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On the Mod p Decomposition of $Q(CP^{\infty})$

Dedicated to Professor Nobuo Shimada on his 60th birthday

By

Akira KONO*

§1. Introduction

Let $j: CP^{\infty} \to BU$ be the natural inclusion and $\xi: Q(BU) \to BU$ be the structure map of the infinite loop space structure defined by the Bott periodicity theorem, where

$$Q(X) = \operatorname{Colim}_n \Omega^n \Sigma^n X$$

for a pointed space X. In [5] Segal defined a map $\lambda: Q(CP^{\infty}) \to BU$ by the composition

$$Q(CP^{\infty}) \xrightarrow{Q(j)} Q(BU) \xrightarrow{\xi} BU$$

and showed that there exists a map $s: BU \to Q(CP^{\infty})$ such that $\lambda \cdot s \simeq 1$. As a corollary of the above fact he showed that there is a space F satisfying $Q(CP^{\infty}) \simeq BU \times F$ and $\pi_*(F)$ is a finite abelian group for any $*$.

Let p be a rational prime. In [1] Adams showed that there exist infinite loop spaces $G_1, \cdots, G_{p-1}$ such that

$$BU_{(p)} \simeq \prod_{k=1}^{p-1} G_k$$

where $BU_{(p)}$ denotes the localization at p (for details see §2). On the other hand by Mimura, Nishida and Toda [4], there exist $X_1, \cdots, X_{p-1}$ such that

$$\Sigma CP^{\infty}_{(p)} \simeq \bigvee_{k=1}^{p-1} X_k .$$

Then

$$Q(CP^{\infty}_{(p)}) \simeq \Omega Q(\Sigma CP^{\infty}_{(p)}) \simeq \Omega Q(\bigvee_{k=1}^{p-1} X_k) \simeq \prod_{k=1}^{p-1} \Omega Q(X_k)$$

(see §3). The purpose of this paper is to show

Theorem 1.1. *There exists F_{k_0} such that*

$$\Omega Q(X_{k_0}) \simeq G_{k_0} \times F_{k_0}$$

and $\pi_*(F_{k_0})$ is finite for each k_0 $(1 \leq k_0 \leq p-1)$.

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* Department of Mathematics, Kyoto University, Kyoto 606, Japan

Noting that $Q(X_{(p)})\simeq Q(X)_{(p)}$ by Lemma 3.1, we have

$$Q(CP^{\infty})_{(p)}\simeq(\prod_{k=1}^{p-1}G_k)\times(\prod_{k=1}^{p-1}F_k).$$

Let n be a positive integer which divides $p-1$. In [6], Sullivan showed that $S_{(p)}^{2n-1}$ is an associative H-space. In [3], McGibbon showed that

$$\sum BS_{(p)}^{2n-1}\simeq\bigvee_{k=1}^{(p-1)/n}X_{nk}$$

(the definition of X_k in [3] is the same as that in [4]). Thus we have

Corollary 1.2. *If n is a positive integer which divides $p-1$, then*

$$Q(BS_{(p)}^{2n-1})\simeq(\prod_{k=1}^{(p-1)/n}G_{nk})\times(\prod_{k=1}^{(p-1)/n}F_{nk}).$$

§2. Mod p Decomposition of the Complex K-Theory

Let p be a rational prime and $K^*(\ )_{(p)}$ be the complex K-theory localized at p. The following is due to Adams (see Lecture 4 of [1] and §9 of [2]):

Tneorem 2.1. *There exist (generalized) cohomology theories $E_1^*(\ ),\cdots,E_{p-1}^*(\ )$ satisfying*

(1) *as a cohomology theory $K^*(\ )_{(p)}=E_1^*(\ )\oplus\cdots\oplus E_{p-1}^*(\ )$, and*

(2) $E_k^a(pt)=\begin{cases} Z_{(p)} & \text{if } -a\equiv 2k \bmod 2(p-1) \\ 0 & \text{otherwise}\end{cases}$

$(1\leqq k\leqq p-1)$.

Let $e_k^*(\ )$ be the associated connective cohomology theory of $E_k^*(\ )$ and G_k be an infinite loop space which represents $e_k^*(\ )$. As a corollary of the above theorem, we have the following:

Corollary 2.2. *There exists a homotopy equivalence*

$$BU_{(p)}\simeq\prod_{k=1}^{p-1}G_k$$

and the homotopy groups of G_k are given by

$$\pi_a(G_k)=\begin{cases} Z_{(p)} & \text{if } a\equiv 2k \bmod 2(p-1) \text{ and } a>0, \\ 0 & \text{otherwise.}\end{cases}$$

The following is due to Mimura, Nishida and Toda (see [4]):

Lemma 2.3. *There exist $X_1,\cdots,X_{p-1}$ such that*

(1) $\sum CP_{(p)}^{\infty}\simeq\bigvee_{k=1}^{p-1}X_k$

and

$$(2)\quad \widetilde{H}_a(X_k)=\begin{cases} Z_{(p)} & \text{if } a-1\equiv 2k \bmod 2(p-1) \text{ and } a>1, \\ 0 & \text{otherwise.} \end{cases}$$

