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Groupoid Dynamical Systems and Crossed Product, I—The Case of W^*-Systems

By

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Abstract

By analogy with W^*-dynamical system, we define a W^*-groupoid dynamical system (M, Γ, ρ) where M is a von Neumann algebra, Γ is a locally compact measured groupoid, and $\rho: \Gamma \to \mathrm{Aut}(M)$ is a continuous groupoid homomorphism. The groupoid crossed product $M \times_\rho \Gamma$ is defined by making use of the non-commutative integration theory of A. Connes, i.e. integration theory over singular quotient spaces, and is shown to have similar properties as the case of a group action. As a special case of this situation, if ρ is a continuous homomorphism from Γ to a locally compact group G, we obtain groupoid dynamical system $(L^\infty(G), \Gamma, \rho)$. In this case, there exists a co-action $\hat{\rho}$ of G on $\mathrm{End}_\Lambda(\Gamma)$ and the groupoid crossed product $L^\infty(G) \times_\rho \Gamma$ is isomorphic to the co-crossed product $\mathrm{End}_\Lambda(\Gamma) *_{\hat{\rho}} G$ of $\mathrm{End}_\Lambda(\Gamma)$ by G in the sense of Nakagami and Takesaki.

§ 1. Introduction

Since the work of Murray and von Neumann, the group measure space construction is well studied (for example, [15], [21]) and discovered to be important for the construction of concrete examples of von Neumann algebras. All known approximately finite dimensional factors are constructed from ergodic non-singular transformation groups. Also, the group measure space construction was extended to crossed product and served as an important tool not only for the construction of examples but also for the structure analysis of von Neumann algebras [2], [7], [29].

On the other hand, the non-commutative integration theory developed by A. Connes ([8], [9], [10], [11]) not only yields examples of operator algebras out of foliated manifolds, but also proposes a method of studying foliated manifolds via operator algebras constructed from the holonomy groupoid. In this paper, we present an extension of crossed product from group to groupoid on the

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basis of the non-commutative integration theory of Connes.

One of our basic ideas is the extension of random operator field (in the sense of Connes), which takes values normally in type I-factor, to the case taking values in a more general W^*-algebra. To formulate this idea, we first summarize the direct integral theory of the field of Hilbert spaces and of operator algebras over singular spaces in Section 2. In Section 3, we discuss the standard representation of a singular direct integral algebra. In Section 4, we shall describe a W^*-groupoid dynamical system and its properties.

Another basic idea in our approach is a generalization of the cocycle construction of skew products to groupoid. This is discussed in Section 4 in the general case, and in the special case of an abelian algebra in Section 5.

Third basic idea is the Poincaré suspension which is a standard method of constructing a unimodular measured groupoid from a non-unimodular measured groupoid (the measured groupoid is called unimodular if the module δ corresponding to the transverse measure is trivial). The Poincaré suspension of a groupoid is known to correspond to Takesaki duality of the corresponding von Neumann algebra (see for example [9], [26]). In the above discussion, the module function δ plays an important role. The module function δ is a groupoid homomorphism from Γ to multiplicative group $\mathbb{R}_+$ and hence $\log\delta$: $\Gamma\to\mathbb{R}$ is a groupoid homomorphsim. In this situation, the Poincaré suspension $\tilde{\Gamma}$ of Γ corresponds to the skew product of $\mathbb{R}$ and Γ by $\log\delta$ in our formalism, and the associated W^*-algebra is isomorphic to the modular crossed product of $\mathrm{End}_\Lambda(\Gamma)$. This situation is generalized to the continuous groupoid homomorphism ρ from a locally compact groupoid Γ to a locally compact (not necessarily abelian) group G in Section 6.

In Section 7, we describe some examples.

In Section 8, we shall give a brief discussion.

All the formalisms work well also in the C^*-algebraic framework which will be discussed in [22]. We shall describe in Appendix A cocycles and groupoid homomorphisms which yields some concrete examples of groupoid homomorphisms via cocycles (see also [25], [27], [30]).

Throughout this paper, we use the Tomita-Takesaki theory (see [29]) and relative modular operators (see [2]). We also use the non-commutative integration theory of Connes on a measured groupoid which admits a faithful proper transverse function $\nu=\{\nu^x\}_{x\in\Gamma^{(0)}}$ and a transverse measure Λ with a module δ (see [9], [20]).

We restrict our attention to loaclly compact separable topological groupoids.

The local compactness assumption holds in many cases where a measured groupoid is constructed as a graph of transformation group or as a holonomy groupoid of a smooth foliation. Actually a measured groupoid which is analytic as a Borel space has an inessential reduction to a locally compact topological groupoid (see [24]). Λ_ν denotes a measure on the unit space $\Gamma^{(0)}$ corresponding to the transverse function ν and transverse measure Λ. In this paper, we call the triplet (ν, Λ, δ), a Haar system of Γ. We assume that all the measures are regular.

§ 2. Direct Integral of Hilbert Spaces over Singular Spaces

In this section, we describe the concept of a direct integral of Hilbert spaces over a singular space. Our scheme of construction and notion are the same as those of Connes [9], Kastler [20], and Bellissard-Testard [5]. Throughout this section, we fix a measured groupoid Γ with a Haar system (ν, Λ, δ). We don't use the local compactness of Γ in this section. We also use the notation "$\mathcal{H}ilb$" to denote the category of Hilbert spaces with unitary mappings as morphisms.

Definition 2.1. We call $\mathcal{H}=\{\mathcal{H}_x\}_{x\in\Gamma^{(0)}}$ a Hilbert Γ-bundle if $\mathcal{H}=\{\mathcal{H}_x\}_{x\in\Gamma^{(0)}}$ is a measurable family of Hilbert spaces over $\Gamma^{(0)}$ and there exists a measurable covariant functor $U\colon \Gamma\to\mathcal{H}ilb$ with $U(x)=\mathcal{H}_x$. It means that

(i) if $\gamma\in\Gamma_x^y$, then $U(\gamma)$ is a unitary mapping of $\mathcal{H}_x$ onto $\mathcal{H}_y$,

(ii) $U(\gamma)U(\gamma_1)=U(\gamma\gamma_1)$ if γ and γ_1 are composable.

(iii) $\gamma\mapsto\langle U(\gamma)\xi_{s(\gamma)}, \eta_{r(\gamma)}\rangle_{r(\gamma)}$ is measurable for measurable section $\xi=\{\xi_x\}_{x\in\Gamma^{(0)}}$, $\eta=\{\eta_x\}_{x\in\Gamma^{(0)}}$ of $\mathcal{H}=\{\mathcal{H}_x\}_{x\in\Gamma^{(0)}}$.

Let $\mathcal{H}^j=\{\mathcal{H}_x^j\}_{x\in\Gamma^{(0)}}$ with covariant functor $U^j, j=1,2$ be Hilbert Γ-bundles. Then $\mathcal{H}^1\otimes\mathcal{H}^2=\{\mathcal{H}_x^1\otimes\mathcal{H}_x^2\}_{x\in\Gamma^{(0)}}$ with $U=U^1\otimes U^2$ is a Hilbert Γ-bundle.

Definition 2.2. A Hilbert Γ-bundle $\mathcal{H}^c$ consisting of $\mathcal{H}_x^c\equiv L^2(\Gamma^x, \nu^x)$ with $U^c(\gamma)\colon L^2(\Gamma^x, \nu^x)\to L^2(\Gamma^y, \nu^y)$ determined by $[U^c(\gamma)\xi](\tilde{\gamma})\equiv\xi(\gamma^{-1}\tilde{\gamma})$ will be called a canonical Hilbert Γ-bundle or a canonical bundle for short. Similarly, a Hilbert Γ-bundle $\mathcal{H}^s\equiv\mathcal{H}^c\otimes\mathcal{H}^c$ will be called a standard Hilbert Γ-bundle or a standard bundle for short. (We call it "standard bundle" because the resulting singular direct integral Hilbert space gives rise to a standard representation of $\mathrm{End}_\Lambda(\Gamma)$, see Corollary 2.8 and Section 3.)

Definition 2.3. A measurable section $\xi=\{\xi_x\}_{x\in\Gamma^{(0)}}$ of a Hilbert Γ-bundle

$\mathcal{H}=\{\mathcal{H}_x\}_{x\in\Gamma^{(0)}}$ is said to be *covariant* if $U(\gamma)\xi_y=\delta(\gamma)^{1/2}\xi_x$, $y=s(\gamma)$, $x=r(\gamma)$, for almost all $\gamma\in\Gamma$ with respect to $(\Lambda_\nu\circ\nu)$.

The remainder of this section is devoted to the study of Hilbert Γ-bundles of the form, $\mathcal{H}^c\otimes\mathcal{H}=\{\mathcal{H}_x^c\otimes\mathcal{H}_x\}_{x\in\Gamma^{(0)}}$, where $\mathcal{H}=\{\mathcal{H}_x\}_{x\in\Gamma^{(0)}}$ is a Hilbert Γ-bundle. Let $\xi=\{\xi(\gamma)\}_{\gamma\in\Gamma}$ for a measurable section in $\tilde{\mathcal{H}}\equiv\mathcal{H}^c\otimes\mathcal{H}$, $\tilde{\mathcal{H}}_x\cong L^2(\Gamma^x, \mathcal{H}_x, \nu^x)$, where $\xi(\gamma)$ is a $\mathcal{H}_{r(\gamma)}$-valued measurable function satisfying

$$\int_{\Gamma^x}||\xi(\gamma)||_x^2 d\nu^x(\gamma)<\infty \tag{2.1}$$

for almost all $x\in\Gamma^{(0)}$ with respect to Λ_ν. Let U be the covariant functor for $\mathcal{H}$. The covariance condition for $\tilde{\mathcal{H}}$ is

$$U(\gamma)\xi(\gamma^{-1}\tilde{\gamma}) = \delta(\gamma)^{1/2}\xi(\tilde{\gamma}), \qquad \gamma, \tilde{\gamma}\in\Gamma^x, \ x\in\Gamma^{(0)} \tag{2.2}$$

for almost all γ, $\tilde{\gamma}$. (see Remark 2.11). Recall that the measure μ on $\Gamma^{(0)}$ is said to be δ-symmetric for ν (see Connes [9], Kastler [20] § 5) if

$$\int_{\Gamma^{(0)}}\left\{\int_{\Gamma^x}f(\gamma)d\nu^x(\gamma)\right\}d\mu(x) = \int_{\Gamma^{(0)}}\left\{\int_{\Gamma^x}\delta(\gamma)^{-1}f(\gamma^{-1})d\nu^x(\gamma)\right\}d\mu(x) \tag{2.3}$$

for any non-negative measurable function f on Γ. It is known that Λ_ν is δ-symmetric for ν.

By the properness of the fixed transverse function ν, there exists $f\in\mathcal{F}^+(\Gamma)$ satisfying

$$\int_{\Gamma^x}f(\tilde{\gamma}^{-1}\gamma)d\nu^x(\tilde{\gamma}) = 1\,, \qquad x = r(\gamma) \tag{2.4}$$

for almost all $\gamma\in\Gamma$ with respect to $(\Lambda_\nu\circ\nu)$ (partition of unity for ν), where $\mathcal{F}^+(\Gamma)$ is the set of measurable functions on Γ with values in $[0, +\infty)$.

In the following, we shall identify a covariant measurable section $\{\xi_x\}_{x\in\Gamma^{(0)}}$ of $\mathcal{H}^c\otimes\mathcal{H}$ with a measurable section $\{\xi(\gamma)\}_{\gamma\in\Gamma}$ whenever $\xi(\gamma)=\xi_x(\gamma)$ for $\gamma\in\Gamma^x$, $x\in\Gamma^{(0)}$.

