

On the Hopf Algebra Structure of the mod 2 Cohomology of a Finite H -Space

By

James LIN*

§ 1. Introduction

The action of the Steenrod algebra on the cohomology of a finite H -space has placed several restrictions on the possible spaces which can be finite H -spaces. It is now known, for example, that the loop space of a finite H -space has no homology torsion. We continue this study here by showing that in the mod 2 cohomology of a finite H -space, every generator of degree $4k+1$ is in the image of Sq^{2k} .

This result has several consequences. For example, if an integer n has dyadic expansion

$$n = 2^{s_1} + \dots + 2^{s_l}$$

where $0 \leq s_i \leq s_{i+1}$ we say n has length l . If $4k+1$ has length l and there is a generator of the mod 2 cohomology of a finite H -space in degree $4k+1$, then there is a generator in degree 2^l-1 . Hence if a finite H -space is highly connected this result implies the non-existence of generators of small length.

Another implication is that there are no $4k$ dimensional primitive indecomposables in the mod 2 cohomology of the loop space. Also, all generators of degree $8k+5$ in the cohomology of a finite H -space are in the image of Sq^2 .

Throughout this paper it is assumed that all finite H -spaces used in this paper are simply connected and have associative mod 2 homology rings. Under this assumption it is easily shown that finite H -spaces that have no two torsion have primitively generated mod 2 cohomology rings. Hence their structure over the Steenrod algebra is restricted by several results of Thomas [4]. It is therefore more interesting to study finite H -spaces that have two torsion in their homology.

Under these circumstances it was shown that the two torsion is of order at most two and arises from odd degree mod 2 cohomology generators whose cup product is non trivial. In chapter 3 of this paper a structure theorem is given for the action of the Steenrod algebra on these generators. In particular if y is

Communicated by N. Shimada, September 7, 1983.

* University of California, San Diego La Jolla, California 92093, USA.

** Partially supported by a National Science Foundation grant.

a generator of the mod 2 cohomology of a finite H -space and the degree of y has the form $2^{s+1}-1+2^{s+2}k$ where $s \geq 0$ and $k > 0$, and y has cup product square non trivial, then y is in the image of $Sq^{2^{s+1}k}$ modulo decomposables. For example, if y is a generator in $H^{19}(X; \mathbf{Z}_2)$ where X is a finite H -space and y^2 is non-zero, then $19=2^2-1+2^4$, hence $y=Sq^8z$ modulo decomposables, where z lies in $H^{11}(X; \mathbf{Z}_2)$. Applying the theorem again, $z=Sq^4w$ modulo decomposables where w lies in $H^7(X; \mathbf{Z}_2)$. So this theorem restricts the structure of the two torsion which appears in the mod 2 cohomology of a finite H -space.

In chapter four a general Hopf algebra structure theorem for the mod 2 cohomology of a finite H -space is presented. If X is a finite H -space, $H^*(X; \mathbf{Z}_2)$ can be considered a universal object on a certain coalgebra which admits a squaring map. Let R be the set of elements of $H^*(X; \mathbf{Z}_2)$ which have reduced coproduct in $\xi H^*(X; \mathbf{Z}_2) \otimes H^*(X; \mathbf{Z}_2)$. Note that this is a coalgebra over the Steenrod algebra, and the squaring map $\xi : H^*(X; \mathbf{Z}_2) \rightarrow H^*(X; \mathbf{Z}_2)$ in fact restricts to R . Such a coalgebra will be called a coalgebra with squares. Let $S(R)$ be the free commutative algebra on R and let I be the ideal in $S(R)$ generated by elements of the form $x^2 - \xi x$. Then $S(R)/I$ is isomorphic to $H^*(X; \mathbf{Z}_2)$ as Hopf algebras over the Steenrod algebra. Hence $H^*(X; \mathbf{Z}_2)$ has a particularly simple form.

The author is grateful to Alex Zabrodsky and John C. Moore for several useful conversations. The author also thanks the Institute for Advanced Study in Jerusalem for their warm hospitality while work was in progress. Throughout the paper it will be assumed that the reader is familiar with the notation of [2].

§ 2. $Q^{1k+1} = Sq^{2k}Q^{2k+1}$

In this chapter, we prove $Q^{1k+1} = Sq^{2k}Q^{2k+1}$ for $k \geq 1$. The proof uses several techniques which appear in [2].

Definition. Let $v(s) = 2^{s+1} - 1$,

$$a(s, k) = 2^{s+2}k + 2v(s-1),$$

$$b(s, k) = 2^{s+2}k + 2v(s).$$

Let I_1 be the two sided ideal of $\mathcal{A}(2)$ generated by Sq^2 , I_2 the two sided ideal generated by $I_1 \cdot I_1$. Let J_r be the right ideal generated by Sq^4, \dots, Sq^{2^r} for $r \geq 2$. Let K_r be the right ideal generated by $J_r + I_2$.

The following theorems were proved in [2] :

Theorem 2.1 [2]. *Let $s \geq 1, k > 1$, then*

- (a_s) $Q^{a(s, k)-1} = Sq^{2^{s+1}k} Q^{b(s-1, k)-1}$.
- (b_s) $Sq^{2^{s+1}} Q^{a(s, k)-1} = 0$.
- (c_s) *If $t \in P_{b(s, k)-1}$, then t is annihilated by K_{s+1} . Dually $Q^{b(s, k)-1} \cap \text{im } K_{s+1} = 0$.*

- (d_s) $Q^{2^s+2+2v(s-1)-1} = Sq^{2^s+1}Q^{2v(s)-1} + Sq^{2v(s-1)}Q^{2^s+2-1}$.
- (e) $Sq^{4l+2}Q^{lk+1} = 0$.

Theorem 2.2 [2]. $Q^{2v(j)-1} = \bar{Q}_{j-1}Q^{2v(j-1)+1}$ where \bar{Q}_{j-1} is $Sq^{0 \dots 2}$, where 2 is in the j^{th} place.

Definition. A doubleton $\beta \in \mathcal{A}(2)$ of degree 2^r is a sum of monomials of the form $b_1Sq^{2^i}b_2Sq^{2^i}b_3$ where $\deg b_2 \equiv \deg b_3 \equiv 0 \pmod{2^{i+1}}$ and $i \leq r$.

It is easy to see that $\bar{Q}_{j-1} = [Sq^{2^j}, \bar{Q}_{j-2}] + \delta$ where δ is a doubleton degree 1.

We now consider several consequences of assuming

$$(2.1) \quad Q^{2^s+2+2v(s-1)-1} = Sq^{2^s+1}Q^{2v(s)-1} \quad \text{for } s \geq 1.$$

Theorem 2.3. If equation (2.1) holds, then Theorem 2.1 is true for $k=1$.

