Publ. RIMS, Kyoto Univ.
20 (1984), 977–1005

Coarse Moduli Space for Polarized Compact Kähler Manifolds

Dedicated to Professor S. Nakano on his 60th birthday

By

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Abstract

Using the results of our previous papers [5] [8] [9] we shall construct the coarse moduli space for non-uniruled polarized compact Kähler manifolds as a separated complex space.

§1. Statement of Result

(1.1) A *polarized compact Kähler manifold* is by definition a pair (X, ω) consisting of a connected compact complex manifold X and a Kähler class $\omega \in H^2(X, \mathbf{R})$ on X. Here a Kähler class is a class represented by a Kähler form, i.e., the fundamental 2-form associated with a Kähler metric on X. An isomorphism of two polarized compact Kähler manifolds (X, ω) and (X', ω') is by definition an analytic isomorphism $\psi: X \to X'$ with $\psi^* \omega' = \omega$.

Definition. i) A *polarized family of compact Kähler manifolds* (parametrized by a complex space S) is a pair $(f, \tilde{\omega})$ consisting of a proper smooth morphism $f: \mathscr{X} \to S$ of complex spaces with connected fibers and an element $\tilde{\omega} \in \Gamma(S, R^2 f_* \mathbf{R})$ such that

(1) $\tilde{\omega}$ induces on each fiber $X_s := f^{-1}(s)$ a Kähler class $\tilde{\omega}_s \in H^2(X_s, \mathbf{R})$, and

(2) $\eta(\tilde{\omega}) = 0$, where $\eta = \eta_f: \Gamma(S, R^2 f_* \mathbf{R}) \to \Gamma(S, R^2 f_* \mathcal{O}_{\mathscr{X}})$ is the homomorphism induced by the natural inclusion $\mathbf{R} \to \mathcal{O}_{\mathscr{X}}$ of sheaves on $\mathscr{X}$. ((2) is a consequence of (1) when S is reduced (cf. (2.3) and Lemma 1 below).) When no confusion may arise we often call $(f, \tilde{\omega})$ simply a polarized family.

ii) An isomorphism of two polarized families $(f: \mathscr{X} \to S, \tilde{\omega})$ and $(f': \mathscr{X}'$

Communicated by S. Nakano, September 26, 1983. Revised February 22, 1984.
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$\to S, \tilde{\omega}')$ is an S-isomorphism $\psi\colon \mathcal{X}\to\mathcal{X}'$ of complex spaces such that $\psi^*\tilde{\omega}' =\tilde{\omega}$ in $\Gamma(S, R^2f_*\mathbb{R})$.

(1.2) We denote by $\mathfrak{M}$ the set of isomorphism classes of polarized compact Kähler manifolds. For the simplicity of notation, however, we usually write $(X, \omega)\in\mathfrak{M}$, identifying a polarized compact Kähler manifold (X, ω) with its isomorphism class.

Let $(f\colon \mathcal{X}\to S, \tilde{\omega})$ be a polarized family of compact Kähler manifolds. Then for each $s\in S$ $(X_s, \tilde{\omega}_s)$ is a polarized compact Kähler manifold. Thus associating with each $s\in S$ the isomorphism class of $(X_s, \tilde{\omega}_s)$ we obtain a natural map $\rho(f, \tilde{\omega})\colon |S|\to\mathfrak{M}$ where $|S|$ denotes the underlying topological space of S. We call $\rho(f, \tilde{\omega})$ the *canonical map associated with* $(f, \tilde{\omega})$.

The *canonical topology* of $\mathfrak{M}$ is by definition the finest topology for which $\rho(f, \tilde{\omega})$ is continuous for any polarized family $(f, \tilde{\omega})$ as above; thus a subset $U\subseteq\mathfrak{M}$ is open if and only if $\rho(f, \tilde{\omega})^{-1}(U)$ is open in $|S|$ for any $(f, \tilde{\omega})$.

(1.3) Let $(f\colon \mathcal{X}\to S, \tilde{\omega})$ be a polarized family of compact Kähler manifolds. Let $\nu\colon T\to S$ be any morphism of complex spaces. We shall then define the pull-back $(f_T, \tilde{\omega}_T)$ of $(f, \tilde{\omega})$ by ν. Let $f_T\colon \mathcal{X}\times_S T\to T$ be the induced morphism. Set $\tilde{\omega}_T=\nu^*\tilde{\omega}$, where $\nu^*\tilde{\omega}$ is considered as an element of $\Gamma(T, R^2f_{T*}\mathbb{R})$ via the natural isomorphism $\nu^*R^2f_*\mathbb{R}\cong R^2f_{T*}\mathbb{R}$. Then the pair $(f_T, \tilde{\omega}_T)$ is again a polarized family of compact Kähler manifolds; indeed, (1) is clear and (2) follows from the naturality of η_f (cf. (2.3) below). We call $(f_T, \tilde{\omega}_T)$ the *pull-back* of $(f, \tilde{\omega})$ by ν.

Let $\mathfrak{F}\subseteq\mathfrak{M}$ be a subset. Then a polarized family $(f, \tilde{\omega})$ as above is said to be a *(polarized)* $\mathfrak{F}$*-family* if $\rho(f, \tilde{\omega})(|S|)\subseteq\mathfrak{F}$. It is clear that if $(f, \tilde{\omega})$ is an $\mathfrak{F}$-family, then $(f_T, \tilde{\omega}_T)$ also is an $\mathfrak{F}$-family.

Let An be the category of complex spaces. Then with the definition of the pull-back as above, for any subset $\mathfrak{F}\subseteq\mathfrak{M}$ we define the contravariant functor $\mathcal{F}\colon \mathrm{An}\to\mathrm{Sets}$ of An into the category of sets by

$\mathcal{F}(S)=$the set of isomorphism classes of polarized $\mathfrak{F}$-families $(f\colon \mathcal{X}\to S, \tilde{\omega})$ parametrized by S.

Definition. Let $\mathfrak{F}\subseteq\mathfrak{M}$ be a subset. Then the *coarse* $\mathfrak{F}$*-moduli space* is a pair (F, ψ) consisting of a complex space F and a morphism of functors $\psi\colon \mathcal{F}\to h_F:=\mathrm{Hom}(\ \ , F)$ of $\mathcal{F}$ into the representable functor h_F such that 1) the underlying topological space of F is canonically homeomorphic to $\mathfrak{F}$ where the topology of $\mathfrak{F}$ is induced from the canonical topology of $\mathfrak{M}$, and 2) for any complex space F_1 and a morphism of functors $\psi_1\colon \mathcal{F}\to h_{F_1}:=\mathrm{Hom}(\ \ , F_1)$

there exists a unique morphism $\tau: F \to F_1$ of complex spaces such that $\psi_1 = h_\tau \psi$.

Note that the canonicity in 1) means that the natural map $\mathfrak{F} \to F$ defined by ψ is homeomorphic.

(1.4) Let X be a compact complex manifold. X is said to be *uniruled* if there exist a compact complex manifold Y, a holomorphic vector bundle E of rank $\geqq 2$ on Y and a generically surjective meromorphic map $\pi: \mathbb{P}(E) \to X$ which is not factored by the projection $\mathbb{P}(E) \to Y$, where $\mathbb{P}(E)$ is the projective bundle associated to E (cf. [8]). Then our main result is stated as follows.

Theorem. *Let* $\mathfrak{A} = \{(X, \omega) \in \mathfrak{M};\ X$ *is not uniruled*$\}$. *Then the coarse* $\mathfrak{A}$*-moduli space* (A, ψ) *exists.*

Remark 1. Actually, in the course of the proof we specify a subset $\mathfrak{B} \subseteqq \mathfrak{M}$ containing $\mathfrak{A}$ such that the coarse $\mathfrak{B}$-moduli space (B, ψ_B) exists except that B is not separated (cf. Theorem 2).

Our proof of theorem follows in main line the method of Narasimhan-Simha [24] in which they have shown the existence of the coarse $\mathfrak{K}$-moduli space (in the reduced category), where $\mathfrak{K} = \{(X, \omega) \in \mathfrak{M}:\ \omega = -c_1^{\mathbb{R}}(X)\}$, $c_1^{\mathbb{R}}(X)$ being the real first chern class of X.

The arrangement of this paper is as follows. Section 2 is preliminary. In Section 3 we give a kählerian analogue of the separation criterion of Matsusaka-Mumford [22] essentially proved in [9]. Let $\mathfrak{T} = \{(X, \omega) \in \mathfrak{M};\ \mathrm{Aut}_0 X$ is a complex torus$\}$. Then in Section 4 we show the cohomological flatness of the relative tangent sheaf for any polarized $\mathfrak{T}$-family (Theorem 1). We construct the local modular family for any (X, ω) in $\mathfrak{T}$ and study the local structure of $\mathfrak{T}$ in Section 5. Then in Section 6 we define the subspace $\mathfrak{B} \subseteqq \mathfrak{M}$ and prove Theorem 2 mentioned above using the results of Section 5. Together with the main result of [8] the proof of Theorem is immediate from Theorem 2 (Section 7). Finally in Section 8, as concrete examples we shall give an explicit description of the moduli spaces for complex tori and K3 surfaces by summarizing the known results in these cases.

The results of this paper were announced in [7] in a somewhat weaker form. In the subsequent paper [12] we shall construct the coarse moduli space for non-uniruled polarized algebraic manifolds as a separated algebraic space.

Our thanks is due to the referee for a simplification of the proof of Proposition 5.

§2. Preliminaries

(2.1) *a*) Let S be a complex space. Let $G \to S$ be a morphism of complex spaces. Then G is called a *complex Lie group over* S if there exist S-morphisms $G \times_S G \to G$, $G \to G$, $S \to G$ defining relative multiplication, inversion, and the identity section respectively satisfying the usual axioms (cf. [23, Def. 0.1] or [10, §1]). Let $f: \mathcal{X} \to S$ be a morphism of complex spaces. Then a relative action of G on $\mathcal{X}$ over S is an S-morphism σ: $G \times_S \mathcal{X} \to \mathcal{X}$ satisfying the usual axiom of actions (cf. [23, Def. 0.3]).

b) Let $f: \mathcal{X} \to S$ and $f': \mathcal{X}' \to S$ be proper smooth morphisms of complex spaces. Let $D_{\mathcal{X} \times_S \mathcal{X}'/S} \to S$ be the relative Douady space associated with $f \times_S f'$: $\mathcal{X} \times_S \mathcal{X}' \to S$. Let $\mathrm{Isom}_S(\mathcal{X}, \mathcal{X}')$ be the Zariski open subset of $D_{\mathcal{X} \times_S \mathcal{X}'/S}$ which represents the functor $\mathcal{I}$: $(\mathrm{An}/S)^\circ \to$ Sets defined by $\mathcal{I}(T)$=the set of T-isomorphisms ψ: $\mathcal{X} \times_S T \to \mathcal{X}' \times_S T$, where An/$S$ is the category of complex spaces over S (cf. [26]).

