Publ. RIMS, Kyoto Univ.
20 (1984), 959–970

Groupoid Dynamical Systems and Crossed Product, II—The Case of C^*-Systems

By

Tetsuya MASUDA*

Abstract

By analogy with C^*-dynamical system, we define a C^*-groupoid dynamical system (A, Γ, ρ) where A is a C^*-algebra, Γ is a locally compact groupoid, and $\rho: \Gamma \to \mathrm{Aut}(A)$ is a continuous groupoid homomorphism. The groupoid crossed product $A \times_\rho \Gamma$ is defined and is shown to have similar properties as the case of a group action. As a special case of this situation, if ρ is a continuous homomorphism from Γ to a locally compact group G, we obtain groupoid dynamical system $(C_0(G), \Gamma, \rho)$. In this case, there exists a co-action $\hat{\rho}$ of G on $C^*(\Gamma)$ and the groupoid crossed product $C_0(G) \times_\rho \Gamma$ is isomorphic to the co-crossed product $C^*(\Gamma) *_{\hat{\rho}} G$ of $C^*(\Gamma)$ by G. The results in this paper is obtained by the analogy with our previous results for the case of W^*-systems.

§1. Introduction

In our previous paper [8], we defined a W^*-groupoid dynamical system and its groupoid crossed product based on the analogy with the case of a group action together with the several basic ideas. In this paper, we shall give the C^*-algebraic framework of groupoid dynamical system and its groupoid crossed product. Because we consider only the regular representation based on the canonical Hilbert Γ-bundle out of the transverse function (see [2]), all the crossed products are in the reduced category. The whole discussion is parallel to those of W^*-algebraic case.

In Section 2, we define C^*-groupoid dynamical system and its groupoid crossed product. In this section, we also describe the general properties of the groupoid crossed product. In Section 3, we shall discuss the C^*-groupoid dynamical system $(C_0(G), \Gamma, \rho)$ defined by a continuous groupoid homomorphism $\rho: \Gamma \to G$ for an auxiliary locally compact (not necessarily abelian) group

Communicated by H. Araki, June 23, 1983. Revised December 26, 1983.
* Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan.
Present address: Mathematical Sciences Research Institute, 2223 Fulton Street, Berkeley California 94720, USA.

G. For the examples, see [8], Section 7.

Throughout this paper, we use a non-commutative integration theory in terms of a locally compact topological groupoid which admits a faithful transverse function $\nu=\{\nu^x\}_{x\in\Gamma^{(0)}}$ (see [2], [4], [10]).

§2. C^*-Groupoid Dynamical Systems

We shall start with the definition of a C^*-groupoid dynamical system.

Definition 2.1. The triplet (A, Γ, ρ) is called a C^*-groupoid dynamical system (or C^*-groupoid system, for short) if A is a C^*-algebra, Γ is a locally compact groupoid with a faithful transverse function $\nu=\{\nu^x\}_{x\in\Gamma^{(0)}}$, and ρ: $\Gamma\to\mathrm{Aut}(A)$ is a continuous homomorphism.

The associated crossed product is defined as the completion of the set $C_c(\Gamma, A)$ of all A-valued continuous functions over Γ with compact support by the C^*-norm defined below. The set $C_c(\Gamma, A)$ is a $*$-algebra by:

$$(2.1)\qquad (f_1 \hat{*} f_2)(\gamma)=\int_{\Gamma^x}\rho_{\tilde{\gamma}}(f_1(\tilde{\gamma}^{-1}\gamma))\,f_2(\tilde{\gamma})\,d\nu^x(\tilde{\gamma}),\quad x=r(\gamma),$$

$$(2.2)\qquad f^{\hat{*}}(\gamma)=\rho_{\gamma}(f(\gamma^{-1})^*),$$

where $f_1, f_2, f\in C_c(\Gamma, A)$. The C^*-norm on $C_c(\Gamma, A)$ is defined by

$$(2.3)\qquad \|f\|=\sup_{x\in\Gamma^{(0)}}\|\pi_x(f)\|,\quad f\in C_c(\Gamma, A),$$

$$(2.4)\qquad [\pi_x(f)\,\xi](\gamma)=\int_{\Gamma^x}\rho_{\tilde{\gamma}}(f(\tilde{\gamma}^{-1}\gamma))\,\xi(\tilde{\gamma})\,d\nu^x(\tilde{\gamma}),$$
$$\gamma\in\Gamma^x,\ \xi\in L^2(\Gamma^x, \nu^x)\otimes H,$$

where H is a faithful representation Hilbert space of A. In view of (2.4), $\|\pi_x(f)\|$, $f\in C_c(\Gamma, A)$ is independent of the choice of representation Hilbert space H so that the norm (2.3) is independent of the choice of representation Hilbert space H.

Definition 2.2. $A\times_\rho\Gamma$ or $C^*(A, \Gamma, \rho)$ denotes the C^*-algebra obtained by the completion of $C_c(\Gamma, A)$ by the C^*-norm given by (2.3).