Proof. $a(s, 1)-1 = 2^s+2+2v(s-1)-1$ so (a_s) is precisely equation (2.1). Now for $s=1$, equation (2.1) becomes

$$Q^9 = Sq^1Q^5$$

and $Sq^4Sq^4 = Sq^7Sq^1 + Sq^6Sq^2$. Since $Sq^2Q^{lk+1} = 0$ and all doubletons of degree 1 annihilate Q^{lk+1} it follows that

$$Sq^1Q^9 = Sq^1Sq^4Q^5 = 0.$$

We assume now by induction that

$$Sq^{2^t+1}Q^{2^t+2+2v(t-1)-1} = 0 \quad \text{for } t < s.$$

Then

$$\begin{aligned} Sq^{2^s+1}Q^{2^s+2+2v(s-1)-1} &= Sq^{2^s+1}Sq^{2^s+1}Q^{2v(s)-1} \quad \text{by (2.1)} \\ &= \delta Q^{2v(s)-1} \quad \text{where } \delta \text{ is a doubleton of} \\ &\quad \text{degree } 2^s \text{ by the Adem relations.} \end{aligned}$$

Consider

$$b_2Sq^{2^i}b_3Q^{2v(s)-1} \subseteq Q^{a(i, k)-1}.$$

By Theorem 2.1 (b_s) and the inductive assumption (for $k=1$), it follows that

$$Sq^{2^i}b_2Sq^{2^i}b_3Q^{2v(s)-1} = 0.$$

We conclude $\delta Q^{2v(s)-1} = 0$. Therefore, by induction we have shown $Sq^{2^s-1}Q^{a(s, 1)-1} = 0$.

To prove (c_s) for $k=1$ note that $b(s, 1) = b(s-1, 3)$ so

$$Q^{b(s, 1)-1} \cap \text{im } K_s = 0 \quad \text{by Theorem 2.1 (c}_{s-1}\text{)}.$$

Note $b(s, 1) = 2v(s+1)$. Now if $Q^{b(s, 1)-1} \cap \text{im } Sq^{2^s+1} \neq 0$, then

$$\begin{aligned} Q^{2v(s+1)-1} &= Sq^{2^s+1}Q^{a(s, 1)-1} \\ &= Sq^{2^s+1}Sq^{2^s+1}Q^{2v(s)-1} = 0 \quad \text{by (b}_s\text{)}. \end{aligned}$$

So we conclude

$$Q^{b(s-1)-1} \cap \text{im } K_{s+1} = 0$$

This completes the proof.

Q. E. D.

Theorem 2.4. *If equation (2.1) holds then for $k \geq 1$*

$$Q^{4k+1} = Sq^{2k} Q^{2k+1}.$$

Proof. Recall if $4k+1$ is not of the form $2v(j)-1$ then $4k+1 = a(s, l)-1$ for some $s \geq 1, l \geq 1$. By Theorems 2.1 and 2.3, $Q^{4k+1} = \alpha Q^{2v(j)-1}$ where $\alpha \in \mathcal{A}(2)$. Now by Theorem 2.2

$$\begin{aligned} Q^{2v(j)-1} &= \bar{Q}_{j-1} Q^{2v(j-1)+1} \\ &= [Sq^{2j}, \bar{Q}_{j-2}] Q^{2v(j-1)+1} \end{aligned}$$

since doubletons of degree one annihilate $Q^{2v(j-1)+1}$. By Theorem 2.3, since $2v(j)-1 = b(j-1, 1)-1$, $Q^{2v(j)-1} \cap \text{im } K_j = 0$. It follows that

$$\begin{aligned} Q^{2v(j)-1} &= \bar{Q}_{j-2} Sq^{2j} Q^{2v(j-1)+1} \\ &= \bar{Q}_{j-3} Sq^{2j-1} Sq^{2j} Q^{2v(j-1)+1} \\ &= \dots \\ &= Sq^2 Sq^4 \dots Sq^{2j} Q^{2v(j-1)+1} \\ &= Sq^{2v(j-1)} Q^{2v(j-1)+1}. \end{aligned}$$

It follows that for every k , there exists $\alpha \in \mathcal{A}(2)$, j an integer such that

$$Q^{4k+1} = \alpha Sq^{2v(j-1)} Q^{2v(j-1)+1}.$$

Now the Adem relations imply for l, m positive integers,

$$Sq^{2l} Sq^m Q^{m+1} = Sq^{m+l} Sq^l Q^{m+1}.$$

Applying this rule to α , which is a polynomial in the even Steenrod squares, we conclude

$$Q^{4k+1} = Sq^{2k} Q^{2k+1}. \quad \text{Q. E. D.}$$

The following factorizations follow by induction using the methods in [2, Prop. 2.7].

Proposition 2.5. (a) $Sq^{10} = Sq^8 Sq^2 + \bar{Q}_1 Sq^4 + Q_0 Q_1 Sq^5 + Q_1 Sq^7$
 $+ Q_2 Sq^3 + Sq^1 Sq^2 Sq^1 Sq^6.$

(b) For $s \geq 2$

$$\begin{aligned} Sq^{2s+2+2v(s-1)} &= \sum_{u=-1}^{s-2} Sq^{2s+2+4v(s-2)-1v(u)} \bar{Q}_{u+1} \\ &+ \bar{Q}_s Sq^{2s+1} + \sum_{u=0}^{s-1} Q_u Q_{u+1} Sq^{2s+2+2s+1-2u+1-2u+2} \\ &+ Q_s Sq^{2s+1-1} + Sq^1 Sq^2 Sq^1 Sq^{2s+2+4v(s-2)-2}. \end{aligned}$$

(c) $Sq^{4k+2} = \sum_{j=-1} Sq^{1k-1v(j)} \bar{Q}_{j+1} + \sum a_i \alpha_i$ where $\deg \alpha_i \equiv 3 \pmod{four}$.
 $Sq^{2'+2i+1l} = Sq^{2i} Sq^{2i+1l}$ + doubletons of degree $\leq i-1$.

Now by [2, Thm. 8.7], we may choose representatives for $Q^{2^{s+2}+2v, s-1}-1$ that have reduced coproduct in $\xi^2 H^* \otimes R$, where

$$R = \{x \in H^* \mid \bar{J}x \in \xi^2 H^* \otimes H^*\}.$$

R is an $\mathcal{A}(2)$ module.