We set $\mathrm{Aut}\, \mathcal{X}/S = \mathrm{Isom}_S(\mathcal{X}, \mathcal{X})$. $\mathrm{Aut}\, \mathcal{X}/S$ has the natural structure of a complex Lie group over S with a natural relative action on $\mathcal{X}$. In the absolute case, i.e., when S is a point, $\mathrm{Aut}\, \mathcal{X}/S$ reduces to the usual complex Lie group $\mathrm{Aut}\, X$ of biholomorphic automorphisms of $X = \mathcal{X}$. We denote by $\mathrm{Aut}_0 X$ the identity component of $\mathrm{Aut}\, X$.

c) Let $(f: \mathcal{X} \to S, \tilde{\omega})$ and $(f': \mathcal{X}' \to S, \tilde{\omega}')$ be polarized families of compact Kähler manifolds. Then define the functor $\mathcal{I}^\omega$: $(\mathrm{An}/S)^\circ \to$ Sets by

$\mathcal{I}^\omega(T)$=the set of isomorphisms of the pull-backs $(f_T, \tilde{\omega}_T)$ and $(f'_T, \tilde{\omega}'_T)$ as polarized families.

Then $\mathcal{I}^\omega$ is represented by an open and closed subset $\mathrm{Isom}_S((\mathcal{X}, \tilde{\omega}), (\mathcal{X}', \tilde{\omega}'))$ of $\mathrm{Isom}_S(\mathcal{X}, \mathcal{X}')$ (cf. [10, 3.2] up to a change of notation). In particular the closure I^- of $\mathrm{Isom}_S((\mathcal{X}, \tilde{\omega}), (\mathcal{X}', \tilde{\omega}'))$ in $D_{\mathcal{X} \times_S \mathcal{X}'/S}$ is analytic in $D_{\mathcal{X} \times_S \mathcal{X}'/S}$.

In the absolute case we use the notations $\mathrm{Isom}((X, \omega), (X', \omega'))$, $\mathrm{Aut}(X, \omega)$ etc. For instance

$$\mathrm{Aut}(X, \omega) = \{g \in \mathrm{Aut}\, X;\, g^*\omega = \omega\} .$$

(2.2) (Grothendieck's criterion for smoothness [16, IV, Th. 3.1]). Let f: $\mathcal{X} \to S$ be a morphism of complex spaces. Let $x \in \mathcal{X}$ and $s = f(x)$. Let A be a local $\boldsymbol{C}$-algebra which is a finite $\mathcal{O}_{S,s}$-algebra. Let m be the maximal ideal of A and I an ideal of A with $mI=0$. Let S_1=Specan A and S_2 the subspace of S_1 defined by I. Let v_2: $S_2 \to \mathcal{X}$ be an S-morphism with $v_2(t)=x$, where t is the unique point of S_1. Then f is smooth at x if and only if for any S_1, S_2, v_2 as above we can always find an extension v_1: $S_1 \to \mathcal{X}$ of v_2.

(2.3) The next result is due to Deligne [3].

Proposition 1. *Let $f: \mathcal{X} \to S$ be a proper smooth morphism of complex spaces with connected fibers. Suppose that X_o is Kähler for some $o \in S$. Then for any $i \geqq 0$ and any morphism $\nu: T \to S$ of complex spaces the natural homomorphism $\nu^* R^i f_* \mathcal{O}_{\mathcal{X}} \to R^i f_{T*} \mathcal{O}_{\mathcal{X}_T}$ is isomorphic in a neighborhood of $\nu^{-1}(o)$. Moreover $R^i f_* \mathcal{O}_{\mathcal{X}}$ is free in a neighborhood of o.*

Proof. See [3, Th. 5.5], where in the proof we may refer to [1, III, Cor. 3.10 and Th. 4.1] instead of (7.8.5) and (6.10.5) of EGA III, respectively.

Since $\nu^* R^2 f_* \mathbb{R} \to R^2 f_{T*} \mathbb{R}$ is isomorphic, it follows that the commutative diagram

$$\begin{array}{ccc} \nu^* R^2 f_* \mathbb{R} & \xrightarrow{\nu^*(\eta_f)} & \nu^* R^2 f_* \mathcal{O}_{\mathcal{X}} \\ \wr\downarrow & & \wr\downarrow \\ R^2 f_{T*} \mathbb{R} & \xrightarrow[\eta_{f_T}]{} & R^2 f_{T*} \mathcal{O}_{\mathcal{X}_T} \end{array}$$

gives the natural identification of $\nu^*(\eta_f)$ and η_{f_T}, where η_f is as in (1.1).

(2.4) Let $\alpha: Z \to S$ be a morphism of complex spaces. Suppose that Z is a dense Zariski open subset of another complex space $\bar{Z}$ and α extends to a proper morphism $\bar{\alpha}: \bar{Z} \to S$. Then there exists a unique maximal Zariski open subset $U \subseteqq S$ such that α is proper over U; in fact, we have only to set $U = S - \bar{\alpha}(\bar{Z} - Z)$. In particular if $Z_s = (\bar{Z})_s$ for some point $s \in S$, then $s \in U$ and α is proper over some neighborhood of s.

§3. A Kählerian Analogue of a Theorem of Matsusaka-Mumford

In this section we prove a refinement of the kählerian analogue of a theorem of Matsusaka-Mumford [22] given in [9].

(3.1) Let X be a compact complex manifold. Then we shall denote a (local) deformation of X by the triple $f: \mathcal{X} \to S$, $X_o = X$, $o \in S$, where f is a proper smooth morphism and we always consider S as a germ of a complex space at o, or one of its representatives. In particular we have the natural isomorphisms

$$(*) \quad H^2(\mathcal{X}, \mathbb{R}) \cong \Gamma(S, R^2 f_* \mathbb{R}) \cong H^2(X, \mathbb{R}).$$

Let $(f: \mathcal{X} \to S, \tilde{\omega})$ be a polarized family of compact Kähler manifolds. Then it is expected that for any point $s \in S$ there exist a neighborhood U of s and a Kähler form β on $\mathcal{X}_U$ which induces $\tilde{\omega}$ in $\Gamma(U, R^2 f_{U*} \mathbb{R})$. We shall show that this is the case when S is nonsingular and of dimension 1 and note that this is even true when dim $S>1$ (cf. Proposition 5 and Remark 2 below). But for the moment the next lemma is enough for our purposes.

Lemma 1. *Let* (X, ω) *be a polarized compact Kähler manifold. Let* f: $\mathcal{X}\to S$, $X_o=X$, $o\in S$, *be a deformation of* X *with* S *nonsingular. Let* $\Omega\in H^2(\mathcal{X}, \mathbf{R})$ *and* $\tilde{\omega}\in\Gamma(S, R^2f_*\mathbf{R})$ *correspond to* ω *with respect to the isomorphism* $(*)$. *Then the following conditions are equivalent.* 1) $\eta_f(\tilde{\omega})=0$ *in* $\Gamma(S, R^2f_*\mathcal{O}_{\mathcal{X}})$, *where* η_f: $\Gamma(S, R^2f_*\mathbf{R})\to\Gamma(S, R^2f_*\mathcal{O}_{\mathcal{X}})$ *is the natural homomorphism,* 2) *the restriction* $\tilde{\omega}_s\in H^2(X_s, \mathbf{R})$ *of* $\tilde{\omega}$ *to* X_s *is a Kähler class and* 3) Ω *is represented by a real closed* C^∞ *2-form* β *on* $\mathcal{X}$ *which is a Kähler form when restricted to each fiber of* f.

Proof. The implication 3)$\to$2) is obvious. By virtue of (2.3) 1) is equivalent to the condition that $\tilde{\omega}_s$ is of type (1,1) when restricted to each fiber of f. From this, the implication 2)$\to$1) follows. We show that 1) implies 3). Fix a Kähler form α on X representing the class ω. Then by Kodaira-Spencer (cf. [18]) there exists a family $\{\alpha_s\}_{s\in S}$ of Kähler forms on X_s with $\alpha_o=\alpha$ which depends differentiably on s. Since α_s are harmonic forms with respect to the metrics g_s associated to α_s, by [28, Lemma, p. 196] there exists a real closed C^∞ 2-form β on $\mathcal{X}$ such that $\beta_o=\alpha$ and that β_s is a harmonic form with respect to g_s. Clearly β represents the class Ω. Then since for each $s\in S$, Ω_s (the restriction of Ω to X_s) is of type (1,1) as a cohomology class by our assumption and since β_s is harmonic, β_s is actually a (1,1)-form on X_s. Moreover (if s is sufficiently near to o) β_s are positive forms on X_s since so is β_o. Thus β_s is a Kähler form on each X_s as desired. q.e.d.

(3.2) Using Lemma 1 we show the next proposition by reducing to an analogous one established in [10].

Proposition 2. *Let* $(f_i$: $\mathcal{X}_i\to S, \tilde{\omega}_i)$, $i=1, 2$, *be polarized families of compact Kähler manifolds. Then* $I^-:=\mathrm{Isom}_S((\mathcal{X}_1, \tilde{\omega}_1), (\mathcal{X}_2, \tilde{\omega}_2))^-$ *is proper over* S, *where* I^- *is the closure of* $\mathrm{Isom}_S((\mathcal{X}_1, \tilde{\omega}_1), (\mathcal{X}_2, \tilde{\omega}_2))$ *in* $D_{\mathcal{X}_1\times_S\mathcal{X}_2/S}$ (cf. (2.1)).

Proof. Since the problem is topological we may assume that S is reduced. Let π: $\hat{S}\to S$ be a resolution of S. Let $(\hat{f}_i$: $\hat{\mathcal{X}}_i\to\hat{S}, \hat{\omega}_i)$ be the pull-back of $(f_i, \tilde{\omega}_i)$ to $\hat{S}$ by π. Then with respect to the natural isomorphism $(D_{\mathcal{X}_1\times_S\mathcal{X}_2/S})\times_S\hat{S}\cong D_{\hat{\mathcal{X}}_1\times_{\hat{S}}\hat{\mathcal{X}}_2/\hat{S}}$, $I^-\times_S\hat{S}$ is naturally considered as a closed analytic subspace of $\mathrm{Isom}_{\hat{S}}((\hat{\mathcal{X}}_1, \hat{\omega}_1), (\hat{\mathcal{X}}_2, \hat{\omega}_2))^-$ (in fact they coincide). Hence it suffices to show the properness for $(\hat{f}_i, \hat{\omega}_i)$. So we may assume from the beginning that S is nonsingular. Then since the problem is local on S, by Lemma 1, 2)$\to$3) we may assume that there exists a real d-closed C^∞ 2-form β_i on $\mathcal{X}_i$ such that $(\beta_i)_s$ is a Kähler form and is a representative of the class $(\tilde{\omega}_i)_s$ for each $s\in S$. Then the proposition follows from Proposition 3 of [10]. q.e.d.

For later reference we record a special case where S is a point and where $\mathcal{X}_1=\mathcal{X}_2$ and $\tilde{\omega}_1=\tilde{\omega}_2$.