Lemma 2.4. *Let $\xi=\{\xi_x\}_{x\in\Gamma^{(0)}}$ be a covariant measurable section of $\tilde{\mathcal{H}}=\mathcal{H}^c\otimes\mathcal{H}$, and μ be a δ-symmetric measure on $\Gamma^{(0)}$. Then the integral,*

$$\int_{\Gamma^{(0)}}\left\{\int_{\Gamma^x}\langle\xi(\gamma), \xi(\gamma)\rangle_x f(\gamma)d\nu^x(\gamma)\right\}d\mu(x) \tag{2.5}$$

is independent of the choice of the partition of unity f i.e. $f\in\mathcal{F}^+(\Gamma)$ satisfying (2.4) *associated with the proper transverse function ν, where $\langle\ ,\ \rangle_x$ denotes the inner product in the fiber $\mathcal{H}_x$ of the Hilbert Γ-bundle $\mathcal{H}$.*

Proof. Let f_1, f_2 be two partitions of unity associated with the transverse function ν, i.e.

$$\int_{\Gamma^x} f_j(\tilde{\gamma}^{-1}\gamma)d\nu^x(\tilde{\gamma}) = 1\,, \quad x = r(\gamma)\,, \tag{2.6}$$

for almost all $\gamma\in\Gamma, j=1, 2$. Then,

$$\begin{aligned}
&\int_{\Gamma^{(0)}}\left\{\int_{\Gamma^x}\langle\xi(\gamma),\,\xi(\gamma)\rangle_x f_1(\gamma)d\nu^x(\gamma)\right\}d\mu(x) \\
&=\int_{\Gamma^{(0)}}\left\{\int_{\Gamma^x}\left[\int_{\Gamma^x}\langle\xi(\gamma),\,\xi(\gamma)\rangle_x f_1(\gamma)f_2(\tilde{\gamma}^{-1}\gamma)d\nu^x(\tilde{\gamma})\right]d\nu^x(\gamma)\right\}d\mu(x) \\
&=\int_{\Gamma^{(0)}}\left\{\int_{\Gamma^x}\left[\int_{\Gamma^x}\langle\xi(\gamma),\,\xi(\gamma)\rangle_x f_1(\gamma)f_2(\tilde{\gamma}^{-1}\gamma)d\nu^x(\gamma)\right]d\nu^x(\tilde{\gamma})\right\}d\mu(x) \\
&=\int_{\Gamma^{(0)}}\left\{\int_{\Gamma^x}\delta(\tilde{\gamma})^{-1}\left[\int_{\Gamma^{s(\tilde{\gamma})}}\langle\xi(\gamma),\,\xi(\gamma)\rangle_{s(\tilde{\gamma})}f_1(\gamma)f_2(\tilde{\gamma}\gamma)d\nu^{s(\tilde{\gamma})}(\gamma)\right]\right. \\
&\qquad\qquad\left.\times\, d\nu^x(\tilde{\gamma})\right\}d\mu(x) \\
&=\int_{\Gamma^{(0)}}\left\{\int_{\Gamma^x}\left[\int_{\Gamma^x}\delta(\tilde{\gamma})^{-1}\langle\xi(\tilde{\gamma}^{-1}\gamma),\,\xi(\tilde{\gamma}^{-1}\gamma)\rangle_{s(\tilde{\gamma})}f_1(\tilde{\gamma}^{-1}\gamma)f_2(\gamma)d\nu^x(\gamma)\right]\right. \\
&\qquad\qquad\left.\times\, d\nu^x(\tilde{\gamma})\right\}d\mu(x)
\end{aligned} \tag{2.7}$$

where we used Fubini's Theorem for the second equality (integrand is non-negative, see Remark 2.11), δ-symmetry for the $\tilde{\gamma}$-integral to obtain the third equality, and the left covariance property of transverse function under the replacement of variable $\tilde{\gamma}\gamma$ by γ for the last equality. Because ξ is a covariant section of $\tilde{\mathscr{H}}$ (see (2.2))

$$\begin{aligned}
\langle\xi(\tilde{\gamma}^{-1}\gamma),\,\xi(\tilde{\gamma}^{-1}\gamma)\rangle_{s(\tilde{\gamma})} &= \langle U(\tilde{\gamma})\xi(\tilde{\gamma}^{-1}\gamma),\,U(\tilde{\gamma})\xi(\tilde{\gamma}^{-1}\gamma)\rangle_x \\
&= \delta(\tilde{\gamma})\langle\xi(\gamma),\,\xi(\gamma)\rangle_x\,.
\end{aligned} \tag{2.8}$$

It follows that the right hand side of (2.7) is equal to

$$\begin{aligned}
&\int_{\Gamma^{(0)}}\left\{\int_{\Gamma^x}\left[\int_{\Gamma^x}\langle\xi(\gamma),\,\xi(\gamma)\rangle_x f_1(\tilde{\gamma}^{-1}\gamma)f_2(\gamma)d\nu^x(\gamma)\right]d\nu^x(\tilde{\gamma})\right\}d\mu(x) \\
&=\int_{\Gamma^{(0)}}\left\{\int_{\Gamma^x}\langle\xi(\gamma),\,\xi(\gamma)\rangle_x f_2(\gamma)d\nu^x(\gamma)\right\}d\mu(x)\,,
\end{aligned} \tag{2.9}$$

where we used (2.6) for f_1. Hence by (2.7) and (2.9), we obtain the assertion. Q.E.D.

Definition 2.5. The singular direct integral Hilbert space of a Hilbert Γ-bundle $\tilde{\mathscr{H}}=\mathscr{H}^c\otimes\mathscr{H}$ is defined to be the set of all covariant measurable sections

$\xi=\{\xi_x\}_{x\in\Gamma^{(0)}}$ of $\tilde{\mathcal{H}}$ with a finite L^2-norm $||\xi||_\Lambda$, where the square of L^2-norm $||\xi||_\Lambda^2$ is defined by (2.5) for $\mu=\Lambda_\nu$. We denote this direct integral Hilbert space by the same notation for the Hilbert Γ-bundle $\tilde{\mathcal{H}}$:

$$\tag{2.10} \begin{aligned} \tilde{\mathcal{H}} &= \int_\Omega^\oplus \tilde{\mathcal{H}}_* d\Lambda(*) \\ &\equiv \{\xi=\{\xi_x\}_{x\in\Gamma^{(0)}}\colon \text{covariant section, } ||\xi||_\Lambda<\infty\}, \end{aligned}$$

where Ω is a symbol for the singular space associated with the groupoid Γ. (We also denote $\Gamma^{(0)}/\Gamma$ or $\Gamma^{(0)}/\sim$ instead of Ω.)

Lemma 2.6. *Let $\mathcal{H}=\{\mathcal{H}_x\}_{x\in\Gamma^{(0)}}$ be a Hilbert Γ-bundle and $\tilde{\mathcal{H}}=\mathcal{H}^c\otimes\mathcal{H}$. Then,*

$$\tag{2.11} \int_\Omega^\oplus \tilde{\mathcal{H}}_* d\Lambda(*) \simeq \int_{\Gamma^{(0)}}^\oplus \mathcal{H}_x d\Lambda_\nu(x)$$

where the right hand side is the usual direct integral.

Proof. Let $\xi=\{\xi_x\}_{x\in\Gamma^{(0)}}$ be a covariant measurable section with $||\xi||_\Lambda<\infty$. For $(\Lambda_\nu\circ\nu)$-almost all $\gamma\in\Gamma$, we define

$$\tag{2.12} \psi_\xi(\gamma) = \delta(\gamma)^{-1/2}U(\gamma)\xi_{s(\gamma)}(\gamma^{-1})\in\mathcal{H}_{r(\gamma)}\,.$$

Then,

$$\tag{2.13} \begin{aligned} \psi_\xi(\gamma\tilde{\gamma}) &= \delta(\gamma\tilde{\gamma})^{-1/2}U(\gamma)U(\tilde{\gamma})\xi_{s(\tilde{\gamma})}(\tilde{\gamma}^{-1}\gamma^{-1}) \\ &= (\delta(\gamma)\delta(\tilde{\gamma}))^{-1/2}U(\gamma)\delta(\tilde{\gamma})^{1/2}\xi_{s(\gamma)}(\gamma^{-1}) \\ &= \psi_\xi(\gamma)\,, \end{aligned}$$

for almost all γ, $\tilde{\gamma}$, where we used the covariance in the second equality. It follows that $\psi_\xi(\gamma)$ depends only on $x=r(\gamma)$ (except for γ in a null set), so we write $\psi_\xi(x)$ instead of $\psi_\xi(\gamma)$. We show that the mapping $\xi\mapsto\psi_\xi$ is the desired isomorphism. Let f be a partition of unity for ν. As the first step, we show that the following two functions

$$\tag{2.14} F(x) = \int_{\Gamma^x} \delta(\gamma)^{-1}\langle\xi(\gamma^{-1}),\, \xi(\gamma^{-1})\rangle_{s(\gamma)} f(\gamma^{-1})d\nu^x(\gamma)\,,$$

$$\tag{2.15} G(x) = \langle\psi_\xi(x),\, \psi_\xi(x)\rangle_x$$

coincide for Λ_ν-almost all $x\in\Gamma^{(0)}$. Let x be in the conull set such that (2.14) is finite (note that $\int_{\Gamma^{(0)}} F(x)d\Lambda_\nu(x)=||\xi||_\Lambda^2<\infty$ and the δ-symmetry of Λ_ν), then there exist $y\in\Gamma^{(0)}$ and $\tilde{\gamma}\in\Gamma_x^y$ such that (2.2) holds for almost all $\gamma\in\Gamma^y$, (2.16) $$\int_{\Gamma^y} f(\gamma_1^{-1}\tilde{\gamma})d\nu^y(\gamma_1) = 1$$

holds and $\psi_\xi(x)=\psi_\xi(\tilde{\gamma}^{-1})$, i.e.

(2.17) $$\delta(\tilde{\gamma})\langle\xi(\tilde{\gamma}),\ \xi(\tilde{\gamma})\rangle_y = \langle\psi_\xi(x),\ \psi_\xi(x)\rangle_x .$$

By replacing γ by $\tilde{\gamma}^{-1}\gamma_1$, $\gamma_1\in\Gamma^y$ in (2.14), and using the left invariance property of transverse function as well as the homomorphic property of δ, we obtain,

(2.18) $$\begin{aligned} F(x) &= \int_{\Gamma^y} \delta(\tilde{\gamma})\delta(\gamma_1)^{-1}\langle\xi(\gamma_1^{-1}\tilde{\gamma}),\ \xi(\gamma_1^{-1}\tilde{\gamma})\rangle_{s(\gamma_1)} f(\gamma_1^{-1}\tilde{\gamma})d\nu^y(\gamma_1) \\ &= \int_{\Gamma^y} \delta(\tilde{\gamma})\langle\xi(\tilde{\gamma}),\ \xi(\tilde{\gamma})\rangle_y f(\gamma_1^{-1}\tilde{\gamma})d\nu^y(\gamma_1) \\ &= \langle\psi_\xi(x),\ \psi_\xi(x)\rangle_x . \end{aligned}$$

Hence,

(2.19) $$\begin{aligned} ||\xi||_A^2 &= \int_{\Gamma^{(0)}}\left\{\int_{\Gamma^x}\langle\xi(\gamma),\ \xi(\gamma)\rangle_x f(\gamma)d\nu^x(\gamma)\right\}dA_\nu(x) \\ &= \int_{\Gamma^{(0)}}\left\{\int_{\Gamma^x}\delta(\gamma)^{-1}\langle\xi(\gamma^{-1}),\ \xi(\gamma^{-1})\rangle_{s(\gamma)} f(\gamma^{-1})d\nu^x(\gamma)\right\}dA_\nu(x) \\ &= \int_{\Gamma^{(0)}}\langle\psi_\xi(x),\ \psi_\xi(x)\rangle_x dA_\nu(x) . \end{aligned}$$

This shows that the mapping $\xi\mapsto\psi_\xi$ is an L^2-isometry. To show the surjectivity of this mapping, we construct the inverse mapping. Let $\psi\in\int_{\Gamma^{(0)}}^{\oplus}\mathscr{H}_x dA_\nu(x)$ i.e. ψ is a measurable section of $\{\mathscr{H}_x\}_{x\in\Gamma^{(0)}}$ satisfying

(2.20) $$\int_{\Gamma^{(0)}}\langle\psi(x),\ \psi(x)\rangle_x dA_\nu(x)<\infty .$$

We define,

(2.21) $$\xi_\psi(\gamma) = \delta(\gamma)^{-1/2}U(\gamma)\psi(s(\gamma)) .$$

Then it is easily checked that ξ_ψ is a covariant section and the mapping $\psi\mapsto\xi_\psi$ actually gives the inverse mapping of $\xi\mapsto\psi_\xi$. This proves the assertion. Q.E.D.

Corollary 2.7. *Let $\mathscr{H}=\{\mathfrak{h}\}_{x\in\Gamma^{(0)}}$ be a Hilbert Γ-bundle with constant fiber $\mathfrak{h}$. Then,*

(2.22) $$\int_{\Omega}^{\oplus}\tilde{\mathscr{H}}_* dA(*) = L^2(\Gamma^{(0)},\ A_\nu)\otimes\mathfrak{h} .$$

Corollary 2.8. *Let $\mathscr{H}^c=\{L^2(\Gamma^x,\ \nu^x)\}_{x\in\Gamma^{(0)}}$ be the canonical bundle. Then*

$\tilde{\mathscr{H}}^c$ *is a standard bundle and*

$$\int_{\Omega}^{\oplus} \tilde{\mathscr{H}}_*^c dA(*) \cong \int_{\Gamma^{(0)}}^{\oplus} L^2(\Gamma^x, \nu^x) dA_\nu(x) = L^2(\Gamma, (A_\nu \circ \nu)) . \tag{2.23}$$

Remark 2.9. In the setting of Corollary 2.8, let $\xi = \{\xi_x\}_{x \in \Gamma^{(0)}}$ be a covariant section. Then ξ is represented by a scalar-valued function ($\xi(\gamma_1, \gamma_2) = \xi_x(\gamma_1, \gamma_2)$, $\gamma_1, \gamma_2 \in \Gamma^x$) on $\tilde{\Gamma} \equiv \coprod_{x \in \Gamma^{(0)}} (\Gamma^x \times \Gamma^x)$ and the covariance condition is written as

$$\xi(\gamma^{-1}\gamma_1, \gamma^{-1}\gamma_2) = \delta(\gamma)^{1/2} \xi(\gamma_1, \gamma_2) , \qquad \gamma, \gamma_1, \gamma_2 \in \Gamma^x, \ x \in \Gamma^{(0)} . \tag{2.24}$$

By symmetry in γ_1 and γ_2, we have two different unitaries from the set of such ξ's onto $\int_{\Gamma^{(0)}}^{\oplus} L^2(\Gamma^x, \nu^x) dA_\nu(x)$. These are shown to have a relation with the symmetry of Tomita algebras and will be discussed in the next section.