Proposition 2.6. *If $Q^{2^{s+2}-2v(s-1)-1}$ is not contained in $\text{im } Sq^{2^{s+1}}$ then there exist elements x with $\bar{J}x \in \xi^2 H^* \otimes R$, $0 \neq \bar{x} \in Q^{2^{s+2}+2v(s-1)-1}$, $Sq^1 x = 0$, and $\bar{x} \in \text{im } Sq^{2^{s+1}}$.*

Proof. By the above remark there is an x' with $\bar{x}' \in \text{im } Sq^{2^{s+1}}$, $\bar{J}x' \in \xi^2 H^* \otimes R$, and $0 \neq \bar{x}' \in Q^{2^{s+2}+2v(s-1)-1}$. Now if $Sq^1 x' \neq 0$ then

$$\{Sq^1 x'\} \in P(H^* // \xi^2 H^*)$$

is decomposable since $Q^{even} = 0$ by [2, Cor. 7.6]. Hence $\{Sq^1 x'\} = \{w\}^2$ where we may assume $w \in R$. If $u = Sq^1 x' - w^2$ then $u \in I(\xi^2 H^*) H^*$ and $\bar{J}u \in \xi^2 H^* \otimes H^*$. Applying the Milnor Moore isomorphism

$$H^* \longrightarrow H^* \otimes H^* \longrightarrow H^* // \xi^2 H^* \otimes \xi^2 H^*$$

implies $u \in \xi^2 H^*$. But $\deg u \equiv 2 \pmod{4}$ and $\xi^2 H^*$ is concentrated in degrees divisible by 4. Therefore $u = 0$ and

$$\begin{aligned} Sq^1 x' &= w^2 \\ &= Sq^1 Sq^{2^{s+1}+2v(s-2)} w. \end{aligned}$$

Hence $x = x' - Sq^{2^{s+1}+2v(s-2)} w$ has $Sq^1 x = 0$. By Theorem 2.1(c) and Proposition 2.5(c) we get

$$Sq^{2^{s+1}+2v(s-2)} w = \sum Sq^{2^{s+1}+1v(s-3)-1v(j)} \bar{Q}_{j-1} u.$$

Now if $j \neq s-3$, a simple argument shows that

$$\text{im } Sq^{2^{s+1}+1v(s-3)-1v(j)} \subseteq \text{im } (Sq^1, Sq^2, \dots, Sq^{2^s}).$$

By Theorem 2.1(c)

$$Sq^{2^{s+1}+2v(s-2)} w = Sq^{2^{s+1}} \bar{Q}_{s-2} u.$$

Hence $x = x' - Sq^{2^{s+1}} \bar{Q}_{s-2} u$. It follows that $\bar{x} \in \text{im } Sq^{2^{s-1}}$. This completes the proof. Q. E. D.

We now prove $Q^0 = Sq^1 Q^5$. This is equation (2.1) for $s=1$. This simple example will illustrate the methods used. Later we will prove equation (2.1) for arbitrary s .

By Proposition 2.5(a)

$$\begin{aligned} Sq^{10} &= Sq^8Sq^2 + \bar{Q}_1Sq^4 + Q_0Q_1Sq^6 + Q_1Sq^7 + Q_2Sq^8 \\ &\quad + Sq^1Sq^2Sq^1Sq^6 \\ &= (Sq^8 + Q_2Sq^1)Sq^2 + (Q_0Q_1 + Q_1Sq^1 + Sq^1Sq^2Sq^1)Sq^6 + \bar{Q}_1Sq^4. \end{aligned}$$

By Theorem 2.1(e), since $Sq^7 = Sq^1Sq^6$, $Sq^8 = Sq^1Sq^2$,

$$0 = Sq^2Q^9 = Sq^6Q^9 = Sq^7Q^9 = Sq^8Q^9.$$

By Theorem 2.2,

$$Sq^4Q^9 \subseteq \bar{Q}_1Q^7 \quad \text{and} \quad \bar{Q}_1\bar{Q}_1 = 0.$$

Our situation is that we can write $Sq^{10} = \bar{Q}_1Sq^4 + \sum a_i b_i$ where if $\bar{x} \in Q^9$ with representative $x \in R$, then $b_i x = 0$ and $Sq^4 x = \bar{Q}_1 y$.

Both x and y may be chosen to be primitive. To see this, note that if $x \in R$ then $\bar{J}x \in \xi^2 H^* \otimes R$. Now because $H^*(X; \mathbf{Z}_2)$ is trivial in degrees 1 and 2, $\xi^2 H^*$ is 11 connected. Hence x is primitive. Similarly, y may be chosen so that $\bar{J}y \in \xi H^* \otimes R$, and ξH^* is 5 connected, so y is primitive.

We now write down the universal example which describes the above relations.

Let

$$K = K(\mathbf{Z}, 9, 7)$$

$$K_0 = K(\mathbf{Z}_2, 11, 13, 15)$$

$$w : K \longrightarrow K_0$$

$$w^*(i_{11}) = Sq^2 i_9$$

$$w^*(i_{13}) = Sq^4 i_9 - \bar{Q}_1 i_7$$

$$w^*(i_{15}) = Sq^6 i_9.$$

Then

$$z = (Sq^8 + Q_2Sq^1)i_{11} + (Q_0Q_1 + Q_1Sq^1 + Sq^1Sq^2Sq^1)i_{13} + \bar{Q}_1 i_{13} \in \ker w^*.$$

In fact

$$w^*(z) = Sq^{10}i_9 = 0.$$

Let E be the fibre of w . It follows that there is an element $v \in H^*(E; \mathbf{Z}_2)$ with $\bar{J}v = u \otimes u$ where $u = p^*(i_9)$

$$\begin{aligned} j^*(v) &= (Sq^8 + Q_2Sq^1)\sigma^*(i_{11}) + (Q_0Q_1 + Q_1Sq^1 + Sq^1Sq^2Sq^1)\sigma^*(i_{13}) \\ &\quad + \bar{Q}_1\sigma^*(i_{13}). \end{aligned}$$

$$\begin{array}{ccc} \Omega K_0 & & \\ \downarrow j & & \\ E & & \\ \downarrow p & \xrightarrow{w} & \\ K & \longrightarrow & K_0. \end{array}$$

The proof of this follows from [2, Prop. 3.1].

