorphic, because $\pi^s_{12}(S^0)=0$ (cf. [12]). Hence (5.11) and (5.12) yield that

$$\varphi=h(2)A(2). \tag{5.13}$$

Thus, by (5.6) and (5.13), we have

$$i_{1,3}h(2)A(2)=f_1h(2), \tag{5.14}$$

and this completes the proof of Theorem 3.

§6. Proof of Theorem B

For the homology of the complex projective space, we denote the standard generators by $b_i\in H_{2i}(CP^m; Z)$ $(1\leqq i\leqq m\leqq\infty)$. Then, $\tilde{H}_*(CP^m_k; Z)$ is a free abelian group with basis $\{b_k, \cdots, b_m\}$.

By the definition of $U(n, k)$ in Section 1, we have that $U(n, k)$ is the stable order of the attaching map of the top cell in CP^{n-1}_{n-k}, or equivalently,

(6.1) $\operatorname{Im} h(n-1, n-k)$ is generated by $U(n, k)b_{n-1}$,

where

(6.2) $h(m, l): \pi^s_{2m}(CP^m_l) \longrightarrow H_{2m}(CP^m_l; Z)$

is the stable Hurewicz homomorphism.

Now using K-theory for CP^m_l and S^{2m} ($m=n-1$, $l=n-k$) just in the same way as Lemma 7, we have immediately the following lemma, where $B(n, s)$ denotes the rational number which is the coefficient of y^{n-1} in the formal power series $(e^y-1)^s$ $(s\geqq 1)$:

Lemma 10. *The number $U(n, n-k)B(n, s)$ is an integer for $k\leqq s$.*

By an elementary arithmetic, we have that $B(2n+1, 3)=3(3^{2n-1}+1-2^{2n})/(2n)!$ and $\nu_2(3^{2n-1}+1-2^{2n})=2$ for $n\geqq 2$. Thus we have

Corollary 11. $\nu_2((2n)!/4)\leqq\nu_2(U(2n+1, 2n-2))$ *for* $n\geqq 2$.

Remark. The lower estimation of $U(m, l)$ is studied by G. Walker [13] by using KO-theory, and Lemma 10 may be a weaker condition than his result. But, for our restricted purpose, Lemma 10 is sufficient.

Let $S^2\to CP^\infty \xrightarrow{q} HP^\infty$ be the usual fibration, and let

(6.3) $t: HP^\infty_+ \longrightarrow CP^\infty_+$

be the Becker-Gottlieb transfer [2] for this fibration. By the cellular approximation, a map

(6.4) $t: HP^n_2 \longrightarrow CP^{2n}_3$ for $n\geqq 2$

is induced from (6.3). Recall that the Euler characteristic of S^2 is equal to 2. Hence according to [2; Theorem 5.5], we have $q_*t_*(\beta_n)=2\beta_n$ for the standard generator $\beta_n\in H_{4n}(HP^\infty)$. Since $q_*: H_{4n}(CP^\infty)\to H_{4n}(HP^\infty)$ is isomorphic, we have

(6.5) $t_*(\beta_n)=2b_{2n}$.

Let $X(n)\in\pi^s_{4n}(HP^n_2)$ be the class of $(3.5)_n$. Then, by $(3.6)_n$ and (6.5), we have

Lemma 12. $h(2n, 3)(t_*(X(n)))=2t_nb_{2n}$, *where t_n is the integer in* (3.3).

By (6.1) this lemma implies that the number $U(2n+1, 2n-2)$ is a divisor of $2t_n$. Thus we have

Corollary 13. $\nu_2(U(2n+1, 2n-2))\leqq\nu_2(a(n)((2n)!/4))$, *where $a(n)=1$ if n is even and $a(n)=2$ if n is odd.*

For even $n \geqq 2$, we have completed the proof of Theorem B by Corollaries 11 and 13.

To prove Theorem B for odd n, we need some notations.

According to [11], there is a stable map

$$F: \Sigma^2 CP^\infty \longrightarrow CP^\infty \tag{6.6}$$

which satisfies

$$F_*(b_n) = (n+1) b_{n+1} \quad \text{for} \quad n \geqq 1 . \tag{6.7}$$

Let

$$F_2: \Sigma^4 CP^2 \longrightarrow CP^4 \tag{6.8}$$

be the restriction of $F \circ F: \Sigma^4 CP^\infty \to CP^\infty$ to $\Sigma^4 CP^2$. Then from (6.7), it follows that

$$F_{2*}(b_n) = (n+1)(n+2) b_{n+2} \ (n \geqq 1) . \tag{6.9}$$

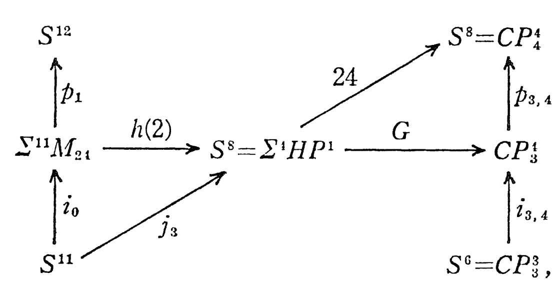
Now we consider the composition

$$G = p_{1,3} \circ F_2 \circ t: S^8 = \Sigma^4 HP^1 \longrightarrow \Sigma^4 CP^2 \longrightarrow CP^4 \longrightarrow CP_3^4 , \tag{6.10}$$

where $t: HP^1 \to CP^2$ is the restriction of the transfer of (6.3). Then we have

Lemma 14. *The composition $G \circ h(2): \Sigma^{11} M_{24} \to CP_3^4$ is null homotopic, where $h(2): \Sigma^{11} M_{24} \to S^8$ is an extension of j_3 (see Section 2).*

Proof. In the following diagram, we explain notations used in this proof:



where i_0 and $i_{3,4}$ are the respective inclusions and p_1 and $p_{3,4}$ are the respective projections.