Remark 2.10. If $\tilde{\nu}$ and $\tilde{\mu}$ are faithful transverse functions, then there exists a Γ-kernel λ such that $\tilde{\mu} = \tilde{\nu} * \lambda$ (see [20], Proposition 4). So, the theory changes isomorphically for the change of faithful transverse function.

Remark 2.11. The set $\tilde{\Gamma} \equiv \coprod_{x \in \Gamma^{(0)}} (\Gamma^x \times \Gamma^x)$ is actually a measurable space with a Borel structure generated by the family

$$\{P_1^{-1}(B), P_2^{-1}(B); B \text{ is measurable in } \Gamma\} \tag{2.25}$$

where P_j: $\tilde{\Gamma} \to \Gamma$, $j=1, 2$, are given by the projections P_j^x: $\Gamma^x \times \Gamma^x \to \Gamma^x$, $j=1, 2$ into first component and second component. Then a measure $A_\nu \circ (\nu \otimes \nu)$ on $\tilde{\Gamma}$ is determined by the integral

$$\begin{aligned} &\int_{\Gamma^{(0)}} \left\{ \int_{\Gamma^x} \left[\int_{\Gamma^x} f(\gamma_1, \gamma_2) d\nu^x(\gamma_1) \right] d\nu^x(\gamma_2) \right\} dA_\nu(x) \\ &= \int_{\Gamma^{(0)}} \left\{ \int_{\Gamma^x} \left[\int_{\Gamma^x} f(\gamma_1, \gamma_2) d\nu^x(\gamma_2) \right] d\nu^x(\gamma_1) \right\} dA_\nu(x) \end{aligned} \tag{2.26}$$

where f is a non-negative measurable function on $\tilde{\Gamma}$. (The equality in (2.26) follows from the fact that there exists A_ν-null set $N \subset \Gamma^{(0)}$ such that for $x \in \Gamma^{(0)} \backslash N$,

$$\begin{aligned} &\int_{\Gamma^x} \left[\int_{\Gamma^x} f(\gamma_1, \gamma_2) d\nu^x(\gamma_1) \right] d\nu^x(\gamma_2) \\ &= \int_{\Gamma^x} \left[\int_{\Gamma^x} f(\gamma_1, \gamma_2) d\nu^x(\gamma_2) \right] d\nu^x(\gamma_1) \end{aligned} \tag{2.27}$$

holds due to Fubini's Theorem.)

Now, the meaning "almost all γ, $\tilde{\gamma}$" in (2.2) is the following sense i.e. there exists Λ_ν-null set $N\subset\Gamma^{(0)}$ such that for $x\in\Gamma^{(0)}\backslash N$, (2.2) holds for almost all $(\gamma, \tilde{\gamma})\in\Gamma^x\times\Gamma^x$ with respect to $\nu^x\otimes\nu^x$.

§ 3. Modular Hilbert Algebra and Random Operator Field

In this section, we shall give the description of a modular Hilbert algebra constructed from the W^*-groupoid dynamical system and the W^*-algebra of random operator field associated to it. Our description in this section is specialized so as to apply for a W^*-groupoid dynamical system and its associated W^*-algebra. It is possible to present the theory in a more general form, which will however be discussed in another occasion.

Definition 3.1. The triplet (M, Γ, ρ) is called a W^*-groupoid dynamical system (or W^*-groupoid system for short) if M is a W^*-algebra, Γ is a locally compact measured groupoid with a Haar system (ν, Λ, δ) and $\rho\colon \Gamma\to\mathrm{Aut}(M)$ is a continuous homomorphism.

Throughout this section, we fix a W^*-groupoid dynamical system (M, Γ, ρ) with a Haar system (ν, Λ, δ) on Γ. As already stated in Section 1, the groupoid Γ is assumed to be σ-finite and further, we assume M_* to be separable. We fix a faithful normal semifinite weight ϕ_0 on M and identify $\pi_{\phi_0}(M)$ with M. We also use the notation $\phi_\gamma=\phi_0\circ\rho_{\gamma^{-1}}$ and $\eta_{\phi_\gamma}\colon N_{\phi_\gamma}\to H_{\phi_\gamma}$ for the GNS-mapping associated with the weight ϕ_γ. The construction of a modular Hilbert algebra is parallel to that of Digerness [13], [14]. Here we consider M to be a topological space by σ-strong $*$ topology.

Definition 3.2. We denote by $\tilde{\mathfrak{A}}$ the linear space of continuous functions $f\colon \Gamma\to M$ with compact support such that

(1) $f(\gamma)\in N_{\phi_0}\cap N^*_{\phi_\gamma}$,

(2) the functions $\gamma\to\phi_0(f(\gamma)^*f(\gamma))$ and $\gamma\mapsto\phi_0\circ\rho_\gamma(f(\gamma^{-1})f(\gamma^{-1})^*)$ are measurable and integrable.

Remark By the triangular inequality and the normality of ϕ_0, we can see that the set $\tilde{\mathfrak{A}}$ is a linear space.

For $f\in\tilde{\mathfrak{A}}$, we define $\|f\|_{(\Lambda,\infty)}=\operatorname{ess.\,sup}_{x\in\Gamma^{(0)}}\|\pi_x(f)\|<\infty$, where $\pi_x(f)\in B(L^2(\Gamma^x, \nu^x)\otimes H_{\phi_0})$ defined by

$$(3.1)\qquad [\pi_x(f)\xi](\gamma)=\int_{\Gamma^x}\rho_{\tilde{\gamma}}(f(\tilde{\gamma}^{-1}\gamma))\xi(\tilde{\gamma})d\nu^x(\tilde{\gamma}),\ \xi\in L^2(\Gamma^x, \nu^x)\otimes H_{\phi_0}\,.$$

For $f_1, f_2 \in \tilde{\mathfrak{A}}$, we define product (convolution) by

$$(3.2) \qquad (f_1 \hat{*} f_2)(\gamma) = \int_{\Gamma^x} \rho_{\tilde{\gamma}}(f_1(\tilde{\gamma}^{-1}\gamma)) f_2(\tilde{\gamma}) d\nu^x(\tilde{\gamma}), \quad x = r(\gamma)$$

and involution by

$$(3.3) \qquad f^{\sharp}(\gamma) = \rho_\gamma(f(\gamma^{-1})^*), \qquad f \in \tilde{\mathfrak{A}}.$$

Lemma 3.3. *The set* $\tilde{\mathfrak{A}}$ *is closed under the involution* (3.3).

Proof. It is easy to see $f^{\sharp}(\gamma) \in N_{\phi_0} \cap N^*_{\phi_\gamma}$ for $(\Lambda_\nu \circ \nu)$-almost all $\gamma \in \Gamma$. Let $\xi, \zeta \in \tilde{H}_x$. Then, by using Fubini's Theorem,

$$(3.4) \qquad \begin{aligned} (\pi_x(f)\xi, \zeta) &= \int_{\Gamma^x} \left(\int_{\Gamma^x} \rho_{\tilde{\gamma}}(f(\tilde{\gamma}^{-1}\gamma)) \xi(\tilde{\gamma}) d\nu^x(\tilde{\gamma}), \zeta(\gamma) \right) d\nu^x(\gamma) \\ &= \int_{\Gamma^x} \left(\xi(\tilde{\gamma}), \int_{\Gamma^x} \rho_{\tilde{\gamma}}(f(\tilde{\gamma}^{-1}\gamma)^*) \zeta(\gamma) d\nu^x(\gamma) \right) d\nu^x(\tilde{\gamma}) \\ &= \int_{\Gamma^x} \left(\xi(\tilde{\gamma}), \int_{\Gamma^x} \rho_\gamma(f^{\sharp}(\gamma^{-1}\tilde{\gamma})) \zeta(\gamma) d\nu^x(\gamma) \right) d\nu^x(\tilde{\gamma}) \\ &= (\xi, \pi_x(f^{\sharp})\zeta). \end{aligned}$$

So, we obtain $\|f^{\sharp}\|_{(\Lambda,\infty)} = \|f\|_{(\Lambda,\infty)}$. We also have $\phi_0(f^{\sharp}(\gamma)^* f^{\sharp}(\gamma)) = \phi_0 \circ \rho_\gamma(f(\gamma^{-1}) f(\gamma^{-1})^*)$, $\phi_0 \circ \rho_\gamma(f^{\sharp}(\gamma^{-1}) f^{\sharp}(\gamma^{-1})^*) = \phi_0(f(\gamma)^* f(\gamma))$. Hence $f^{\sharp} \in \tilde{\mathfrak{A}}$ if $f \in \tilde{\mathfrak{A}}$. Q.E.D.

Remark 3.4. For $f \in \tilde{\mathfrak{A}}$, $\pi_x(f^{\sharp}) = \pi_x(f)^*$.

Lemma 3.5. *Let* $f_1, f_2 \in \tilde{\mathfrak{A}}$. *Then* $\pi_x(f_1 \hat{*} f_2) = \pi_x(f_1)\pi_x(f_2)$.

Proof. Let $\xi \in \tilde{H}_x$. Then, by using Fubini's Theorem,

$$(3.5) \qquad \begin{aligned} [\pi_x(f_1)\pi_x(f_2)\xi](\gamma) &= \int_{\Gamma^x} \rho_{\tilde{\gamma}}(f_1(\tilde{\gamma}^{-1}\gamma)) [\pi_x(f_2)\xi](\tilde{\gamma}) d\nu^x(\tilde{\gamma}) \\ &= \int_{\Gamma^x} \left\{ \int_{\Gamma^x} \rho_{\tilde{\gamma}}(f_1(\tilde{\gamma}^{-1}\gamma)) \rho_{\hat{\gamma}}(f_2(\hat{\gamma}^{-1}\tilde{\gamma})) \xi(\hat{\gamma}) d\nu^x(\hat{\gamma}) \right\} d\nu^x(\tilde{\gamma}) \\ &= \int_{\Gamma^x} \left\{ \int_{\Gamma^x} \rho_{\tilde{\gamma}}(f_1(\tilde{\gamma}^{-1}\gamma)) \rho_{\hat{\gamma}}(f_2(\hat{\gamma}^{-1}\tilde{\gamma})) d\nu^x(\tilde{\gamma}) \right\} \xi(\hat{\gamma}) d\nu^x(\hat{\gamma}) \\ &= \int_{\Gamma^x} \left\{ \int_{\Gamma^{s(\hat{\gamma})}} \rho_{\hat{\gamma}\tilde{\gamma}}(f_1(\tilde{\gamma}^{-1}\hat{\gamma}^{-1}\gamma)) \rho_{\hat{\gamma}}(f_2(\tilde{\gamma})) d\nu^{s(\hat{\gamma})}(\tilde{\gamma}) \right\} \xi(\hat{\gamma}) d\nu^x(\hat{\gamma}) \\ &= \int_{\Gamma^x} \rho_{\hat{\gamma}}((f_1 \hat{*} f_2)(\hat{\gamma}^{-1}\gamma)) \xi(\hat{\gamma}) d\nu^x(\hat{\gamma}) \\ &= [\pi_x(f_1 \hat{*} f_2)\xi](\gamma). \end{aligned}$$

Q.E.D.

