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# On the Stable Hurewicz Image of Some Stunted Projective Spaces, II

*Dedicated to Professor N. Shimada on his* 60*th birthday*

By

Mitsunori IMAOKA* and Kaoru MORISUGI*

## § 1. Introduction

In the previous paper [3], we investigated the order of the cokernel of the stable Hurewicz homomorphism on the stunted projective space $HP_2^\infty$. In this paper, we consider the analogous problem for the quaternionic quasi-projective spaces.

Let $QP^n$ $(1 \leqq n \leqq \infty)$ be the $(4n-1)$-dimensional quaternionic quasi-projective space, and $QP_k^n = QP^n/QP^{k-1}$ $(2 \leqq k \leqq n)$ (denoted by $Q_{n,n-k+1}$ in James [4]) be the stunted quasi-projective space. For the complex projective space $CP^n$ and the quaternionic projective space $HP^n$, it is known (cf. James [5]) that $QP^\infty$ is a cofiber of the projection $q: CP^\infty \to HP^\infty$. Thus there is a cofiber sequence

$$CP^\infty \xrightarrow{q} HP^\infty \longrightarrow QP^\infty \xrightarrow{\Delta} \Sigma CP^\infty. \tag{1.1}$$

As is well known, the induced homomorphism $q_*: H_*(CP^\infty; Z) \to H_*(HP^\infty; Z)$ is epimorphic, and so the induced homomorphism $\Delta_*: H_*(QP^\infty; Z) \to H_{*-1}(CP^\infty; Z)$ is monomorphic. We denote by $b_i \in H_{2i}(CP^\infty; Z)$ $(i \geqq 1)$ the standard generators. Then we define $\gamma_i \in H_{4i-1}(QP^\infty; Z)$ $(i \geqq 1)$ to be the element which satisfies

$$\Delta_* \gamma_i = b_{2i-1}. \tag{1.2}$$

Thus the reduced homology group of $QP_k^n$ is a free abelian group with basis $\{\gamma_i \mid k \leqq i \leqq n\}$, that is,

$$\tilde{H}_*(QP_k^n; Z) = Z\{\gamma_k, \gamma_{k+1}, \cdots, \gamma_n\} \qquad (1 \leqq k \leqq n \leqq \infty). \tag{1.3}$$

Consider the Atiyah-Hirzebruch spectral sequence

$$E_{p,q}^2 = \tilde{H}_p(QP^\infty) \otimes \pi_q^s(S^0) \Longrightarrow \pi_{p+q}^s(QP^\infty), \tag{1.4}$$

and let $d^r: E_{p,q}^r \to E_{p-r,q+r-1}^r$ $(r \geqq 2)$ be the differential in it. For an element

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* Department of Mathematics, Faculty of Education, Wakayama University, Wakayama 640, Japan.

$\gamma \in E^2_{p,q}$, we denote its class in $E^r_{p,q}$ $(r \geqq 2)$ simply by $\gamma$. In (3.1), we define a homotopy class $\alpha'(n) \in \pi^s_{4n-5}(S^0)$. Then our main theorem is stated as follows:

**Theorem I.** *Let $t'(n)=(2n-1)!/15$ if $n$ is odd, and $t'(n)=(2n-1)!/6$ if $n$ is even. Then*

$$t'(n)\gamma_n \in E^{4n-4}_{4n-1,0} \text{ and } d^{4n-4}(t'(n)\gamma_n)=\gamma_1 \otimes \alpha'(n).$$

Let

$$h_{n,k}(Q) : \pi^s_{4n-1}(QP^\infty_k) \longrightarrow H_{4n-1}(QP^\infty_k ; Z) \tag{1.5}$$

be the stable Hurewicz homomorphism, and $|h_{n,k}(Q)|$ the order of the cokernel of $h_{n,k}(Q)$. Then $\operatorname{Im} h_{n,k}(Q)$ is the subgroup generated by $|h_{n,k}(Q)|\gamma_n$, equivalently, $|h_{n,k}(Q)|$ is equal to the stable order of the attaching map of the $(4n-1)$-dimensional cell of $QP^\infty_k$. By Walker [12] (see also Mukai [9] or [7]), $|h_{n,1}(Q)|$ is determined. Note that $|h_{n,k}(Q)|$ is equal to the so called stable quaternionic James number.

Now, for an integer $i$, $\nu_2(i)$ denotes the index of 2 in the prime power decomposition of $i$. Then, by Theorem I, we have

**Theorem II.** *Let $n \geqq 2$. Then*

$$\nu_2(|h_{n,2}(Q)|)=\nu_2((2n-1)!/a(n+1)),$$

*where $a(i)=1$ if $i$ is even, and $a(i)=2$ if $i$ is odd.*

Concerning the stable Hurewicz homomorphism $h(m, l) : \pi^s_{2m}(CP^\infty_l) \to H_{2m}(CP^\infty_l ; Z)$, we denote by $U(n, k)$ the order of the cokernel of $h(n-1, n-k)$. Then as an application of Theorems I and II, we have the following:

**Corollary III.** *Let $n \geqq 2$. Then*

$$\nu_2(U(2n, 2n-2))=\nu_2(U(2n, 2n-3))=\nu_2((2n-1)!/2).$$

We remark that $U(2n, 2n-3)$ is already known by Walker [12; Corollary 6.2]. Also, the odd primary components of $U(n+2, n)$ and $\nu_2(U(m+2, m))$ for $m \not\equiv 0 \bmod 4$ are determined by Knapp [6; Proposition 7.41].

We have also determined $\nu_2(U(2n+1, 2n-1))$ as an application of the stable map $g : \Sigma^8 QP^\infty \to QP^\infty$ (see Appendix).