Corollary [6]. *For any polarized compact Kähler manifold* (X, ω), $\mathrm{Aut}(X, \omega)$ *has only a finite number of connected components.*

(3.3) Let $D=\{t\in \boldsymbol{C};\ |t|<1\}$ be the unit disc. Let $D'=D-\{0\}$. Let $(f_i\colon \mathcal{X}_i\to D, \tilde{\omega}_i)$, $i=1, 2$, be polarized families of compact Kähler manifolds over D. The next proposition is a kählerian analogue of Theorem 2 of Matsusaka-Mumford [22], a little weaker version of which was established in [9].

Proposition 3. *Let* $\varphi\colon \mathcal{X}_1\to\mathcal{X}_2$ *be a bimeromorphic map over* D *which induces over* D' *an isomorphism of the induced polarized families* $(f_{1,D'}, \tilde{\omega}_{1,D'})$ *and* $(f_{2,D'}, \tilde{\omega}_{2,D'})$. *Then if* $X_{1,0}$ *is not ruled,* φ *must be isomorphic.*

Here a compact complex manifold X is said to be *ruled* if there exist a compact complex manifold Y and a holomorphic vector bundle E on Y of rank$\geqq 2$ such that X is bimeromorphic to the associated projective bundle $\boldsymbol{P}(E)$.

Before the proof we first derive from this proposition the following:

Proposition 4. *Let* $(f_i\colon \mathcal{X}_i\to S, \tilde{\omega}_i)$, $i=1, 2$, *be polarized families of compact Kähler manifolds with* $X_{1,o}$ *non-ruled for some* $o\in S$. *Then* $I=\mathrm{Isom}_S((\mathcal{X}_1, \tilde{\omega}_1), (\mathcal{X}_2, \tilde{\omega}_2))$ *is proper over some neighborhood of* o *in* S.

Proof. Since $I^- = \mathrm{Isom}_S((\mathcal{X}_1, \tilde{\omega}_1), (\mathcal{X}_2, \tilde{\omega}_2))$ is proper over S by Proposition 2, it suffices to show that $I_o^-=I_o$ (cf. (2.4)). So assuming that $I_o^-\neq I_o$ we shall derive a contradiction. Take a morphism $h\colon D\to I^-$ with $h(0)\in I_o^- - I_o$ and $h^{-1}(I_o^- - I)=\{0\}$, where $D=\{t\in \boldsymbol{C};\ |t|<1\}$. Let $\bar{h}$ be the composition of h with the natural projection $I^-\to S$. Let $(\hat{f}_i\colon \hat{\mathcal{X}}_i\to D, \hat{\omega}_i)$, $i=1, 2$, be the pull-back of $(f_i, \tilde{\omega}_i)$ by $\bar{h}$. Then by our choice of h there exists a bimeromorphic D-map $\varphi\colon \hat{\mathcal{X}}_1\to\hat{\mathcal{X}}_2$ which is isomorphic over $D'=D-\{0\}$, but not isomorphic over the whole D (cf. [10]). Moreover if $\varphi'\colon \hat{\mathcal{X}}_1'\to\hat{\mathcal{X}}_2'$ is the induced isomorphism with $\hat{\mathcal{X}}_i'=\hat{f}_i^{-1}(D')$, then we have $\varphi'^*\hat{\omega}_2=\hat{\omega}_1$. It then follows from Proposition 3 that φ must be isomorphic, which is a contradiction. q.e.d.

Corollary. *Under the above assumptions the set* $S_1:=\{s\in S; (X_{1s}, \tilde{\omega}_{1s})\cong$

$(X_{2s}, \tilde{\omega}_{2s})\}$ *is an analytic subset of* S *in a neighborhood of* o.

Proof. S_1 is the image of $\mathrm{Isom}_S((\mathcal{X}_1, \tilde{\omega}_1), (\mathcal{X}_2, \tilde{\omega}_2))$ by a proper map. q.e.d.

(3.4) Let (X, ω) be a polarized compact Kähler manifold. Then a *polarized* (local) *deformation* of (X, ω) is a polarized family of compact Kähler manifolds $(f\colon \mathcal{X}\to S, \tilde{\omega})$ such that $(X_o, \tilde{\omega}_o)=(X, \omega)$ for a point $o\in S$ with S considered as an analytic germ at o.

Now Proposition 3 follows from Theorem 4.3 of [9] if $\tilde{\omega}_i$ is induced from a Kähler class Ω_i on $\mathcal{X}_i$. This is indeed guaranteed by the following:

Proposition 5. *Let* (X, ω) *be a polarized compact Kähler manifold. Let* $(f\colon \mathcal{X}\to S, \tilde{\omega})$, $(X_o, \tilde{\omega}_o)=(X, \omega)$, $o\in S$, *be a polarized deformation of* (X, ω). *Suppose that* $\dim S=1$ *and* S *is nonsingular. Then the class* $\Omega\in H^2(\mathcal{X}, \boldsymbol{R})$ *which corresponds to* ω, *or* $\tilde{\omega}$, *in the isomorphism* (*) *is a Kähler class.*

The proof which follows is suggested by the referee. (The original proof of ours is by way of the harmonic theory of Kodaira-Spencer.)

We proceed in steps. Let $\mathcal{E}^{0,q}_{\mathcal{X}}$ be the sheaf of germs of C^∞ $(0, q)$-forms on $\mathcal{X}$ and set $\mathcal{E}_{\mathcal{X}}=\mathcal{E}^{0,0}_{\mathcal{X}}$, the sheaf of germs of C^∞ functions on $\mathcal{X}$. Let t be the local parameter of S with center o. Let $\mathcal{I}=t\mathcal{O}_{\mathcal{X}}$ and $\mathcal{J}=(t, \bar{t})\mathcal{E}_{\mathcal{X}}$. Let $\hat{\mathcal{O}}_{\mathcal{X}}=\varprojlim \mathcal{O}_{\mathcal{X}}/\mathcal{I}^n$ and $\hat{\mathcal{E}}^{0,q}_{\mathcal{X}}=\varprojlim \mathcal{E}^{0,q}_{\mathcal{X}}/\mathcal{J}^n\mathcal{E}^{0,q}_{\mathcal{X}}$. Then the usual $\bar{\partial}$-operator $\mathcal{E}^{0,q}_{\mathcal{X}}\to \mathcal{E}^{0,q+1}_{\mathcal{X}}$ induces a similar homomorphism $\hat{\mathcal{E}}^{0,q}_{\mathcal{X}}\to\hat{\mathcal{E}}^{0,q+1}_{\mathcal{X}}$ (still denoted by $\bar{\partial}$) with respect to the natural homomorphism $r^{\cdot}\colon \mathcal{E}^{0,\cdot}_{\mathcal{X}}\to\hat{\mathcal{E}}^{0,\cdot}_{\mathcal{X}}$. This then gives rise to the "formal Dolbeault complex"

$$(\sharp)\quad 0\to\hat{\mathcal{O}}_{\mathcal{X}}\to\hat{\mathcal{E}}_{\mathcal{X}}\to\hat{\mathcal{E}}^{0,1}_{\mathcal{X}}\to\cdots\to\hat{\mathcal{E}}^{0,q}_{\mathcal{X}}\to\cdots$$

which turns out to be a fine resolution of $\hat{\mathcal{O}}_{\mathcal{X}}$; this is indeed a special case of the formal Dolbeault lemma proved by Bingener [31] and the author [13] independently. As a consequence of this we note the following:

Lemma 2. *The natural homomorphism* $H^q\Gamma(X_o, \mathcal{E}^{0,\cdot}_{\mathcal{X}}|_{X_o})\to H^q\Gamma(X_o, \hat{\mathcal{E}}^{0,\cdot}_{\mathcal{X}})$ *induced by* $r^{\cdot}$ *is injective for any* $q\geqq 0$.

Proof. Let $(R^qf_*\mathcal{O}_{\mathcal{X}})^\wedge=\varprojlim R^qf_*\mathcal{O}_{\mathcal{X}}/\mathcal{I}^nR^qf_*\mathcal{O}_{\mathcal{X}}$. Let $\hat{f}\colon \hat{\mathcal{X}}\to\hat{S}$ be the formal completion of $f\colon \mathcal{X}\to S$ along X_o. Let $u\colon (R^qf_*\mathcal{O}_{\mathcal{X}})_o\to R^q\hat{f}_*\hat{\mathcal{O}}_{\mathcal{X}}$ be the composition of the natural injection $(R^qf_*\mathcal{O}_{\mathcal{X}})_o\to(R^qf_*\mathcal{O}_{\mathcal{X}})^\wedge$ and isomorphism $(R^qf_*\mathcal{O}_{\mathcal{X}})^\wedge\to R^q\hat{f}_*\hat{\mathcal{O}}_{\mathcal{X}}$ (cf. [1, VI, 4.5]). The lemma then follows from the commutative diagram

$$\begin{array}{ccc} (R^q f_* \mathcal{O}_{\mathcal{X}})_o & \xrightarrow{u} & R^q \hat{f}_* \hat{\mathcal{O}}_{\mathcal{X}} \\ \wr\downarrow & & \wr\downarrow \\ H^q \Gamma(X_o, \mathcal{E}^{0,\cdot}_{\mathcal{X}} | X_o) & \rightarrow & H^q \Gamma(X_o, \hat{\mathcal{E}}^{0,\cdot}_{\mathcal{X}}) \end{array}$$

where the vertical arrows are the usual and the formal Dolbeault isomorphisms (induced by (♯) in the formal case). q.e.d.

Using Lemma 2 we shall next show the following:

Lemma 3. *Let β be a $\bar\partial$-closed C^∞ (0, 2)-form on $\mathcal{X}$ whose restriction to each fiber of f vanishes identically. Then $\beta=\bar\partial\alpha$ for some C^∞ (0, 1)-form α (in a neighborhood of X_o).*

Proof. First we note that β is written in the form $\beta=\varphi\wedge d\bar t$ for some C^∞ (0, 1)-form φ on $\mathcal{X}$. Indeed, from the local consideration it is clear that we may write $\beta=\varphi_1\wedge d\bar t$ for some $\varphi_1\in\Gamma(X, \mathcal{E}^{0,1}_{\mathcal{X}}/\mathcal{E}_{\mathcal{X}}d\bar t)$ in the obvious sense and then it suffices to take φ to be any lift of φ_1 with respect to the natural surjection $\Gamma(X, \mathcal{E}^{0,1}_{\mathcal{X}})\to\Gamma(X, \mathcal{E}^{0,1}_{\mathcal{X}}/\mathcal{E}_{\mathcal{X}}d\bar t)$. Similar remark applies also to any C^∞ (0, q)-form ψ, $q\geqq 1$, with $d\bar t\wedge\psi=0$ and will be used without further mention.