Lemma 3.6. *For* $f_1, f_2 \in \tilde{\mathfrak{A}}$,

$$(3.6)\qquad \int_\Gamma \phi_0((f_1\hat{*}f_2)(\gamma)^*(f_1\hat{*}f_2)(\gamma))d(\Lambda_\nu\circ\nu)(\gamma)$$
$$\leqq \|f_1\|^2_{(\Lambda,\infty)}\int_\Gamma \phi_0(f_2(\gamma)^*f_2(\gamma))d(\Lambda_\nu\circ\nu)(\gamma)\,,$$

$$(3.7)\qquad \int_\Gamma \phi_0\circ\rho_\gamma((f_1\hat{*}f_2)(\gamma^{-1})(f_1\hat{*}f_2)(\gamma^{-1})^*)d(\Lambda_\nu\circ\nu)(\gamma)$$
$$\leqq \|f_2\|^2_{(\Lambda,\infty)}\int_\Gamma \phi_0\circ\rho_\gamma(f_1(\gamma^{-1})f_1(\gamma^{-1})^*)d(\Lambda_\nu\circ\nu)(\gamma)\,.$$

Proof.

$$(3.8)\qquad \int_\Gamma \phi_0((f_1\hat{*}f_2)(\gamma)^*(f_1\hat{*}f_2)(\gamma))d(\Lambda_\nu\circ\nu)(\gamma)$$
$$=\int_{\Gamma^{(0)}}\left\{\int_{\Gamma^x}\|\eta_{\phi_0}((f_1\hat{*}f_2)(\gamma))\|^2 d\nu^x(\gamma)\right\}d\Lambda_\nu(x)$$
$$=\int_{\Gamma^{(0)}}\|\pi_x(f_1)\eta_{\phi_0}(f_2)\|^2_{\tilde H_x}d\Lambda_\nu(x)$$
$$\leqq\int_{\Gamma^{(0)}}\|\pi_x(f_1)\|^2\|\eta_{\phi_0}(f_2)\|^2_{\tilde H_x}d\Lambda_\nu(x)$$
$$\leqq \|f_1\|^2_{(\Lambda,\infty)}\int_{\Gamma^{(0)}}\left\{\int_{\Gamma^x}\|\eta_{\phi_0}(f_2(\gamma))\|^2 d\nu^x(\gamma)\right\}d\Lambda_\nu(x)$$
$$=\|f_1\|^2_{(\Lambda,\infty)}\int_\Gamma \phi_0(f_2(\gamma)^*f_2(\gamma))d(\Lambda_\nu\circ\nu)(\gamma)\,,$$

$$(3.9)\qquad \int_\Gamma \phi_0\circ\rho_\gamma((f_1\hat{*}f_2)(\gamma^{-1})(f_1\hat{*}f_2)(\gamma^{-1})^*)d(\Lambda_\nu\circ\nu)(\gamma)$$
$$=\int_\Gamma \phi_0((f_1\hat{*}f_2)^\sharp(\gamma)^*(f_1\hat{*}f_2)^\sharp(\gamma))d(\Lambda_\nu\circ\nu)(\gamma)$$
$$=\int_\Gamma \phi_0((f_2^\sharp\hat{*}f_1^\sharp)(\gamma)^*(f_2^\sharp\hat{*}f_1^\sharp)(\gamma))d(\Lambda_\nu\circ\nu)(\gamma)$$
$$\leqq \|f_2^\sharp\|^2_{(\Lambda,\infty)}\int_\Gamma \phi_0(f_1^\sharp(\gamma)^*f_1^\sharp(\gamma))d(\Lambda_\nu\circ\nu)(\gamma)$$
$$=\|f_2\|^2_{(\Lambda,\infty)}\int_\Gamma \phi_0\circ\rho_\gamma(f_1(\gamma^{-1})f_1(\gamma^{-1})^*)d(\Lambda_\nu\circ\nu)(\gamma)\,,$$

where we used Lemma 3.3 and (3.8). Q.E.D.

Lemma 3.7. *The linear space* $\tilde{\mathfrak{A}}$ *is an involutive algebra by* (3.2) *and* (3.3). *Further, the mapping* $\pi_x\colon \tilde{\mathfrak{A}}\to B(\tilde H_x)$ *is a* $*$*-homomorphism.*

Proof. Let $f_1, f_2\in\tilde{\mathfrak{A}}$. By Lemma 3.6, $(f_1\hat{*}f_2)(\gamma)\in N_{\phi_0}\cap N^*_{\phi_\gamma}$ for almost all $\gamma\in\Gamma$. By Lemma 3.5, we obtain $\|\pi_x(f_1\hat{*}f_2)\|\leqq\|\pi_x(f_1)\|\,\|\pi_x(f_2)\|$ and hence, $\|f_1\hat{*}f_2\|_{(\Lambda,\infty)}\leqq\|f_1\|_{(\Lambda,\infty)}\|f_2\|_{(\Lambda,\infty)}$. Again by Lemma 3.6, condition (2) in Definition 3.2 holds for $f=f_1\hat{*}f_2$. So, we obtain $f_1\hat{*}f_2\in\tilde{\mathfrak{A}}$. Hence, by Lemma 3.3

and $(f_1 \hat{*} f_2)^{\#} = f_2^{\#} \hat{*} f_1^{\#}$, we conclude that $\tilde{\mathfrak{A}}$ is an involutive algebra. By Remark 3.4 and Lemma 3.5, π_x is a $*$-homomorphism. Q.E.D.

Definition 3.8. We define a GNS-mapping η_Λ: $\tilde{\mathfrak{A}} \to \tilde{H} \equiv L^2(\Gamma, (\Lambda_\nu \circ \nu)) \otimes H_{\phi_0} = L^2(\Gamma, H_{\phi_0}, (\Lambda_\nu \circ \nu))$ by

$$(3.10) \qquad [\eta_\Lambda(f)](\gamma) = \eta_{\phi_0}(f(\gamma)), \qquad f \in \tilde{\mathfrak{A}}.$$

We set $\mathfrak{A} = \eta_\Lambda(\tilde{\mathfrak{A}})$ and define a Hilbert algebra structure in $\mathfrak{A}$ by the involutive algebraic structure of $\tilde{\mathfrak{A}}$ and the pre-Hilbert structure as a subset of $\tilde{H}$.

We note that the algebraic structure of $\mathfrak{A}$ is well-defined because η_{ϕ_0} and hence η_Λ is faithful.

Notation 3.9. We denote by $\mathcal{T}(\Gamma)$ the set of all continuous functions f: $\Gamma \to \boldsymbol{C}$ with compact support satisfying the conditions of Definition 3.2 with M replaced by $\boldsymbol{C}$. The set $\mathcal{T}(\Gamma)$ is actually a modular Hilbert algebra associated with $\mathrm{End}_\Lambda(\Gamma)$ (see Remark 4.2, [9], [20]). Hence, $\mathcal{T}(\Gamma) \hat{*} \mathcal{T}(\Gamma)$ is dense in $L^2(\Gamma, (\Lambda_\nu \circ \nu))$. We denote by λ the $*$-homomorphism π: $\mathcal{T}(\Gamma) \to B(L^2(\Gamma, (\Lambda_\nu \circ \nu)))$ defined by (3.1).

Lemma 3.10. *The set $\mathfrak{A} \hat{*} \mathfrak{A}$ is dense in $L^2(\Gamma, (\Lambda_\nu \circ \nu)) \otimes H_{\phi_0}$.*

Proof. Let $a_k, b_k \in N_{\phi_0}$, $F_k \in \mathcal{T}(\Gamma)$, $k=1, 2$. Then, it is easy to see that the mappings $\gamma \mapsto F_k(\gamma)\rho_\gamma(b_k^*)a_k = f_k(\gamma)$, $k=1, 2$ are in $\tilde{\mathfrak{A}}$. Now, we assume that $\xi \in L^2(\Gamma, (\Lambda_\nu \circ \nu)) \otimes H_{\phi_0}$ satisfies $(\xi, \eta_\Lambda(f_1 \hat{*} f_2)) = 0$ for all f_1, f_2 of the above form. Hence,

$$(3.11) \qquad \int_{\Gamma^{(0)}} \left\{ \int_{\Gamma^x} \left[\int_{\Gamma^x} F_1(\tilde{\gamma}^{-1}\gamma) F_2(\tilde{\gamma})(\xi(\gamma), \rho_\gamma(b_1^*)a_1\rho_{\tilde{\gamma}}(b_2^*)\eta_{\phi_0}(a_2)) d\nu^x(\tilde{\gamma}) \right] d\nu^x(\gamma) \right\} d\Lambda_\nu(x) = 0.$$

By taking $*$-strong nets $b_k \to 1$, $k=1, 2$ and $a_1 \to 1$, we obtain

$$(3.12) \qquad \int_\Gamma (F_1 \hat{*} F_2)(\gamma)(\xi(\gamma), \eta_{\phi_0}(a_2)) d(\Lambda_\nu \circ \nu)(\gamma) = 0.$$

By the density of $\mathcal{T}(\Gamma) \hat{*} \mathcal{T}(\Gamma)$ in $L^2(\Gamma, (\Lambda_\nu \circ \nu))$, we obtain

$$(3.13) \qquad (\xi(\gamma), \eta_{\phi_0}(a_2)) = 0 \qquad \text{for } (\Lambda_\nu \circ \nu)\text{-almost all } \gamma \in \Gamma.$$

By the density of $\eta_{\phi_0}(N_{\phi_0})$ in H_{ϕ_0}, we obtain $\xi = 0$ in $L^2(\Gamma, (\Lambda_\nu \circ \nu)) \otimes H_{\phi_0}$. Q.E.D.

We use the notation $U(\phi_1, \phi_2)$ for the intertwining unitary mapping $H_{\phi_2} \simeq H_{\phi_1}$ identifying representative vectors of states in the natural positive cone in the standard representation Hilbert spaces H_{ϕ_k}, $k=1, 2$ constructed from faithful normal semifinite weights ϕ_k, $k=1, 2$ (see [3], [16]). We also use the notation $U_{\phi_0}(\gamma)$ for the canonical unitary implementation of $\rho_\gamma \in \mathrm{Aut}(M)$ on the Hilbert space H_{ϕ_0}.

Now, we define

$$(3.14) \qquad [\Delta_\Lambda \xi](\gamma) = \delta(\gamma)^{-1} \Delta_{\phi_\gamma, \phi_0} \xi(\gamma),$$

$$(3.15) \qquad [J_\Lambda \xi](\gamma) = \delta(\gamma)^{-1/2} U_{\phi_0}(\gamma) J_{\phi_0} \xi(\gamma^{-1}), \ \xi \in L^2(\Gamma, (\Lambda_\nu \circ \nu)) \otimes H_{\phi_0}.$$

The operator Δ_Λ is positive self-adjoint and generates one parameter group of unitaries $t \mapsto \Delta_\Lambda^{it}$, $[\Delta_\Lambda^{it} \xi](\gamma) = \delta(\gamma)^{-it} \Delta_{\phi_\gamma, \phi_0}^{it} \xi(\gamma)$. The operator J_Λ is conjugate linear and conjugate unitary.

Lemma 3.11. *The set $\mathfrak{A}$ is a modular Hilbert algebra.*

Proof. By Lemma 3.10, $\mathfrak{A} \hat{*} \mathfrak{A}$ is dense in $L^2(\Gamma, (\Lambda_\nu \circ \nu)) \otimes H_{\phi_0}$. By Lemma 3.4 (or by (3.5)), (3.10) and $[\eta_\Lambda(f_1 \hat{*} f_2)](\gamma) = \pi_x(f_1)[\eta_\Lambda(f_2)](\gamma)$, $x = r(\gamma)$, we obtain $(\xi \hat{*} \zeta_1, \zeta_2) = (\zeta_1, \xi^\# \hat{*} \zeta_2)$ for ξ, ζ_1, $\zeta_2 \in \mathfrak{A}$. By Lemma 3.6 (or by (3.6)), the mapping $\zeta \mapsto \xi \hat{*} \zeta$ is bounded by $\|\xi\|_{(\Lambda, \infty)}$. Now, we show the preclosedness of the mapping $\xi \mapsto \xi^\#$. By (3.14) and (3.15),

$$\begin{aligned}
(3.16) \qquad [\eta_\Lambda(f)^\#](\gamma) &= \eta_{\phi_0}(\rho_\gamma(f(\gamma^{-1})^*)) \\
&= U_{\phi_0}(\gamma) U(\phi_0, \phi_{\gamma^{-1}}) \eta_{\phi_{\gamma^{-1}}}(f(\gamma^{-1})^*) \\
&= U_{\phi_0}(\gamma) U(\phi_0, \phi_{\gamma^{-1}}) J_{\phi_{\gamma^{-1}}, \phi_0} \Delta_{\phi_{\gamma^{-1}}, \phi_0}^{1/2} \eta_{\phi_0}(f(\gamma^{-1})) \\
&= U_{\phi_0}(\gamma) J_{\phi_0} \Delta_{\phi_{\gamma^{-1}}, \phi_0}^{1/2} \eta_{\phi_0}(f(\gamma^{-1})) \\
&= \delta(\gamma)^{-1/2} U_{\phi_0}(\gamma) J_{\phi_0} [\Delta_\Lambda^{1/2} \eta_\Lambda(f)](\gamma^{-1}) \\
&= [J_\Lambda \Delta_\Lambda^{1/2} \eta_\Lambda(f)](\gamma).
\end{aligned}$$

Hence, $\xi \mapsto \xi^\#$ is preclosed. Q.E.D.

Remark 3.12. In view of (3.14), it is easy to see that the associated modular automorphism group $\{\sigma_t^\Lambda\}_{t \in \mathbf{R}}$ is

$$(3.17) \qquad [\sigma_t^\Lambda(f)](\gamma) = \delta(\gamma)^{-it} (D\phi_\gamma : D\phi_0)_t \sigma_t^{\phi_0}(f(\gamma)), \qquad f \in \tilde{\mathfrak{A}}.$$

Remark 3.13. We denote by $\mathfrak{A}_0$ the linear space of continuous functions $f: \Gamma \to M$ with compact support. Then $\mathfrak{A}_0$ is an involutive algebra by the same operation as $\tilde{\mathfrak{A}}$. Then $\pi(\tilde{\mathfrak{A}}) \subset \pi(\mathfrak{A}_0) \subset (J_\Lambda \pi(\tilde{\mathfrak{A}}) J_\Lambda)'$, where $\pi = \int_{\Gamma^{(0)}}^{\oplus} \pi_x d\Lambda_\nu(x)$ is the representation of $\mathfrak{A}_0$ on $L^2(\Gamma, (\Lambda_\nu \circ \nu))$ and the prime denotes the commutant. Hence, $\pi(\mathfrak{A}_0)$ also generates the left von Neumann algebra. The proof of Lemma 3.4 also shows that the linear space $\mathfrak{A}_0$ is an involutive algebra by (3.2) and (3.3).