Throughout this paper, we make free use of notations used in [3]. The notations of the collapsing map $p_{k,l}$, the inclusion map $i_{k,l}$ and the map $\partial_k$ mentioned in [3; (1.5)] are used also for the quasi-projective spaces; that is,

$$p_{k,l} : QP^n_k \to QP^n_l, \quad i_{k,l} : QP^k_m \to QP^l_m \quad (1 \leqq m \leqq k \leqq l \leqq n \leqq \infty) \text{ and}$$
$$\partial_k : QP^n_{k+1} \to \Sigma QP^k \quad (n \geqq k).$$

This paper is organized as follows:

In Section 2, we consider a stable map $g: \Sigma^8 QP^\infty \to QP^\infty$ from [7] and investigate some properties which are necessary for the proof of Theorem I. In Section 3 we prove Theorems I and II, and Section 4 is devoted to the proof of Corollary III. In Appendix we determine $\nu_2(U(2n+1, 2n-1))$.

## §2. The Stable Map $g$

By [7], there is a stable map

(2.1) $$g: \Sigma^8 QP^\infty \longrightarrow QP^\infty$$

which satisfies

(2.2) $$g_*(\gamma_n) = ((2n+3)!/(2n-1)!)\gamma_{n+2}.$$

Restricting this map to $\Sigma^8 QP^1$, we have a map

(2.3) $$g_1: S^{11} = \Sigma^8 QP^1 \longrightarrow QP^3.$$

Then from (2.2) it follows that

(2.4) $$g_{1*}(\gamma_1) = 120\gamma_3.$$

By James [4; (2.10)], the following is known:

(2.5) *The attaching map of the $(4l-1)$-dimensional cell $(l\geqq 2)$ to the $(4l-5)$-dimensional cell is $lj_3$,*

where $j_{4k-1} \in \pi^s_{4k-1}(S^0)$ is a $(4k-1)$-dimensional generator of the image of the stable $J$-homomorphism. Especially, $QP^3_2$ is a mapping cone of $3j_3$. Since the order of $j_3$ is equal to 24, we have a unique map

(2.6) $$g': S^{11} \longrightarrow QP^3_2 \text{ satisfying } g'_*(\iota_{11}) = 8\gamma_3,$$

where $\iota_{11} \in H_{11}(S^{11})$ is a generator. Then we have

**Lemma 2.1.** *$\partial_1 g' = 16j_7$, where $\partial_1: QP^3_2 \to \Sigma QP^1 = S^1$.*

*Proof.* Recall that there is a map $\Delta: QP^\infty \to \Sigma CP^\infty$ in (1.1) which satisfies (1.2). Then $\Delta$ induce a map from the spectral sequence (1.4) to the spectral sequence

(2.7) $$E^2_{p,q} = \tilde{H}_p(CP^\infty) \otimes \pi^s_q(S^0) \Longrightarrow \pi^s_{p+q}(CP^\infty).$$

But, according to Mosher [8; Proposition 4.11], we have

(2.8) $$d^8(8b_5) = b_1 \otimes 16j_7$$

in (2.7), where $b_n \in H_{2n}(CP^\infty; Z)$ is a standard generator. Thus by (1.2) and the naturality of spectral sequences, we have

(2.9) $$d^8(8\gamma_3)=\gamma_1\otimes 16j_7\,,$$

and (2.6) and (2.9) imply the lemma. q.e.d.

By (2.4) and (2.6) we have $p_{1,2}g_1=15g'$. Hence we have the following lemma by this equality and Lemma 2.1:

**Lemma 2.2.** (i) *There is an extension $h'(1)\,:\,\Sigma^7M_{15}\to S^0$ of $16j_7$ such that the following diagram is commutative up to sign*:

$$\begin{array}{ccccccc}
\Sigma^{10}M_{15} & \xrightarrow{p_1} & S^{11} & \xrightarrow{15} & S^{11} & \xrightarrow{i_0} & \Sigma^{11}M_{15} \\
\downarrow{\scriptstyle h'(1)} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle h'(1)} \\
S^3 & \xrightarrow{i_{1,3}} & QP^3 & \xrightarrow{p_{1,2}} & QP^3_2 & \xrightarrow{\partial_1} & S^4\,,
\end{array}$$

*where $M_t$ is the mod $t$ Moore spectrum, and $i_0\,:\,S^0\to M_t$ and $p_1\,:\,M_t\to S^1$ are the inclusion and the projection respectively.*

(ii) *There is a coextension $A'(1)\,:\,\Sigma^8M_{15}\to M_{15}$ of $h'(1)$ which satisfies $p_1A'(1)i_0=16j_7$.*

Since the order of $j_3$ and $j_7$ are equal to 24 and 240 respectively and since $\langle 12, 20j_7, 12\rangle=0$ by [11; (3.10)], we have

**Lemma 2.3.** (i) *There is an extension $h'(2)\,:\,\Sigma^3M_{12}\to S^0$ of $2j_3$.*
(ii) *There is a map $A'(2)\,:\,\Sigma^8M_{12}\to M_{12}$ such that $p_1A'(2)i_0=20j_7$.*

Now, consider the diagram

$$\begin{array}{ccccc}
S^{15} & & QP^3_2 & \xleftarrow{i_{2,3}} & QP^2_2=S^7 \\
\uparrow{\scriptstyle p_1} & & \uparrow{\scriptstyle p_{1,2}} & & \\
\Sigma^{14}M_{12} & \xrightarrow{g_1h'(2)} & QP^3 & & \\
\uparrow{\scriptstyle i_0} & & \uparrow{\scriptstyle i_{1,3}} & & \\
S^{11} & & S^3=QP^1\,. & &
\end{array} \tag{2.10}$$