Now we shall show by induction on $n\geqq 1$ the following statement: $(*)_n$. There exist C^∞ (0, 1)-forms α_n, φ_n on $\mathcal{X}$ such that β is written in the form $\beta=\bar t^n\bar\partial\varphi_n+\bar\partial\alpha_n$ and that $\alpha_{n+1}\equiv\alpha_n$ mod $\mathcal{J}^{n+1}$. First of all, when $n=1$, we have only to take $\varphi_1=\varphi$ and $\alpha_1=-\varphi\bar t$. Suppose next that $(*)_k$ are proved for any $1\leqq k\leqq n$. Then $\bar\partial\beta=n\bar t^{n-1}d\bar t\wedge\bar\partial\varphi_n=0$, which implies that $\bar\partial\varphi_n=\psi_n\wedge d\bar t$ for some C^∞ (0, 1)-form ψ_n on $\mathcal{X}$. Then we have

$$\beta = \bar t^n d\bar t+\bar\partial\alpha_n = \bar\partial(\alpha_n-1/(n+1)\bar t^{n+1}\psi_n)+\bar t^{n+1}\bar\partial(1/(n+1)\psi_n)\,.$$

Hence $\alpha_{n+1}:=\alpha_n-1/(n+1)\bar t^{n+1}\psi_n$ and $\varphi_{n+1}:=1/(n+1)\psi_n$ satisfy $(*)_{n+1}$. Thus $(*)_n$ are true for all $n\geqq 1$. Now $\hat\alpha:=\{\alpha_n\}$ determines an element of $\Gamma(X_o, \hat{\mathcal{E}}^{0,1}_{\mathcal{X}})$ with $\bar\partial\hat\alpha=\hat\beta$, where $\hat\beta$ is the natural image of β in $\Gamma(X_o, \hat{\mathcal{E}}^{0,1}_{\mathcal{X}})$. The lemma thus follows from Lemma 2. q.e.d.

Proof of Proposition 5. We have to show that Ω is represented by a Kähler form. By Lemma 1 there exists a real d-closed C^∞ 2-form β representing Ω such that when restricted to each fiber it is a Kähler form and hence in particular of type (1,1). Let $\beta=\beta^{2,0}+\beta^{1,1}+\beta^{0,2}$ be the type decomposition of β, where $\beta^{0,2}=\bar\beta^{2,0}$ and $\bar\partial\beta^{0,2}=0$. Moreover $\beta^{0,2}$ restricted to each fiber vanishes. Hence by Lemma 2 we may write $\beta^{0,2}=\bar\partial\alpha$ for some C^∞ (0, 1)-form α on $\mathcal{X}$. Then $\beta':=\beta-d(\alpha+\bar\alpha)$ is a d-closed (1,1)-form representing Ω. Moreover $\beta^{1,1}_s=\beta'_s$ for each s. Thus β' gives a Kähler form on each fiber. Finally replacing β' by $\beta'+Ndt\wedge d\bar{t}$ for some sufficiently large constant $N>0$ we get a Kähler form representing Ω.

Remark 2. As pointed out by the referee the above proof also works for the case dim $S>1$ (with S nonsingular) by a suitable modification of the proof of Lemma 3.

§4. Cohomological Flatness of the Relative Tangent Sheaf

In this section we shall show the cohomological flatness of $\Theta_{\mathscr{X}/S}$ for any polarized family ($f: \mathscr{X}\to S$, $\tilde{\omega}$) of compact Kähler manifolds such that $\mathrm{Aut}_0 X_s$ is a complex torus for any $s\in S$, where $\Theta_{\mathscr{X}/S}$ is the relative tangent sheaf associated to f, i.e., the sheaf of germs of holomorphic vector fields on $\mathscr{X}$ which are tangent to the fibers of f.

(4.1) We begin with a general definition. Let $f: \mathscr{X}\to S$ be a morphism of complex spaces and $\mathscr{F}$ an f-flat coherent analytic sheaf on $\mathscr{X}$. Then $\mathscr{F}$ is said to be *cohomologically flat* (*in dimension zero*) *at* $s\in S$ with respect to f, if the following equivalent conditions are satisfied (cf. [1, III, Cor. 3.7]):

i) For any morphism $\nu: T\to S$ of complex spaces, the natural map $\nu^* f_*\mathscr{F} \to f_{T*}\tilde{\nu}^*\mathscr{F}$ is isomorphic at any point $t\in T$ with $\nu(t)=s$, where $\tilde{\nu}: \mathscr{X}\times_S T\to\mathscr{X}$ is the natural morphism.

ii) Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_S$ at s. Then the restriction maps $f_*(\mathscr{F}/\mathfrak{m}^k\mathscr{F})_s\to H^0(X_s, \mathscr{F}/\mathfrak{m}\mathscr{F})$ are surjective for all $k>0$.

Moreover in this case $f_*\mathscr{F}$ is free in a neighborhood of s. Further, when S is reduced, $\mathscr{F}$ is cohomologically flat at s if and only if $d(s):=\dim H^0(X_s, \mathscr{F}/\mathfrak{m}\mathscr{F})$, $s\in S$, is constant in a neighborhood of s (cf. [1, Th. 4.12]). We say that $\mathscr{F}$ is *cohomologically flat with respect to* f if so is $\mathscr{F}$ at any point of S. Clearly, if $\mathscr{F}$ is cohomologically flat with respect to f, then $\tilde{\nu}^*\mathscr{F}$ is cohomologically flat with respect to f_T, where $\tilde{\nu}$ is as in i) above.

Now the purpose of this section is to prove the following:

Theorem 1. *Let X be a compact connected Kähler manifold. Let $f: \mathscr{X}\to S$, $X_o=X$, $o\in S$, be a deformation of X. Suppose that* $\mathrm{Aut}_0 X$ *is a complex torus.**) *Then the relative tangent sheaf $\Theta_{\mathscr{X}/S}$ is cohomologically flat* (*in dimension zero*) *at o. In particular* $\dim H^0(X_s, \Theta_{X_s})$ *is independent of s in a neighborhood of o.*

Remark 3. The last assertion is shown first by Matsusaka under the assumption that there exists a relatively ample line bundle L on $\mathscr{X}$ and that

*) The case where $\mathrm{Aut}_0 X$ reduces to the identity is included.

$(X, c_1^R(L_o))$ is in $\mathfrak{B}$ (cf. §6, below), where $c_1^R(L_o)$ is the real chern class of L_o. (See Corollary to Proposition 11 of [21].)

(4.2) First we note that for a proper smooth morphism $f: \mathcal{X}\to S$ the cohomological flatness of $\Theta_{\mathcal{X}/S}$ has the following significance.

Proposition 6. *Let $f: \mathcal{X}\to S$ be a proper smooth morphism of complex spaces. Let s be a fixed point of S. Then the following conditions are equivalent.*

1) *$\Theta_{\mathcal{X}/S}$ is cohomologically flat at s with respect to f.*

2) *α: Aut $\mathcal{X}/S\to S$ is smooth at any point of $\mathrm{Aut}_0 X_s$ with respect to the natural inclusion $\mathrm{Aut}_0 X_s \subseteq (\mathrm{Aut}\ \mathcal{X}/S)_s$.*

We first prove a lemma.

Lemma 4. *α is smooth at any point of $\mathrm{Aut}_0 X$ if it is smooth at $e(s)$, where e is the identity section of α.*

Proof. Let $a\in \mathrm{Aut}_0 X_s$ be arbitrary. We first show that there exists a holomorphic section ε: $S\to$Aut $\mathcal{X}/S$ with $\varepsilon(s)=a$. Let U be a neighborhood of $e(s)$ in Aut $\mathcal{X}/S$ such that α is smooth on U. Then we can find an integer $m>0$ such that the image of $U_S^m := U\times_S \cdots \times_S U$ (m-times) by the natural S-morphism π: $U_S^m\to$Aut $\mathcal{X}/S$ (induced by the relative multiplication of Aut $\mathcal{X}/S$) contains a. Take any point $u\in U_S^m$ with $\pi(u)=a$. Since $U_S^m\to S$ is smooth, we can find a holomorphic section ε^m: $S\to U_S^m$ with $\varepsilon^m(s)=u$. Let $\varepsilon=\pi\varepsilon^m$: $S\to$Aut $\mathcal{X}/S$. Then ε is a desired holomorphic section with $\varepsilon(s)=a$. $\varepsilon(S)$ then defines by translation an S-automorphism ε^* of Aut $\mathcal{X}/S$ as a complex space over S such that $\varepsilon^*(e(s))=a$. The lemma follows.

Proof of Proposition 6. (cf. Wavrik [29]) 1) $\to$ 2). Suppose that $\Theta_{\mathcal{X}/S}$ is cohomologically flat. We shall show that α is smooth. By Lemma 4 we have only to check this at $e(s)$. We use Grothendieck's criterion (2.2): in the notation there for a given S-morphism ν_2: $S_2\to$Aut $\mathcal{X}/S$ with $\nu_2(t)=e(s)$, we want to find its extension ν_1: $S_1\to$Aut $\mathcal{X}/S$. Let $\mathcal{X}_i := \mathcal{X}\times_S S_i$ and e_i: $S_i\to$Aut $\mathcal{X}_i/S_i$ the identity section, $i=1, 2$. Let $\mathfrak{m}_i$ be the maximal ideal of S_i at t. Note the isomorphism Aut $\mathcal{X}/S\times_S S_i\cong$Aut $\mathcal{X}_i/S_i$. Then ν_2 corresponds to a holomorphic section $\nu_2'\in\Gamma(S_2$, Aut $\mathcal{X}_2/S_2)$ with $\nu_2'(t)=e_2(t)$.

Now we have the natural isomorphism of sets $\Gamma_e(S_i$, Aut $\mathcal{X}_i/S_i)\cong H^0(\mathcal{X}_i, \mathfrak{m}_i\Theta_{\mathcal{X}_i/S_i})$, where $\Gamma_e(S_i$, Aut $\mathcal{X}_i/S_i)=\{\psi\in\Gamma(S_i$, Aut $\mathcal{X}_i/S_i); \psi(t)=e_i(t)\}$ (cf. [29, §3]). On the other hand, by the cohomological flatness of $\Theta_{\mathcal{X}/S}$ the natural morphism $H^0(\mathcal{X}_1, \mathfrak{m}_1\Theta_{\mathcal{X}_1/S_1})\to H^0(\mathcal{X}_2, \mathfrak{m}_2\Theta_{\mathcal{X}_2/S_2})$ is surjective. Hence we can find an extension $v_1' \in \Gamma_e(S_1, \text{Aut } \mathcal{X}_1/S_1)$ of v_2', which in turn gives a desired extension v_1 of v_2.