Remark. If the module function is continuous, then $\mathfrak{A}_0$ is $\{\sigma_t^A\}$-invariant (see (3.17)). In Definition 3.2, we can replace $\tilde{\mathfrak{A}}$ by the set of all continuous functions with compact support satisfying (1).

Next, we shall give a description of the associated left von Neumann algebra as a random field of operators associated with a Hilbert Γ-bundle. We define a Hilbert Γ-bundle $\mathcal{H}$ by

$$\mathcal{H}_x = L^2(\Gamma^x, \nu^x) \otimes H_{\phi_0} \tag{3.18}$$

with a unitary representation of Γ by

$$[U(\gamma)\xi](\tilde{\gamma}) = \xi(\gamma^{-1}\tilde{\gamma}), \qquad \xi \in \mathcal{H}_x. \tag{3.19}$$

By Lemma 2.6 and Corollary 2.8, we obtain

$$\int_\Omega^\oplus \tilde{\mathcal{H}}_* dA(*) \simeq L^2(\Gamma, (A_\nu \circ \nu)) \otimes H_{\phi_0} \equiv \tilde{H}. \tag{3.20}$$

By (2.15) and (2.24),

$$\psi(\gamma) = \delta(\tilde{\gamma})^{-1/2} \xi(\tilde{\gamma}^{-1}, \tilde{\gamma}^{-1}\gamma), \text{ (independent of } \tilde{\gamma} \in \Gamma^{r(\gamma)}), \tag{3.21}$$

$$\xi(\gamma_1, \gamma_2) = \delta(\gamma_1)^{-1/2} \psi(\gamma_1^{-1}\gamma_2) \tag{3.22}$$

give the isomorphic correspondence of (3.20) ($\xi \in \tilde{\mathcal{H}}$, $\psi \in \tilde{H}$).

Lemma 3.14. *In the description of $\tilde{H}$ by the covariant section of $\tilde{\mathcal{H}}$,*

$$[\Delta_A^{it}\xi](\gamma_1, \gamma_2) = \delta(\gamma_2^{-1}\gamma_1)^{it} \Delta_{\phi_{\gamma_1^{-1}\gamma_2}, \phi_0}^{it} \xi(\gamma_1, \gamma_2), \tag{3.23}$$

$$[J_A\xi](\gamma_1, \gamma_2) = U_{\phi_0}(\gamma_1^{-1}\gamma_2) J_{\phi_0} \xi(\gamma_2, \gamma_1), \tag{3.24}$$

$$[\pi(f)\xi](\gamma_1, \gamma_2) = \int_{\Gamma^x} \rho_{\gamma_1^{-1}\tilde{\gamma}}(f(\tilde{\gamma}^{-1}\gamma_2) \xi(\gamma_1, \tilde{\gamma}) d\nu^x(\tilde{\gamma}), \qquad x = r(\gamma_1), \tag{3.25}$$

where $f \in \mathfrak{A}_0$ and ξ is a covariant section of $\tilde{\mathcal{H}} = \mathcal{H}^c \otimes \mathcal{H}$.

Proof. The assertion is the direct consequence of (3.14), (3.15), (3.1), (3.21) and (3.22). Q.E.D.

Now we define a Hilbert Γ-bundle $\mathcal{H}^0$ by

$$\mathcal{H}_x^0 = L^2(\Gamma^x, \nu^x) \otimes H_{\phi_0} \tag{3.26}$$

with a unitary representation of Γ by

$$[U^0(\gamma)\xi](\tilde{\gamma}) = U_{\phi_0}(\gamma) \xi(\gamma^{-1}\tilde{\gamma}), \qquad \xi \in \mathcal{H}_x^0. \tag{3.27}$$

By Corollary 2.7, we obtain

$$\int_{\Omega}^{\oplus} \tilde{\mathscr{H}}_{*}^{0} d\Lambda(*) \cong L^2(\Gamma, (\Lambda_\nu \circ \nu)) \otimes H_{\phi_0} \equiv \tilde{H}. \tag{3.28}$$

In this case, by (2.12) and (2.23),

$$\psi(\gamma) = \delta(\tilde{\gamma})^{-1/2} U_{\phi_0}(\tilde{\gamma}) \xi(\tilde{\gamma}^{-1}, \tilde{\gamma}^{-1}\gamma), \tag{3.29}$$

$$\xi(\gamma_1, \gamma_2) = \delta(\gamma_1)^{-1/2} U_{\phi_0}(\gamma_1) \psi(\gamma_1^{-1}\gamma_2) \tag{3.30}$$

give the isomorphic correspondence of (3.28). By looking at (3.21), (3.22) and (3.29), (3.30), we obtain a unitary mapping W from the covariant sections of $\tilde{\mathscr{H}}$ to the covariant sections of $\tilde{\mathscr{H}}^0$ by

$$[W\xi](\gamma_1, \gamma_2) = U_{\phi_0}(\gamma_1) \xi(\gamma_1, \gamma_2). \tag{3.31}$$

Lemma 3.15. *In the description of $\tilde{H}$ by the covariant section of $\tilde{\mathscr{H}}^0$,*

$$[J_\Lambda \xi](\gamma_1, \gamma_2) = J_{\phi_0} \xi(\gamma_2, \gamma_1) \tag{3.32}$$

$$[\pi(f)\xi](\gamma_1, \gamma_2) = \int_{\Gamma^x} \rho_{\tilde{\gamma}}(f(\tilde{\gamma}^{-1}\gamma_2)) \xi(\gamma_1, \tilde{\gamma}) d\nu^x(\tilde{\gamma}), \qquad x = r(\gamma_1), \tag{3.33}$$

where $f \in \mathfrak{A}_0$ and ξ is a covariant section of $\tilde{\mathscr{H}}^0$.

Proof. By Lemma 3.14 and (3.31). Q.E.D.

Lemma 3.16. *The W^*-algebra associated with the Hilbert algebra $\mathfrak{A}$ is isomorphic to the W^*-algebra of random operator fields:*

(3.34) $\{T = \{T_x\}_{x \in \Gamma^{(0)}}$: essentially bounded measurable field of operators on $\tilde{\mathscr{H}}^0$ such that $T_x \in B(L^2(\Gamma^x, \nu^x) \otimes L^2(\Gamma^x, \nu^x)) \otimes M$ and $(Ad_{U^s(\gamma)} \otimes \rho_\gamma)(T_x) = T_y,\ \gamma \in \Gamma_x^{\ y}\}$,

where $U^s(\gamma) = U^c(\gamma) \otimes U^c(\gamma)$ is the unitary representation of Γ on the standard bundle. In this expression, $\|T\| = \operatorname{ess.sup}_{x \in \Gamma^{(0)}} \|T_x\|$.

Proof. By (3.33), $\pi(f) = \{\pi_x(f)\}_{x \in \Gamma^{(0)}}$ for $f \in \mathfrak{A}_0$ belongs to (3.34). On the other hand, since the action of $1 \otimes B(L^2(\Gamma^x, \nu^x))$ on $L^2(\Gamma^x, \nu^x) \otimes L^2(\Gamma^x, \nu^x)$ commutes with the operators given by integral kernel which involves only the first component of the tensor product $L^2(\Gamma^x, \nu^x) \otimes L^2(\Gamma^x, \nu^x)$, by using (3.32) and (3.33) it is easy to see that the each element in (3.34) commutes with the operators of the form $J_\Lambda \pi(f) J_\Lambda$, $f \in \mathfrak{A}_0$. So we have only to show that the set (3.34) is a W^*-algebra. The set (3.34) is actually a weakly closed $*$-subalgebra of $\int_{\Gamma^{(0)}}^{\oplus} B(L^2(\Gamma^x, \nu^x) \otimes L^2(\Gamma^x, \nu^x)) \otimes M d\Lambda_\nu(x)$ on $\int_{\Gamma^{(0)}}^{\oplus} L^2(\Gamma^x, \nu^x) \otimes L^2(\Gamma^x, \nu^x) \otimes H_{\phi_0} d\Lambda_\nu(x)$ due to the weak continuity of the covariance condition. Q.E.D.

§ 4. Crossed Product by Groupoid

In a close analogy with the W^*-dynamical system defined by a locally compact group, we defined a W^*-groupoid dynamical system (M, Γ, ρ) in Section 3. In this section, we shall discuss about the corresponding crossed product by a groupoid which is given as the left von Neumann algebra associated with the modular Hilbert algebra discussed in Section 3.

Definition 4.1. $M\times_\rho\Gamma$ or $W^*(M, \Gamma, \rho)$ will denote the left von Neumann algebra associated with the modular Hilbert algebra $\mathfrak{A}$ of Lemma 3.11 and will be called the groupoid crossed product.

Next, we shall give the description of this crossed product in a different manner. Note that the W^*-algebra $\mathrm{End}_\Lambda(\Gamma)$ is isomorphic to the W^*-algebra generated by the left multiplication of the modular Hilbert algebra $\mathcal{T}(\Gamma)$ (see Notation 3.9) associated with the measured groupoid Γ with Haar system (ν, Λ, δ) on the (standard) representation Hilbert space $L^2(\Gamma, (\Lambda_\nu\circ\nu))$.

Remark 4.2. In the usual description of modular Hilbert algebra $\hat{\mathfrak{A}}$ or Tomita algebra associated with $\mathrm{End}_\Lambda(\Gamma)$, the convolution is defined by

$$(4.1)\qquad (f_1(*)f_2)(\gamma)=\int_{\Gamma^x} f_1(\tilde{\gamma})f_2(\tilde{\gamma}^{-1}\gamma)d\nu^x(\tilde{\gamma}), \qquad x=r(\gamma),$$

and the involution is defined by

$$(4.2)\qquad f^{(\sharp)}(\gamma)=\delta(\gamma)^{-1}\overline{f(\gamma^{-1})}$$

(for example, see Kastler [20]). This modular Hilbert algebra is isomorphic to our description by $R\colon \hat{\mathfrak{A}}\to\mathcal{T}(\Gamma)$ where

$$(4.3)\qquad [Rf](\gamma)=\delta(\gamma)^{-1/2}f(\gamma^{-1}).$$

Lemma 4.3. *The W^*-algebra $M\times_\rho\Gamma$ is isomorphic to the W^*-algebra generated by $\lambda(f)\otimes 1$, $f\in\mathcal{T}(\Gamma)$, and $\pi_0(a)$, $a\in M$ on $\tilde{H}\equiv L^2(\Gamma, H_{\phi_0}, (\Lambda_\nu\circ\nu))$, where*

$$(4.4)\qquad [\lambda(f)\xi](\gamma)=\int_{\Gamma^x}\delta(\tilde{\gamma})^{-1/2}f(\tilde{\gamma}^{-1})\xi(\tilde{\gamma}^{-1}\gamma)d\nu^x(\tilde{\gamma}),$$

$$x=r(\gamma),\ \xi\in L^2(\Gamma, (\Lambda_\nu\circ\nu)),$$

$$(4.5)\qquad [\pi_0(a)\xi](\gamma)=\rho_{\gamma^{-1}}(a)\xi(\gamma), \qquad a\in M,\ \xi\in\tilde{H}.$$

Proof. We define an involutive unitary operator $\tilde{R}$ on $\tilde{H}$ by

$$(4.6)\qquad [\tilde{R}\xi](\gamma)=\delta(\gamma)^{-1/2}\xi(\gamma^{-1}).$$

Then,

$$[\tilde{R}(\lambda(f)\otimes 1)\tilde{R}^*\xi)](\gamma)=\int_{\Gamma^x} f(\tilde{\gamma}^{-1}\gamma)\xi(\tilde{\gamma})d\nu^x(\tilde{\gamma}), \qquad x=r(\gamma), \tag{4.7}$$

$$[\tilde{R}\pi_0(a)\tilde{R}^*\xi](\gamma)=\rho_\gamma(a)\xi(\gamma), \qquad a\in M, \ \xi\in\tilde{H}. \tag{4.8}$$

Hence by (3.1) and (3.15), the operators $\tilde{R}(\lambda(f)\otimes 1)\tilde{R}^*$ and $\tilde{R}\pi_0(a)\tilde{R}^*$ commute with $J_\Delta\pi(\mathfrak{A}_0)J_\Delta$. Therefore they are in $M\times_\rho\Gamma$. We define a unitary operator U_ρ on $\tilde{H}$ by

$$[U_\rho\xi](\gamma)=U_{\phi_0}(\gamma)^*\xi(\gamma), \qquad \xi\in\tilde{H}, \tag{4.9}$$

where $U_{\phi_0}(\gamma)$ is the canonical unitary implementation of $\rho_\gamma\in\mathrm{Aut}(M)$. Then,

$$[U_\rho\tilde{R}(\lambda(f)\otimes 1)\tilde{R}^*U_\rho^*\xi](\gamma)=\int_{\Gamma^x} U_{\phi_0}(\gamma^{-1}\tilde{\gamma})f(\tilde{\gamma}^{-1}\gamma)\xi(\tilde{\gamma})d\nu^x(\tilde{\gamma}), \quad x=r(\gamma), \ f\in\mathcal{C}(\Gamma), \tag{4.10}$$

$$[U_\rho\tilde{R}\pi_0(a)\tilde{R}^*U_\rho^*\xi](\gamma)=a\xi(\gamma), \qquad a\in M, \tag{4.11}$$

$$[U_\rho\pi(f)U_\rho^*\xi](\gamma)=\int_{\Gamma^x} U_{\phi_0}(\gamma^{-1}\tilde{\gamma})f(\tilde{\gamma}^{-1}\gamma)\xi(\tilde{\gamma})d\nu^x(\tilde{\gamma}), \ x=r(\gamma), \quad f\in\mathfrak{A}_0, \ \xi\in\tilde{H}. \tag{4.12}$$

Hence the operator $U_\rho\pi(f)U_\rho^*$, $f\in\mathfrak{A}_0$, is the weak limit of linear combinations of the operators of the form $U_\rho\tilde{R}(\lambda(f)\otimes 1)\pi_0(a)\tilde{R}^*U_\rho^*$, $f\in\mathcal{C}(\Gamma)$, $a\in M$. Hence we obtain the assertion. Q.E.D.