Example 2.3. The definition of a groupoid algebra given by A. Connes is as follows. The set $C_c(\Gamma)$ is a $*$-algebra by

$$(2.5)\qquad (f_1 * f_2)(\gamma)=\int_{\Gamma^x}f_1(\tilde{\gamma})\,f_2(\tilde{\gamma}^{-1}\gamma)\,d\nu^x(\tilde{\gamma}),\quad x=r(\gamma),$$

$$(2.6)\qquad f^{*}(\gamma)=\overline{f(\gamma^{-1})},$$

where $f_1, f_2, f \in C_c(\Gamma)$. The C^*-algebra $C^*(\Gamma)$ is defined by the completion of $C_c(\Gamma)$ with respect to the norm on $C_c(\Gamma)$ defined by

$$\|f\| = \sup_{x \in \Gamma^{(0)}} \|\pi_x(f)\|, \quad f \in C_c(\Gamma), \tag{2.7}$$

$$[\pi_x(f)\, \xi]\,(\gamma) = \int_{\Gamma^x} f(\gamma^{-1}\, \tilde{\gamma})\, \xi(\tilde{\gamma})\, d\nu^x(\tilde{\gamma}), \quad \gamma \in \Gamma^x,\ \xi \in L^2(\Gamma^x, \nu^x). \tag{2.8}$$

Now, we define for $A = \boldsymbol{C}$ bijection R: $C_c(\Gamma, A) \to C_c(\Gamma)$ by $[Rf](\gamma) = f(\gamma^{-1})$. Then, R is a $*$-algebra isomorphism between $C_c(\Gamma, A)$ with $A = \boldsymbol{C}$ and $C_c(\Gamma)$ preserving C^*-norm (cf. (2.3), (2.4) and (2.7), (2.8)). So, our definition with $A = \boldsymbol{C}$ actually gives the usual Connes algebra $C^*(\Gamma)$.

Example 2.4. Let (A, G, α) be a C^*-dynamical system. The crossed product $A \times_\alpha G$ associated with the C^*-dynamical system (A, G, α) is defined as the C^*-completion of $L^1(G, A)$ with the $*$-algebra operations defined by

$$(f_1 * f_2)\,(g) = \int_G f_1(h)\, \alpha_h(f_2\,(h^{-1}\, g))\, dh, \tag{2.9}$$

$$f^\sharp(g) = \Delta_G(g)^{-1}\, \alpha_g(f(g^{-1})^*), \tag{2.10}$$

for $f_1, f_2, f \in L^1(G, A)$, and with the C^*-norm defined through the $*$-representation

$$[\pi(f)\xi]\,(g) = \int_G \alpha_{g^{-1}}\,(f(h))\, \xi(h^{-1}\, g)\, dh, \tag{2.11}$$

where $f \in L^1(G, A)$, $\xi \in L^2(G) \otimes H$ and H is any faithful representation Hilbert space of A. It is known that due to inequality $\|f\|_{C^*} \leqq \|f\|_{L^1}$ (which follows from (2.11)), the C^*-completion of $C_c(G, A)$ gives $A \times_\alpha G$. For the purpose of comparison with our formulation, we define $A \times_\alpha G$ in a different manner. We define the $*$-algebra operations in $C_c(G, A)$ by

$$(f_1 * f_2)\,(g) = \int_G \alpha_h(f_1\,(h^{-1}\, g))\, f_2(h)\, dh, \tag{2.12}$$

$$f^\sharp(g) = \alpha_g(f(g^{-1})^*), \tag{2.13}$$

for $f_1, f_2, f \in C_c(G, A)$ and the C^*-norm through the $*$-representation

$$[\pi(f)\, \xi]\,(g) = \int_G \alpha_h(f(h^{-1}\, g))\, \xi(h)\, dh, \tag{2.14}$$

where $f \in C_c(G, A)$, $\xi \in L^2(G) \otimes H$. Then we obtain $A \times_\alpha G$ by taking the C^*-completion of $C_c(G, A)$. In fact, the mapping R defined by

$$[Rf](g) = \Delta_G(g)^{-1/2}\,\alpha_g(f(g^{-1})),\quad f\in L^1(G, A) \tag{2.15}$$

is a $*$-isomorphism of $L^1(G, A)$ ($*$ and $\sharp$ given by (2.9) and (2.10)) onto $L^1_{\mathrm{sym}}(G, A)$ ($*$ and $\sharp$ given by (2.12) and (2.13)), which is the L^1-space with respect to the symmetric Haar measure $d\mu(g)=\Delta_G(g)^{-1/2}dg$. The unitary mapping $\tilde{R}$: $L^2(G)\otimes H\to L^2(G)\otimes H$ defined by

$$[\tilde{R}\xi](g) = \Delta_G(g)^{-1/2}\,\xi(g^{-1}),\quad \xi\in L^2(G)\otimes H \tag{2.16}$$

implements this isomorphism and intertwines $\pi(f)$ (π given by (2.11)) with $\{\pi_x\pi(R(f))$ (π given by (2.14)).

Proposition 2.5. (1) *Let $f\in C_c(\Gamma, A)$. Then the family of operators $\{\pi_x(f)\}_{x\in\Gamma^{(0)}}$ is covariant in the sense that*

$$(Ad_{U(\gamma)}\otimes\rho_\gamma)(\pi_x(f)) = \pi_y(f),\quad \gamma\in\Gamma_x^y, \tag{2.17}$$

where $Ad_{U(\gamma)}=U(\gamma)\cdot U(\gamma)^*$ and $[U(\gamma)\,\xi](\tilde{\gamma})=\xi(\gamma^{-1}\tilde{\gamma})$, $\tilde{\gamma}\in\Gamma^y$, $\xi\in L^2(\Gamma^y, \nu^y)$.

