Publ. RIMS, Kyoto Univ.
20 (1984), 897–902

Boundary Behavior of the Bergman Kernel Function on Pseudoconvex Domains

By

Takeo OHSAWA*

Introduction

Let $D \subset \mathbb{C}^n$ be a bounded domain of holomorphy and let $H^2(D)$ be the set of square integrable holomorphic functions on D. The Bergman kernel function (cf. [1]) is defined by

$$K_D(z) := \sup_{f \in H^2(D) - \{0\}} |f(z)|^2 / \|f\|_D^2 ,$$

where

$$\|f\|_D^2 = \int_D |f(z)|^2 dv \quad (dv \text{ denotes the Lebesgue measure on } \mathbb{C}^n).$$

$K_D(z)$ is regarded as a function measuring how large the space $H^2(D)$ can be. We are interested in the growth of K_D near the boundary. Our motivation is the following theorem which has been proved by Hörmander and Diederich independently (see [2] and [4]).

Theorem 1. *If the boundary of D is strictly pseudoconvex, then $K_D(z) \sim d(z)^{-n-1}$. Here $d(z) = \inf_{x \in \partial D} |z - x|$ and $A \sim B$ means that both A/B and B/A are bounded.*

From Theorem 1 and the definition of K_D it can be easily seen that $K_D(z) \lesssim d(z)^{-n-1}$ if ∂D is locally Lipschitz, where $A \gtrsim B$ means that B/A is bounded. Further it has been shown by Pflug [6], [7] that if D has a C^2-pseudoconvex boundary, then

$$K_D(z) \gtrsim d(z)^{-2+\varepsilon}.$$

Here ε is any positive number.

* Received February 6, 1982. Revised December 21, 1983.
Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan.

The present note may well be understood as a continuation of the above works. Let φ be a defining function of a bounded domain D with a C^2-pseudoconvex boundary in $\mathbb{C}^n$, let x be a boundary point, and let

$$N_x := \left\{(\xi^1, \cdots, \xi^n) \in \mathbb{C}^n ; \sum_{i=1}^{n} \frac{\partial \varphi}{\partial z_i}(x)\xi^i = 0,\ \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial z_j}(x)\xi^j \xi^j = 0\right\},$$

$$\nu_x = \dim_{\mathbb{C}} N_x .$$

Under this situation we have the following theorem.

Main Theorem. *Let D be a bounded domain in $\mathbb{C}^n$ with C^4-pseudoconvex boundary. Fix $x \in \partial D$. Then, for any positive number ε,*

$$\inf_{z\in D} K_D(z)\,|z-x|^{n-\nu_x+1-\varepsilon} > 0 .$$

Moreover if we set $S_x := \{y;\ \nu_y = \nu_x\}$ and

$$m_\varepsilon^x(y) := \inf_{z\in D} K_D(z)\,|z-y|^{n-\nu_x+1-\varepsilon},$$

then the function $y \mapsto m_\varepsilon^x(y)$ is continuous on S_x.

Corollary. *Let $\nu_D := \sup_{x\in\partial D} \nu_x$. Then, for any $\varepsilon > 0$,*

$$K_D(z) \gtrsim d(z)^{-n+\nu_D-1+\varepsilon} .$$

The main tool in the proof is a vanishing theorem on complete Kähler manifolds.

The author thanks to Professors K. Diederich and P. Pflug for their encouragement. Also many thanks to the referee for valuable criticisms.

§ 1. Localization lemma

Let $D \subset \mathbb{C}^n$ be a bounded domain of holomorphy, let x be a boundary point, and let V and U with $V \subset\subset U$ be two open neighbourhoods of x in $\mathbb{C}^n$.

