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Stable Self Maps of the Quaternionic (Quasi-) Projective Space

Dedicated to Professor N. Shimada for his 60th birthday

By

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§1. Introduction

Let HP^n (resp. CP^n) be the quaternionic (resp. complex) n-dimensional projective space. QP^n denotes the quaternionic quasi-projective space of dimension $4n-1$. For a pointed space X we denote the n-th reduced suspension by $\Sigma^n X$ and denote the associated suspension spectrum by $\Sigma^\infty X$. We denote the group of stable homotopy classes of stable maps from $\Sigma^\infty X$ to $\Sigma^\infty Y$ by $[X, Y]^s$. $\pi_*^s(X)$ denotes the stable homotopy group of X. Throughout this paper we shall denote the homotopy class of a map f by the same letter f for abbreviation.

Our results are summarized by the following theorem:

Theorem A. *Let $X=HP^\infty$ or ΣQP^∞ and $n\geqq 0$. Let $h\colon \pi_{4n+4}^s(X)\to H_{4n+4}(X; Z)$ be the stable Hurewicz homomorphism of X. Then for any $x\in \mathrm{Im}\, h$, there exists a stable self map $f_x\colon \Sigma^{4n}X\to X$ such that $h(f_x\circ i)=x$, where $i\colon S^{4n+4}\to \Sigma^{4n}X$ is the inclusion map of the bottom sphere.*

For the CP^∞ case the above theorem is classically known [9].

In order to state our results more precisely, first we recall that

$$\tilde{H}_*(CP^\infty; Z)\cong Z\{\alpha_1, \alpha_2, \cdots\}\ ,$$

$$\tilde{H}_*(HP^\infty; Z)\cong Z\{\beta_1, \beta_2, \cdots\}\ ,$$

where α_i (resp. β_i) is a standard generator of $H_{2i}(CP^\infty; Z)$ (resp. $H_{4i}(HP^\infty; Z)$)

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$\cong Z$. Let $q\colon CP^{\infty}\to HP^{\infty}$ be the canonical map. Then we can choose α_i and β_i such that $q_*\alpha_{2i}=\beta_i$.

Our results are as follows.

Theorem 1. *Let $n\geqq 0$ and $s\geqq 1$. Then there is a stable map $f(n, s)\colon \Sigma^{4n}HP^{\infty}\to HP^{\infty}$ such that*

$$f(n, s)_*\beta_k = a(n+s-1)\,((2k+2n)!/(2k)!)\,\Big(\sum_{i=0}^{s-1}(-1)^i\binom{2s}{i}(s-i)^{2k}\Big)\beta_{n+k},$$

where $a(i)=1$ if i is even and $a(i)=2$ if i is odd. Moreover $\{f(n, s)_*\}_{s\geqq 1}$ forms a basis of the image from $[\Sigma^{4n}HP^{\infty}, HP^{\infty}]^s$ to Hom$(H_*(HP^{\infty}; Z), H_{4n+*}(HP^{\infty}; Z))$.

Corollary 2 (cf. [7], [6]). *Let $h\colon \pi_{4n}^s(HP^{\infty})\to H_{4n}(HP^{\infty}; Z)$ be the stable Hurewicz homomorphism of HP^{∞}. Then* Image *h is generated by $f(n-1, 1)_*\beta_1 =((2n)!/a(n))\beta_n$.*

Remark. It is already known that Im h is generated by $((2n)!/a(n))\beta_n$ [7], [6]. Our result is that these come from maps from $\Sigma^{4n}HP^{\infty}$ to HP^{∞}. Moreover such maps can be chosen as follows. Let $f=f(2, 1)$ and $f'=f(1, 1)$. Then if n is odd, we can take the $((n-1)/2)$-fold iterated composition of f, and if n is even, we can take the composite of the map f' with the $((n-2)/2)$-fold iterated composition of f.