(2) *If ρ, σ: $\Gamma\to\mathrm{Aut}(A)$ are cohomologous in the sense that there exists continuous mapping τ: $\Gamma^{(0)}\to\mathrm{Aut}\,(A)$ such that $\rho_\gamma=\tau_{r(\gamma)}\circ\sigma_\gamma\circ\tau^{-1}_{s(\gamma)}$. Then $A\times_\rho\Gamma\cong A\times_\sigma\Gamma$.*

(3) *If ρ, σ: $\Gamma\to\mathrm{Aut}\,(A)$ are one-cocycle equivalent in the sense that there exists a unitary valued mapping u: $\Gamma\to M(A)$ such that $\gamma\mapsto u_\gamma a$ and $\gamma\mapsto au_\gamma$ are continuous for all $a\in A$ and*

$$\rho_\gamma(a) = u_\gamma\sigma_\gamma(a)u_\gamma^*,\quad a\in A\,, \tag{2.18}$$

$$u_{\gamma_1\gamma_2} = u_{\gamma_1}\sigma_{\gamma_1}(u_{\gamma_2}),\quad s(\gamma_1) = r(\gamma_2)\,, \tag{2.19}$$

then $A\times_\rho\Gamma\cong A\times_\sigma\Gamma$.

(4) *If Γ is the graph groupoid of topological transformation group (X, G, α), then $A\times_\rho\Gamma$ is isomorphic to a crossed product of $C_0(X)\otimes A$ by G with the action*

$$\hat{\rho}_g[f](x) = \rho_{(x,g)}(f(\alpha_{g^{-1}}(x))),\quad f\in C_0(X)\otimes A,\quad g\in G\,. \tag{2.20}$$

(Note that $C_0(X)=C(X)$ if X is compact.)

Proof. (1) Let $\hat{\gamma}\in\Gamma_x^y$ and $\xi\in L^2(\Gamma^y, \nu^y)\otimes H$. Then

$$\begin{aligned}&[(Ad_{U(\hat{\gamma})}\otimes\rho_{\hat{\gamma}})(\pi_x(f))\,\xi](\gamma)\\ &=\int_{\Gamma^x}\rho_{\hat{\gamma}}\circ\rho_{\tilde{\gamma}}(f(\tilde{\gamma}^{-1}(\hat{\gamma}^{-1}\gamma)))\,\xi(\hat{\gamma}\tilde{\gamma})\,d\nu^x(\tilde{\gamma})\\ &=\int_{\Gamma^x}\rho_{\hat{\gamma}\tilde{\gamma}}(f((\hat{\gamma}\tilde{\gamma})^{-1}\gamma))\,\xi(\hat{\gamma}\tilde{\gamma})\,d\nu^x(\tilde{\gamma})\end{aligned} \tag{2.21}$$

$$=\int_{\Gamma^y} \rho_{\gamma_1}(f(\gamma_1^{-1}\gamma))\,\xi(\gamma_1)\,d\nu^y(\gamma_1)$$
$$=[\pi_y(f)\,\xi](\gamma).$$

This shows (2.17).

(2) We define a mapping Φ by

$$\Phi[f](\gamma)=\tau^{-1}_{r(\gamma)}(f(\gamma)),\quad f\in C_c(\Gamma, A). \tag{2.22}$$

Then Φ gives a bijective mapping of $C_c(\Gamma, A)$ onto itself and the following relations hold:

$$\Phi[f_1]\hat{*}_\sigma\,\Phi[f_2]=\Phi[f_1\hat{*}_\rho f_2],\quad f_1, f_2\in C_c(\Gamma, A), \tag{2.23}$$

$$\Phi[f]^{(\#,\sigma)}=\Phi[f^{(\#,\rho)}],\quad f\in C_c(\Gamma, A), \tag{2.24}$$

where $\hat{*}_\sigma$, $(\#, \sigma)$ and $\hat{*}_\rho$, $(\#, \rho)$ are convolution and involution of $C_c(\Gamma, A)$ with respect to the actions σ and ρ respectively. Moreover,

$$\begin{aligned}[\pi^\sigma_x(\Phi[f])\,\xi](\gamma)&=\int_{\Gamma^x}\sigma_{\tilde\gamma}\circ\tau^{-1}_{s(\tilde\gamma)}(f(\tilde\gamma^{-1}\gamma))\,\xi(\tilde\gamma)\,d\nu^x(\tilde\gamma)\\ &=\int_{\Gamma^x}\tau^{-1}_x(\rho_{\tilde\gamma}(f(\tilde\gamma^{-1}\gamma)))\,\xi(\tilde\gamma)\,d\nu^x(\tilde\gamma),\\ &=[\{(1\otimes\tau^{-1}_x)\,\pi^\rho_x(f)\}\,\xi](\gamma)\end{aligned} \tag{2.25}$$

where $f\in C_c(\Gamma, A)$, $\xi\in L^2(\Gamma^x, \nu^x)\otimes H$. Hence we obtain $\|\pi^\sigma_x(\Phi[f])\|=\|\pi^\rho_x(f)\|$ for any $x\in\Gamma^{(0)}$ where π^σ and π^ρ are representations relevant for σ and ρ, respectively. This implies the desired isomorphism.