2)$\to$1). (cf. [29, Prop. 4.3]) Suppose that $\Theta_{\mathcal{X}/S}$ is not cohomologically flat. We shall show that α is not smooth at $e(s)$. Let $\mathfrak{m}$ be the maximal ideal of S at s. By our assumption there exist an integer $k>0$ and $\theta \in H^0(X_s, \Theta_{X_s})$ which is not in the image of the natural homomorphism $f_*(\Theta_{\mathcal{X}/S}/\mathfrak{m}^k\Theta_{\mathcal{X}/S}) \to H^0(X_s, \Theta_{X_s})$. Replacing S by the subspace defined by $\mathfrak{m}^k$ we may assume that $\mathfrak{m}^k=0$ on S. Let $T=S\times \text{Spec } \mathbf{C}[\varepsilon]$ and $T'=\{s\}\times \text{Spec } \mathbf{C}[\varepsilon]$, where ε is the dual number; $\varepsilon^2=0$. Consider T as a complex space over S via the natural projection $T \to S$. Let $o \in \text{Spec } \mathbf{C}[\varepsilon]$ be the unique point and $\mathfrak{n}$ the maximal ideal of T at (s, o). Then $\varepsilon\theta$ is naturally considered as a section of $f_{T'*}\mathfrak{n}\Theta_{\mathcal{X}_{T'}/T'}$ and it is not in the image of the natural map $f_{T*}\mathfrak{n}\Theta_{\mathcal{X}_T/T} \to f_{T'*}\mathfrak{n}\Theta_{\mathcal{X}_{T'*}/T'}$. Since, in the same notation as above, $f_{Y*}(\mathfrak{n}\Theta_{\mathcal{X}_Y/Y})_{(s,o)} \cong \Gamma_e(Y, \text{Aut } \mathcal{X}_Y/Y)$ $(Y=T, T')$ by [29], this implies that α is not smooth at $e(s)$ by (2.2). q.e.d.

(4.3) In general let S be any Artin space, i.e., a complex space with a unique point, say o. Let $f\colon \mathcal{X} \to S$ be a proper smooth morphism with connected fibers. Set $X=X_o$. Since S is an Artin space, Aut $\mathcal{X}$ and Aut S have the natural structure of complex Lie groups (cf. [4]). Moreover f induces the natural homomorphism Aut $\mathcal{X} \to$ Aut S, and hence $\text{Aut}_0\mathcal{X} \to \text{Aut}_0 S$. (Consider for instance S as a distinguished connected component of the Douady space of $\mathcal{X}$ (cf. [17, Lemma 3]) and consider the induced action of Aut $\mathcal{X}$ on S.) Denote the last homomorphism by f_*. Let G and N be the image and the kernel of f_* respectively. Let $u\colon \text{Aut}_0\mathcal{X} \to \text{Aut}_0 X$ be the natural restriction homomorphism to $X=\mathcal{X}_{\text{red}}$. Then by associating to any $g\in G$ the coset of $u(f_*^{-1}(g))$ we get a map $w\colon G\to E$ of G into the coset space $E := \text{Aut}_0 X/u(N)$. It is easy to see that $u(N)$ is closed in $\text{Aut}_0 X$ and w is holomorphic.

Lemma 5. *Suppose that X is Kähler and that* $\text{Aut}_0 X$ *is a complex torus. Then w is a constant map.*

First we recall some terminology from [6]. (See [6] for the more detail.) A *meromorphic Lie group* is a complex Lie group with a bimeromorphic equivalence class of its compactifications. An algebraic group is a special case of meromorphic Lie groups if we consider its algebraic compactifications as the corresponding equivalence class of compactifications. A homomorphism $G_1 \to G_2$ of meromorphic Lie groups is said to be *meromorphic* if it extends to a meromorphic map $G_1^* \to G_2^*$ between any compactifications G_i^* of G_i in the equivalence classes.

Proof of Lemma 5. Since X is Kähler, $\mathrm{Aut}_0\mathcal{X}$ has the natural structure of a meromorphic Lie group (cf. [5, Th. 5.3], [6, Th. 3.5]). (See also Corollary to Theorem of [11], where a gap in the argument in [5] is filled.) On the other hand, since S is an Artin space, Aut S has the natural structure of a linear algebraic group, e.g., as a closed subgroup of $GL(A)$, where $A:=\mathcal{O}_{S,o}$ is considered a finite dimensional $\boldsymbol{C}$-vector space. Then by [6, Lemma 2.4], with respect to these meromorphic Lie group structures f_* is a meromorphic homomorphism. In particular the image G of f_* is a linear algebraic subgroup of Aut S. Moreover by the standard arguments we see readily that for any algebraic compactification G^* of G, w extends to a meromorphic map $w^*: G^* \rightarrow E$. It follows then that w is a constant map since E is a complex torus and G is linear algebraic (cf. [6, Lemma 3.8]). q.e.d.

(4.4) Let X be a compact complex manifold. Let $f: \mathcal{X} \rightarrow S$, $X_o = X$, $o \in S$, be the Kuranishi family of X [19]. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_S$ at o. Let S_n be the subspace of S defined by $\mathfrak{m}^{n+1}$. Let $\mathcal{X}_n = \mathcal{X} \times_S S_n$. Let $f_n: \mathcal{X}_n \rightarrow S_n$ be the induced morphism.

Lemma 6. *The restriction homomorphism u_n*: Aut $\mathcal{X}_n \rightarrow$ Aut X *is surjective for any* $n \geqq 0$.

Proof. Take any $h \in$ Aut X. The versality of Kuranishi family implies that there exist morphisms $\tilde{h}: \mathcal{X}_n \rightarrow \mathcal{X}_n$ and $\bar{h}: S_n \rightarrow S_n$ such that $\tilde{h}|_{X_o = X} = h$ and that the following diagram is cartesian

$$\begin{array}{ccc} \mathcal{X}_n & \xrightarrow{\tilde{h}} & \mathcal{X}_n \\ f_n \downarrow & & \downarrow f_n \\ S_n & \xrightarrow{\bar{h}} & S_n \end{array} .$$

Thus it suffices to check that $\bar{h}$, and hence $\tilde{h}$ also, is an automorphism. Since S_n is an Artin space, this follows if we see that $\bar{h}$ is an embedding, or the differential $\bar{h}_*$ of $\bar{h}$ is injective. In fact, if $\bar{h}_*$ is not injective, there is an embedding $\iota: S_\varepsilon = \mathrm{Spec}\, \boldsymbol{C}[\varepsilon] \rightarrow S_n$ such that $\mathcal{X}_n \times_{S_n} S_\varepsilon \rightarrow S_\varepsilon$ is a trivial family, contradicting the fact that f is the Kuranishi family, where $\boldsymbol{C}[\varepsilon]$, $\varepsilon^2 = 0$, is the ring of dual numbers. q.e.d.

Proof of Theorem 1. Clearly we may assume that f is the Kuranishi family of X. Further by the (second) definition of the cohomological flatness it suffices then to show the theorem for each f_n in the notation above. Let $\mathrm{Aut}_0\mathcal{X}_n/S_n$ be the connected component of Aut $\mathcal{X}_n/S_n$ containing $\mathrm{Aut}_0 X$. Then by Proposition 6 we have only to show the smoothness of $\mathrm{Aut}_0\mathcal{X}_n/S_n$ over S_n. First, by Lemmas 5 and 6 we see immediately that $\mathrm{Aut}_0X=u_n(N_n)$, where N_n is the kernel of $f_{n|\mathrm{Aut}_0\mathcal{X}_n}$. By definition N_n is just the group of automorphisms of $\mathcal{X}_n$ *over* S_n inducing those of Aut_0X on X, or in other words, the group of holomorphic sections of α_n: $\mathrm{Aut}_0\mathcal{X}_n/S_n\to S_n$. Thus for any $a\in\mathrm{Aut}_0X$ if we take a local holomorphic section v of $N_n\to\mathrm{Aut}_0X$ defined in a neighborhood V of a, v defines a holomorphic family of sections of α_n parametrized by $V\subseteq\mathrm{Aut}_0X=(\mathrm{Aut}_0\mathcal{X}_n/S_n)_o$, which in turn gives an S_n-isomorphism of a neighborhood of a in $\mathrm{Aut}_0\mathcal{X}_n/S_n$ to $S_n\times V$. Thus α_n is smooth. q.e.d.

Remark 4. Pushing the above arguments a little further, we can show more generally the following: Let f: $\mathcal{X}\to S$, $X_o=X$, $o\in S$, be a deformation of a compact connected Kähler manifold X. Then dim $T(X_s)$ is lower semicontinuous in a neighborhood of o, where $T(X_s)=T(\mathrm{Aut}_0X_s)$ is the Albanese torus of Aut_0X_s (cf. [6]). The proof will be given elsewhere.

(4.5) Since Theorem 1 is important, we shall give an alternative proof in the following special case using the notion of polarized Kuranishi family (to be defined in §5).

Proposition 7. *Let* (f: $\mathcal{X}\to S$, $\tilde{\omega}$) *be any polarized family of compact Kähler manifolds with S reduced. Suppose that* Aut_0X_o *is a complex torus for some* $o\in S$. *Then* $\Theta_{\mathcal{X}/S}$ *is cohomologically flat at* o.

We first prove a lemma. Let (X, ω) be a polarized compact Kähler manifold in general. Let (f: $\mathcal{X}\to T$, $\tilde{\omega}$), $(X_o, \tilde{\omega}_o)=(X, \omega)$, $o\in T$, be the Kuranishi family of (X, ω) (cf. Remark 6 in §5). Let r_i: $T\times T\to T$ be the projection to the i-th factor. Let (f_i: $\mathcal{X}_i\to T\times T$, $\tilde{\omega}_i$), $i=1, 2$, be the pull-back of $(f, \tilde{\omega})$ via r_i. Let $I=\mathrm{Isom}_{T\times T}((\mathcal{X}_1, \tilde{\omega}_1), (\mathcal{X}_2, \tilde{\omega}_2))$ and let $\gamma: I\to T\times T$ be the natural morphism. Let J be the connected component of I containing Aut_0X with respect to the natural identification of $\mathrm{Aut}(X, \omega)$ with $I_{(o\ o)}$. Let $Q=\gamma(J)$.

Lemma 7. *If* Aut_0X *is a complex torus, then* Q *is an analytic subset of* $T\times T$ *and* $r_i|_Q$: $Q\to T$ *is a finite morphism.*

Proof. Since $J_{(o,o)}=\mathrm{Aut}_0X$, by (2.4) $\gamma_J=\gamma_{|J}$: $J\to T\times T$ is proper (if we restrict T). Hence Q is analytic. Then for the finiteness of $r_{i|Q}$ it suffices to show that $Q_o:=r_i^{-1}(o)\cap Q_{\mathrm{red}}$ reduces to the point (o, o). Consider Q_o as a subspace of $T=r_i^{-1}(o)$. Then $X_t\cong X_{t'}$ for any t, $t'\in Q_o$. Then by Schuster [26] f_{Q_0}: $X\times_T Q_o\to Q_o$ is a trivial family. Hence the constant map τ: $Q_o\to\{o\}$

is a versal map associated to f_{Q_0} considered as a deformation of X. Since the differential τ_* of τ is zero and τ_* is unique, the inclusion $Q_o \to T$ must coincide with τ, i.e., $Q_o=(o, o)$. q.e.d.

In the above proof one can actually show that $r_i^{-1}(o) \cap Q=(o, o)$.