Now suppose Q^9 is not contained in the image of Sq^4 . Then there exists an $x \in R^9$, with $x \in \text{im } Sq^4$, x primitive, and a y primitive such that $Sq^4 x = \bar{Q}_1 y$. Hence there is a commutative diagram

$$\begin{array}{ccccc}
 & & \Omega K_0 & & \\
 & & \downarrow j & & \\
 & & E & & \\
 \tilde{f} \nearrow & & \downarrow p & & \\
 X & \xrightarrow{f} & K & \xrightarrow{w} & K_0
 \end{array}$$

where $f^*(i_9) = x$, $f^*(i_7) = y$. It follows that the H -deviation of \tilde{f} factors through ΩK_0 so we have $D\tilde{f} = j\hat{D}$ and

$$\begin{aligned}
 \bar{J}\tilde{f}^*(v) &= (\tilde{f}^* \otimes \tilde{f}^*)(\bar{J}v) + D\tilde{f}^*(v) \\
 &= x \otimes x + \hat{D}^* j^*(v) \\
 &\in x \otimes x + \text{im}(Sq^8 + Q_2 Sq^1) \\
 &\quad + \text{im}(Q_0 Q_1 + Q_1 Sq^1 + Sq^1 Sq^2 Sq^1) \\
 &\quad + \text{im } \bar{Q}_1.
 \end{aligned}$$

Now if $x \in \text{im } Sq^4$, there exists a primitive $t \in P_9$ with $\langle t, x \rangle \neq 0$ and $t Sq^1 = 0$. By Browder, if $\text{deg } \alpha$ odd, $\alpha \in \mathcal{A}(2)$, $t\alpha = 0$ [1]. It follows that

$$\begin{aligned}
 \langle t^2, \tilde{f}^*(v) \rangle &= \langle t \otimes t, \bar{J}\tilde{f}^*(v) \rangle \\
 &= \langle t \otimes t, x \otimes x \rangle + \langle t \otimes t, \text{im}(Sq^8 + Q_2 Sq^1) \rangle \\
 &\quad + \langle t \otimes t, \text{im}(Q_0 Q_1 + Q_1 Sq^1 + Sq^1 Sq^2 Sq^1) \rangle \\
 &\quad + \langle t \otimes t, \text{im } \bar{Q}_1 \rangle \\
 &= \langle t, x \rangle^2 + \langle t \otimes t, \text{im } \bar{Q}_1 \rangle.
 \end{aligned}$$

Now $\bar{Q}_1 = [Sq^4, Sq^2] + Sq^4$ doubletons

$$\bar{J}\bar{Q}_1 = Q_1 \otimes Q_1.$$

So $t\bar{Q}_1 = 0$ implies

$$\langle t^2, \tilde{f}^*(v) \rangle = \langle t, x \rangle^2 \neq 0.$$

This is a contradiction because $t^2 = 0$ if $t \in P_{\text{odd}}$ by Browder [1].

We conclude that our original assumption that Q^9 is not contained in $\text{im } Sq^4$ must be false. Therefore we have proven

Theorem 2.7. $Q^9 = Sq^4 Q^5$.

To prove that $Q^{2^s+2+2v(s-1)-1} = Sq^{2^s+1} Q^{2v(s)-1}$ is analogous to the above proof,

we must construct the universal example, verify that primary operations vanish on our element and then examine the properties of the H -deviation. The only difference is that in general elements in the domain of the operation are not necessarily primitive. We therefore use the methods of Cartan formulae described in [5, Sec. 4]. We assume the reader is familiar with this and will use these results freely.

Now consider the factorization from Proposition 2.5(b).

$$\begin{aligned}
 Sq^{2^{s+2}+2v(s-1)} &= \sum_{u=-1}^{s-2} Sq^{2^{s+2}+1v(s-2)-4v(u)} \bar{Q}_{u+1} + \bar{Q}_s Sq^{2^{s+1}} \\
 &\quad + \sum_{u=0}^{s-1} Q_u Q_{u+1} Sq^{2^{s+2}+2^{s+1}-2u-1-2u+2} \\
 &\quad + Q_s Sq^{2^{s+1}-1} + Sq^1 Sq^2 Sq^1 Sq^{2^{s+2}+1v(s-2)-2}.
 \end{aligned}$$

Henceforth we assume $Q^{2^{s+2}+2v(s-1)-1}$ is not contained in $\text{im } Sq^{2^{s+1}}$. Our goal will be to produce a contradiction. We will find a $t \in P_{2^{s+2}+2v(s-1)-1}$ with $t^2 \neq 0$. This will contradict the result of Browder.

By Proposition 2.6 there is an x with $0 \neq \bar{x} \in Q^{2^{s+2}+2v(s-1)-1}$ and $Sq^1 x = 0$ and $\bar{x} \in \text{im } Sq^{2^{s+1}}$. It follows that there is a primitive t with

$$\langle t, x \rangle \neq 0 \quad \text{and} \quad t Sq^{2^{s+1}} = 0.$$

Further, $\bar{A}x \in \xi^2 H^* \otimes R$.

By Theorem 2.1(e) we have

- (a) $\bar{Q}_{u+1}x = 0$ for all u .
- (b) $Sq^{2^{s+1}-1}x = Sq^1 Sq^{2^{s+1}-2}x = 0$.
- (c) $Sq^{2^{s+2}+4v(s-2)-2}x = 0$.

By Theorem 2.2, there is a $y \in R$ with

- (d) $Sq^{2^{s+1}}x = \bar{Q}_s y$. (Theorem 2.2 states that this is true modulo decomposables. A simple argument using the Milnor Moore isomorphism implies strict equality.)

Proposition 2.8. $Q_{u+1} Sq^{2^{s+2}+2^{s+1}-2u+1-2u+2}x = 0$ for all u such that $0 \leq u \leq s-1$.

Proof. The case $u=0$ follows from Theorem 2.1(e). Recall that $Q_0 = Sq^1$, $Q_1 = [Sq^2, Sq^1]$ and $Q_i = [Sq^{2^i-1}, Q_{i-1}]$. Hence Q_{u+1} is a sum of monomials with Sq^1 and Sq^2 in them. If $b_1 Sq^1 b_2 Sq^2 b_3$ is such a monomial, then degree b_3 must be divisible by 4. By Theorem 2.1(e), $Sq^2 b_3 Sq^{2^{s+2}+2^{s+1}-2u+1-2u+2}x = 0$.

Now consider a monomial of the form $c_1 Sq^2 c_2 Sq^1 c_3$. If this is a summand of Q_{u+1} then degree c_3 is divisible by 4. The Adem relation

$$Sq^1 Sq^{4l} = Sq^{4l} Sq^1 + Sq^2 Sq^1 Sq^{4l-2}$$

and Theorem 2.1(e) imply for $u > 0$,

$$\begin{aligned} & c_1 Sq^2 c_2 Sq^1 c_3 Sq^{2^{s+2}+2^{s+1}-2^{u+1}-2^{u+2}} x \\ &= c_1 Sq^2 c_2 c_3 Sq^{2^{s+2}+2^{s+1}-2^{u+1}-2^{u+2}} Sq^1 x \\ &= 0 \quad \text{because } Sq^1 x = 0. \end{aligned}$$

The theorem follows from the above arguments. Q. E. D.