(3) We define a mapping Ψ by

$$\Psi[f](\gamma)=u_\gamma f(\gamma),\quad f\in C_c(\Gamma, A). \tag{2.26}$$

This gives a bijective mapping of $C_c(\Gamma, A)$ onto itself and the following relations hold:

$$\Psi[f_1]\hat{*}_\rho\,\Psi[f_2]=\Psi[f_1\hat{*}_\sigma f_2],\quad f_1, f_2\in C_c(\Gamma, A), \tag{2.27}$$

$$\Psi[f]^{(\#,\rho)}=\Psi[f^{(\#,\sigma)}],\quad f\in C_c(\Gamma, A), \tag{2.28}$$

where we use (2.18), (2.19) and $u_{1_x}=1$, which follows from (2.19). We define a family of unitary operators $U=\{U_x\}_{x\in\Gamma^{(0)}}$ by

$$[U_x\xi](\gamma)=u_\gamma\,\xi(\gamma),\quad \xi\in L^2(\Gamma^x, \nu^x)\otimes H. \tag{2.29}$$

Then, we obtain

$$\pi^\rho_x(\Psi[f])\,\xi=U_x\,\pi^\sigma_x(f)\,U^*_x\,\xi,\quad f\in C_c(\Gamma, A),\quad \xi\in L^2(\Gamma^x, \nu^x)\otimes H, \tag{2.30}$$

for all $x\in\Gamma^{(0)}$. Hence $||\pi_x^\rho(\Psi[f])||=||\pi_x^\sigma(f)||$ and we obtain the desired isomorphism.

(4) For $f_1, f_2, f\in C_c(\Gamma, A)$,

$$(2.31)\qquad (f_1 \hat{*}_\rho f_2)(x, g) = \int_G \rho_{(x,h)}(f_1(\alpha_{h^{-1}}(x), h^{-1} g)) f_2(x, h)\, dh$$
$$= \int_G [\hat{\rho}_h[f_1](h^{-1} g)](x) f_2(x, h)\, dh\,,$$
$$= [(f_1 *_{\hat{\rho}} f_2)(g)](x)$$

$$(2.32)\qquad f^{(\sharp,\rho)}(x, g) = \rho_{(x,g)}(f(\alpha_{g^{-1}}(x), g^{-1})^*)$$
$$= [\hat{\rho}_g[f](g^{-1})]^*(x)$$
$$= [f^{(\sharp,\hat{\rho})}(g^{-1})](x)\,,$$

where ρ and $\hat{\rho}$ in the subscripts for $*$ and in the subscripts for $\sharp$ indicate the convolution and involution in $A\times_\rho\Gamma$ and in $C_0(X, A)\times_{\hat{\rho}}G$, respectively. Furthermore,

$$(2.33)\qquad [\pi_x^\rho(f)\,\xi](x, g) = \int_G \rho_{(x,h)}(f(\alpha_{h^{-1}}(x), h^{-1} g))\,\xi(x, h)\, dh$$
$$= \int_G [\hat{\rho}_h[f](h^{-1} g)](x)\,\xi(x, h)\, dh$$
$$= \{[\pi_x^{\hat{\rho}}(f)\,\xi](g)\}(x)\,,$$

for $\xi\in L^2(\Gamma, H)=L^2(G, L^2(X)\otimes H)$. In view of Example 2.4, these formulas agree with the defining relations (2.12), (2.13), (2.14) of C^*-crossed product $C_0(X, A)\times_{\hat{\rho}}G$ through the action (2.20). Hence we obtain the assertion by the density of $C_c(X, A)$ in $C_0(X, A)$. Q.E.D.

Remark 2.6. In the situation of (4), if ρ: $\Gamma\to\mathrm{Aut}(A)$ is of G-split type (see Remark 4.8 of [8]), then the action (2.20) of G on $C_0(X)\otimes A$ is of product type.

Now remember the definition of a locally compact transformation groupoid which is introduced in analogy with the skew product, see [8], §5.

Lemma 2.7. *Let* (Ω, Γ, ρ) *be a locally compact transformation groupoid and* $\tilde{\Gamma}=\Omega\times_\rho\Gamma$ *be the associated graph. Then,*

$$(2.34)\qquad C^*(\tilde{\Gamma}) = C_0(\Omega)\times_\rho\Gamma\,.$$

(*Note that* $C_0(\Omega)=C(\Omega)$ *if* Ω *is compact.*)

Proof. By definition, $C_0(\Omega)\times_\rho\Gamma$ is defined by the C^*-completion of the $*$-algebra $C_c(\Gamma, C_0(\Omega))$. By definition of the relevant C^*-norm, we may assume that $C_0(\Omega)\times_\rho\Gamma$ is generated by $C_c(\Omega\times\Gamma)=C_c(\Gamma,\ C_c(\Omega))\subset C_c(\Gamma,\ C_0(\Omega))$. In view of the definition of groupoid crossed product after Definition 2.1, the $*$-algebraic structure and the C^*-norm on $C_c(\Omega\times\Gamma)$ is

$$(2.35)\quad (f_1 \hat{*} f_2)(\omega, \gamma) = \int_{\Gamma^x} f_1(\rho_{\tilde{\gamma}^{-1}}(\omega), \tilde{\gamma}^{-1}\gamma) f_2(\omega, \tilde{\gamma})\, d\nu^x(\tilde{\gamma}), \quad x = r(\gamma),$$

$$(2.36)\quad f^{\hat{*}}(\omega, \gamma) = \overline{f(\rho_{\gamma^{-1}}(\omega), \gamma^{-1})},$$

$$(2.37)\quad \|f\| = \sup_{x\in\Gamma^{(0)}} \|\pi_x(f)\|, \quad f\in C_c(\Omega\times\Gamma),$$

$$(2.38)\quad [\pi_x(f)\,\xi](\omega, \gamma) = \int_{\Gamma^x} f(\rho_{\tilde{\gamma}^{-1}}(\omega), \tilde{\gamma}^{-1}\gamma)\,\xi(\omega, \tilde{\gamma})\, d\nu^x(\tilde{\gamma}),$$

where $\xi\in L^2(\Omega)\otimes L^2(\Gamma^x, \nu^x)$ with respect to a suitable measure on Ω. ($C_0(\Omega)$ is a concrete C^*-algebra on $L^2(\Omega)$.) In view of (2.38), $\|\pi_x(f)\| = \sup_{\omega\in\Omega} \|\pi_{(\omega,x)}(f)\|$, where