Proof of Proposition 7. Since S is reduced, it suffices to show that dim $H^0(X_s, \Theta_{X_s})$ is independent of s in a neighborhood of o. This clearly follows from the same assertion for the Kuranishi family $(f: \mathcal{X} \to S, \tilde{\omega})$ of $(X, \omega)=(X_o, \tilde{\omega}_o)$ as above. So we shall prove the latter, using the above notations. For any $(t, t') \in Q$, $r_J^{-1}(t, t')$ is a union of connected components of $\mathrm{Isom}(X_t, X_{t'}) \neq \phi$ and hence $\dim r_J^{-1}(t, t')=\dim \mathrm{Aut}_0 X_t = d(t) := \dim H^0(X_t, \Theta_{X_t})$. Since $p_1 := r_{1|Q}$ is finite by Lemma 7, there exist only a finite number of holomorphic sections, say $c_1, \cdots, c_m$, of p_1 defined on T_{red}. Let $Q_i = c_i(T_{\mathrm{red}}) \subseteq Q$.

Suppose now that $d(t)$ is not constant on T. Then by the first remark the dimension of the general fiber of r_i: $r_J^{-1}(Q_i) \to Q_i$ is less than $d(o)$ on an irreducible component of Q_i. It follows that there exists a Zariski open subset V of $J_{(o,o)}$ such that for any $v \in V$ and any i there is no holomorphic section of r_i passing through v. On the other hand, let $(\tilde{v}, \bar{v})$ be the automorphism of $(f, \tilde{\omega})$ associated to $v \in J_{(o,o)} = \mathrm{Aut}_0 X$ as in the proof of Lemma 6. Then $\tilde{v}$ is naturally regarded as a holomorphic section of r_J defined over the graph $\Gamma_{\bar{v}} \subseteq T \times T$ of $\bar{v}$ and passing through v. Since $\Gamma_{\bar{v}}$ defines a holomorphic section of p_1, this is a contradiction. Hence $d(t)$ is constant. q.e.d.

Remark 5. The above proof actually works also for the Kuranishi family itself (not necessarily polarized). Indeed, the proof depends only on the fact that r_J is proper and the corresponding fact can be proved by a method similar to Proposition 2 using the C^∞ extension theorem of Kähler metrics due to Kodaira-Spencer [18].

§5. Local Modular Family

In this section we shall construct the local modular family for a polarized compact Kähler manifold with $\mathrm{Aut}_0 X$ a complex torus and give some of its basic properties.

(5.1) We start with the following:

Definition. Let $(f: \mathcal{X} \to S, \tilde{\omega})$ be a polarized family of compact Kähler manifolds. Let $s \in S$. Then we say that $(f, \tilde{\omega})$ is *locally complete* at s if for any polarized deformation $(f': \mathcal{X}' \to S', \tilde{\omega}')$, $(X'_{s'}, \tilde{\omega}'_{s'}) = (X_s, \tilde{\omega}_s)$, $s' \in S'$, of $(X_s, \tilde{\omega}_s)$ there exists a morphism $\tau: S' \to S$, $\tau(s') = s$, such that $(f', \tilde{\omega}')$ is isomorphic to the pull-back $(f_{S'}, \tilde{\omega}_{S'})$ of $(f, \tilde{\omega})$ to S'. If, further, τ is unique for any $(f', \tilde{\omega}')$ as above, we say that $(f, \tilde{\omega})$, considered as a polarized deformation of $(X_s, \tilde{\omega}_s)$, is the *local modular family* of $(X_s, \tilde{\omega}_s)$. In this case τ is called the *universal map* associated to $(f', \tilde{\omega}')$.

Proposition 8. *For any polarized compact Kähler manifold (X, ω) such that $\mathrm{Aut}_0 X$ is a complex torus its local modular family $(f: \mathcal{X} \to S, \tilde{\omega})$, $(X_o, \tilde{\omega}_o) = (X, \omega)$, $o \in S$, exists. Moreover it is locally complete at every point in a neighborhood of o.*

Proof. Let $f_1: \mathcal{X}_1 \to S_1$, $X_{1,o} = X$, $o \in S_1$, be the Kuranishi family of X [19]. Let $\tilde{\omega}_1 \in \Gamma(S_1, R^2 f_{1*} \mathbf{R})$ be the element corresponding to ω with respect to the isomorphism $(*)$ in (3.1) for f_1. Let S be the subspace of S_1 defined by the equation $\eta(\tilde{\omega}_1) = 0$, where $\eta: \Gamma(S_1, R^2 f_{1*} \mathbf{R}) \to \Gamma(S_1, R^2 f_{1*} \mathcal{O}_{\mathcal{X}_1})$ is the natural homomorphism. (Note that since $R^2 f_{1*} \mathcal{O}_{\mathcal{X}_1} \cong \mathcal{O}_{S_1}^{\oplus r}$ for some $r \geqq 0$ by Proposition 1, the equation is just a system of r ordinary equations.) Let $f: \mathcal{X} \to S$ and $\tilde{\omega} \in \Gamma(S, R^2 f_* \mathbf{R})$ be the restriction of f_1 and $\tilde{\omega}_1$ to S respectively. Then we claim that $(f, \tilde{\omega})$ thus obtained is 1) a polarized deformation of (X, ω), 2) locally complete at any point of s in a neighborhood of o, and finally, 3) the local modular family of (X, ω).

1) By our construction $(f, \tilde{\omega})$ clearly satisfies the condition (2) of Definition in (1.1). Thus it suffices to show that $\tilde{\omega}_s$ is a Kähler class on X_s for any $s \in S$ in a neighborhood of o. Let $\pi: \hat{S} \to S_{\mathrm{red}}$ be a resolution of S_{red}. Let $(\hat{f}: \hat{\mathcal{X}} \to \hat{S}, \hat{\omega})$ be the pull-back of $(f, \tilde{\omega})$ to $\hat{S}$. Then $(\hat{f}, \hat{\omega})$ satisfies the condition 1) of Lemma 1 at each point of $\pi^{-1}(o)$. Hence by that lemma $\hat{\omega}_{\hat{s}}$ is a Kähler class on $X_{\hat{s}}$ for any $\hat{s}$ in a neighborhood of $\pi^{-1}(o)$. It follows that $\tilde{\omega}_s$ is a Kähler class for any s in a neighborhood of o.

2) Let $(f': \mathcal{X}' \to S', \tilde{\omega}')$, $(X'_{s'}, \tilde{\omega}_{s'}) = (X_s, \tilde{\omega}_s)$, $s' \in S'$, be any polarized deformation of $(X_s, \tilde{\omega}_s)$. By virtue of the local completeness of the original Kuranishi family f_1 (if s is sufficiently near to o) we get a versal map $\tau: S' \to S$ induced by $f': \mathcal{X}' \to S'$ considered as a deformation of $X'_{s'} = X_s$ with the differential τ_* unique when $s = o$. Then since $\eta_{f'}(\tilde{\omega}') = 0$, by (2.3) we conclude that τ actually factors through $S \subseteqq S_1$. It is then immediate to see that the pull-back $(f_{S'}, \tilde{\omega}_{S'})$ of $(f, \tilde{\omega})$ to S' is isomorphic to $(f', \tilde{\omega}')$.

3) Finally in case $s = o$ the uniqueness of τ above follows from the cohomological flatness of $\Theta_{\mathcal{X}/S}$ shown in Theorem 1 by (completely the same

argument as) Wavrik [29] or Palamodov [25]. q.e.d.

Remark 6. In the general case (i.e., the case where $\mathrm{Aut}_0 X$ is not necessarily a complex torus) $(f, \tilde{\omega})$ defined above is not necessarily the local modular family (cf. [29] [25]). However, in view of the properties 1) and 2) proved above we shall call it the (polarized) *Kuranishi family* of (X, ω).

Remark 7. The Zariski tangent space of the parameter space S at o is naturally identified with the kernel $H^1(X, \Theta_X)_\omega$ of the linear map $H^1(X, \Theta_X)\to H^2(X, \mathcal{O}_X)$ defined by the cup product with ω, considered as an element of $H^1(X, \Omega_X^1)$.

(5.2) Let (X, ω) be a polarized compact Kähler manifold such that $\mathrm{Aut}_0 X$ is a complex torus. Let $(f\colon \mathfrak{X}\to S, \tilde{\omega})$, $(X_o, \tilde{\omega}_o)=(X, \omega)$, $o\in S$, be the local modular family of (X, ω). Then the next lemma shows the topological significance of considering local modular family in the study of the local structure of $\mathfrak{M}$.

Lemma 8. *Take S sufficiently small so that $(f, \tilde{\omega})$ is locally complete at any point of S. Then the canonical map $\rho=\rho(f, \tilde{\omega})\colon |S|\to\mathfrak{M}$ is an open map so that the image $\bar{S}:=\rho(|S|)$ is an open neighborhood of (X, ω) in $\mathfrak{M}$.*

Proof. Let $U\subseteq S$ be any open subset. It suffices to show that for any polarized family of compact Kähler manifolds $(f'\colon \mathfrak{X}'\to S', \tilde{\omega}')$, $U':=\rho'^{-1}(\rho(U))$ is open in S', where $\rho'=\rho(f', \tilde{\omega}')$. Let $s'\in U'$ be an arbitrary point. Take a point $s\in U$ with $\rho'(s')=\rho(s):=m_1$. Let (X_1, ω_1) be the polarized compact Kähler manifold corresponding to m_1. Let $(f_1\colon \mathfrak{X}_1\to S_1, \tilde{\omega}_1)$, $(X_{1,o_1}, \tilde{\omega}_{1,o_1})=(X_1, \omega_1)$, $o_1\in S_1$, be the local modular family of (X_1, ω_1). (We may assume that $\mathrm{Aut}_0 X_1$ is also a complex torus.) Let $\tau\colon V\to S_1$, $\tau(s)=o_1$ (resp. $\tau'\colon V'\to S_1$, $\tau'(s')=o_1$) be the corresponding universal map, where V (resp. V') is a small neighborhood of s (resp. s').

On the other hand, by the local completeness of $(f, \tilde{\omega})$ at s we have a versal map $\tau_1\colon V_1\to S$ with $\tau_1(o_1)=s$, where V_1 is a sufficiently small neighborhood of o_1 in S_1. Then by the modularity of $(f_1, \tilde{\omega}_1)$ at o_1, $\tau\tau_1$ must be the identity of V_1. Hence if we take V' sufficiently small, $\tau'(V')\subseteq V_1\subseteq\tau(V)$. Now let $\rho_1=\rho(f_1, \tilde{\omega}_1)$. Then we have $\rho_1\tau=\rho_{|V}$ and $\rho_1\tau'=\rho'_{|V'}$. Hence $\rho'(V')\subseteq\rho_1\tau(V)=\rho(V)$, and so $V'\subseteq\rho'^{-1}(\rho(U))=U'$. Since s' was arbitrary, U' is open. q.e.d.