Remark 4.4. We define $\mathfrak{A}_1=\{\delta^{-1/2}\tilde{f},\ f\in\mathcal{C}(\Gamma)\}$, where $[\delta^{-1/2}\tilde{f}](\gamma)=\delta(\gamma)^{-1/2}f(\gamma^{-1})$. Then $\mathfrak{A}_1$ is a $*$-algebra in a usual manner and the generators of $M\times_\rho\Gamma$ are $\pi_0(a)$, $a\in M$, and $\lambda_0(f)\otimes 1, f\in\mathfrak{A}_1$, where

$$[\lambda_0(f)\xi](\gamma)=\int_{\Gamma^x} f(\tilde{\gamma})\xi(\tilde{\gamma}^{-1}\gamma)d\nu^x(\tilde{\gamma}), \qquad x=r(\gamma), \ \xi\in L^2(\Gamma, (\Lambda_\nu\circ\nu)). \tag{4.13}$$

The generators of this form look similar to that of usual crossed product. It is also noticed that the representation space H_{ϕ_0} of M may be replaced by an arbitrary representation space. (The resulting W^*-algebra is algebraically isomorphic to the one above.)

It is also noticed that if $\tilde{\nu}$ is also a faithful proper transverse function, then $\tilde{\nu}$ is of the form $\nu*\lambda$, where λ is a suitable Γ-kernel (see [20], Proposition 4). Then, the corresponding W^*-algebra $\mathrm{End}_\Lambda(\Gamma)$ is isomorphic. Hence the groupoid product $M\times_\rho\Gamma$ is isomorphic.

Lemma 4.5.

(1) *If* $\rho, \sigma: \Gamma \to \mathrm{Aut}(M)$ *are mutually cohomologous in the sense that there exists a continuous mapping* $\tau: \Gamma^{(0)} \to \mathrm{Aut}(M)$ *such that* $\rho_\gamma = \tau_{r(\gamma)} \circ \sigma_\gamma \circ \tau^{-1}_{s(\gamma)}$. *Then* $M \times_\rho \Gamma \cong M \times_\sigma \Gamma$.

(2) *If* $\rho, \sigma: \Gamma \to \mathrm{Aut}(M)$ *are mutually one-cocycle equivalent in the sense that there exists a strongly continuous unitary valued mapping* $u: \Gamma \to M$ *such that*

$$\rho_\gamma(a) = u_\gamma \sigma_\gamma(a) u_\gamma^*, \qquad a \in M \tag{4.14}$$

$$u_{\gamma_1 \gamma_2} = u_{\gamma_1} \sigma_{\gamma_1}(u_{\gamma_2}), \quad s(\gamma_1) = r(\gamma_2) \tag{4.15}$$

then $M \times_\rho \Gamma \cong M \times_\sigma \Gamma$.

Proof. (1) Let $W^\tau = \{W^\tau_x\}_{x \in \Gamma^{(0)}}$ be the canonical implementation of $\{\tau_x\}_{x \in \Gamma^{(0)}}$ on H_{ϕ_0} and define unitary operator $\tilde{W}^\tau$ on $\tilde{H}$ by

$$[\tilde{W}^\tau \xi](\gamma) = W^\tau_{r(\gamma)} \xi(\gamma), \qquad \xi \in \tilde{H}. \tag{4.16}$$

In view of Remark 3.13, we define

$$\Phi^\tau[f](\gamma) = \tau^{-1}_{r(\gamma)}(f(\gamma)), \qquad f \in \mathfrak{A}_0^{(\rho)} \tag{4.17}$$

where $\mathfrak{A}_0^{(\rho)}$is the set of all M-valued function on Γ such that $\pi^\rho(f)$ is bounded. Then we obtain

$$\tilde{W}^\tau \pi^\sigma(\Phi^\tau[f])(\tilde{W}^\tau)^* = \pi^\rho(f), \tag{4.18}$$

where π^σ and π^ρ are the corresponding $*$-representations on $\tilde{H}$ with respect to the actions σ and ρ, respectively. Hence $f \in \mathfrak{A}_0^{(\rho)}$ if and only if $\Phi^\tau[f] \in \mathfrak{A}_0^{(\sigma)}$. We also obtain $\Phi^{\tau^{-1}} \circ \Phi^\tau[f] = f, f \in \mathfrak{A}_0^{(\rho)}$, $\Phi^\tau \circ \Phi^{\tau^{-1}}[\tilde{f}] = \tilde{f}$, $\tilde{f} \in \mathfrak{A}_0^{(\sigma)}$. Hence we obtain the desired isomorphism.

(2) We define unitary operator U on $\tilde{H}$ by

$$[U\xi](\gamma) = u_{\gamma^{-1}} \xi(\gamma), \ \xi \in \tilde{H}. \tag{4.19}$$

Then by (4.4), $U(\lambda(f) \otimes 1)U^* = \lambda(f) \otimes 1, f \in \mathcal{T}(\Gamma)$, and $U \pi_0^\sigma(a) U^* = \pi_0^\rho(a)$, $a \in M$. Hence by Lemma 4.3, we obtain the isomorphism.

Remark 4.6. Under the situation (1), we obtain

$$\Phi^\tau[f_1] *_\sigma \Phi^\tau[f_2] = \Phi^\tau[f_1 *_\rho f_2], \tag{4.20}$$

$$\Phi^\tau[f^{(\#, \rho)}] = \Phi^\tau[f]^{(\#, \sigma)}, \tag{4.21}$$

where $*_\sigma$, $(\#, \sigma)$ and $*_\rho$, $(\#, \rho)$ are the convolution and the involution with respect to the actions σ and ρ, respectively.

Lemma 4.7. *If Γ is the graph groupoid $X\times_\alpha G$ of topological transformation group (X, G, α) with a nonsingular Borel measure on X and $r(x, g)=x$, $s(x, g)=\alpha_{g^{-1}}(x)$ for $(x, g)\in\Gamma$, then $M\times_\rho\Gamma$ is isomorphic to a crossed product of $L^\infty(X)\otimes M$ by G with the action*

$$\tilde{\rho}_g[f](x) = \rho_{(x,g)}(f(\alpha_{g^{-1}}(x))), \qquad f\in L^\infty(X)\otimes M,\ g\in G. \tag{4.22}$$

Proof. By Lemma 4.3, generators of $M\times_\rho\Gamma$ are:

$$[\tilde{R}(\lambda(f)\otimes 1)\tilde{R}^*\xi](x, g) = \int_G f(\alpha_{h^{-1}}(x), h^{-1}g)\xi(x, h)dh, \qquad f\in\mathcal{T}(\Gamma), \tag{4.23}$$

$$[\tilde{R}\pi_0(a)\tilde{R}^*\xi](x, g) = \rho_{(x,g)}(a)\xi(x, g), \qquad a\in M,\ \xi\in\tilde{H}, \tag{4.24}$$

where dh in (4.23) is the left Haar measure of G (which we assume to give the transverse function of Γ). Now, tıansform the generators by the involutive unitary operator V on $\tilde{H}$ defined by

$$[V\xi](x, g) = \Delta_G(g)^{-1/2}\xi(x, g^{-1}), \qquad \xi\in\tilde{H}, \tag{4.25}$$

where Δ_G is the modular function of G. Then we obtain

$$\begin{aligned}[V\tilde{R}(\lambda(f)\otimes 1)\tilde{R}^*V^*\xi](x, g) \\ = \int_G f(\alpha_{k^{-1}g}(x), k^{-1})\Delta_G(k)^{-1/2}\xi(x, k^{-1}g)dk, \quad f\in\mathcal{T}(\Gamma),\end{aligned} \tag{4.26}$$

$$[V\tilde{R}\pi_0(a)\tilde{R}^*V^*\xi](x, g) = \rho_{(x,g^{-1})}(a)\xi(x, g), \qquad a\in M,\ \xi\in\tilde{H}. \tag{4.27}$$

The W^*-algebra generated by the family of operators (4.26) is the crossed product $L^\infty(X)\times_\alpha G$, hence it is also generated by the following families of operators:

$$[\lambda_1(f)\xi](x, g) = f(\alpha_g x)\xi(x, g), \qquad f\in L^\infty(X), \tag{4.28}$$

$$[\lambda_2(h)\xi](x, g) = \xi(x, h^{-1}g), \qquad h\in G,\ \xi\in\tilde{H}. \tag{4.29}$$

On the other hand, in view of (4.22), $(L^\infty(X)\otimes M)\times_{\tilde{\rho}} G$ is generated by (4.29) and

$$\begin{aligned}[\tilde{\pi}(f)\xi](x, g) &= [\tilde{\rho}_{g^{-1}}(f)(x)\xi(x)](g) \\ &= \rho_{(x,g^{-1})}(f(\alpha_g(x)))\xi(x, g), \qquad f\in L^\infty(X)\otimes M,\ \xi\in\tilde{H}.\end{aligned} \tag{4.30}$$

Now, $f\in 1\otimes M$ in (4.30) corresponds to (4.27) and $f\in L^\infty(X)\otimes 1$ in (4.30) corresponds to (4.28). Hence we obtain the isomorphism. Q.E.D.

Remark 4.8. Let α be an action of G on X and $\Gamma=X\times_\alpha G$. If ρ: $\Gamma\to\mathrm{Aut}(M)$ is of G-split type in the sense that there exists a continuous homomorphism $\beta\colon G\to\mathrm{Aut}(M)$ such that the diagram

$$(4.31)\qquad \begin{array}{ccc} \Gamma & \xrightarrow{\ \rho\ } & \mathrm{Aut}(M) \\ {\scriptstyle p}\big\downarrow & \nearrow_{\beta} & \\ G & & \end{array}$$

commutes, where $p\colon \Gamma\ni(x, g)\mapsto g\in G$ is the canonical projection, then the action of G on $L^\infty(X)\otimes M$ is of product type $\alpha\otimes\beta$.

§ 5. Generalized Skew Product

The concept of skew product was originally introduced by Anzai [1] in the construction of non-isomorphic ergodic measure-preserving automorphisms. We present here a new viewpoint for this concept: a dynamical system constructed by skew product is viewed as a special case of a transformation groupoid with groupoid which is a graph of some transformation group.

Let Γ be a locally compact groupoid with faithful proper transverse function ν. If we say that Γ is "measured", then we consider a Haar system (ν, Δ, δ).

Definition 5.1. We call the triple (Ω, Γ, ρ) a locally compact transformation groupoid if Ω is a locally compact space and the mapping $\rho\colon \Gamma\times\Omega\ni(\gamma, \omega)\mapsto\rho_\gamma(\omega)\in\Omega$ is a continuous action of Γ on Ω. Furthermore (Ω, Γ, ρ) is called measured if Γ is measured, Ω is equipped with a Borel measure μ and the action ρ preserves μ-null sets.

Remark 5.2. Let (Ω, Γ, ρ) be a locally compact measured transformation groupoid, then we obtain a W^*-groupoid dynamical system $(L^\infty(\Omega, \mu), \Gamma, \rho)$ in a natural manner. But the converse is not true in general.

Definition 5.3. For a given locally compact transformation groupoid (Ω, Γ, ρ), we construct the product system as a locally compact groupoid $\tilde\Gamma=\Omega\times_\rho\Gamma$. The groupoid $\tilde\Gamma$ is defined as the product of Ω and Γ as a topological space with the unit space $\tilde\Gamma^{(0)}=\Omega\times\Gamma^{(0)}$ and $\{\tilde\gamma=(\omega, \gamma)\}=\tilde\Gamma$ is given by the following groupoid structure: $r(\tilde\gamma)=(\omega, r(\gamma))\in\tilde\Gamma^{(0)}$, $s(\tilde\gamma)=(\rho_{\gamma^{-1}}(\omega), s(\gamma))\in\tilde\Gamma^{(0)}$ and $\tilde\gamma^{-1}=(\rho_{\gamma^{-1}}(\omega), \gamma^{-1})$, $\tilde\gamma_1\tilde\gamma_2=(\omega_1, \gamma_1\gamma_2)$ for $\tilde\gamma_j=(\omega_j, \gamma_j)$, $j=1, 2$ with $\rho_{\gamma_1^{-1}}(\omega_1)=\omega_2$. We have a natural continuous homomorphism of groupoids $\pi\colon \tilde\Gamma\ni\tilde\gamma=(\omega, \gamma)\mapsto\gamma\in\Gamma$. The transverse function $\tilde\nu=\{\tilde\nu^{(\omega,x)}\}_{(\omega,x)\in\tilde\Gamma^{(0)}}$ of $\tilde\Gamma$ is naturally constructed from the pull back of the transverse function ν on Γ by π i.e. $d\tilde\nu^{(\omega,x)}(\tilde\gamma)=d\nu^x(\gamma)$ if $\tilde\gamma=(\omega, \gamma)$. By construction, $\tilde\nu$ is proper if ν is.