$$(2.39)\quad [\pi_{(\omega,x)}(f)\,\xi](\gamma) = \int_{\Gamma^x} f(\rho_{\tilde{\gamma}^{-1}}(\omega), \tilde{\gamma}^{-1}\gamma)\,\xi(\tilde{\gamma})\, d\nu^x(\tilde{\gamma}),$$

where $\xi\in L^2(\Gamma^x, \nu^x)$. Hence $\|f\| = \sup_{(\omega,x)\in\Omega\times\Gamma^{(0)}} \|\pi_{(\omega,x)}(f)\|$. These expressions agree with the definition of the $*$-algebraic structure and the C^*-norm of $C_c(\tilde{\Gamma})$, $\tilde{\Gamma}=\Omega\times_\rho\Gamma$. Q.E.D.

§3. Groupoid Crossed Product and Co-action

In this section, we shall discuss the co-action on a groupoid algebra by a locally compact group arising from a groupoid homomorphism. First, we recall the definition of co-action and the associated crossed product in the C^*-algebraic framework (see [3], [5], [6], [7], [9]). Let G be a locally compact group. The Kac-Takesaki operator W is a unitary operator on $L^2(G\times G)$ defined by

$$(3.1)\quad [W\xi](g, h) = \xi(g, gh), \quad \xi\in L^2(G\times G).$$

Then we define an isomorphism δ_G: $W_r^*(G)\to W_r^*(G)\otimes W_r^*(G)$ by

$$(3.2)\quad \delta_G(x) = W^*(x\otimes 1)\,W, \quad x\in W_r^*(G)$$

where $W_r^*(G)$ is the W^*-algebra generated by the left regular representation $\lambda(g)$ of $g\in G$.

Definition 3.1. The co-action of G on a C^*-algebra A is defined as the isomorphism

$$\delta: A \to \tilde{M}_L(A \otimes C_r^*(G)) \tag{3.3}$$

satisfying

$$(\delta \otimes 1) \circ \delta = (1 \otimes \delta_G) \circ \delta \tag{3.4}$$

where

$$\tilde{M}_L(A \otimes_{\min} B) = \{a \in M(A \otimes_{\min} B) : a(1 \otimes b) + (1 \otimes c)\, a \in A \otimes_{\min} B,\ L_\psi(a) \in A \text{ for } b, c \in B, \psi \in B^*\} \tag{3.5}$$

and L_ψ denotes the left slice mapping by ψ. The co-action δ is said to be non-degenerate if $\{L_\psi(\delta(a)) : a \in A, \psi \in A(G)\}$ generates A, where $A(G)$ is the Fourier algebra of G (see Theorem 5 of [6]). If this is the case, the co-crossed product C^*-algebra $A *_\delta G$ by the co-action δ is defined as the C^*-algebra generated by $(1 \otimes f)\, \delta(a)$ on $H \otimes L^2(G)$, $a \in A$, $f \in C_0(G)$, where H is a faithful representation Hilbert space of A.

Now, let Γ be a locally compact topological groupoid with a faithful transverse function $\nu = \{\nu^x\}_{x \in \Gamma^{(0)}}$ and G be a locally compact group.

Theorem 3.2. *Let $\rho: \Gamma \to G$ be a continuous homomorphism. Then there exists a continuous co-action $\hat{\rho}$ of G on $C^*(\Gamma)$ such that the associated co-crossed product $C^*(\Gamma) *_{\hat{\rho}} G$ is isomorphic to the groupoid crossed product $C_0(G) \times_\rho \Gamma$.*

Proof. By choosing a suitable faithful Borel measure μ on $\Gamma^{(0)}$, $C^*(\Gamma)$ is a concrete C^*-algebra acting on a Hilbert space $L^2(\Gamma, (\mu \circ \nu))$. The action of $f \in C_c(\Gamma)$ on $L^2(\Gamma, (\mu \circ \nu))$ is

$$[\pi(f)\, \xi]\, (\gamma) = \int_{\Gamma^x} f(\tilde{\gamma}^{-1} \gamma)\, \xi(\tilde{\gamma})\, d\nu^x(\tilde{\gamma}), \quad x = r(\gamma), \tag{3.6}$$

where $\xi \in L^2(\Gamma, (\mu \circ \nu))$. Now, we define a mapping $\hat{\rho}: C_c(\Gamma) \to B(L^2(\Gamma, (\mu \circ \nu)) \otimes L^2(G))$ by

$$[\hat{\rho}(f)\, \xi]\, (\gamma, g) = \int_{\Gamma^x} f(\tilde{\gamma}^{-1} \gamma)\, \xi(\tilde{\gamma}, \rho(\tilde{\gamma}^{-1}\gamma)\, g)\, d\nu^x(\tilde{\gamma}), \quad x = r(\gamma), \tag{3.7}$$

where $f \in C_c(\Gamma)$, $\xi \in L^2(\Gamma, (\mu \circ \nu)) \otimes L^2(G)$. Then $\hat{\rho}(f) = W_\rho(\pi(f) \otimes 1)\, W_\rho^*$ where W_ρ is a unitary operator (analogue of Kac-Takesaki operator) defined by

$$[W_\rho\, \xi]\, (\gamma, g) = \xi(\gamma, \rho(\gamma)\, g), \quad \xi \in L^2(\Gamma, (\mu \circ \nu)) \otimes L^2(G). \tag{3.8}$$