Then the following lemma can be proved using the totally similar method to the proof of Lemma 8 in [3], and we omit its proof:

**Lemma 2.4.** $$p_{1,2}\circ g_1\circ h'(2)=0.$$

Thus by (2.10) there is a map $\varphi'\,:\,\Sigma^{14}M_{12}\to S^3$ satisfying

(2.11) $$i_{1,3}\varphi'=g_1\circ h'(2)\,.$$

Then we have

**Lemma 2.5.** $e(\varphi' i_0)=1/12,$

*where $e(\alpha)$ is the Adams $e'_R$-invariant of $\alpha=\varphi' i_0$.*

*Proof.* We put $\psi'=\varphi' i_0$. From (2.11), we have the following commutative diagram:

$$\begin{array}{ccc} S^{16} & \xrightarrow{g_1} & \Sigma^5 QP^3 \\ \uparrow 2j_3 & & \uparrow i_{1,3} \\ S^{19} & \xrightarrow{\psi'} & S^8 . \end{array} \tag{2.12}$$

Let $ph : \widetilde{KO}(\ ) \to \widetilde{H}^*(\ ; Q)$ be the Pontrjagin character and $ph_{20}$ be the 20-dimensional component of $ph$. We denote a generator of $\widetilde{KO}(S^{8l})$ by $g_{8l}$. By Adams [2] (see also Walker [12]), Adams $e'_R$-invariant is a functional Pontrjagin character. So in our case we have

$$e(\psi')=(ph_{20})_{\psi'}(g_8), \tag{2.13}$$

where $(ph_{20})_{\varphi'} : \widetilde{KO}(S^8) \to \widetilde{H}^*(S^{20}; Q)/\mathrm{Im}\, ph_{20} \cong Q/Z$ is a functional Pontrjagin character of $\psi'$. From (2.12) we have the commutative diagram

$$\begin{array}{ccc} \widetilde{KO}(\Sigma^5 QP^3) & \xrightarrow{i^*_{1,3}} & \widetilde{KO}(S^8) \\ \downarrow g_1^* & & \downarrow (ph_{20})_{\psi'} \\ \widetilde{KO}(S^{16}) & \xrightarrow{(ph_{20})_{2j_3}} & H^{20}(S^{20}; Q)/\mathrm{Im}\, ph_{20} \cong Q/Z . \end{array} \tag{2.14}$$

Let $\xi$ be a canonical quaternionic line bundle over $HP^2$, and $\xi \otimes \xi^*$ denote the tensor product over the quaternion of $\xi$ and its conjugate bundle $\xi^*$. Then, as is well known, $\Sigma QP^3$ is a Thom space of $\xi \otimes \xi^*$. We put $\zeta=\xi \otimes \xi^* \oplus 4_R$, where $4_R$ is the real 4-dimensional trivial bundle. Thus we have $\Sigma^5 QP^3=(HP^2)^\zeta$. Then there is a Thom class $U \in \widetilde{KO}(\Sigma^5 QP^3)$ and we have $i^*_{1,3}(U)=g_8$. Moreover we have

$$g_1^*(U)=g_{16} \quad \text{up to sign.} \tag{2.15}$$

Indeed, in order to prove (2.15) we may show that $ph_{16}(g_1^* U)=\iota_{16}$ for a generator $\iota_{16} \in H^{16}(S^{16}; Z)$, because $ph_{16} : \widetilde{KO}(S^{16}) \to H^{16}(S^{16}; Z)$ is isomorphic. Applying [1; Theorem 5.1] we see that

$$ph_{16}(U)=(1/120)\bar{\gamma}_3 \quad \text{up to sign,} \tag{2.16}$$

where $\bar{\gamma}_3 \in H^{16}(\Sigma^5 QP^3; Z)=H^{11}(QP^3; Z)$ is the dual of $\gamma_3 \in H_{11}(QP^3; Z)$. By (2.4), $g_1^*(\bar{\gamma}_3)=120\iota_{16}$. Thus we have $ph_{16}(g_1^* U)=g_1^* ph_{16}(U)=\iota_{16}$ up to sign, hence (2.15).

Now, by (2.14), (2.15) and the naturality of the functional operation, we have

$$(ph_{20})_{\varphi'}(g_8)=(ph_{20})_{\varphi'}(i_{1,3}^* U)=(ph_{20})_{2j_3}(g_1^* U)=(ph_{20})_{2j_3}(g_{16}) . \tag{2.17}$$

Since $(ph_{20})_{2j_3}(g_{16})=e(2j_3)=1/12$, (2.13) and (2.17) give the desired result.

q. e. d.

Let

$$M'(\varepsilon)=M_{15} \text{ if } \varepsilon=1, \text{ and } M'(\varepsilon)=M_{12} \text{ if } \varepsilon=2. \tag{2.18}$$

Then, using Lemmas 2.2—2.5, we can prove the following theorem by the similar way of the proof of Theorem 3 in [3]:

**Theorem 2.6.** *Let $\varepsilon=1$ or 2, and let $k(\varepsilon)=7$ (resp. 3) if $\varepsilon=1$ (resp. 2). Then the following diagram is commutative*:

$$\begin{array}{ccccc}
\Sigma^{11+k(\varepsilon)}M'(\varepsilon) & \xrightarrow{h'(\varepsilon)} & S^{11} & \xrightarrow{g_1} & QP^3 \\
\downarrow{\scriptstyle A'(\varepsilon)} & & & & \uparrow{\scriptstyle i_{1,3}} \\
\Sigma^{3+k(\varepsilon)}M'(\varepsilon) & & \xrightarrow{h'(\varepsilon)} & & S^3 .
\end{array}$$