Lemma 5.4. *Let (Ω, Γ, ρ) be a locally compact measured transformation groupoid. Define $\tilde{\Lambda}$ by $\tilde{\Lambda}_{\tilde{\nu}} = \mu \otimes \Lambda_\nu$ and $\tilde{\delta}$ by*

$$(5.1) \qquad \tilde{\delta}(\omega, \gamma) = \delta(\gamma) \frac{d\mu(\omega)}{d\mu \circ \rho_{\gamma^{-1}}(\omega)}.$$

Then $(\tilde{\nu}, \tilde{\Lambda}, \tilde{\delta})$ is a Haar system for the product groupoid $\tilde{\Gamma} = \Omega \times_\rho \Gamma$.

Proof. It is readily seen that $\tilde{\delta}$ is a homomorphism of $\tilde{\Gamma}$ and $\tilde{\Lambda}_{\tilde{\nu}} \circ \tilde{\nu}$ is $\tilde{\delta}$-symmetric (c.f. (2.7)). Lemma then follows from the characterization of transverse measure by its δ-symmetry (see A. Connes [9] or D. Kastler [20]).
Q.E.D.

Proposition 5.5. *Let (Ω, Γ, ρ) be a locally compact measured transformation groupoid and $\tilde{\Gamma} = \Omega \times_\rho \Gamma$ be the associated product. Then*

$$(5.2) \qquad \mathrm{End}_{\tilde{\Lambda}}(\tilde{\Gamma}) \cong L^\infty(\Omega, \mu) \times_\rho \Gamma.$$

Proof. By Remark 3.13, $L^\infty(\Omega) \times_\rho \Gamma$ is generated by the set of all mappings $f: \Gamma \to L^\infty(\Omega)$ with $\|f\|_{(\Lambda,\infty)} < \infty$ where,

$$(5.3) \qquad \|f\|_{(\Lambda,\infty)} = \operatorname*{ess.\,sup}_{x \in \Gamma^{(0)}} \|\pi_x(f)\|,$$

$$(5.4) \qquad [\pi_x(f)\xi](\omega, \gamma) = \int_{\Gamma^x} f((\omega, \tilde{\gamma})^{-1}(\omega, \gamma))\xi(\omega, \tilde{\gamma}) d\nu^x(\tilde{\gamma})$$
$$= \int_{\Gamma^x} f(\rho_{\tilde{\gamma}^{-1}}(\omega), \tilde{\gamma}^{-1}\gamma)\xi(\omega, \tilde{\gamma}) d\nu^x(\tilde{\gamma}), \qquad x = r(\gamma),$$

$\xi \in L^2(\Omega, \mu) \otimes L^2(\Gamma^x, \nu^x)$. In view of (5.3),

$$(5.5) \qquad \|\pi_x(f)\| = \operatorname*{ess.\,sup}_{\omega \in \Omega} \|\pi_{(\omega,x)}(f)\|,$$

$$(5.6) \qquad [\pi_{(\omega,x)}(f)\xi](\gamma) = \int_{\Gamma^x} f(\rho_{\tilde{\gamma}^{-1}}(\omega), \tilde{\gamma}^{-1}\gamma)\xi(\tilde{\gamma}) d\nu^x(\tilde{\gamma}), \qquad x = r(\gamma),$$

$\xi \in L^2(\Gamma^x, \nu^x)$. Hence $\|f\|_{(\Lambda,\infty)} = \operatorname*{ess.\,sup}_{(\omega,x) \in \Gamma^{(0)}} \|\pi_{(\omega,x)}(f)\|$ and f agrees with a generator of $\mathrm{End}_{\tilde{\Lambda}}(\tilde{\Gamma})$.
Q.E.D.

Remark 5.6. For the purpose of the discussion of W^*-algebra, we need not assume that the space Ω is locally compact and the homomorphism ρ is continuous. We only have to assume that $(L^\infty(\Omega, \mu), \Gamma, \rho)$ is a W^*-groupoid dynamical system.

Remark 5.7. Let Γ be a (locally compact) measured groupoid with a Haar system (ν, Λ, δ). Then the Poincaré suspension $\tilde{\Gamma}$ of Γ is equal to $\mathbb{R} \times_{\log \delta} \Gamma$, the graph of transformation groupoid ($\boldsymbol{R}$, Γ, $\log\delta$), and the associated W^*-algebra $\mathrm{End}_{\tilde{A}}(\tilde{\Gamma})\cong L^\infty(\boldsymbol{R})\times_{\log\delta}\Gamma$ is isomorphic to the modular crossed product of $\mathrm{End}_A(\Gamma)$. (See [9], [26], [31]. See also Remark 6.2 (1).) Here $\log\delta$ is an action of Γ on $\boldsymbol{R}$ such that $t\mapsto t+\log\delta(\gamma)$.

§ 6. Groupoid Crossed Product and Co-action

In this section, we consider the following situation i.e. we consider a W^*-groupoid dynamical system (M, Γ, ρ) with $M=L^\infty(G)$ where G is a locally compact group, together with the action ρ determined by the left translation of G through a continuous groupoid homomorphism ρ: $\Gamma\mapsto G$ i.e.

(6.1) $$[\rho_\gamma(X)](g) = X(\rho(\gamma)^{-1}g)\,, \qquad X\in L^\infty(G)\,.$$

Throughout this section, we use the notion of co-dynamical system in the sense of Nakagami and Takesaki, see [23].

Theorem 6.1. (1) *Let ρ: $\Gamma\to G$ be a continuous homomorphism. Then there exists a coactoin $\hat{\rho}$ of G on* $\mathrm{End}_A(\Gamma)$ *i.e. $\hat{\rho}$ is an injective $*$-homomorphism $\hat{\rho}$:* $\mathrm{End}_A(\Gamma)\to\mathrm{End}_A(\Gamma)\otimes W_r^*(G)$ *such that the following diagram commutes;*

(6.2) $$\begin{array}{ccc} \mathrm{End}_A(\Gamma) & \xrightarrow{\hat{\rho}} & \mathrm{End}_A(\Gamma)\otimes W_r^*(G) \\ \downarrow{\scriptstyle\hat{\rho}} & & \downarrow{\scriptstyle\hat{\rho}\otimes 1} \\ \mathrm{End}_A(\Gamma)\otimes W_r^*(G) & \xrightarrow[1\otimes\delta_G]{} & \mathrm{End}_A(\Gamma)\otimes W_r^*(G)\otimes W_r^*(G) \end{array}$$

where $W_r^*(G)$ is the W^*-algebra generated by the left regular representation $\lambda(g)$ of $g\in G$ and $\delta_G(\lambda(g))=\lambda(g)\otimes\lambda(g)$, $g\in G$. Furthermore $L^\infty(G)\times_\rho\Gamma\cong\mathrm{End}_A(\Gamma)*_{\hat{\rho}}G$ where $*_{\hat{\rho}}$ denotes the crossed product by co-action $\hat{\rho}$.

(2) *If G is abelian, $\hat{\rho}$ gives an action of $\hat{G}$ on* $\mathrm{End}_A(\Gamma)$ *such that $L^\infty(G)\times_\rho\Gamma\cong\mathrm{End}_A(\Gamma)\times_{\hat{\rho}}\hat{G}$. Let G, H be locally compact abelian groups with ψ: $G\to H$ a continuous homomorphism. Then $\hat{\rho}\circ{}^t\psi=(\psi\circ\rho)^\wedge$ as an action of $\hat{H}$ on* $\mathrm{End}_A(\Gamma)$.

Proof. (1) We define a unitary operator F on $L^2(\Gamma, (A_\nu\circ\nu))\otimes L^2(G)$ by

(6.3) $$[F\xi](\gamma, g) = \xi(\gamma, \rho(\gamma)^{-1}g)\,, \quad \xi\in L^2(\Gamma, (A_\nu\circ\nu))\otimes L^2(G)\,.$$

By Remark 4.4 (4.8), $\mathrm{End}_A(\Gamma)$ is generated by the following set of operators,

(6.4) $$[(\lambda_0(f)\otimes 1)\xi](\gamma, g) = \int_{\Gamma^x} f(\tilde{\gamma})\xi(\tilde{\gamma}^{-1}\gamma, g)d\nu^x(\tilde{\gamma}), \qquad x = r(\gamma)\,,$$

$\xi\in L^2(\Gamma, (A_\nu\circ\nu))\otimes L^2(G)$, $f\in\mathfrak{A}_1$. Now we set $\hat{\rho}_0(f)=F(\lambda_0(f)\otimes 1)F^*$. Then by (6.3) and (6.4), we obtain

$$(6.5)\qquad [\hat{\rho}_0(f)\xi](\gamma, g) = \int_{\Gamma^x} f(\tilde{\gamma})\xi(\tilde{\gamma}^{-1}\gamma, \rho(\tilde{\gamma})^{-1}g)d\nu^x(\tilde{\gamma}), \qquad x = r(\gamma),$$

$\xi \in L^2(\Gamma, (\Lambda_\nu \circ \nu)) \otimes L^2(G)$, $f \in \mathfrak{A}_1$. Hence, the mapping $\hat{\rho}_0$ extends to an injective $*$-isomorphism $\mathrm{End}_\Lambda(\Gamma) \to B(L^2(\Gamma, (\Lambda_\nu \circ \nu)) \otimes L^2(G))$ which is denoted by $\hat{\rho}$. The fact $\hat{\rho}(\mathrm{End}_\Lambda(\Gamma)) \subset \mathrm{End}_\Lambda(\Gamma) \otimes W^*_r(G)$ is easily seen by the commutativity of $\hat{\rho}(\mathrm{End}_\Lambda(\Gamma))$ and the commutant $(J_\Lambda\ \mathrm{End}_\Lambda(\Gamma)J_\Lambda) \otimes$ (right translations) of $\mathrm{End}_\Lambda(\Gamma) \otimes W^*_r(G)$. In view of (6.5) it is easy to see that the diagram (6.2) for $\hat{\rho}_0(f)$, $f \in \mathfrak{A}_1$ commutes. Hence we obtain the commutativity of the diagram (6.2) by the density of $\mathfrak{A}_1$ in $\mathrm{End}_\Lambda(\Gamma)$. The W^*-algebra $F(L^\infty(G) \times_\rho \Gamma)F^*$ is generated by the family of operators $\hat{\rho}_0(f)$, $f \in \mathfrak{A}_1$ of (6.5) and

$$(6.6)\qquad [F\pi_0(X)F^*\xi](\gamma, g) = X(g)\xi(\gamma, g), \qquad X \in L^\infty(G),$$

by Remark 4.4, $\xi \in L^2(\Gamma, (\Lambda_\nu \circ \nu)) \otimes L^2(G)$. These are exactly the generators of the co-crossed product $\mathrm{End}_\Lambda(\Gamma) *_{\hat{\rho}} G$.

(2) We define a Plancherel transform U: $L^2(\Gamma, (\Lambda_\nu \circ \nu)) \otimes L^2(G) \to L^2(\Gamma, (\Lambda_\nu \circ \nu)) \otimes L^2(\hat{G})$ by

$$(6.7)\qquad [U\xi](\gamma, p) = \int_G \overline{\langle g, p \rangle}\xi(\gamma, g)dg, \qquad p \in \hat{G}.$$

Then we obtain

$$(6.8)\qquad [U\hat{\rho}(f)U^*\xi](\gamma, p) = \int_{\Gamma^x} \langle \rho(\tilde{\gamma})^{-1}, p \rangle f(\tilde{\gamma})\xi(\tilde{\gamma}^{-1}\gamma, p)d\nu^x(\tilde{\gamma}), \qquad x = r(\gamma),$$

$$(6.9)\qquad [UF\pi_0(X)F^*U^*\xi](\gamma, p) = \int_{\hat{G}} \hat{X}(q)\xi(\gamma, p-q)dq$$

where $f \in \mathfrak{A}_1$, $\xi \in L^2(\Gamma, (\Lambda_\nu \circ \nu)) \otimes L^2(\hat{G})$ and $X \in L^\infty(G)$ is such that

$$(6.10)\qquad X(g) = \int_{\hat{G}} \langle g, p \rangle \hat{X}(p)dp, \qquad \hat{X} \in L^1(\hat{G}).$$

In view of (6.8) and (6.9) together with the definition of crossed product, the $\hat{G}$-action $\hat{\rho}$ is generated by

$$(6.11)\qquad [\hat{\rho}_p(f)](\gamma) = \langle \rho(\gamma), p \rangle f(\gamma)$$

or the adjoint action of the unitary representation $[V(p)]_{p \in \hat{G}}$ of $\hat{G}$ on $L^2(\Gamma, (\Lambda_\nu \circ \nu))$ determined by

$$(6.12)\qquad [V(p)\xi](\gamma) = \langle \rho(\gamma), p \rangle \xi(\gamma), \qquad \xi \in L^2(\Gamma, (\Lambda_\nu \circ \nu)).$$

Let $q\in\widehat{H}$. Then $\langle\rho(\gamma), {}^t\psi(q)\rangle=\langle\psi\circ\rho(\gamma), q\rangle$, $\gamma\in\Gamma$. Hence, we obtain (2). Q.E.D.