Localization lemma. *There is a positive number δ such that for any point $y \in D \cap V$,*

$$(*) \qquad \delta K_{D\cap U}(y) \leqq K_D(y) \leqq K_{D\cap U}(y) .$$

Proof. Let χ: $\mathbb{C}^n \to \mathbb{R}$ be a C^∞ function such that $\chi = 1$ on a neighbourhood of V and $\chi = 0$ outside U. Let $z_0 \in V \cap D$ be any point and let f be a holomorphic function in $H^2(D \cap U)$ such that $\|f\|_{D\cap U} = 1$ and $|f(z_0)|^2 = K_{D\cap U}(z_0)$. We set $\alpha = \bar\partial(f\chi)$ on U and $\alpha = 0$ outside U. Then α is a $\bar\partial$-closed (0, 1)-form defined on D satisfying

$$\int_D |z-z_0|^{-2n} |\exp(|z-z_0|^2)|\alpha|^2 dv < C,$$

where C is a constant independent of z_0. Thus, by a well known theorem of Hörmander (cf. Theorem 2.2.1 in [4]), there is a function β satisfying $\bar{\partial}\beta=\alpha$ and

$$\int_D |z-z_0|^{-2n} \exp(-|z-z_0|^2)|\beta|^2 dv < C.$$

Hence $\chi f-\beta$ is a holomorphic function defined on D satisfying $\chi(z_0)f(z_0)-\beta(z_0)=f(z_0)$ and

$$\int_D |\chi f-\beta|^2 dv < 2(Cd^{2n}\exp(d^2)+1),$$

where d denotes the diameter of D. Therefore if we set $\delta=\frac{1}{2}(Cd^{2n}\exp(d^2)+1)^{-1}$, we obtain (*).

§ 2. Proof of Main Theorem

In what follows D is a bounded domain in C^n with C^4-pseudoconvex boundary and a defining function φ, and x is a boundary point.

Proposition 1. *Let F be a holomorphic function on a neighbourhood of $\bar{D}$ such that $\{F=0\}$ is nonsingular, $\{F=0\}\cap\partial D=\{x\}$, and the zero-order of $\varphi|\{F=0\}$ is two for every tangent direction at x. Then, for every positive number ε,*

$$\int_D |F|^{-n-1+\varepsilon} dv < \infty.$$

Proof is easy.

We need the following proposition. (cf. [3]).

Proposition 2 (Diederich-Fornaess). *There are positive numbers L and η_0 such that for any positive number $\eta<\eta_0$, the function $-(d(z)\exp(-L|z|^2))^\eta$ is strictly plurisubharmonic on $D\cap\{z_0;\ d(z)$ is C^2 at $z_0\}$.*

In what follows we take a coordinate $z=(z_1, \cdots, z_n)$ so that x is the origin. We put $\nu=\nu_x$, $z'=(z_1, \cdots, z_\nu, 0, \cdots, 0)$, and $z''=(0, \cdots, 0, z_{\nu+1}, \cdots, z_n)$. We may assume that the linear subspace $H=\{z;\ z'=0\}$ is transversal to N_x and ∂D.

We set $B(r)=\{z;|z|<r\}$. We may assume that $B(1)\cap D\cap H$ is simply connected.

We put

$$p(\varphi)=p(\varphi)(z'')=\sum_{i=\nu+1}^{n}\frac{\partial\varphi}{\partial z_i}(0)z_i+\sum_{i,j=\nu+1}^{n}\frac{\partial^2\varphi}{\partial z_i\partial z_j}(0)z_iz_j\,.$$

Let f_ε be a branch of $p(\varphi)^{(-n+\nu-1+\varepsilon)/2}$ over $H\cap B(1)\cap D$, where ε is a positive number satisfying $\varepsilon<2/(n-\nu+1)$. Then, by Proposition 1,

$$\int_{H\cap B(1)\cap D}|f_\varepsilon|^{2+\varepsilon^2}dv<\infty\,.$$

We shall extend $f_\varepsilon|_{H\cap B(1/2)\cap D}$ to a square integrable holomorphic function on $B(1/2)\cap D$. First, in virtue of Proposition 2, we may assume that for any sufficiently small η, $-(-\varphi)^\eta$ is strictly plurisubharmonic on D. If we set $\sigma(z_1,\cdots,z_n)=z''$, then we can find a positive number C such that

$$\sigma^{-1}(H\cap D)\supset\{z\in D;\ -\varphi(z)>C|z'|\}\,.$$

Let χ be a C^∞ function on $\boldsymbol{R}$ such that

$$\chi(t)=\begin{cases}1 & \text{if}\quad t\geqq 2C\\ 0 & \text{if}\quad t<C\,.\end{cases}$$