## § 3. Proofs of Theorems I and II

Using the maps in Lemmas 2.2 and 2.3, we define elements $\alpha'(n)\in\pi_{4n-5}^s(S^0)$ ($n\geqq 2$) as follows:

$$\alpha'(n)=\begin{cases} h'(1)A'(1)^{m-1}i_0 & \text{if } n=2m+1 \quad (m\geqq 1), \\ h'(2)A'(2)^{m-1}i_0 & \text{if } n=2m \quad (m\geqq 1), \end{cases} \tag{3.1}$$

where $i_0 : S^t\to\Sigma^t M'(\varepsilon)$ are the respective inclusions. Then $\alpha'(2)=2j_3$ and $\alpha'(3)=16j_7$. Moreover we have the following proposition by the definition of $\alpha'(n)$ and using [2; Theorem 11.1]:

**Proposition 3.1.** *Let $m\geqq 1$.*

(i) *The order of $\alpha'(2m+1)$ is equal to 15, and*

$$\alpha'(2m+3)\in\langle 16j_7,\ 15,\ \alpha'(2m+1)\rangle .$$

(ii) *The order of $\alpha'(2m)$ is equal to 12, and*

$$\alpha'(2m+2)\in\langle\alpha'(2m),\ 12,\ 20j_7\rangle .$$

Let $t'(n)$ be the following integer:

$$t'(n)=\begin{cases} (2n-1)!/15 & \text{if } n \text{ is odd}, \\ (2n-1)!/6 & \text{if } n \text{ is even}. \end{cases} \tag{3.2}$$

Then, from the construction of the spectral sequence (1.4), it is easy to see that

Theorem I is equivalent to the following theorem, so we shall prove it:

**Theorem 3.2.** *There is a stable map* $X'(n): S^{4n-1}\to QP_2^n$ *for* $n\geqq 2$ *which satisfies*

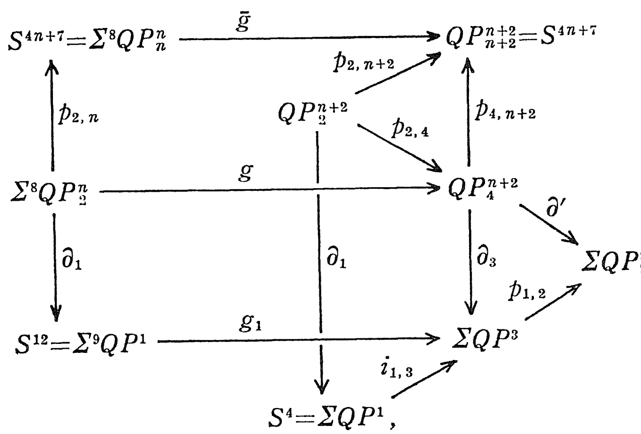
$$h_{n,2}(Q)(X'(n))=t'(n)\gamma_n \tag{3.3$_n$}$$

*and*

$$\partial_1 X'(n)=\alpha'(n)\,, \tag{3.4$_n$}$$

*where* $h_{n,2}(Q): \pi_{4n-1}^s(QP_2^n)\to H_{4n-1}(QP_2^n; Z)$ *is the stable Hurewicz homomorphism and* $\partial_1: QP_2^n\to \Sigma QP^1=S^4$.

*Proof.* Consider the following diagram:

(3.5) 


where the maps $g$ and $\bar{g}$ are defined from the map in (2.1) by restricting it to $\Sigma^8 QP^k$ ($k=1$, $n-1$, $n$), $g_1$ is the map in (2.3) and $\partial'=p_{1,2}\partial_3$. Then all the squares and the triangles in (3.5) are commutative, and the sequences

$$QP_2^{n+2}\xrightarrow{p_{2,4}} QP_4^{n+2}\xrightarrow{\partial'}\Sigma QP_2^3 \quad\text{and}\quad \Sigma QP^1\xrightarrow{i_{1,3}}\Sigma QP^3\xrightarrow{p_{1,2}}\Sigma QP_2^3$$

in (3.5) are cofiberings.

We prove the theorem by induction on $n$.

For $n=2$, we take $X'(2)$ as the identity map of $S^7$. Then $(3.3)_2$ is obvious. Since $QP^2$ is a mapping cone of $2j_3$ by (2.5) and $\alpha'(2)=2j_3$, $(3.4)_2$ also holds. For $n=3$, we take $X'(3)$ as the map $g'$ in (2.6). Then $(3.3)_3$ and $(3.4)_3$ follow from (2.6) and Lemma 2.1 respectively.

We assume that the theorem holds for $n\geqq 2$, and we may prove it for $n+2$. By Theorem 2.6, we have

$$g_1\alpha'(n)=i_{1,3}\alpha'(n+2)\,. \tag{3.6}$$

Using (3.6) and the diagram (3.5), we can construct a required map $X'(n+2)$ quite similarly to the construction of the map $X(n+2)$ in Theorem 5 of [3].

q. e. d.

*Proof of Theorem II.* For integers $k$ and $l$, $k|l$ means that $k$ is a divisor of $l$. Then, by Theorem I, we have

(3.7) $$|h_{n,2}(Q)|\,|t'(n).$$

Thus we have

(3.8) $$\nu_2(|h_{n,2}(Q)|)\leqq\nu_2(t'(n))=\nu_2((2n-1)!/a(n+1)),$$

where $a(i)=1$ if $i$ is even and $a(i)=2$ if $i$ is odd.