Now, by (6.5) and (6.9), we have $G_*(b_1) = 24 b_4$. Hence, by using [12; Proposition 1.9, (3.10)], we have

$$p_{3,4} \circ G \circ h(2) \in p_1^* \langle 24, j_3, 24 \rangle = 0 . \tag{6.11}$$

Thus, there is a map $\bar{\alpha}: \Sigma^{11} M_{24} \to S^6$ such that $G \circ h(2) = i_{3,4} \bar{\alpha}$. Moreover, since $\pi_5^s(S^0) = 0$, $\bar{\alpha}$ factors $\alpha: S^{12} \to S^6$, that is, $G \circ h(2) = i_{3,4} \alpha p_1$. Hence it is sufficient to show that

(6.12) $$\alpha=0.$$

We remark that

(6.13) $\pi_6^s(S^0)$ is a group of order 2 and generated by $(j_3)^2$ (cf. [12]).

Hence $\alpha=0$ or $(j_3)^2$. Now we suppose that $\alpha=(j_3)^2$. Then $j_3\alpha=(j_3)^3\neq 0$ (cf. [12]), and

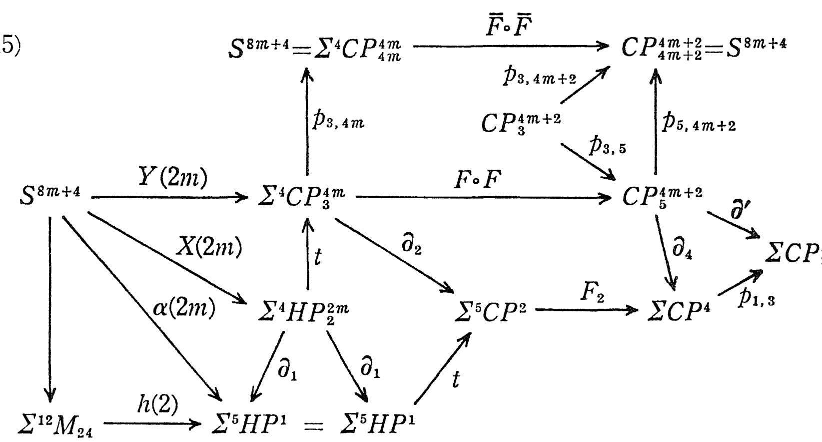
(6.14) $$j_3\alpha p_1\neq 0,$$

because the homomorphism $p_1^*: \pi_{12}^s(S^6)\to\{\Sigma^{11}M_{24}, S^6\}$ is monomorphic by (6.13). However, since CP_3^4 is a mapping cone of a generator η of $\pi_1^s(S^0)$ and since $j_3\eta=0$, there is an extension $\bar{j}_3: CP_3^4\to S^3$ of $j_3: CP_3^3=S^6\to S^3$. But $\bar{j}_3G\in\pi_8^s(S^3)=0$, and so $j_3\alpha p_1=\bar{j}_3Gh(2)=0$. This contradicts (6.14), and we have (6.12).

q. e. d.

Let $Y(2m)$ be the composition $t\circ X(2m): S^{8m+4}\to\Sigma^4HP_2^{2m}\to\Sigma^4CP_3^{4m}$ (see the definition of $X(2m)$ to (3.5)). Then we have the following commutative diagram:

(6.15)



where the maps F and $\bar{F}$ are defined from the map in (6.6) by restricting it, F_2 is the map in (6.8), and t are the maps in (6.4) and (6.10)

By Lemma 14, we have

$$p_{1,3}F_2t\alpha(2m)=0.$$

Moreover, the sequence

$$CP_3^{4m+2}\xrightarrow{p_{3,5}}CP_5^{4m+2}\xrightarrow{\partial'}\Sigma CP_3^4$$

in (6.15) is a cofibering. Therefore, by chasing the diagram (6.15), we have a map

(6.16) $$Y(2m+1): S^{8m+4}\longrightarrow CP_3^{4m+2}$$

which satisfies

$$p_{3,5}Y(2m+1)=(F\circ F)Y(2m)\,. \tag{6.17}$$

Then it follows that

$$p_{3,4m+2}Y(2m+1)=(\bar{F}\circ\bar{F})p_{3,4m}Y(2m)\,. \tag{6.18}$$

By $(3.6)_{2m}$ and (6.5), $Y(2m)_*(\iota_{8m+4})=t_*X(2m)_*(\iota_{8m+4})=2t_{2m}b_{4m}$ for a generator $\iota_l\in H_l(S^l;Z)$ $(l=8m+4)$. Also by (6.7), $(\bar{F}\circ\bar{F})_*(b_{4m})=(F\circ F)_*(b_{4m})=(4m+2)(4m+1)b_{4m+2}$. Thus these equalities and (6.18) give

$$h(2n,3)(Y(n))=((2n)!/12)b_n \qquad (n=2m+1)\,. \tag{6.19}$$

By (6.1), this implies that

$$U(2n+1,2n-2)\,|\,((2n)!/12) \qquad (n=2m+1)\,, \tag{6.20}$$

and we have completed the proof of Theorem B by Corollary 11 and (6.20).

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