(5.3) Let (X, ω) be a polarized compact Kähler manifold such that $\mathrm{Aut}_0 X$ is a complex torus. Let $(f\colon \mathfrak{X}\to S, \tilde{\omega})$, $(X_o, \tilde{\omega}_o)=(X, \omega)$, $o\in S$, be the local modular family of (X, ω). Let $(f': \mathcal{X}' \to S', \tilde{\omega}')$ be any polarized deformation of $(X'_{o'}, \tilde{\omega}'_{o'})$, $o' \in S'$, with an isomorphism $b: (X'_{o'}, \tilde{\omega}'_{o'}) \cong (X, \omega)$. Then there exists a morphism $(\tilde{\tau}, \tau): (f', \tilde{\omega}') \to (f, \tilde{\omega})$ of polarized deformations, i.e., morphisms $\tilde{\tau}: \mathcal{X}' \to \mathcal{X}$ and $\tau: S' \to S$ with $\tau f' = f\tilde{\tau}$ and $\tau(o') = o$, such that $\tilde{\tau}_{o'} = b$, where $\tilde{\tau}_{o'}: (X'_{o'}, \tilde{\omega}'_{o'}) \to (X_o, \tilde{\omega}_o) = (X, \omega)$ is the isomorphism induced by $\tilde{\tau}$. Moreover τ is unique by the modularity of $(f, \tilde{\omega})$. We call this τ the *universal map associated with* $(f', \tilde{\omega}')$ *with respect to the isomorphism* b.

When $(f', \tilde{\omega}') = (f, \tilde{\omega})$ (so that $\mathcal{X} = \mathcal{X}'$ and $S' = S$), b is an automorphism of (X, ω) and the resulting $\tilde{\tau}$ and τ are automorphisms of $\mathcal{X}$ and S respectively (cf. the proof of Lemma 6). Thus letting $\delta_0(b) = \tau$ we get a map $\delta_0: \mathrm{Aut}(X, \omega) \to \mathrm{Aut}\, S\ (= \mathrm{Aut}\,(S, o))$, which turns out to be a homomorphism.

Lemma 9. $\mathrm{Aut}_0 X$ *is contained in the kernel of* δ_0.

Proof. Let $b \in \mathrm{Aut}_0 X$ be arbitrary. Since $\mathrm{Aut}\,\mathcal{X}/S$ is smooth along $\mathrm{Aut}_0 X \subseteq (\mathrm{Aut}\,\mathcal{X}/S)_o$ by Proposition 6, there exists a holomorphic section $\mu: S \to \mathrm{Aut}\,\mathcal{X}/S$ with $\mu(o) = b$. This defines an automorphism $\tilde{\tau}$ of $\mathcal{X}$ *over* S. By the definition of δ_0, this implies that $\delta_0(b)$ is the identity of S. q.e.d.

Set $H = H(X, \omega) := \mathrm{Aut}(X, \omega)/\mathrm{Aut}_0 X$. Then H is a finite group by Corollary to Proposition 2. By Lemma 9 δ_0 induces a homomorphism $\delta: H \to \mathrm{Aut}\, S$. By the definition of δ for any $h \in H$ there exists an automorphism $(\tilde{h}, \bar{h})$ of $(f, \tilde{\omega})$;

$$(\dagger) \qquad \begin{array}{ccc} \mathcal{X} & \xrightarrow{\tilde{h}} & \mathcal{X} \\ f\downarrow & & \downarrow f \\ S & \xrightarrow{\bar{h}} & S \end{array},$$

such that $q(\tilde{h}_o) = h$, where $\tilde{h}_o \in \mathrm{Aut}(X, \omega)$ is the automorphism induced by $\tilde{h}$, $q: \mathrm{Aut}(X, \omega) \to H(X, \omega)$ is the quotient homomorphism and where $\bar{h} = \delta(h)$.

On the other hand, the quotient space S/H admits a natural complex space structure so that the quotient map $\pi: S \to S/H$ is a morphism of complex spaces. Then the canonical map $\rho(f, \tilde{\omega}): |S| \to \bar{S}$ obviously factors as

$$|S| \xrightarrow{|\pi|} |S/H| \xrightarrow{\bar{\rho}} \bar{S}.$$

In the next section we shall show that $\bar{\rho}$ is homeomorphic under a suitable assumption on (X, ω).

§6. Definition and the Analytic Structure of $\mathfrak{B}$

(6.1) Let (X, ω) be a polarized compact Kähler manifold such that $\mathrm{Aut}_0 X$

is a complex torus. Let $(f\colon \mathcal{X}\to S, \tilde{\omega})$, $(X_o, \tilde{\omega}_o)=(X, \omega)$, $o\in S$, be the local modular family of (X, ω). Let $p_i\colon S\times S\to S$, $i=1, 2$, be the projection to the i-th factor. Let $(f_i\colon \mathcal{X}_i\to S\times S, \tilde{\omega}_i)$ be the pull-back of $(f, \tilde{\omega})$ via p_i. Set $I=\mathrm{Isom}_{S\times S}((\mathcal{X}_1, \tilde{\omega}_1), (\mathcal{X}_2, \tilde{\omega}_2))$. Let $\gamma\colon I\to S\times S$ be the natural morphism. γ is up to isomorphism uniquely determined by (X, ω) as a germ over $(o, o)\in S\times S$. By the definition we have the natural identification $I_{(o,o)}=\mathrm{Aut}(X, \omega)$. Using I we shall define the subset $\mathfrak{B}$ of $\mathfrak{M}$ as follows.

Definition. $\mathfrak{B}=\{(X, \omega)\in\mathfrak{M};\ \gamma$ is proper over $S\times S$ in a neighborhood of $(o, o)\}$.

In particular if $(X, \omega)\in\mathfrak{B}$, $\mathrm{Aut}(X, \omega)$ is compact and $\mathrm{Aut}_0 X$ is a complex torus. Further if $\mathrm{Aut}_0\mathcal{X}/S$ is the connected component of $\mathrm{Aut}\,\mathcal{X}/S$ containing $\mathrm{Aut}_0 X\subseteq(\mathrm{Aut}\,\mathcal{X}/S)_o$, $\mathrm{Aut}_0\mathcal{X}/S$ is proper, smooth and with connected fibers over S (cf. Theorem 1 and Proposition 6). We also remark that by Proposition 4 if $(X, \omega)\in\mathfrak{M}-\mathfrak{B}$, then X is ruled.

Now the purpose of this section is to prove the following:

Theorem 2. *The coarse $\mathfrak{B}$-moduli space (B, ψ) exists except that B is not separated.*

(6.2) In this and the next subsections we shall fix a polarized compact Kähler manifold (X, ω) in $\mathfrak{B}$ and its local modular family $(f\colon \mathcal{X}\to S, \tilde{\omega})$, $(X_o, \tilde{\omega}_o)=(X, \omega)$, $o\in S$. We shall show that $\bar{\rho}\colon |S/H|\to\bar{S}$ defined in (5.3) is homeomorphic. We set

$$R=\{(s_1, s_2)\in|S|\times|S|;\ (X_{s_1}, \tilde{\omega}_{s_1})\cong(X_{s_2}, \tilde{\omega}_{s_2})\}\ .$$

Then R defines an equivalence relation on $|S|$, and the quotient $|S|/R$ of $|S|$ with respect to this equivalence relation endowed with the quotient topology is naturally homeomorphic to $\bar{S}$ (cf. Lemma 8). Let $H=H(X, \omega)$ be as in (5.3). Then the set of connected components of $\mathrm{Aut}(X, \omega)$ is canonically indexed by H. Namely $\mathrm{Aut}(X, \omega)=\coprod_{h\in H}A^h$ with $A^h=q^{-1}(h)$, where $q\colon\mathrm{Aut}(X, \omega)\to H(X, \omega)$ is the quotient homomorphism. Let I^h be the connected component of I containing A^h with respect to the natural identification $I_{(o,o)}=\mathrm{Aut}(X, \omega)$. Then $I=\coprod_{h\in H}I^h$. (Recall that we are considering I as a germ over (o, o).)

Lemma 10. *For any $h\in H$, $\gamma(I^h)=R^h$, where $R^h\subseteq S\times S$ is the graph of $\bar{h}:=\delta(h)\in\mathrm{Aut}\,S$. Moreover the induced morphism $\gamma^h\colon I^h\to R^h$ is proper, smooth and with connected fibers. In particular $R=\cup_h|R^h|$.*

Proof. Let $(\hat{f}_i, \hat{\omega}_i)$ be the pull-back of $(f_i, \tilde{\omega}_i)$ to I^h via $\gamma|_{I^h}$. We have the canonical I^h-isomorphism λ: $(\hat{\mathcal{X}}_1, \hat{\omega}_1)\to(\hat{\mathcal{X}}_2, \hat{\omega}_2)$. On the other hand, for any $a\in I^h_{(o,o)}$ we have the canonical identification $(\hat{X}_{2,a}, \hat{\omega}_{2,a})=(X, \omega)$. In particular λ induces an isomorphism $\tilde{\lambda}_a$: $(\hat{X}_{1,a}, \hat{\omega}_{1,a})\cong(X, \omega)$. Let τ: $(I^h, a)\to(S, o)$ be the universal map associated with $(\hat{f}_1, \hat{\omega}_1)$ with respect to $\tilde{\lambda}_a$ (cf. (5.3)). Then we may write $\tau=\bar{h}p_1\gamma$ and also $\tau=p_2\gamma$; indeed as the corresponding morphism $\tilde{\tau}$: $\hat{\mathcal{X}}_1\to\mathcal{X}$ which covers τ as in (5.3) we can take, in the first case, the composition of the natural maps $\hat{\mathcal{X}}_1\to\mathcal{X}_1\to\mathcal{X}\overset{\tilde{h}}{\to}\mathcal{X}$ where $\tilde{h}$ is as in (†), and in the second case, λ followed by the natural projection $\hat{\mathcal{X}}_2\to\mathcal{X}_2\to\mathcal{X}$. Thus $\bar{h}p_1\gamma=p_2\gamma$. Since a was arbitrary, this implies that $\gamma(I^h)\subseteq R^h$.

For the other inclusion, we have only to show the existence of a holomorphic section b: $R^h\to I^h$. Let $(\tilde{h}, \bar{h})$ be as above, i.e., any automorphism of $(f, \tilde{\omega})$ defined by any element of A^h (cf. (†)). Then $\tilde{h}$ induces an isomorphism $\hat{h}$ of $(f_1, \tilde{\omega}_1)$ and $(f_2, \tilde{\omega}_2)$ over R^h with $\hat{h}_{(o,o)}\in A^h$, which in turn gives a desired holomorphic section b. Further the existence of such a section implies that I^h is a relative principal homogeneous space over R^h with respect to the relative action of $\mathrm{Aut}_0(\mathcal{X}_1/S\times S)|_{R^h}=p_1^*\mathrm{Aut}_0\mathcal{X}/S|_{R^h}$ over S. Then, since $\mathrm{Aut}_0\mathcal{X}/S$ is smooth and proper over S with connected fibers, and hence so is $p_1^*(\mathrm{Aut}_0\mathcal{X}/S)|_{R^h}$ over R^h, the same is true for $I^h\to R^h$. Finally, by what we have proved above, the last equality follows from the relation $R=\gamma(I)$. q.e.d.