We set $\lambda(z)=\chi(-\varphi(z)/|z'|)$. Let Φ be a plurisubharmonic function on $B(1)\cap D$ defined by

$$\Phi(z)=-\log(-\log|z'|)-(-\varphi)^\eta+2\nu\log|z'|+|z|^2\,.$$

Let dv_Φ and $|\ \ |_\Phi$ denote respectively the volume form and the length of forms with respect to $\partial\bar\partial\Phi$. We put

$$\tilde\alpha=\begin{cases}\bar\partial(\lambda\sigma^*f_\varepsilon)\wedge dz_1\wedge\cdots\wedge dz_n & \text{on}\quad \sigma^{-1}(H\cap D\cap B(1))\\ 0 & \text{otherwise,}\end{cases}$$

and $\alpha=\tilde\alpha|_{D\cap B(1/2)}$. Then α is a $\bar\partial$-closed $(n,1)$-form on $D\cap B(1/2)$.

Proposition 3. *Let Φ and α be as above, then*

$$(**)\qquad \int_{D\cap B(1/2)-H}e^{-\Phi}|\alpha|_\Phi^2dv_\Phi<\infty\,.$$

Proof. Choosing η so small that $-(-\varphi)^{2\eta}$ is strictly plurisubharmonic, we may assume that

$$\partial\bar{\partial}(-(-\varphi)^{\eta})\geqq\eta^{2}(-\varphi)^{\eta-2}\partial\varphi\wedge\bar{\partial}\varphi .$$

Hence we have

$$\partial\bar{\partial}\Phi\geqq\frac{\partial|z'|\wedge\bar{\partial}|z'|}{|z'|^{2}(\log|z'|)^{2}}+\eta^{2}(-\varphi)^{\eta-2}\partial\varphi\wedge\bar{\partial}\varphi+\sum_{i=1}^{n}dz_{i}\wedge d\bar{z}_{i}$$

$$\text{on}\quad D\cap B(1)-H .$$

Therefore,

$$|\bar{\partial}\lambda(z)|_{\Phi}^{2}=\left|\chi'(-\varphi(z)/|z'|)\left(-\frac{\bar{\partial}\varphi}{|z'|}+\frac{\varphi\bar{\partial}|z'|}{|z'|^{2}}\right)\right|_{\Phi}^{2}$$

$$\leqq\frac{2K^{2}C^{2}}{\eta^{2}}\{(-\varphi)^{-\eta}+(\log|z'|)^{2}\} .$$

Here, $K=\sup_{t\in R}\chi'(t)$ and η is chosen to be smaller than one. Since $|dz_{1}\wedge\cdots\wedge dz_{n}|_{\Phi}^{2}dv_{\Phi}=|dz_{1}\wedge\cdots\wedge dz_{n}|^{2}dv$ we have

$$|\alpha|_{\Phi}^{2}dv_{\Phi}\leqq C_{1}(\eta)|f_{\varepsilon}(\sigma(z))|^{2}\{(-\varphi)^{-\eta}+(\log|z'|)^{2}\}dv ,$$

for some constant $C_{1}(\eta)$ depending on η. On the support of α we have $|z'|<-\varphi(z)/C$. Hence

$$|\alpha|_{\Phi}^{2}dv_{\Phi}\leqq C_{2}(\eta)|f_{\varepsilon}(\sigma(z))|^{2}(-\varphi)^{-\eta}dv ,$$

for some constant $C_{2}(\eta)$. Therefore, for some constants $C_{3}(\eta)$, m and M, we have

$$\int_{D\cap B(1/2)-H}e^{-\Phi}|\alpha|_{\Phi}^{2}dv_{\Phi}$$

$$\leqq C_{3}(\eta)\int_{D\cap B(1)\cap H}|f_{\varepsilon}|^{2}|\varphi(z'')|^{-2\eta}\left(\int_{B(M\varphi(z''))-B(m\varphi(z''))}|z'|^{-2\nu}dv_{1}\right)dv_{2} .$$