Hence $\hat{\rho}$ extends to a $*$-isomorphism $C^*(\Gamma) \to B(L^2(\Gamma, (\mu \circ \nu)) \otimes L^2(G))$ which is also denoted by $\hat{\rho}$. To see that equality (3.4) holds, let $W_\rho^{(j)}$, $j=1, 2$ be unitary operators on $L^2(\Gamma, (\mu\circ\nu))\otimes L^2(G\times G)$ defined by

$$[W_\rho^{(1)}\xi](\gamma, g, h) = \xi(\gamma, \rho(\gamma)g, h), \tag{3.9}$$

$$[W_\rho^{(2)}\xi](\gamma, g, h) = \xi(\gamma, g, \rho(\gamma)h), \tag{3.10}$$

$\xi\in L^2(\Gamma, (\mu\circ\nu))\otimes L^2(G\times G)$. Then $(\hat{\rho}\otimes 1)\circ\hat{\rho}(f)=W_\rho^{(1)}\, W_\rho^{(2)}(\pi(f)\otimes 1\otimes 1)\, W_\rho^{(2)*}\, W_\rho^{(1)*}$, $(1\otimes\delta_G)\circ\hat{\rho}(f)=(1\otimes W^*)\, W_\rho^{(1)}(\pi(f)\otimes 1\otimes 1)\, W_\rho^{(1)*}(1\otimes W)$. Equality (3.4) for $\delta=\hat{\rho}$ is obtained by the direct computation.

Let $f\in C_c(\Gamma)$ and $\phi\in C_c(G)$. Then

$$\begin{aligned}(3.11)\qquad &[\hat{\rho}(f)\,(1\otimes\lambda(\phi))\,\xi]\,(\gamma, g)\\ &=\int_{\Gamma^x} f(\tilde{\gamma}^{-1}\gamma)\,[(1\otimes\lambda(\phi))\,\xi]\,(\tilde{\gamma}, \rho(\tilde{\gamma}^{-1}\gamma)g)\,d\nu^x(\tilde{\gamma})\\ &=\int_{\Gamma^x}\left[\int_G f(\tilde{\gamma}^{-1}\gamma)\,\phi(h)\,\xi(\tilde{\gamma}, h^{-1}\rho(\tilde{\gamma}^{-1}\gamma)g)\,dh\right]d\nu^x(\tilde{\gamma})\\ &=\int_{\Gamma^x}\left[\int_G f(\tilde{\gamma}^{-1}\gamma)\,\phi(\rho(\tilde{\gamma}^{-1}\gamma)k)\,\xi(\tilde{\gamma}, k^{-1}g)\,dk\right]d\nu^x(\tilde{\gamma})\\ &=[(\pi\otimes\lambda)\,(f*\phi)\,\xi]\,(\gamma, g),\quad x=r(\gamma),\end{aligned}$$

where $\xi\in L^2(\Gamma, \mu\circ\nu)\otimes L^2(G)$, $f\in C_c(\Gamma)$ and $f*\phi\in C_c(\Gamma, C_c(G))$ is defined by

$$(f*\phi)\,(\gamma, g) = f(\gamma)\phi(\rho(\gamma)g). \tag{3.12}$$

Hence $\hat{\rho}(f)\ (1\otimes\lambda(\phi))=(\pi\otimes\lambda)\ (f*\phi)$. Similarly, $(1\otimes\lambda(\phi))\hat{\rho}(f)=(\pi\otimes\lambda)(\phi*f)$ where $\phi*f\in C_c(\Gamma, C_c(G))$ is defined by

$$(\phi*f)\,(\gamma, g) = f(\gamma)\Delta_G(\rho(\gamma))\phi(g\rho(\gamma)). \tag{3.13}$$

Hence $(1\otimes b)\hat{\rho}(a)+\hat{\rho}(a)(1\otimes c)\in C^*(\Gamma)\otimes C_r^*(G)$ for $a\in C^*(\Gamma)$, b, $c\in C_r^*(G)$.

Next, let $f\in C_c(\Gamma)$ and $\psi\in C_r^*(G)^*$. Then,

$$L_\psi(\hat{\rho}(f)) = \psi*f\in C_c(\Gamma), \tag{3.14}$$

where

$$(\psi*f)\,(\gamma) = \psi(\lambda(\rho(\gamma)^{-1}))\,f(\gamma). \tag{3.15}$$

This shows $L_\psi(\hat{\rho}(C^*(\Gamma)))\subset C^*(\Gamma)$. Therefore $\hat{\rho}(C^*(\Gamma))\subset\tilde{M}_L(C^*(\Gamma)\otimes C_r^*(G))$. By (3.15), the set $\{\psi*f\colon \psi\in C_r^*(G)^*, f\in C_c(\Gamma)\}$ exhausts $C_c(\Gamma)$. Hence $\{L_\psi(\hat{\rho}(f))\colon \psi\in C_r^*(G)^*, f\in C^*(\Gamma)\}$ generates $C^*(\Gamma)$ and the co-action is non-degenerate.