Let QP^n be the quaternionic quasi-projective space of dimension $4n-1$. Then using the result of Kono [5], we have

Theorem 3. *There is a stable map $g(n, s)\colon \Sigma^{4n}QP^{\infty}\to QP^{\infty}$ such that*

$$g(n, s)_*\gamma_k = a(n+s-1)\,((2n+2k-1)!/(2k-1)!)\,\Big(\sum_{i=0}^{s-1}(-1)^i\binom{s}{i}(s-i)^{2k-1}\Big)\gamma_{n+k},$$

where γ_k is a standard generator of $H_{4k-1}(QP^{\infty}; Z)\cong Z$.

Corollary 4 (cf. [10]). *$g(n-1, 1)_*\gamma_1=a(n-1)((2n-1)!)\gamma_n$ generates the image of the stable Hurewicz homomorphism of QP^{∞}.*

Remark. It may be known to experts that Im h is generated by $a(n-1)((2n-1)!)\gamma_n$. Our result is that these come from maps from $\Sigma^{4n}QP^{\infty}$ to QP^{∞}. Moreover such maps can be chosen as follows. Let $g=g(2, 1)$ and $g'=g(1, 1)$. Then if n is odd, we can take the $((n-1)/2)$-fold iterated composition of g, and if n is even, we can take the composite of the map g' with the $((n-2)/2)$-fold iterated composition of g.

Let ξ_k be the canonical quaternionic line bundle over HP^{k-1}. Then using S-duality we have

Theorem 5. *There is a stable map between Thom complexes*

$$h(n, s): \Sigma^{4n}(HP^{n+k-1})^{-(n+k)\xi_{n+k}} \to (HP^{k-1})^{-k\xi_k},$$

such that

$$h(n, s)_*\beta_l$$
$$= a(n+s-1)\,((2n-2l-1)!/(2k-1)!)\,(\sum_{i=0}^{s-1}(-1)^i\binom{s}{i}(s-i)^{-2l-1})\beta_l,$$

where as spectra,

$$(HP^{k-1})^{-k\xi_k} = S^{-4k}\cup e^{-4k+4}\cup\cdots\cup e^{-4},$$

$$\Sigma^{4n}(HP^{n+k-1})^{-(n+k)\xi_{n+k}} = S^{-4k}\cup e^{-4k+4}\cup\cdots\cup e^{-4}\cup\cdots\cup e^{4n-4},$$

and β_l is corresponding to the $4l$-dimensional cell.

The applications of the results in this paper will appear in [3].

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§2. The Segal-Becker Theorem and the Construction of $f(n, s)$

Let BSp (resp. BU) be the classifying space of stable quaternionic (resp. complex) vector bundles. Let $j: HP^\infty \to BSp$ (resp. $j_c: CP^\infty \to BU$) be the classifying map of the canonical quaternionic (resp. complex) line bundle ξ (resp η). Recall that $H_*(BSp; Z)\cong Z[\beta_1, \beta_2, \cdots]$ and $H_*(BU; Z)\cong Z[\alpha_1, \alpha_2, \cdots]$, where we abbreviate $j_*\beta_i$ by β_i and similarly $j_{c*}\alpha_i$ by α_i. Let $\bar{j}: \Omega^\infty\Sigma^\infty HP^\infty \to BSp$ be the canonical extension of j which is induced by the infinite loop structure of BSp. Then Segal [8] and Becker [2] constructed a map $\tau: BSp \to \Omega^\infty\Sigma^\infty HP^\infty$ such that $\bar{j}\circ\tau = \mathrm{id}_{BSp}$ and in rational homology $\tau_*\circ\bar{j}_* = \mathrm{id}$ of $H_*(\Omega^\infty\Sigma^\infty HP^\infty; Q)$. The complex case is quite similar. We denote the splitting map by $\tau_c: BU \to \Omega^\infty\Sigma^\infty CP^\infty$.

Lemma 2.1. *Let $\tilde{\tau}: \Sigma^\infty BSp \to \Sigma^\infty HP^\infty$ be the adjoint map of τ. Then $\tilde{\tau}_*\beta_n = \beta_n$ and $\tilde{\tau}_*$*(decomp.)$=0$.