On the other hand, Walker [12] estimates the lower bound of the James numbers. Using his result [12; Theorem 0.2], we have

(3.9) $$(1/a(n-s)((2n-1)!)s)\left(\sum_{i=0}^{s-1}(-1)^i\binom{2s}{i}(s-i)^{2n}\right)|h_{n,k}(Q)|\in Z \quad \text{for} \quad k\leqq s\leqq n.$$

Especially, for $k=s=2$, since $\sum_{i=0}^{s-1}(-1)^i\binom{2s}{i}(s-i)^{2n}=4^n-4$ we have

(3.10) $$a(n+1)(4^{n-1}-1)|h_{n,2}(Q)|/(2n-1)!\in Z.$$

Thus we have

(3.11) $$\nu_2(|h_{n,2}(Q)|)\geqq\nu_2((2n-1)!/a(n+1)),$$

and (3.8) and (3.11) complete the proof. q. e. d.

## §4. Proof of Corollary III

According to Walker [12; Theorem 0.1(i)], the following proposition holds:

**Proposition 4.1.** *Let* $K(n, s)=\sum_{i=0}^{s-1}(-1)^i\binom{2s}{i}(s-i)^{2n}$ $(n\geqq s\geqq 1)$, *and* $k=2l$ *or* $2l-1$ $(l\geqq 1)$. *Then, for* $n-l+1\leqq s\leqq n$,

(i) $$(1/s((2n-1)!))K(n, s)U(2n, k)\in Z,$$

(ii) $$(\varepsilon/(2n)!)K(n, s)U(2n+1, k)\in Z,$$

*where* $\varepsilon=2$ *if* $k\equiv 1 \bmod 4$ *and* $s=n-l+1$, *and otherwise* $\varepsilon=1$.

Especially, for $k=2n-2$ or $2n-3$, we have

**Corollary 4.2.** *Let* $n\geqq 2$. *Then*

$$\nu_2(U(2n, 2n-i))\geqq\nu_2((2n-1)!/2) \qquad (i=2 \text{ or } 3).$$

Since $U(2n, 2n-3)|U(2n, 2n-2)$ by definition, to prove Corollary III we have only to show the following:

**Proposition 4.3.** $\nu_2(U(2n, 2n-2)) \leqq \nu_2((2n-1)!/2)$.

The remainder of this section is devoted to the proof of Proposition 4.3.

First we prove Proposition 4.3 for even $n$. We put $n=2m$ $(m\geqq 1)$. By Theorem 3.2, there is a map $X'(2m): S^{8m-1}\to QP_2^{2m}$ satisfying $h_{2m,2}(Q)(X'(2m)) =((4m-1)!/6)\gamma_{2m}$ and $\partial_1 X'(2m)=\alpha'(2m)$. We define a map $Y'(2m)$ as follows:

(4.1) $$Y'(2m)=\Delta\circ X'(2m) : S^{8m-1}\to QP_2^{2m}\to \Sigma CP_2^{4m-1},$$

where $\Delta$ is the map in (1.1). Then by (1.2) we have

(4.2) $$Y'(2m)_*(\iota_{8m-1})=((4m-1)!/6)b_{4m-1}$$

for a generator $\iota_{8m-1}\in H_{8m-1}(S^{8m-1})$. This implies that

$$h_{2m,2}(Y'(2m))=((4m-1)!/6)b_{4m-1}.$$

Thus it follows that

(4.3) $$U(4m, 4m-2) \mid (4m-1)!/6,$$

and we have the proposition for $n=2m$.

Next we shall prove the proposition for odd $n$. We put $n=2m+1$ $(m\geqq 1)$. By Toda [10], there is a stable map

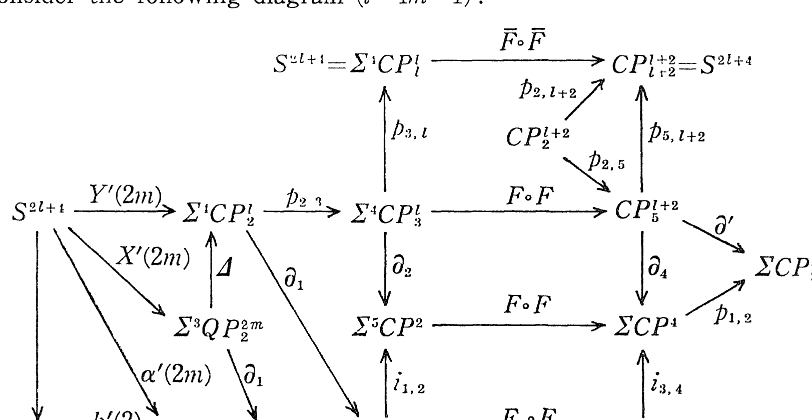
(4.4) $$F : \Sigma^2 CP^\infty \longrightarrow CP^\infty$$

which satisfies

(4.5) $$F_*(b_i)=(i+1)b_{i+1} \quad \text{for} \quad i\geqq 1.$$

We consider the following diagram $(l=4m-1)$:

(4.6)




where the maps $F$, $\bar{F}$, $F_1$ are defined from the map in (4.4) by restricting it,

$\partial'=p_{1,2}\partial_4$ and $Y'(2m)$ is the map in (4.1). Here the squares and the triangles in (4.6) are commutative, and the sequence

$$CP_2^{4m+1} \xrightarrow{p_{2,5}} CP_5^{4m+1} \xrightarrow{\partial'} \Sigma CP_2^4$$

is a cofibering.