We now construct the universal example which describes the relations (a), (b), (c), (d) and Proposition 2.8.

Let $K = K(\mathbb{Z}_2, 2^{s+2} + 2v(s-1) - 1, 2^{s+2} - 1)$. Let i be the fundamental class in degree $2^{s+2} + 2v(s-1) - 1$, η be the fundamental class of degree $2^{s+2} - 1$

$$\begin{aligned} K_0 &= \sum_{u=-1}^{s-2} K(\mathbb{Z}_2, 2^{s+2} + 2v(s-1) + 2v(u+1) - 1) \\ &\quad \times K(\mathbb{Z}_2, 2v(s+1) - 1) \\ &\quad \times \sum_{u=0}^{s-1} K(\mathbb{Z}_2, 2^{s+3} + 2v(s) - 2^{u+1} - 2) \\ &\quad \times K(\mathbb{Z}_2, 2v(s+1) - 2, 2^{s+3} + 8v(s-2) - 1). \end{aligned}$$

$$w : K \longrightarrow K_0$$

$$w^*(i_{2^{s+2}+2v(s-1)+2v(u+1)-1}) = \bar{Q}_{u+1} i$$

$$w^*(i_{2v(s+1)-1}) = Sq^{2^{s+1}} i - \bar{Q}_s \eta$$

$$w^*(i_{2^{s+3}+2v(s)-2^{u+1}-2}) = Q_{u+1} Sq^{2^{s+2}+2^{s+1}-2^{u+1}-2^{u+2}} i$$

$$w^*(i_{2v(s+1)-2}) = Sq^{2^{s+1}-1} i$$

$$w^*(i_{2^{s+3}+8v(s-2)-1}) = Sq^{2^{s+2}+4v(s-2)-2} i.$$

Let E be the fibre of w

$$\begin{array}{ccc} \Omega K_0 & & \\ \downarrow j & & \\ E & & \\ \downarrow p & w & \\ K & \longrightarrow & K_0. \end{array}$$

The argument in [2, Sec. 3] implies there is a $v \in H^*(E; \mathbb{Z}_2)$ with

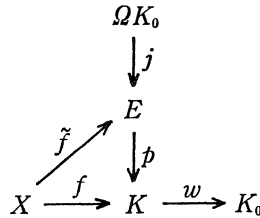
$$\bar{A}v = u \otimes u, \quad u = p^*(i)$$

and

$$\begin{aligned} j^*(v) &= \sum_{u=-1}^{s-2} Sq^{2^{s+2}+4v(s-2)-1v(u)} \sigma^*(i_{2^{s+2}+2v(s-1)+2v(u+1)-1}) \\ &\quad + \bar{Q}_s \sigma^*(i_{2v(s+1)-1}) \end{aligned}$$

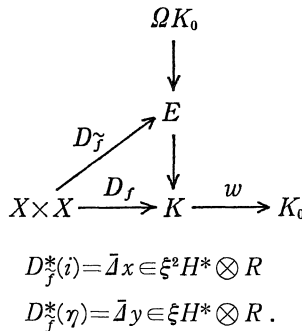
$$\begin{aligned}
 &+ \sum_{u=0}^{s-1} Q_u \sigma^*(i_{2s+3+2v(s)-2} u + 1 - 2) \\
 &+ Q_s \sigma^*(i_{2v(s+1)-2}) \\
 &+ Sq^1 Sq^2 Sq^1 \sigma^*(i_{2s+3+8v(s-2)-1}).
 \end{aligned}$$

There is a commutative diagram



and $f^*(i) = x, f^*(\eta) = y, \bar{D}\tilde{f}^*(v) = x \otimes x + D_f^*(v)$.

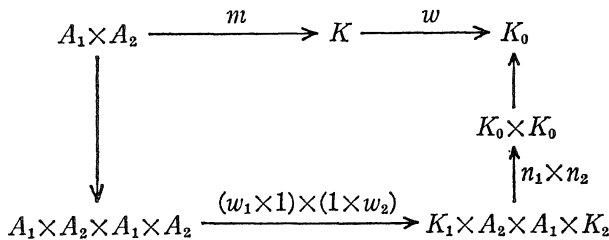
In [2, 5, Sec. 4] the theory of Cartan maps describes how to compute $D_f^*(v)$. Note there is a commutative diagram



Hence there exist generalized Eilenberg MacLane spaces A_1 concentrated in even degrees and A_2 concentrated in odd degrees such that D_f factors

$$X \times X \xrightarrow{f_1 \times f_2} A_1 \times A_2 \xrightarrow{m} K$$

and m is a Cartan map. That is, there exist K_1, K_2 and maps w_i, n_i such that there is a commutative square



By [2, Theorem 4.1] if E_i is the fibre of w_i , there is a commutative diagram

$$\begin{array}{ccccc}
 & & & & \Omega K_0 \\
 & & & & \downarrow \\
 & & & & E \\
 & & \nearrow \tilde{f}_1 \times \tilde{f}_2 & \xrightarrow{l} & \\
 X \times X & \xrightarrow{f_1 \times f_2} & E_1 \times E_2 & & \\
 & & \downarrow & & \downarrow p \\
 & & A_1 \times A_2 & \xrightarrow{m} & K
 \end{array}$$

Hence $D_{\tilde{f}}$ and $l(\tilde{f}_1 \times \tilde{f}_2)$ both lift D_f so

$$D_{\tilde{f}}^*(v) = (\tilde{f}_1 \times \tilde{f}_2)^* l^*(v) + \hat{D}^* j^*(v)$$

for some $\hat{D}: X \times X \rightarrow \Omega K_0$.

By Theorem 4.1 of [2]

$$(\tilde{f}_1^* \otimes \tilde{f}_2^*) l^*(v) \subseteq \text{im } f_1^* \otimes H^* + H^* \otimes \text{im } f_2^*.$$

Now $\text{im } f_i^* \subseteq \xi H^*$, hence this part has no summand in an odd degree. We therefore concentrate on the contribution of $(\tilde{f}_1^* \otimes \tilde{f}_2^*) l^*(v)$ in $H^* \otimes \text{im } f_2^*$ in bidegrees $(2^{s+2} + 2v(s-1) - 1, 2^{s+2} + 2v(s-1) - 1)$.