Proof. It is enough to show in rational homology. Then it is easy to prove the above Lemma by using the Segal-Becker theorem and the definition of the adjoint map. Q.E.D.

Let $K(X)$ (resp. $KSp(X)$) be the reduced complex K-theory (resp. the reduced symplectic K-theory). The following lemma is well-known:

Lemma 2.2. *Let $c': KSp(\Sigma^{4n}HP^\infty)\to K(\Sigma^{4n}HP^\infty)$ be the complexification homomorphism. Then c' is monic and the image of c' is a free abelian group generated by $a(n+s-1)t^{2n}z^s$ for $s\geqq 1$, where $a(i)$ is 2 if i is odd and is 1 if i is even, $t\in K(S^2)$ is a standard generator, $t^n\in K(S^{4n})$, $z=c'(\xi)-2\in K(HP^\infty)$ and $z^n\in K(HP^\infty)$.*

Now we shall construct the map $f(n, s)$. Define $f(n, s)$ as the adjoint of the following composite;

$$\Sigma^{4n}HP^\infty \xrightarrow{f'(n,s)} BSp \xrightarrow{\tau} \Omega^\infty\Sigma^\infty HP^\infty ,$$

where $f'(n, s)$ is the map corresponding to $a(n+s-1)t^{2n}z^s$ under the complexification homomorphism c'.

§3. Proofs

Proof of Theorem 1. From Lemmas 2.1 and 2.2 it easily follows that the image from $[\Sigma^{4n}HP^\infty, HP^\infty]^s$ to $\mathrm{Hom}(H_*(HP^\infty; Z), H_{4n+*}(HP^\infty; Z))$ is generated by $\{f(n, s)_*\}_{s\geq 1}$ (cf. [6]). Let $C^{2i}\in H^{4i}(BU; Z)$ and $P^i\in H^{4i}(BSp; Z)$ be the dual of α_{2i} and β_i respectively. Then as is well-known $c'^*C^{2i}=2P^i$. Let $f(n, s)_*\beta_k=m\beta_{n+k}$ for some integer m. Then by Lemma 2.1 and the Kronecker pairing,

$$\begin{aligned} m &= \langle f'(n, s)_*\beta_k, P^{n+k}\rangle = \langle\beta_k, f'(n, s)^*P^{n+k}\rangle \\ &= (1/2)\langle\beta_k, C^{2n+2k}(a(n+s-1)t^{2n}z^s)\rangle . \end{aligned}$$

Now

$$\begin{aligned} C^{2n+2k}(t^{2n}z^s) &= (2k+2n)!\; ch_{2n+2k}(t^{2n}z^s) \\ &= (2k+2n)!\; ch_{2n}(t^{2n})ch_{2k}(z^s) \\ &= (2k+2n)!\; ch_{2k}(z^s) \\ &= ((2k+2n)!/(2k)!)2\Big(\sum_{i=0}^{s-1}(-1)^i\binom{2s}{i}(s-i)^{2k}\Big)x^k , \end{aligned}$$

where $ch: K(\ \ )\to H^{ev}(\ \ ; Q)$ is the Chern character, ch_{2n} is the $2n$-component of ch and $x^k\in H^{4k}(HP^\infty; Z)$ is a generator dual to β_k. Here the above last equation is easily verified (cf. [10]). Since $ch_{2k}(z^s)=0$ for $s>k$ and $ch_{2k}(z^k)$

$=x^k$, so $f(n, s)_*$, $s\geqq 1$, are linearly independent. Therefore we have proved Theorem 1.

Proof of Corollary 2. It is obvious from Theorem 1 that $f(n-1, 1)_*\beta_1 =((2n)!/a(n))\beta_n$ belongs to the image of h. Conversely by standard arguments of K-theory and Chern character it is easy to show that if $m\beta_n\in \mathrm{Im}\, h$ for some integer m then $(2n)!/a(n)$ divides m (for example see, [7]).