Here dv_{1} is the Lebesgue measure on $\{z''=\text{constant}\}$ and dv_{2} is the Lebesgue measure on H. Hence,

$$\int_{D\cap B(1/2)-H}e^{-\Phi}|\alpha|_{\Phi}^{2}dv_{\Phi}$$

$$\leqq C_{4}(\eta)\int_{D\cap B(1)\cap H}|f_{\varepsilon}|^{2}|\varphi(z'')|^{-3\eta}dv_{2}$$

$$\leqq C_{4}(\eta)\left(\int_{D\cap B(1)\cap H}|f_{\varepsilon}|^{2+\varepsilon^{2}}dv_{2}\right)^{2/(2+\varepsilon^{2})}\left(\int_{D\cap B(1)\cap H}|\varphi(z'')|^{-3(2+\varepsilon^{2})\eta/\varepsilon^{2}}dv_{2}\right)^{\varepsilon^{2}/(2+\varepsilon^{2})}$$

for some constant $C_{4}(\eta)$. Hence, if η is sufficiently small relative to ε^{2}, we have (**).

In [5] we have proved the following

Proposition 4. *Let X be a complex manifold which admits a complete Kähler metric, and let Φ be a strictly plurisubharmonic function on X of class C^4. Then, for any $\bar\partial$-closed $(n, 1)$-form α with $\int_X e^{-\Phi}|\alpha|^2_\Phi dv_\Phi<\infty$, we can find an $(n, 0)$-form β satisfying $\bar\partial\beta=\alpha$ and $\int_X e^{-\Phi}|\beta|^2_\Phi dv_\Phi\leqq\int_X e^{-\Phi}|\alpha|^2_\Phi dv_\Phi$.*

Since $D\cap B(1/2)-H$ admits a complete Kähler metric

$$\sum_{i=1}^n dz_i\wedge d\bar z_i+\partial\bar\partial(-\log(-\log|z'|))+\partial\bar\partial(-1/\varphi)+\partial\bar\partial(1/2-|z|^2)^{-1},$$

Proposition 4 is applicable and we can find β such that $\bar\partial\beta=\alpha$ and

$$\int_{D\cap B(1/2)-H} e^{-\Phi}|\beta|^2_\Phi dv_\Phi\leqq\int_{D\cap B(1/2)-H} e^{-\Phi}|\alpha|^2_\Phi dv_\Phi\,.$$

If we set

$$fdz_1\wedge\cdots\wedge dz_n = \lambda\sigma^*f_\varepsilon dz_1\wedge\cdots\wedge dz_n-\beta\,,$$

we obtain a square integrable holomorphic function f on $D\cap B(1/2)-H$ which naturally extends across H and gives the desired extension of f_ε.

Since the constructions of f_ε and f are uniform with respect to the choices of x and H, we obtain the Main Theorem. Q.E.D.

Question. Is it possible to drop ε in Main Theorem?

References

[1] Bergman, S., *The kernel function and the conformal mapping,* Math. Surveys, No. 5 (1950).

[2] Diederich, K., Das Randverhalten der Bergmanschen Kernfunction und Metrik in streng pseudo-konvexen Gebieten, *Math. Ann.*, **187** (1970), 9–36

[3] Diederich, K. and Fornaess, J.E., Pseudoconvex domains: Bounded strictly plurisubharmonic exhaustion functions, *Invent. Math.* **25** (1977), 129–141.

[4] Hörmander, L., L^2-estimates and existence theorems for the $\bar\partial$-operator. *Acta Math.* **113** (1965), 89–152.

[5] Ohsawa, T., On complete Kähler domains with C^1-boundary, *Publ. RIMS, Kyoto Univ.* **16** (1980), 929–940.

[6] Pflug, P., Quadratintegrable holomorphe Funktionen und die Serre-Vermutung, *Math. Ann.* **216** (1975), 285–288.

[7] ———, Various applications of the existence of well growing holomorphic functions, *Functional analysis, Holomorphy and approximation theory,* J.A. Barroso (ed.), North-Holland Publishing Company, 1982.