Lastly, we show $C^*(\Gamma)*_{\hat{\rho}}G\cong C_0(G)\times_\rho\Gamma$. By the definition of the co-crossed product, $C^*(\Gamma)*_{\hat{\rho}}G$ is the C^*-algebra generated by $(1\otimes\phi)\ \hat{\rho}(f)$,

$\phi\in C_0(G)$, $f\in C^*(\Gamma)$ and hence generated by $\hat{\rho}(f)(1\otimes\phi)$ by taking adjoint. By the density of $C_c(\Gamma)$ in $C^*(\Gamma)$, $C^*(\Gamma)*_{\hat{\rho}}G$ is generated by $\hat{\rho}(f)(1\otimes\phi)$, $f\in C_c(\Gamma)$, $\phi\in C_c(G)$. Now, we define an unitary operator $\tilde{W}_\rho$: $L^2(G)\otimes L^2(\Gamma, (\mu\circ\nu))\to L^2(\Gamma, (\mu\circ\nu))\otimes L^2(G)$ by

$$(3.16) \qquad [\tilde{W}_\rho\, \xi](\gamma, g) = \xi(\rho(\gamma)g, \gamma), \quad \xi\in L^2(G)\otimes L^2(\Gamma, (\mu\circ\nu)).$$

Then,

$$(3.17) \quad [\tilde{W}_\rho^*\hat{\rho}(f)(1\otimes\phi)\tilde{W}_\rho\, \xi](g, \gamma) = \int_{\Gamma^x} f(\tilde{\gamma}^{-1}\gamma)\, \phi(\rho(\tilde{\gamma})^{-1}g)\, \xi(g, \tilde{\gamma})\, d\nu^x(\tilde{\gamma}), \quad x = r(\gamma),$$

where $f\in C_c(\Gamma)$, $\phi\in C_c(G)$, $\xi\in L^2(G)\otimes L^2(\Gamma, (\mu\circ\nu))$. Now, in view of the definition of $C^*(\tilde{\Gamma})$, $\tilde{\Gamma}=G\times_\rho\Gamma$, the right hand side of (3.17) is equal to $[\tilde{\pi}(\tilde{f})\,\xi](g, \gamma)$, where $\tilde{\pi}$ is the representation of $\tilde{\Gamma}$ on $L^2(\tilde{\Gamma}, ((dg\otimes\mu)\circ\tilde{\nu}))=L^2(G)\otimes L^2(\Gamma, (\mu\circ\nu))$ and $\tilde{f}(g, \gamma)=f(\gamma)\,\phi(g)$. The linear combinations of such $\tilde{f}$ belong to $C_c(G)\otimes_{\mathrm{alg}}C_c(\Gamma)$ which is dense in $C_c(\tilde{\Gamma})$ with respect to the L^1-norm topology on $C_c(\tilde{\Gamma})$ defined by

$$(3.18) \qquad \|f\|_{L^1} = \max\{\sup_{x\in\tilde{\Gamma}^{(0)}}\int_{\tilde{\Gamma}^x}|f(\gamma)|\,d\tilde{\nu}^x(\gamma),\ \sup_{x\in\tilde{\Gamma}^{(0)}}\int_{\tilde{\Gamma}^x}|f(\gamma^{-1})|\,d\tilde{\nu}^x(\gamma)\},$$

$f\in C_c(\tilde{\Gamma})$ (see [10]). This implies the norm density of $\{\tilde{\pi}(\tilde{f})\colon \tilde{f}(g, \gamma)=f(\gamma)\,\phi(g), f\in C_c(\Gamma), \phi\in C_c(G)\}$ in $C^*(\tilde{\Gamma})$. Q.E.D.

Remark 3.3. If G is abelian, then we obtain a C^*-dynamical system $(C^*(\Gamma), \hat{G}, \hat{\rho})$ where the $\hat{G}$-action $\hat{\rho}$ is defined by the relation

$$(3.19) \qquad \hat{\rho}_k[f](\gamma) = \langle\rho(\gamma), k\rangle f(\gamma),\ k\in\hat{G}, f\in C_c(\Gamma).$$

By using L^1-norm, this action is shown to be continuous. The continuity of the action also follows from the non-degeneracy of $\hat{\rho}$ as a co-action, see [6].

Now, we shall give an explicit correspondence between $C^*(\tilde{\Gamma})$, $\tilde{\Gamma}=G\times_\rho\Gamma$ and $C^*(\Gamma)\times_{\hat{\rho}}\hat{G}$ for abelian G. By the definition of a crossed product (see Example 2.4) and the density of $C_c(\Gamma)$ in $C^*(\Gamma)$, $C^*(\Gamma)\times_{\hat{\rho}}\hat{G}$ is given by the C^*-completion of $C_c(\hat{G}\times\Gamma)$ with the $*$-algebraic structure

$$(3.20) \quad (f_1*f_2)(k, \gamma) = \int_{\hat{G}\times\Gamma^x}\langle\rho(\tilde{\gamma}^{-1}\gamma), l\rangle f_1(-l+k, \tilde{\gamma}^{-1}\gamma)\, f_2(l, \tilde{\gamma})\, dl d\nu^x(\tilde{\gamma}), \quad x = r(\gamma),$$

$$(3.21) \qquad f^{\sharp}(k, r) = \langle \rho(r), k \rangle \overline{f(-k, r^{-1})},$$

and with the C^*-norm given by the following regular representation

$$(3.22) \quad [\pi(f)\,\xi](k, r) = \int_{\hat{G}\times\Gamma^x} \langle \rho(\tilde{r}^{-1}r), l \rangle f(-l+k, \tilde{r}^{-1}r)\,\xi(l, \tilde{r})\, dl d\nu^x(\tilde{r}), \quad x = r(r),$$

where $f_1, f_2, f \in C_c(\hat{G}\times\Gamma)$ and $\xi \in L^2(\hat{G})\otimes L^2(\Gamma, (\mu\circ\nu))$. Now, we define twisted inverse Plancherel transformations as follows:

$$(3.23) \qquad [Ff](g, r) = \int_{\hat{G}} \langle -\rho(r)+g, k \rangle f(k, r)\, dk, \quad f \in C_c(\hat{G}\times\Gamma),$$

$$(3.24) \quad [\tilde{F}\xi](g, r) = \int_{\hat{G}} \langle -\rho(r)+g, k \rangle \xi(k, r)\, dk, \quad \xi \in L^2(\hat{G})\otimes L^2(\Gamma, (\mu\circ\nu)).$$