Let $F_m R = \{z \in R \mid \bar{A}^m z = 0\}$. Let $C(m)$ be the $\mathcal{A}(2)$ subHopf algebra of H^* generated by $F_m R$. Then H^* is filtered

$$Z_2 \subseteq C(1) \subseteq C(2) \subseteq \dots \subseteq H^*.$$

Assume $x \in C(m), x \in C(m-1)$. By coassociativity $\bar{J}x \in C(m-1) \otimes C(m-1)$. We can clearly choose t so that $\langle t, C(m-1) \rangle = 0$.

Theorem 2.9. $\langle t \otimes t, (\tilde{f}_1^* \otimes \tilde{f}_2^*) l^*(v) \rangle = 0$.

Proof. We follow closely [2, Sec. 4]. Given a fundamental class i

$$m^* w^*(i) = \sum_j \hat{a}_j \otimes a_j + \sum_k b_k \otimes \hat{b}_k$$

where $\hat{a}_j \in \ker f_1^*, \hat{b}_k \in \ker f_2^*, f_1^*(b_k) \in \xi H^*, f_2^*(a_j) \in R$.

Define

$$\begin{array}{ll}
 K_1 = \coprod K(Z_2, \text{deg } \hat{a}_j) & K_2 = \coprod K(Z_2, \text{deg } \hat{b}_k) \\
 w_1: A_1 \longrightarrow K_1 & w_1^*(i_{\text{deg } \hat{a}_j}) = \hat{a}_j \\
 w_2: A_2 \longrightarrow K_2 & w_2^*(i_{\text{deg } \hat{b}_k}) = \hat{b}_k \\
 n_1: K_1 \times A_2 \longrightarrow K_0 & n_1^*(i) = \sum i_{\text{deg } \hat{a}_j} \otimes a_j \\
 n_2: A_1 \times K_2 \longrightarrow K_0 & n_2^*(i) = \sum b_k \otimes i_{\text{deg } \hat{b}_k}.
 \end{array}$$

n_1, n_2 induce maps

$$\begin{aligned}
 k_1 &: \Omega K_1 \wedge E_2 \longrightarrow \Omega K_0 \\
 k_2 &: E_1 \wedge \Omega K_2 \longrightarrow \Omega K_0 \\
 k_1^*(\sigma^*(i)) &= \sum_j \sigma^*(i_{\text{deg } \hat{a}_j}) \otimes p_{\frac{1}{2}}^*(a_j) \\
 k_2^*(\sigma^*(i)) &= \sum_k p_1^*(b_k) \otimes \sigma^*(i_{\text{deg } \hat{b}_k}).
 \end{aligned}$$

From this we can compute $k_1^*j^*(v)$, $k_2^*j^*(v)$

$$\begin{aligned}
 k_1^*j^*(v) &= \sum \kappa'_j \otimes p_{\frac{1}{2}}^*(a'_j) \\
 k_2^*j^*(v) &= \sum p_1^*(a'_i) \otimes \kappa''_i
 \end{aligned}$$

where $j^*(v'_j) = \kappa'_j$, $j^*(v''_i) = \kappa''_i$. From this

$$\begin{aligned}
 (\tilde{f}_1^* \otimes \tilde{f}_2^*)l^*(v) &= \sum \tilde{f}_1^*(v'_j) \otimes f_2^*(a'_j) + \sum f_1^*(a'_i) \otimes f_2^*(v''_i) \\
 &\text{modulo image } (f_1^* \otimes f_2^*).
 \end{aligned}$$

Clearly the second summation belongs to $\xi H^* \otimes H^*$. These operations annihilate $t \otimes t$ for degree reasons.

Now consider the first summand. Let S_1 be the $\mathcal{A}(2)$ span of terms in the second bidegree of $m^*(i)$ and let S_2 be the $\mathcal{A}(2)$ span of terms in the second bidegree of $m^*(\eta)$. By construction $a'_j \in S_1 + S_2$.

Case 1. If $a'_j \in S_1$ then $f_2^*(a'_j) \in C(m-1)$ and hence a term of this form annihilates $t \otimes t$.

Case 2. If $a''_i \in S_2$ and $\langle t, f_2^*(a''_i) \rangle \neq 0$ we must have degree $f_2^*(a''_i) = 2^{s+2} + 2v(s-1) - 1$.

Tracing backwards through the construction, we must have $j^*(v'_j) \otimes p_{\frac{1}{2}}^*(a'_j)$ appearing in

$$\bar{Q}_s k_1^* \sigma^*(i_{2v(s+1)-1}).$$

Since $\bar{J}\bar{Q}_s = Q_s \otimes Q_s$ and $tQ_s = 0$ we can reduce to the situation where the secondary operations comes from

$$j^*(v'_j) = \bar{Q}_s \sigma^*(i_{\text{deg } \hat{a}_j}).$$

But then $f_2^*(a'_j)$ must be an element in the second bidegree of a summand of $\bar{J}y$. Because degree y is less than degree x , degree $f_2^*(a'_j) < \text{degree } x$. Hence

$$\langle t, f_2^*(a'_j) \rangle = 0.$$

Finally, $\text{im } \tilde{f}_1^* \otimes f_2^* \subseteq \xi H^* \otimes R$ so this also annihilates $t \otimes t$.

We conclude $\langle t \otimes t, (\tilde{f}_1^* \otimes \tilde{f}_2^*)l^*(v) \rangle = 0$.

Q. E. D.

It follows that

$$\begin{aligned}
 \langle t^2, \tilde{f}^*(v) \rangle &= \langle t \otimes t, \bar{J}\tilde{f}^*(v) \rangle \\
 &= \langle t \otimes t, x \otimes x + (\tilde{f}_1^* \otimes \tilde{f}_2^*)l^*(v) + \hat{D}j^*(v) \rangle
 \end{aligned}$$

$$\begin{aligned} &= \langle t, x \rangle^2 + \langle t \otimes t, \hat{D}j^*(v) \rangle \text{ by Theorem 2.9} \\ &= \langle t, x \rangle^2 + \langle t \otimes t, \sum_{u=-1}^{s-2} \text{im } Sq^{2^{s+2}+4v(s-2)-4v(u)} \rangle \\ &\quad + \langle t \otimes t, \text{im } \bar{Q}_s \rangle + \langle t \otimes t, \sum_{u=0}^s \text{im } Q_u \rangle \\ &\quad + \langle t \otimes t, \text{im } Sq^1 Sq^2 Sq^1 \rangle. \end{aligned}$$

Since $t\alpha=0$ for $\text{deg } \alpha$ odd, only the first two terms besides $\langle t, x \rangle^2$ can possibly be nonzero.