Now we put $G=p_{1,2}\circ i_{3,1}\circ(F_1\circ F_1) : S^7\to\Sigma CP_2^4$. Then

**Lemma 4.4.** $G\circ h'(2)=0$.

Assume that the lemma holds. Then, by chasing the diagram (4.6), it follows that there is a map $Y'(2m+1) : S^{8m+2}\to CP_2^{4m+1}$ satisfying

$$p_{2,5}\circ Y'(2m+1)=(F\circ F)\circ p_{2,3}\circ Y'(2m)\,. \tag{4.7}$$

Then by (4.7) and the commutativity of (4.6) we have

$$p_{2,4m+1}\circ Y'(2m+1)=(\bar{F}\circ\bar{F})\circ p_{2,4m-1}\circ Y'(2m)\,. \tag{4.8}$$

By (4.2), (4.5) and (4.8), we see that

$$h(4m+1, 2)(Y'(2m+1))=((4m+1)\,!/6)b_{4m+1}\,, \tag{4.9}$$

where $h(n, 2)$ is the stable Hurewicz homomorphism. Thus we have

$$U(4m+2, 4m)\,|\,((4m+1)\,!/6)\,, \tag{4.10}$$

and complete the proof of the proposition for $n=2m+1$.

*Proof of Lemma* 4.4. We consider the following diagram:

$$\begin{array}{ccccccccc}
S^{11} & & S^7 & \xrightarrow{i_{3,4}} & \Sigma CP_3^4 & \xrightarrow{p_{3,4}} & \Sigma CP_4^4=S^9 & \xrightarrow{\eta} & S^8 \\
\uparrow{\scriptstyle p_1} & & \uparrow{\scriptstyle 6} & & \uparrow{\scriptstyle p_{2,3}} & \nearrow{\scriptstyle p_{2,4}} & & & \\
\Sigma^{10}M_{12} & \xrightarrow{h'(2)} & S^7 & \xrightarrow{G} & \Sigma CP_2^4 & & & & \\
\uparrow{\scriptstyle i_0} & \nearrow{\scriptstyle 2j_3} & & & \uparrow{\scriptstyle i_{2,4}} & & & & \\
S^{10} & & CP_3^4 & \xrightarrow{\partial'} & S^5, & & & &
\end{array} \tag{4.11}$$

Here the triangles commute obviously, and the square commute, because $(F_1\circ F_1)_*b_1=6b_3$ by (4.5). $\eta$ denotes a generator of $\pi_1^s(S^0)$, and $\partial'=p_{1,2}\partial_2$. The sequence

$$CP_3^4 \xrightarrow{\partial'} S^5 \xrightarrow{i_{2,4}} \Sigma CP_2^4 \xrightarrow{p_{2,3}} \Sigma CP_3^4$$

is a cofibering. As is well known,

$$CP_3^4 \text{ is a mapping cone of } \eta\,, \tag{4.12}$$

and $\partial'$ factors an odd multiple of $j_3$, that is, (4.13) $\partial'=(2l+1)j_3\circ p_{3,4}$ for some integer $l$.

By (4.12) the sequence

$$S^7 \xrightarrow{i_{3,4}} \Sigma CP_3^4 \xrightarrow{p_{3,4}} \Sigma CP_4^4 \xrightarrow{\eta} S^5$$

in (4.11) is a cofibering.

Now, by (4.12) and that $\eta^3=12j_3$, we have

(4.14) $$p_{2,3}\circ G\circ h'(2)\circ i_0=i_{4,4}\circ 12j_3=0\,.$$

Since $\pi_5^s(S^0)=0$, (4.14) yields

(4.15) $$G\circ h'(2)\circ i_0=0\,.$$

Thus there is a map $\phi : S^{11}\to\Sigma CP_2^4$ such that $\phi\circ p_1=G\circ h'(2)$. Then

(4.16) $$p_{2,3}\circ\phi=0\,.$$

In fact, since $p_{2,4}\circ\phi\in\pi_2^s(S^0)$, $p_{2,4}\circ\phi=0$ or $\eta^2$. But $\eta\circ p_{2,4}\circ\phi=\eta\circ p_{3,4}\circ p_{2,3}\circ\phi=0$ and $\eta^3\neq 0$. Thus $p_{2,4}\circ\phi=0$, and we have (4.16), since $\pi_4^s(S^0)=0$.

Therefore there is a map $\varphi : S^{11}\to S^5$ satisfying

(4.17) $$i_{2,4}\circ\varphi\circ p_1=G\circ h'(2)\,.$$

Since $\pi_6^s(S^0)$ is a group of order 2 and generated by $(j_3)^2$, $\varphi=0$ or $(j_3)^2$. But the kernel of $i_{2,4*}:\{S^{11}, S^5\}\to\{S^{11}, \Sigma CP_2^4\}$ is generated by $(j_3)^2$ by (4.13). Thus $i_{2,4}\circ\varphi=0$, and we have the lemma by (4.17). q.e.d.

## Appendix. The Number $U(2n+1, 2n-1)$

As is well-known, $U(n, n-1)=(n-1)!$ (cf. [10], [8]) and this is given by applying the map $F$ in (4.4). We have determined in [3] the values of $\nu_2(U(2m+1, 2m-2))$ and in Corollary III the values of $\nu_2(U(2m, 2m-2))=\nu_2(U(2m, 2m-3))$ for $m\geqq 2$ by using the maps $f : \Sigma^8HP^\infty\to HP^\infty$ and $g : \Sigma^8QP^\infty\to QP^\infty$ in [7] respectively. Using Proposition 4.1(ii) and the above fact that $U(n, n-1)=(n-1)!$, we have immediately that $\nu_2(U(4m+1, 4m-1))=\nu_2(U(4m+1, 4m))$. In this appendix we shall determine $\nu_2(U(4m+3, 4m+1))$ for $m\geqq 0$ by using the map $g_1$ in (2.3) and a stable map $\Sigma^8CP^\infty\to CP^\infty$. We denote the map $g_1$ simply by $g$ in this appendix. Consequently we obtain all values of $\nu_2(U(n, n-i))$ for $1\leqq i\leqq 3$ and $i<n$.