Then, $F(C_c(\hat{G}\times\Gamma)) \subset C_c(\Gamma, C_0(G))$ (the image is dense in L^1-norm) and further F gives a $*$-homomorphism, where $C_c(\Gamma, C_0(G))$ is a $*$-algebra by

$$(3.25) \qquad (f_1 * f_2)(g, r) = \int_{\Gamma^x} f_1(-\rho(\tilde{r})+g, \tilde{r}^{-1}r) f_2(g, \tilde{r})\, d\nu^x(\tilde{r}), \quad x = r(r),$$

$$(3.26) \qquad f^{\sharp}(g, r) = \overline{f(-\rho(r)+g, r^{-1})},$$

where $f_1, f_2, f \in C_c(\Gamma, C_0(G))$. It also holds that the unitary operator $\tilde{F}$ defined by (3.24) intertwines the $*$-representations of $C_c(\hat{G}\times\Gamma)$ and $C_c(\Gamma, C_0(G))$ defined by (3.22) and

$$(3.27) \qquad [\tilde{\pi}(f)\,\xi](g, r) = \int_{\Gamma^x} f(-\rho(\tilde{r})+g, r^{-1}r)\,\xi(g, \tilde{r})\, d\nu^x(\tilde{r}), \quad x = r(r),$$

where $f \in C_c(\Gamma, C_0(G))$ and $\xi \in L^2(G)\otimes L^2(\Gamma, (\mu\circ\nu))$. On the other hand, the operations (3.25), (3.26), (3.27) agree with the definition of $C_0(G)\times_{\rho}\Gamma$. Hence the mapping F defined by (3.23) gives the concrete isomorphism $C^*(\Gamma)\times_{\hat{\rho}}\hat{G} \to C_0(G)\times_{\rho}\Gamma$.

Remark 3.4. If we consider the case that Γ is a locally compact abelian group, then $C_0(G)\times_{\rho}\Gamma \cong C^*(\Gamma)\times_{\hat{\rho}}\hat{G} \cong C_0(\hat{\Gamma})\times_{\hat{\rho}}G$. This duality can be viewed as the Plancherel transformation of abelian groupoid, see Bellissard-Testard [1] (see also Remark 6.2 (3) of [8]).

Proposition 3.5. *Assume that G is abelian. If ρ, σ: $\Gamma \to G$ are cohomologous in the sense that there exists a continuous mapping τ: $\Gamma^{(0)} \to G$ such that $\rho(r)$*

$=\tau(r(\gamma))\ \sigma(\gamma)\ \tau(s\ (\gamma))^{-1}$, *then the two* $\hat{G}$*-actions* $\hat{\rho}$, $\hat{\sigma}$ *are one-cocycle equivalent i.e. there exists a unitary valued mapping* u: $\hat{G}\to M(C^*(\Gamma))$ *such that* $k\to u_k a$ *and* $k\mapsto au_k$ *are continuous for* $a\in C^*(\Gamma)$ *and*

$$\hat{\rho}_k(a) = u_k \hat{\sigma}_k(a) u_k^*, \quad a\in C^*(\Gamma)\,, \tag{3.28}$$

$$u_{k+l} = u_k \hat{\sigma}_k(u_l)\,. \tag{3.29}$$

Proof. The unitary operator u_k on $L^2(\Gamma, (\mu\circ\nu))$ defined by

$$[u_k\ \xi]\ (\gamma) = \langle \tau\ (r\ (\gamma)),\ k\rangle \xi(\gamma), \quad \xi\in L^2(\Gamma,\ (\mu\circ\nu)) \tag{3.30}$$

satisfies the condition. Q.E.D.

Acknowledgements

The author would like to express his hearty thanks to Professor H. Araki for his kind advise, invaluable discussions and criticisms, and careful inspection of the manuscript. The author also would like to thank Dr. Y. Katayama for various discussions.

References

[1] Bellissard, J. and Testard, D., Almost periodic Hamiltonians: an algebraic approach, *preprint* 1981.

[2] Connes, A., Sur la theorie non commutative de l'integration, *Lecture Notes in Math.*, **725** (1979), Springer, 19–143.

[3] Imai, S. and Takai, H., On a duality for C^*-crossed product by a locally compact group, *J. Math. Soc. Japan*, **30** (1978), 495–504.

[4] Kastler, D., On A. Connes' non commutative integration theory, *Comm. Math. Phys.*, **85** (1982), 99–120.

[5] Katayama, Y., Remarks on a C^*-dynamical system, *Math. Scand.*, **49** (1981), 250–258.

[6] ———, Takesaki's duality for a non-degenerate co-action, *preprint*.

[7] Landstad, M.B., Duality theory for covariant systems, *Trans. Amer. Math. Soc.*, **248** (1979), 223–267.

[8] Masuda, T., Groupoid dynamical systems and crossed product I–The case of W^*-systems, *Publ. RIMS, Kyoto Univ.*, **20** (1984), 929–957.

[9] Nakagami, Y. and Takesaki, M., Duality for crossed products of von Neumann algebras, *Lecture Notes in Math.*, **731** (1979), Springer.

[10] Renault, J., A groupoid approach to C^*-algebras, *Lecture Notes in Math.*, **793** (1980), Springer.