Step 1. $\langle t \otimes t, \sum \text{im } Sq^{2^{s+2}+4v(s-2)-4v(u)} \rangle$. We must consider terms of the form

$$t Sq^{2^{s+1}+2v(s-2)-2v(u)+l} \otimes t Sq^{2^{s+1}+2v(s-2)-2v(u)-l}$$

where $-2v(u) \leq l \leq 2v(u)$.

By Theorem 2.1(c) and the assumption $t Sq^{2^{s+1}}=0$, t is annihilated by K_{s+1} . By checking degrees, Theorem 2.1(c) implies t is annihilated by doubletons of degree $\leq s$. Now if $\text{degree } 2^{s+1}+2v(s-2)-2v(u)+l \equiv 2 \pmod 4$ Proposition 2.5(c) implies

$$\begin{aligned} \text{im } Sq^{2^{s+1}+2v(s-2)-2v(u)+l} &\subseteq \text{im } Sq^{2^i} + \text{im } Sq^{2^{s+1}} + \text{doubletons of degree } \leq s \\ &\text{for some } i \leq s. \end{aligned}$$

If $\text{degree } 2^{s+1}+2v(s-2)-2v(u)+l \equiv 0 \pmod 4$, Proposition 2.5(c) implies

$$\begin{aligned} \text{im } Sq^{2^{s+1}+2v(s-2)-2v(u)+l} &\subseteq \text{im } Sq^{2^i} + \text{doubletons of degree } \leq i-1 \\ &\text{for some } i \leq s. \end{aligned}$$

In all cases

$$\langle t \otimes t, \text{im } Sq^{2^{s+2}+4v(s-2)-4v(u)} \rangle = 0.$$

Step 2. $\langle t \otimes t, \text{im } \bar{Q}_s \rangle$.

$$\bar{Q}_s = [Sq^{2^{s+1}}, \bar{Q}_{s-1}], \quad (t \otimes t)\bar{Q}_s = t\bar{Q}_s \otimes t + t \otimes t\bar{Q}_s.$$

If $t\bar{Q}_s \neq 0$ then $\bar{Q}_s Q^{2^{s+1}-1} = Sq^{2^{s+1}} \bar{Q}_{s-1} Q^{2^{s+1}-1}$. Hence $t\bar{Q}_s \neq 0$ implies $t Sq^{2^{s+1}} \neq 0$. But by assumption $t Sq^{2^{s+1}} = 0$. Therefore $\langle t \otimes t, \text{im } \bar{Q}_s \rangle = 0$.

We conclude $\langle t^2, \tilde{f}^*(v) \rangle = \langle t, x \rangle^2$ which implies $t^2 \neq 0$. This is the desired contradiction.

We have proved the following theorem :

Theorem 2.10. $Q^{2^{s+2}-2v(s-1)-1} = Sq^{2^{s+1}} Q^{2v(s)-1}$ for $s \geq 1$. Hence by Theorem 2.4, $Q^{1^{k+1}} = Sq^{2^k} Q^{2^{k+1}}$.

§ 3. $\xi H^*(X; \mathbb{Z}_2)$

In this chapter we prove a theorem about algebra generators of $H^*(X; \mathbb{Z}_2)$

whose cup product square is nontrivial. The results derive from techniques developed by Zabrodsky [6]. Because $H_*(\Omega X; \mathbf{Z})$ is two torsion free it was shown that

$$\text{Tor}_{H^*(X; \mathbf{Z}_2)}(\mathbf{Z}_2, \mathbf{Z}_2) \text{ is isomorphic to } H^*(\Omega X; \mathbf{Z}_2)$$

as a coalgebra [2].

If $H^*(X; \mathbf{Z}_2) = \bigotimes_i \wedge(x_i) \otimes_j \mathbf{Z}_2[y_j]/(y_j)^{2^{f_j}}$ then there is an exterior coalgebra $\wedge(\sigma^*(y_j)) \subseteq H^*(\Omega X; \mathbf{Z}_2)$. In $H_*(\Omega X, \mathbf{Z}_2)$ this corresponds to a subHopf algebra which is the kernel of $\xi: H_*(\Omega X; \mathbf{Z}_2) \rightarrow H_*(\Omega X; \mathbf{Z}_2)$.

Since

$$\text{Tor}_{H^*(X; \mathbf{Z}_2)}(\mathbf{Z}_2, \mathbf{Z}_2) = \Gamma(\sigma^*(x_i)) \otimes \wedge(\sigma^*(y_j)) \otimes \Gamma(\varphi_{2^{f_j}}(y_j))$$

it follows that all generators of kernel ξ map monomorphically to $H_*(X; \mathbf{Z}_2)$. This implies the following theorem:

Theorem 3.1. *Let $y \in H^*(X; \mathbf{Z}_2)$ have degree $v(s) + 2^{s+2}k$, $s \geq 0, k > 0$ and suppose $y^2 \neq 0$ and $\bar{\Delta}y \in \xi H^*(X; \mathbf{Z}_2) \otimes H^*(X; \mathbf{Z}_2)$. Then $y = Sq^{2^{s+1}k}z$ for some z , and $y \in \text{im } Sq^{2^i}$ for $i \leq s$.*

Proof. y projects nontrivially to $QH^*(X; \mathbf{Z}_2)$ because $\{y\} \in H^*(X; \mathbf{Z}_2) // \xi H^*(X; \mathbf{Z}_2)$ is odd degree primitive. Hence $\sigma^*(y)$ is nontrivial and is dual to an element $t \in H_*(\Omega X; \mathbf{Z}_2)$ with $t^2 = 0$. Associated to the factorization

$$Sq^{v(s) + 2^{s+2}k} = \sum_{u=-1} Sq^{2^{s+2}k + 2v(s-1) - 2v(u)} Q_{u+1}$$

is a secondary operation ϕ . Now since $H_*(\Omega X; \mathbf{Z}_2)$ is even dimensional, $Q_{u+1}\sigma^*(y) = 0$ for all u . According to Zabrodsky [6], since ΩX is homotopy commutative,

$$(3.1) \quad \bar{\Delta}\phi(\sigma^*y) \in \sigma^*y \otimes \sigma^*y + \sum_{u=-1} \text{im } Sq^{2^{s+1}k + v(s-1) - v(u)} \otimes Sq^{2^{s+1}k + v(s-1) - v(u)}.$$

The proof now proceeds by induction. For $s=0$ the theorem follows because $Q^{4k+1} = Sq^{2k}Q^{2k+1}$. Assume the theorem is true for $s' < s$. Then if degree $y = v(s) + 2^{s+2}k$. Suppose $y = Sq^{2^i}z$ for some $i \leq s$. Pick the smallest such i . Then $z^2 \neq 0$ since $y^2 \neq 0$. By induction $z = Sq^{2^i + \dots + 2^{s-1} + 2^{s+1}k}w$ for some w . But then by applying Adem relations $Sq^{2^i}z = y$ is in the image of $Sq^1, \dots, Sq^{2^{i-1}}$. This contradicts the assumption that the smallest i was chosen.