§3. Proof of Theorem 1.1

Let X be a connected, simply connected CW complex. Then $(\Sigma X)_{(p)}\simeq\Sigma X_{(p)}$ and $(\Omega X)_{(p)}\simeq\Omega X_{(p)}$ (cf. [4]). The following is easily proved:

Lemma 3.1. *For any pointed CW complex X, $Q(X)_{(p)}\simeq Q(X_{(p)})$.*

The homotopy equivalence in Lemma 2.3 induces homotopy equivalences

$$Q(CP^{\infty}_{(p)})\simeq\Omega Q(\Sigma CP^{\infty}_{(p)})\simeq\prod_{k=1}^{p-1}\Omega Q(X_k).$$

Let $j_k:\Omega Q(X_k)\to Q(CP^{\infty}_{(p)})$ and $j'_k:G_k\to BU_{(p)}$ be the natural inclusions and $q_k:Q(CP^{\infty}_{(p)})\to\Omega Q(X_k)$ and $q'_k:BU_{(p)}\to G_k$ be the natural projections $(1\leqq k\leqq p-1)$. Put

$$\lambda_k=q'_k\circ\lambda_{(p)}:Q(CP^{\infty}_{(p)})\to BU_{(p)}\to G_k$$

and

$$s_k=s_{(p)}\circ j'_k:G_k\to BU_{(p)}\to Q(CP^{\infty}_{(p)}).$$

Then we have

$$(*)\qquad \lambda_k\circ s_k=q'_k\circ\lambda_{(p)}\circ s_{(p)}\circ j'_k\simeq q'_k\circ j'_k\simeq 1_{G_k}.$$

Now to prove Theorem 1.1, we need only show the following (see [5]):

Theorem 3.2. *For each k_0, the composition*

$$(\lambda_{k_0}\circ j_{k_0})\circ(q_{k_0}\circ s_{k_0})$$

is a homotopy equivalence.

The following is proved by a standard argument (cf. [5]):

Lemma 3.3. *The homotopy groups of $\Omega Q(X_k)$ are given by*

$$\pi_a(\Omega Q(X_k))=\begin{cases} Z_{(p)}\oplus p\text{-torsion} & \text{if } a\equiv 2k \bmod 2(p-1) \text{ and } a>0, \\ p\text{-torsion} & \text{otherwise.} \end{cases}$$

To prove Theorem 3.2, we need the following algebraic lemma:

Lemma 3.4. *Let R be a (commutative) ring (with unity), $f:A\to B$ and $s:B\to A$ be an R-module homomorphism such that $f\circ s=1_B$. Suppose that there is a direct sum decomposition $A=A_1\oplus\cdots\oplus A_n$ of R-modules with the projection $p_k:A\to A_k$ and the inclusion $i_k:A_k\to A$ $(1\leqq k\leqq n)$. If B is a free R-module and there is an integer k_0 $(1\leqq k_0\leqq n)$ such that A_k is a torsion R-module for each*

$k\neq k_0$, then $f_k\circ s_k=\delta_{k,k_0}\circ 1_B$, where $f_k=f\circ i_k$ and $s_k=p_k\circ s$.

Proof of Lemma 3.4. If $k\neq k_0$, then $f_k=0$, since A_k is a torsion R-module and B is a free R-module. Therefore $f_k\circ s_k=0$ if $k\neq k_0$ and

$$f_{k_0}\circ s_{k_0}=\sum_{k=1}^{n} f_k\circ s_k=\sum_{k=1}^{n} f\circ i_k\circ p_k\circ s=f\circ(\sum_{k=1}^{n} i_k\circ p_k)\circ s=f\circ s=1_B .$$

Proof of Theorem 3.2. Fix an integer k_0 $(1\leqq k_0\leqq p-1)$. Let a be a positive integer such that $a\equiv 2k_0 \bmod 2(p-1)$. Then to prove Theorem 3.2, we need only show

$$U=\lambda_{k_0^*}\circ j_{k_0^*}\circ q_{k_0^*}\circ s_{k_0^*} : \pi_a(G_{k_0})\to\pi_a(G_{k_0})$$

is an isomorphism. Put $R=Z_{(p)}$, $A=\pi_a(Q(CP^\infty_{(p)}))$, $A_k=\pi_a(\Omega Q(X_k))$, $B=\pi_a(G_{k_0})$, $f=\lambda_{k_0^*}$ and $s=s_{k_0^*}$. Then clearly $j_{k*}=i_k$ and $q_{k*}=p_k$. Since $\lambda_{k_0^*}\circ s_{k_0^*}=1_B$ by (*), A_k is a p-torsion group if $k\neq k_0$ by Lemma 3.3 and B is a free $Z_{(p)}$-module by Corollary 2.2, $U=1_B$ by Lemma 3.4.

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