Let $\eta\in\pi_1^s(S^0)$ be the generator and $\bar\eta : \Sigma M_2\to S^0$ be any extension of $\eta$ to the mod 2 Moore spectrum $M_2=S^0\cup_2 e^1$. We prepare the following diagram:

(A.1)
$$\begin{array}{ccccccc}
S^{13} & & QP_2^2=S^7 & \xrightarrow{i_{2,3}} & QP_2^3 & \xrightarrow{p_{2,3}} & QP_3^3=S^{11} \\
\uparrow{\scriptstyle p_1} & & & & \uparrow{\scriptstyle p_{1,2}} & \nearrow{\scriptstyle p_{1,3}} & \\
\Sigma^{12}M_2 & \xrightarrow{\bar{\eta}} & S^{11} & \xrightarrow{g} & QP^3 & & \\
\uparrow{\scriptstyle i_0} & & & & \uparrow{\scriptstyle i_{1,3}} & & \\
S^{12} & & & & S^3=QP^1 . & &
\end{array}$$

Then we have

**Lemma A.2.** $p_{1,2}\circ g\circ\bar{\eta}=0$.

*Proof.* Using (2.4), we have $p_{1,3}\circ g\circ\bar{\eta}\in p_1^*\langle 120, \eta, 2\rangle=0$. Thus $p_{1,2}\circ g\circ\bar{\eta}$ factors a map $\Sigma^{12}M_2\to QP_2^2=S^7$. Since $\pi_5(S^0)=0$, there is a map $\alpha : S^{13}\to QP_2^2=S^7$ with $i_{2,3}\circ\alpha\circ p_1=p_{1,2}\circ g\circ\bar{\eta}$. But $\pi_6^s(S^0)$ is generated by $j_3^2$ and $QP_2^3$ is the mapping cone of $3j_3$. Therefore $i_{2,3}\circ\alpha=0$, and we have the desired result. q. e. d.

By the above lemma we have a map $\varphi : \Sigma^{12}M_2\to QP^1=S^3$ which satisfies

(A.3) $$i_{1,3}\circ\varphi=g\circ\bar{\eta} .$$

**Lemma A.4.** *There is a coextension $h : S^8\to M_2$ of $120j_7$ such that $\bar{\eta}\circ h=\varphi\circ i_0 : S^{12}\to S^3=QP^1$.*

*Proof.* For an element $\alpha$ we denote its $d_R$-invariant by $d_R(\alpha)$. By (A.3) we have a commutative diagram:

(A.5)
$$\begin{array}{ccc}
\widetilde{KO}(\Sigma^5QP^3) & \xrightarrow{g^*} & \widetilde{KO}(S^{16}) \\
\downarrow{\scriptstyle i_{1,3}^*} & & \downarrow{\scriptstyle \eta^*} \\
\widetilde{KO}(S^8) & \xrightarrow{(\varphi\circ i_0)^*} & \widetilde{KO}(S^{17})\cong Z_2 .
\end{array}$$

Let $U\in\widetilde{KO}(\Sigma^5QP^3)$ be a Thom class (see the proof of Lemma 2.5) and $g_{8i}\in\widetilde{KO}(S^{8i})$ be a generator. Then by (A.5) and (2.15) we have

$$d_R(\varphi i_0)=(\varphi i_0)^*(g_8)=(\varphi i_0)^*i_{1,3}^*(U)=\eta^*(g^*U)=\eta^*(g_{16})\neq 0 .$$

On the other hand by [2] the Toda bracket $\langle\eta, 2, 120j_7\rangle$ consists of elements in $\pi_9^s(S^0)$ whose $d_R$-invariants are non-zero. Thus we can take a coextension $h$ of $120j_7$ satisfying $\bar{\eta}\circ h=\varphi\circ i_0$. q. e. d.

Since $\langle 2, 120j_7, 2\rangle=0$, there is an extension $A : \Sigma^8M_2\to M_2$ of $h$. Then it follows that $p_1\circ A\circ i_0=120j_7$. Using these maps, we define a $\mu$-series $\mu_{8m-1}\in\pi_{8m+1}^s(S^0)$ $(m\geqq 0)$ as follows:

(A.6) $\mu_{8m+1}=\bar{\eta}\circ A^m\circ i_0 : S^{8m+1}\longrightarrow \Sigma^{8m+1}M_2\longrightarrow \Sigma M_2\longrightarrow S^0.$

Now we have the following theorem in which we consider only for the 2-localized version and we denote the 2-primary component of $\pi_i^s(Y)$ by ${}_2\pi_i^s(Y)$.