We conclude $y \in \text{im } Sq^{2^i}$ for $i \leq s$. It follows that $\sigma^*y \in \text{im } Sq^{2^i}$ for $i \leq s$, because, if so, $\sigma^*y = Sq^{2^i}w$ and $w \in \bigotimes_j \wedge(\sigma^*(y_j))$ which implies $w = \sigma^*z$.

Now by (3.1), if

$$\sigma^*y \in \sum \text{im } Sq^{2^{s+1}k + v(s-1) - v(u)}$$

then $\langle t^2, \phi(\sigma^*y) \rangle \neq 0$ which would imply $t^2 \neq 0$. This can't happen, so

$$\sigma^*y \in \sum \text{im } Sq^{2^{s+1}k + v(s-1) - v(u)}.$$

The only term not in the image of Sq^{2^i} for $i \leq s$ is $Sq^{2^{s+1}k}$. Hence $\sigma^*y = Sq^{2^{s+1}k}\sigma^*z$ which implies $y = Sq^{2^{s+1}k}z$. Q. E. D.

§ 4. Applications

There are several applications of the fact that $Q^{1k+1} = Sq^{2k}Q^{2k+1}$. We describe a few which will be used in subsequent papers to deduce the action of the Steenrod algebra on $H^*(\Omega X)$.

Definition 4.1. Let n have dyadic expansion

$$n = 2^{s_1} + 2^{s_2} + \dots + 2^{s_l}$$

where $0 < s_i < s_{i+1}$. We will call l the "length of n ".

Proposition 4.1. *If $4k+1$ has length l and Q^{1k+1} is nontrivial, then $Q^{2^{l-1}}$ is nontrivial.*

Proof. From Theorem 2.3 and Theorem 2.1(a) we must have

$$Q^{2^{v(l)-1}} \text{ nontrivial.}$$

By Theorem 2.2, $Q^{2^{l-1}}$ is nontrivial. Q. E. D.

Proposition 4.2. $PH^{1k}(\Omega X) = \xi(\sigma^*Q^{2k+1})$.

Proof. It is known that

$$PH^{1k}(\Omega X) = \sigma^*Q^{1k+1}.$$

But $Q^{1k+1} = Sq^{2k}Q^{2k+1}$. Q. E. D.

Proposition 4.3. $Q^{8k+5} = Sq^2Q^{8k+3}$

$$\begin{aligned} \text{Proof. } Q^{8k+5} &= Sq^{1k+2}Q^{1k+3} \\ &= Sq^2Sq^{1k}Q^{1k+3} \end{aligned}$$

since $Sq^{1k+2} = Sq^2Sq^{1k}$ modulo a Sq^1 doubleton. Q. E. D.

Definition 4.3. A coalgebra with squares is a coalgebra C together with a map $\xi: C \rightarrow C$ of coalgebras such that degree $\xi x =$ twice degree of x .

Let $\hat{\xi}: H^*(X; \mathbb{Z}_2) \rightarrow H^*(X; \mathbb{Z}_2)$ be the cup product squaring map. Define

$$R = \{x \in H^*(X; \mathbb{Z}_2) \mid \bar{J}x \in \hat{\xi}H^*(X; \mathbb{Z}_2) \otimes H^*(X; \mathbb{Z}_2)\}.$$

Note that since $\hat{\xi}H^*(X; \mathbb{Z}_2)$ is a subHopf algebra, $\hat{\xi}H^*(X; \mathbb{Z}_2) \subseteq R$.

Further, by coassociativity,

$$\bar{J}R \subseteq R \otimes R$$

and in fact R is a coalgebra over the Steenrod algebra with squares.

Let $S(R)$ be the free commutative algebra on R . Then $S(R)$ is a Hopf algebra and there is a diagram

$$\begin{array}{ccc}
 R & \longrightarrow & H^*(X; \mathbf{Z}_2) \\
 & \searrow & \nearrow \theta \\
 & S(R) &
 \end{array}$$

where the dotted arrow is a map of Hopf algebras.

Theorem 4.4. *Let $I \subseteq S(R)$ be the ideal generated by elements of the form $x^2 - \xi x$. Then θ induces an isomorphism $\tilde{\theta}$ of Hopf algebras over the Steenrod algebra*

$$\begin{array}{ccc}
 & S(R) & \\
 & \swarrow & \searrow \theta \\
 S(R)/I & \xrightarrow{\tilde{\theta}} & H^*(X; \mathbf{Z}_2)
 \end{array}$$

Proof. $\tilde{\theta}$ is a map of Hopf algebras. Because the map

$$R \longrightarrow H^*(X; \mathbf{Z}_2) \longrightarrow QH^*(X; \mathbf{Z}_2)$$

is an epimorphism, $\tilde{\theta}$ is an epimorphism, with kernel ξR . Filter both $S(R)/I$ and $H^*(X; \mathbf{Z}_2)$ by powers of the augmentation ideal. Then $E_0(S(R)/I)$ is a primitively generated Hopf algebra on generators $R/\xi R$. Since generators of a Borel decomposition of $H^*(X; \mathbf{Z}_2)$ can be chosen in R , $E_0(H^*(X; \mathbf{Z}_2))$ is also primitively generated on $R/\xi R$. It follows that $E_0(\tilde{\theta})$ is an isomorphism. Hence $\tilde{\theta}$ is an isomorphism. Q. E. D.

References

- [1] Browder, W., Torsion in H -spaces, *Ann. of Math.*, **74** (1961), 24-51.
- [2] Lin, J., Two torsion and the loop space conjecture, *Ann. of Math.*, **115** (1982), 35-91.
- [3] Milnor, J. and Moore, J. C., On the structure of Hopf algebras, *Ann. of Math.*, **81** (1965), 211-264.
- [4] Thomas, E., Steenrod squares and H -spaces, I, II, *Ann. of Math.*, **77** (1963), 306-317; **81** (1965), 473-495.
- [5] ———, Whitney Cartan product formulas, *Math. Z.* **118** (1970), 115-138.
- [6] Zabrodsky, A., Cohomology operations and homotopy commutative H -spaces in the Steenrod algebra and its applications, *Springer Verlag*, **168** (1970), 308-317.