Proof of Theorem 3. Consider the following diagram;

$$\begin{array}{ccccc}
\Sigma^{4n}CP^\infty & \xrightarrow{a(n+s-1)t^{2n}(\eta-1)^s} & BU & \xrightarrow{\tau_c} & \Omega^\infty\Sigma^\infty CP^\infty \\
\downarrow{\scriptstyle \Sigma^{4n}q} & \text{(I)} & \downarrow{\scriptstyle q} & \text{(II)} & \downarrow{\scriptstyle \Omega^\infty\Sigma^\infty q} \\
\Sigma^{4n}HP^\infty & \xrightarrow{f'(n, s)} & BSp & \xrightarrow{\tau} & \Omega^\infty\Sigma^\infty HP^\infty \\
a(n+s-1)t^{2n}z^s & \searrow & \downarrow{\scriptstyle c'} & & \\
 & & BU & &
\end{array}$$

The commutativity of the diagram (I) easily follows from the fact that c': $KSp(\Sigma^{4n}CP^\infty)\to K(\Sigma^{4n}CP^\infty)$ is monic because of the freeness of $KSp(\Sigma^{4n}CP^\infty)$. The commutativity of the diagram (II) follows from [5]. Therefore the above diagram commutes. Thus taking adjoints we have the following commutative diagram in the stable category:

$$\begin{array}{ccc}
\Sigma^{4n}CP^\infty & \xrightarrow{f_c(n, s)} & CP^\infty \\
\downarrow{\scriptstyle \Sigma^{4n}q} & & \downarrow{\scriptstyle q} \\
\Sigma^{4n}HP^\infty & \xrightarrow{f(n, s)} & HP^\infty
\end{array}$$

As is well-known [4], there is a cofibering;

$$CP^\infty \to HP^\infty \to QP^\infty ,$$

thus extending the above diagram in the vertical direction we have a map $g(n, s)$: $\Sigma^{4n}QP^\infty\to QP^\infty$ such that the following diagram commutes up to sign in the stable category.

$$\begin{array}{ccc}
\Sigma^{4n}HP^\infty & \xrightarrow{f(n, s)} & HP^\infty \\
\downarrow & & \downarrow \\
\Sigma^{4n}QP^\infty & \xrightarrow{g(n, s)} & QP^\infty \\
\downarrow & & \downarrow \\
\Sigma^{4n+1}CP^\infty & \xrightarrow{f_c(n, s)} & \Sigma CP^\infty
\end{array}$$

By the similar argument of the proof of Theorem 1 it is easy to show that

$$f_c(n, s)_*\alpha_k = a(n+s-1)\ ((2n+k)!/k!)\ (\sum_{i=0}^{s-1}(-1)^i\binom{s}{i}(s-i)^k)\alpha_{2n+k}\ .$$

Since $H_{4k-1}(QP^\infty; Z)\cong H_{4k-2}(CP^\infty; Z)$, the rest of the proof is clear. Thus we have the desired result.

Proof of Corollary 4. It is obvious from Theorem 3 that $g(n-1,\ 1)_*\gamma_1 = a(n-1)((2n-1)!)\gamma_n$ belongs to the image of h. Conversely if $m\gamma_n \in \mathrm{Im}\ h$ for some integer m then from Theorem 0.2 in [10] it holds that $a(n-1)\ ((2n-1)!)$ divides m.

Proof of Theorem 5. This theorem easily follows by the well-known S-duality theorem [1] and our Theorem 3. By the cellular approximation and Theorem 3 we have a map

$$g(n, s)\colon \Sigma^{4n}QP^k \to QP^{n+k}\ .$$

Since QP^k is S-dual to the stable Thom complex $(HP^{k-1})^{-k\xi_k}$, we have a map

$$h(n, s)\colon \Sigma^{4n}(HP^{n+k-1})^{-(n+k)\xi_{n+k}} \to (HP^{k-1})^{-k\xi_k}\ .$$

Now the rest of the proof is easily verified.

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