**Theorem A.** *For $m\geqq 0$ there is an element*

$$X_m\in {}_2\pi_{8m+4}^s(CP_2^{4m+2})$$

*which satisfies*

$$h(X_m)=((4m+2)!/2)b_{4m+2} \quad \text{and} \quad \partial_1 X_m=\mu_{8m+1},$$

*where $h : {}_2\pi_i^s(Y)\rightarrow H_i(Y; Z_{(2)})$ is the stable Hurewicz homomorphism and $\partial_1 : {}_2\pi_i^s(Y)\rightarrow {}_2\pi_{i-1}^s(CP^1)={}_2\pi_{i-3}^s(S^0)$ for $Y=CP_2^{4m+2}$ and $i=8m+4$.*

By the above theorem and Proposition 4.1(ii) we obtain

**Corollary B.** $\nu_2(U(4m+3, 4m+1))=\nu_2((4m+2)!/2)$ *for $m\geqq 0$.*

The rest of this paper is devoted to the proof of Theorem A. Let $f_c(n, s) : \Sigma^{4n}CP^\infty\rightarrow CP^\infty$ be the stable map in [7; Section 3]. We put $F'=f_c(2, 1) : \Sigma^8 CP^\infty\rightarrow CP^\infty$ which may be equal to the 4-fold composition of $F$ in (4.1). Then it follows that

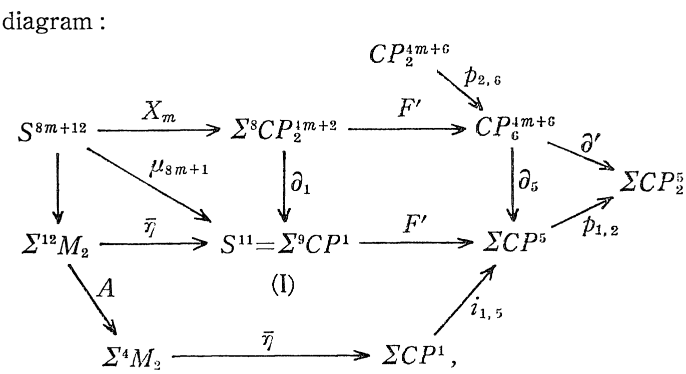
(A.7) $$F'_*(b_i)=((i+4)!/i!)b_{i+4},$$

and the following diagram is commutative:

(A.8) $$\begin{array}{ccc} \Sigma^8 QP^\infty & \xrightarrow{\ g\ } & QP^\infty \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} \\ \Sigma^9 CP^\infty & \xrightarrow{\ F'\ } & \Sigma CP^\infty, \end{array}$$

where $\Delta$ is the map in (1.1).

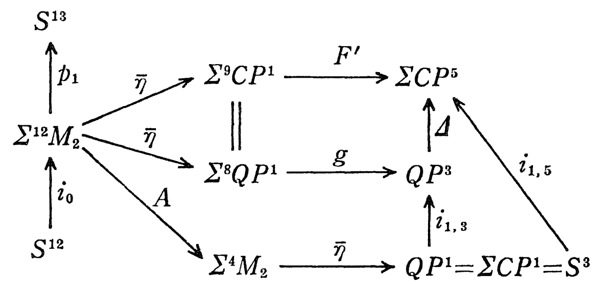
Now we prove the theorem by induction on $m$. First we take $X_0$ to be the identity map. Since $CP^2$ is the mapping cone of $\eta$, the theorem clearly holds for $m=0$. Assume that the theorem holds for $m$. Then we can consider the following diagram:

(A.9)


where the square and the triangles are commutative. If the part (I) in (A.9) is commutative, then by the same reason in the proof of Theorem 5 in [3] or Theorem 3.2, we can construct a stable map $X_{m+1}: S^{8m+12} \to CP_2^{4m+6}$ satisfying the assertions in the theorem and the proof is completed.

**Lemma A.10.** *The part (I) in the diagram (A.9) is commutative, that is,* $F'\bar{\eta} = i_{1,5} \circ \bar{\eta} \circ A$.

*Proof.* Consider the following diagram:




Then we have $\Delta g = F'$ by (A.8). By Lemma A.4 there is a map $\alpha: S^{13} \to QP^1 = S^3$ such that $i_{1,3} \circ \alpha \circ p_1 = g \circ \bar{\eta} - i_{1,3} \circ \bar{\eta} \circ A$. But ${}_2\pi_{10}^s(S^0)$ is generated by $\eta\mu_9$ (cf. [11]) and $CP^2$ is the mapping cone of $\eta$. Therefore $i_{1,2} \circ \alpha = 0$ and we have $g \circ \bar{\eta} = i_{1,3} \circ \bar{\eta} \circ A$. Thus we have the desired result. q. e. d.

This completes the proof of Theorem A.

## References

[1] Adams, J. F., On the groups $J(X)$-II, *Topology*, **3** (1965), 137-171.

[2] ———, On the groups $J(X)$-IV, *Topology*, **5** (1966), 21-71.

[3] Imaoka, M. and Morisugi, K., On the stable Hurewicz image of some stunted projective spaces, I, *Publ. RIMS, Kyoto Univ.*, **20** (1984) (this volume), 839-852.

[4] James, I. M., Spaces associated with Stiefel manifolds, *Proc. London Math. Soc.*, **9** (1959), 115-140.

[5] ———, The topology of Stiefel manifolds, *London Math. Soc. Lecture Note Series*, **24**.

[6] Knapp, K., Some applications of $K$-theory to framed bordism: $E$-invariant and transfer, *Habilitationsschrift*, Bonn, 1979.

[7] Morisugi, K., Stable self maps of the quaternionic (quasi-)projective space, *preprint*.

[8] Mosher, R. E., Some stable homotopy of complex projective space, *Topology*, **7** (1968), 179-193.

[9] Mukai, J., The order of attaching class in the suspended quaternionic quasi-projective space, *Publ. RIMS, Kyoto Univ.*, **20** (1984) (this volume), 717-725.

[10] Toda, H., A topological proof of theorems of Bott and Borel-Hirzebruch for homotopy groups of unitary groups, *Memoirs Univ. of Kyoto*, **32** (1957), 103-119.

[11] ———, Composition methods in homotopy groups of spheres, *Annals of Math. Studies*, **49**, Princeton Univ. Press, 1962.

[12] Walker, G., Estimates for the complex and quaternionic James numbers, *Quart. J. Math. Oxford* (2), **32** (1981), 467-489.