Remark 6.2. (1) Let Γ be a locally compact measured groupoid with a Haar system (ν, Λ, δ) such that δ is continuous. Consider $G=\boldsymbol{R}$, $\rho=\log\delta$. Then $\tilde{\Gamma}=\boldsymbol{R}\times_{\log\delta}\Gamma$ is the Poincaré suspension of the measured groupoid Γ (see C. Series [26]). In this case, the co-action $\hat{\rho}$ gives rise to action of $\widehat{\boldsymbol{R}}\cong\boldsymbol{R}$, which coincides with the modular automorphisms of $\mathrm{End}_\Lambda(\Gamma)$. (See [9], [26], [31].)

(2) In the same situation, suppose that $\mathrm{End}_\Lambda(\Gamma)$ is a factor of type III_1. It is known that the $\boldsymbol{Z}$-crossed product of $\mathrm{End}_\Lambda(\Gamma)$ through the restriction of the modular action by $\boldsymbol{Z}\ni 1\mapsto(2\pi/L)\in\boldsymbol{R}$ is a factor of type III_λ, $\lambda=\exp(-(2\pi/L))$ (see A. Connes [7], [8]). This $\boldsymbol{Z}$-action is interpreted as the co-aciton of S^1 given by the groupoid homomorphism $\Gamma\to S^1$ obtained by the composition of $\log\delta\colon\Gamma\to\boldsymbol{R}$ and the quotient homomorphism $\boldsymbol{R}\to S^1$ by period L. This S^1 with $\boldsymbol{R}$-flow corresponds with the flow of weight associated with the above factor of type III_λ (see Hamachi-Oka-Oshikawa [17], [18]).

(3) If we consider the special case that Γ is a locally compact abelian group, then $L^\infty(G)\times_\rho\Gamma\cong W_\gamma^*(\Gamma)\times_{\hat{\rho}}\widehat{G}\cong L^\infty(\widehat{\Gamma})\times_{\hat{\rho}}\widehat{G}$. This duality can be viewed as the Plancherel transformation of abelian groupoid, see Bellissard-Testard [5].

Lemma 6.3. *Suppose G is abelian. If* $\rho, \sigma\colon\Gamma\to G$ *are cohomologous in the sense that there exists a continuous mapping* $\tau\colon\Gamma^{(0)}\to G$ *such that* $\rho(\gamma)=\tau(r(\gamma))\sigma(\gamma)\tau(s(\gamma))^{-1}$. *Then the two* $\widehat{G}$*-actions* $\hat{\rho}, \hat{\sigma}$ *are one-cocycle equivalent i.e. there exists a strongly continuous unitary-valued mapping* $u\colon\widehat{G}\to\mathrm{End}_\Lambda(\Gamma)$ *such that*

$$\hat{\rho}_k(a)=u_k\hat{\sigma}_k(a)u_k^*\,,\qquad a\in\mathrm{End}_\Lambda(\Gamma)\,,\tag{6.13}$$

$$u_{k+l}=u_k\hat{\sigma}_k(u_l)\,,\qquad k, l\in\widehat{G}\,.\tag{6.14}$$

Proof. We define a unitary operator by

$$[u_k\xi](\gamma)=\langle\tau(r(\gamma)), k\rangle\xi(\gamma)\,,\qquad \xi\in L^2(\Gamma, (\Lambda_\nu\circ\nu))\,.\tag{6.15}$$

Then it is easy to see that the operator u_k commutes with the commutant $J_\Lambda\mathrm{End}_\Lambda(\Gamma)J_\Lambda$ of $\mathrm{End}_\Lambda(\Gamma)$. The intertwining property (6.13) and the cocycle condition (6.14) easily follows from (6.11) and (6.12) Q.E.D.

Remark 6.4 In the situation of Remark 6.2 (1), a coboundary change of cocycle corresponds to a change of measure on the unit space within the same measure class and the corresponding cocycle in Lemma 6.3 is exactly the one-cocycle derivative of A. Connes.

§ 7. Examples

In this section, we collect some examples of W^*-groupoid dynamical systems (also valid for a C^*-algebraic framework) and their crossed products. The first one is an example of A. Connes.

Example 7.1. (See [6], [8], [11].) Let Γ be a locally compact measured groupoid constructed as the graph of an Anosov foliation $\mathfrak{F}_A$ on a unit sphere bundle $T_1(\Omega)$ of a compact Riemann surface Ω with genus $g \geqq 2$. This foliation $\mathfrak{F}_A$ on $T_1(\Omega)$ is defined as the orbits of $H \equiv \{\begin{bmatrix} e^t & s \\ 0 & e^{-t} \end{bmatrix} : s, t \in \boldsymbol{R}\}$ on $T_1(\Omega)$ under the identification $L \backslash PSL(2, \boldsymbol{R}) \simeq T_1(\Omega)$, where $L \simeq \pi_1(\Omega)$, a uniform lattice subgroup of $PSL(2, \boldsymbol{R})$, and the H-action is the right multiplication on $L \backslash PSL(2, \boldsymbol{R})$. It follows that the groupoid Γ is actually a graph of transformation group and is locally compact. Any transverse function comes from a Haar measure of H in each orbit and the transverse measure is defined by the restriction of the Haar measure of $PSL(2, \boldsymbol{R})$ to the fundamental domain of L. It is known that this groupoid Γ defines a hyperfinite factor of type III_1. Now, let $(K, \boldsymbol{R}, \theta)$ be an ergodic measure preserving $\boldsymbol{R}$-dynamical system on a compact manifold K with a probability measure. We construct a new foliated manifold $(K \times T_1(\Omega), \tilde{\mathfrak{F}}_A)$ defined by the product type action of H where the H-action on K is defined by the composition of the homomorphism $\pi: H \ni \begin{bmatrix} e^t, & s \\ 0, & e^{-t} \end{bmatrix} \mapsto t \in \boldsymbol{R}$ and θ. Then the graph $\tilde{\Gamma}$ of this foliation is identical with $K \times_\rho \Gamma$, $\rho(x, h) = \theta \circ \pi(h)$ for $(x, h) \in \tilde{\Gamma}$, and the resulting von Neumann algebra $\mathrm{End}_{\tilde{\Lambda}}(\tilde{\Gamma}) \simeq L^\infty(K) \times_\rho \Gamma$ defines a type III-factor with the smooth flow of weight isomorphic to $(K, \boldsymbol{R}, \theta)$ (see [12]). If we take $K = S^1$, θ is the translation action of $\boldsymbol{R}$ on S^1 via a quotient homomorphism with the kernel $L\boldsymbol{Z}$ $(L > 0)$ and $\tilde{\Gamma} = S^1 \times_\rho \Gamma$, then $\mathrm{End}_{\tilde{\Lambda}}(\tilde{\Gamma}) \simeq L^\infty(S^1) \times_\rho \Gamma \simeq \mathrm{End}_\Lambda(\Gamma) \times_{\hat{\rho}} \boldsymbol{Z}$, where the action $\hat{\rho}$ (see Section 6) agrees with the restriction of modular action on $\mathrm{End}_\Lambda(\Gamma)$ to $\boldsymbol{Z}$ and the resulting W^*-algebra is the hyperfinite factor of type III_λ, $\lambda = \exp(-2\pi/L)$ (see also [7]). This mechanism works for a more general measured groupoid defined by a non-singular transformation group, see [18], Theorem 4. It is noticed that in the above example of A. Connes, the W^*-algebra $W^*(K \times T_1(\Omega), \tilde{\mathfrak{F}}_A)$ for the foliated manifold $(K \times T_1(\Omega), \tilde{\mathfrak{F}}_A)$ is actually the usual crossed product of $L^\infty(K \times T_1(\Omega))$ by H (see Lemma 4.7). But in general, we can not expect that $\mathrm{End}_{\tilde{\Lambda}}(K \times_\rho \Gamma)$ is expressed as a crossed product.

Example 7.2. Let G be a locally compact group with closed subgroups H and K. We assume that the group has a continuous factorization $G\cong(G/K)\times K$ as a topological space i.e. we assume that K-quotient homomorphism is continuous (for the K-quotient homomorphism, see Appendix A). Let (M, K, α) be a W^*-dynamical system and $(\tilde{M}, G, \tilde{\alpha})=\underset{K\uparrow G}{\mathrm{ind}}(M, K, \alpha)$ be the induced system in the sense of Takesaki (see [29]). Let $\Gamma=(G/K)\times H$ be a locally compact measured groupoid defined by the topological transformation group $((G/K), H,$ left multiplication). Then the crossed product of $\tilde{M}$ by H through $\tilde{\alpha}$ is isomorphic to the groupoid crossed produtc $M\times_\rho\Gamma$ where $\rho\colon\Gamma\to\mathrm{Aut}(M)$ is the composition of K-quotient homomoprhism and the K-action α. It must be noticed that this groupoid Γ may fail to be principal in general. (Many of concrete and important foliations arise in this way, though without continuity of the quotient homomorphism.)

Example 7.3 (Discrete modular lift of non-unimodular groupoid). Let Γ be a locally compact measured groupoid with Haar system (ν, Λ, δ). Assume that the range of $\log\delta$ is contained in $k\boldsymbol{Z}$. (For example, Γ is the graph of topological transformation group defined by the H_d-action on $T_1(\Omega)$ in Example 7.1, where $H_d=\boldsymbol{R}\times_s\boldsymbol{Z}\subset\boldsymbol{R}\times_s\boldsymbol{R}\cong H$. In this case $\Gamma=T_1(\Omega)\times_\alpha H_d$ defines the Powers factor.) Now we define $\tilde{\Gamma}=(k\boldsymbol{Z})\times_{\log\delta}\Gamma$ and take a suitable measure on $k\boldsymbol{Z}$. Then it is easily checked that $\tilde{\Gamma}$ is unimodular groupoid (the same mechanism as the construction of Poincaré suspension works), and $W^*(\tilde{\Gamma})\cong W^*(\Gamma)\times_{\hat{\rho}}S^1$ is a crossed product W^*-algebra by compact modular action.

§ 8. Discussion

The concept of an action of a groupoid on W^*-algebra is discussed also by Jones and Takesaki [19] for the purpose of classifying discrete automorphisms of an operator algebra of the form $L^\infty(X)\otimes M$. Although they discuss the groupoid with discrete countable equivalence relations, they don't discuss the resulting "crossed product algebra" in terms of groupoid. Our approach is algebraically the same with theirs, but direction is different. We are discussing our theory in the locally compact category with continuous groupoid homomorphisms.

Nevertheless, it seems to be possible to discuss under a more non-restrictive condition.

Appendix A

In this section, we show the relation between the continuous groupoid homomorphisms and the continuous cocycles of skew product (see [1]).

Let (X, G, α) be a topological transformation group and let Ω be an auxiliary topologiacl space with a topological structure group H. The continuous mapping $\Psi: X\times G\to H$ is called a cocycle of (X, G, α) if

(A.1) $$\Psi(x, g_1g_2) = \Psi(\alpha_{g_2}(x), g_1)\Psi(x, g_2), \qquad x\in X, g_1, g_2\in G.$$

Then the topological space $\Omega\times X$ is a G-space (skew product) by $\tilde{\alpha}$, where

(A.2) $$\tilde{\alpha}_g(\omega, x) = (\Psi(x, g)\omega, \alpha_g(x)).$$

Put

(A.3) $$\rho(x, g) = \Psi(x, g^{-1})^{-1}.$$

Then ρ gives a continuous groupoid homomorphism $\Gamma=X\times_\alpha G\to H$. (Further, all the continuous groupoid homomorphism $\Gamma\to H$ arises in this way.) In this situation, (Ω, Γ, ρ) is a transformation groupoid and $\Omega\times_\rho\Gamma=(\Omega\times X)\times_{\tilde{\alpha}}G$.

By making use of this fact, we obtain an example of groupoid homomorphism as follows. Let G be a locally compact group with a closed subgroup K. Assume G has a continuous factorization $G\cong(G/K)\times K$ as a topological space. (For example, Iwasawa decomposition gives such an example.) Now G and G/K are left G-space such that there exists a continuous groupoid homomorphism $\Psi: (G/K)\times G\to K$. We call this homomorphism K-quotient homomorphism. So, we obtain groupoid homomorphism $\rho: \Gamma=(G/K)\times H\to K$ where H is any closed subgroup of G. As a concrete example, let $G=SL(2, \mathbf{R})$ with $K=S^1$. Then G/K is the upper half plane H_+ with the G-action by fractional linear transformations, and the cocycle is given by

(A.4) $$\Psi(z, g) = \arg(cz+d), \qquad z\in H_+,$$

where $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL(2, \mathbf{R})$ ($cz+d$ is known as the automorphic factor).

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