From the last relation we get

Corollary. $\bar{\rho}$: $|S/H|\to\bar{S}$ *is homeomorphic.*

In view of Lemma 10, in what follows we consider R as a complex subspace (not necessarily reduced) of $S\times S$ by the equality $R=\cup_h R^h$. Then γ induces a morphism $I\to R$ of complex space (which will still be denoted by γ). Then γ is factored as

$$I=\coprod_h I^h\to\tilde{R}:=\coprod_h R^h\overset{\bar{\gamma}}{\to}R$$

where the first arrow is given by $\coprod_h\gamma^h$ and $\bar{\gamma}|_{R^h}$ is the natural inclusion $R^h\to R$. Note that $p_i\bar{\gamma}$: $\tilde{R}\to S$ is a finite unramified covering.

(6.3) We can now define a complex space structure on $\bar{S}$ (not necessarily reduced) by transplanting that of S/H via the homeomorphism $\bar{\rho}$: $|S/H|\to\bar{S}$. We want to see that at any $(X', \omega')\in\bar{S}$ this complex structure on $\bar{S}$ is canonically isomorphic to the one which is defined as above starting from the local modular family of (X', ω'). For this purpose we first note the following fact which is easily seen from our construction. Let $s\in S$. Let $H(X, \omega)_s=\{h\in H(X, \omega); \bar{h}s=s\}$. Then there exists a natural isomorphism ψ_s: $H(X, \omega)_s \xrightarrow{\sim} H(X_s, \tilde{\omega}_s)$ defined by $\psi_s(h)=q_s(I^h_{(s,s)})$, $h\in H(X, \omega)_s$, with respect to the natural identification $I_{(s,s)}=\mathrm{Aut}(X_s, \tilde{\omega}_s)$, where q_s: $\mathrm{Aut}(X_s, \tilde{\omega}_s)\to H(X_s, \tilde{\omega}_s)=\mathrm{Aut}(X_s, \tilde{\omega}_s)/\mathrm{Aut}_0 X_s$ is the natural homomorphism.

In view of this remark our remaining task is to show the next proposition, which is indeed a special case of a theorem of Palamodov (cf. [26, 7°]). Since his proof is rather involved (using his own construction of the Kuranishi family), we shall give a simpler proof in our special case.

Proposition 9. *For any $s\in S$ which is sufficiently near to $o\in S$, $(f, \tilde{\omega})$ gives the local modular family for $(X_s, \tilde{\omega}_s)$ also.*

Proof. Take S small so that $(f, \tilde{\omega})$ is locally complete at any point of S and that $(X_s, \tilde{\omega}_s)\in\mathfrak{B}$ for any $s\in S$. Fix any $s\in S$. Let $(f'\colon \mathcal{X}'\to S', \tilde{\omega}')$, $(X'_{o'}, \tilde{\omega}'_{o'})=(X_s, \tilde{\omega}_s)$, $o'\in S'$, be the local modular family for $(X_s, \tilde{\omega}_s)$. Let τ: $(S, s)\to(S', o')$ be the universal map obtained by considering $(f, \tilde{\omega})$ as a deformation of $(X_s, \tilde{\omega}_s)$. Let τ': $(S', o')\to(S, s)$ be a versal map induced by the local completeness of $(f, \tilde{\omega})$ at s. By the modularity of $(f', \tilde{\omega}')$, $\tau\tau'$ is the identity of S' at o'. In particular τ is surjective.

Thus it suffices to show that τ is an embedding, which in turn would follow if we show that the natural inclusion $\Delta_S\subseteq(\tau\times\tau)^{-1}(\Delta_{S'})$ is an equality where for $Y=S$, S', Δ_Y is the diagonal of $Y\times Y$.

Let I and I^e be as in (6.2) (for $(f, \tilde{\omega})$), where e is the identity of $H=H(X, \omega)$. Then by Lemma 10 I^e is smooth over Δ_S with $I^e_{(s,s)}=\mathrm{Aut}_0 X_s$ with respect to the natural identification $I_{(s,s)}=\mathrm{Aut}(X_s, \tilde{\omega}_s)$. Let $(f'_i\colon \mathcal{X}'_i\to S'\times S', \tilde{\omega}'_i)$ be the pull-back of $(f', \tilde{\omega}')$ to $S'\times S'$ via $p_i\colon S'\times S'\to S'$, the projection to the i-th factor. Let

$$'I = \mathrm{Isom}_{S'\times S'}((\mathcal{X}'_1, \tilde{\omega}'_1), (\mathcal{X}'_2, \tilde{\omega}'_2))\,.$$

We then get the natural decomposition $'I=\coprod {}'I^h$, $h\in H(X_s, \tilde{\omega}_s)$ as in (6.2). Further $'I^e$ is the unique connected component of $'I^e$ with $'I^e=\mathrm{Aut}_0 X_s$, where we denote the unit element of $H(X_s, \tilde{\omega}_s)$ again by e. Let $'\tilde{I}={}'I\times_{(S'\times S')}(S\times S)$ and $'\tilde{I}^e={}'I^e\times_{(S'\times S')}(S\times S)$, where $S\times S$ is over $S'\times S'$ by $\tau\times\tau$. Then $'\tilde{I}^e$ is smooth over $(\tau\times\tau)^{-1}(\Delta_{S'})$. Since we must have a natural isomorphism $I\simeq{}'\tilde{I}$ which induces the identity $I_{(s,s)}={}'\tilde{I}_{(s,s)}$ $(=\mathrm{Aut}(X_s, \tilde{\omega}_s))$ we get $I^e={}'\tilde{I}^e$ over $S\times S$. Thus $'\tilde{I}^e$ is smooth also over Δ_S. Hence $(\tau\times\tau)^{-1}(\Delta_{S'})=\Delta_S$ as was desired. q.e.d.

In what follows when we speak of the local modular family ($f\colon \mathcal{X}\to S$, $\tilde{\omega}$) we always assume that the condition of the above proposition is satisfied for any $s\in S$.

(6.4) *Proof of Theorem* 2. i) *Existence of complex structure.* By Lemma 8 and by Proposition 9 we see that $\mathfrak{B}$ is open in $\mathfrak{M}$. Let (X, ω) be any point of $\mathfrak{B}$. Let ($f\colon \mathcal{X}\to S$, $\tilde{\omega}$) be the local modular family of (X, ω). By Lemma 8 $\bar{S}=\rho(f, \tilde{\omega})(|S|)$ is an open neighborhood of (X, ω) in $\mathfrak{B}$, and by Corollary to Lemma 10 $\rho(f, \tilde{\omega})$ induces a homeomorphism $\bar{\rho}\colon |S/H|\to\bar{S}$. Then we define a complex space structure on $\bar{S}$ by transplanting the natural one on S/H via $\bar{\rho}$ so that $\rho(f, \tilde{\omega})$ is induced from a morphism of complex spaces $S\to\bar{S}$, which we shall still denote by $\rho(f, \tilde{\omega})$. By Proposition 9 together with a remark just before that proposition it follows that for each $(X', \omega')\in\bar{S}$ the complex space structure of $\bar{S}$ in a neighborhood of (X', ω') by the above procedure is canonically isomorphic to the complex space structure on $\bar{S}$ just obtained. Since $\mathfrak{B}$ is covered by open subsets of the form $\rho(f, \tilde{\omega})(|S|)$ with varying (X, ω) and $(f, \tilde{\omega})$, it follows readily that we get a global complex space structure on $\mathfrak{B}$ in this way. Let B be the resulting complex space (not necessarily separated).

ii) *Existence of the functorial morphism* $\mathcal{B}\to h_B$. Let ($g\colon \mathcal{Y}\to T$, $\tilde{\omega}$) be any polarized $\mathfrak{B}$-family of compact Kähler manifolds. Then for any $t\in T$ there exist a neighborhood $t\in U_t$ and the universal map $\tau_t\colon U_t\to S^t$, where ($f^t\colon \mathcal{X}^t\to S^t$, $\tilde{\omega}^t$) is the local modular family of $(Y_t, \tilde{\omega}_t)$. Composing this with $\rho(f^t, \tilde{\omega}^t)\colon S^t\to\bar{S}^t=S^t/H(Y_t, \tilde{\omega}_t)\subseteq B$ we have a morphism $\bar{\tau}_t\colon U_t\to B$. For any $t'\in U_t$, if we take a neighborhood $U_{t'}$ of t' in T and obtain a morphism $\bar{\tau}_{t'}\colon U_{t'}\to B$ by the same procedure as above, then from Proposition 9 it follows that $\bar{\tau}_t$ and $\bar{\tau}_{t'}$ coincide at t'. Since T is covered by U_t as above with t varying on T, from this we see readily that $\bar{\tau}_t$, $t\in T$, patch together to give a global morphism $\bar{\tau}\colon T\to B$. Then the correspondence $(g, \tilde{\omega})\to\bar{\tau}$ defines a desired morphism of functors $\psi\colon \mathcal{B}\to h_B:=\mathrm{Hom}(\quad, B)$.

iii) We show that the pair (B, ψ) is the desired coarse $\mathfrak{B}$-moduli space. Let B_1 be any complex space with a functorial morphism $\psi_1\colon \mathcal{B}\to h_{B_1}$ Take any $(X, \omega)\in\mathfrak{B}$. Let ($f\colon \mathcal{X}\to S$, $\tilde{\omega}$) be the local modular family of (X, ω). Then ψ_1 defines a unique morphism $\tau_1\colon S\to B_1$ associated to $(f, \tilde{\omega})$. We shall show that τ_1 factors through $\rho(f, \tilde{\omega})\colon S\to\bar{S}$. Take any automorphism $(\tilde{h}, \bar{h})$ of $(f, \tilde{\omega})$ as in (†). This then defines an isomorphism over S of $(f, \tilde{\omega})$ and the pull-back $\bar{h}^*(f, \tilde{\omega})$ of $(f, \tilde{\omega})$ by $\bar{h}$. Since the universal map associated to $\bar{h}^*(f, \tilde{\omega})$ is $\tau_1\bar{h}$, we must have $\tau_1\bar{h}=\tau_1$. Our assertion thus follows, $(\tilde{h}, \bar{h})$ being arbitrary. Let $\bar{\tau}_1=\bar{\tau}_1(X, \omega)\colon \bar{S}\to B_1$ be the induced morphism. Then by the same argument as in ii) all the $\bar{\tau}_1(X, \omega)$ defined for varying (X, ω) and $(f, \tilde{\omega})$ as above patch together to define a global morphism $\bar{\tau}_1\colon B\to B_1$. By our construction of ψ we conclude easily that $\psi_1=\mathrm{Hom}(\ \ , \bar{\tau}_1)\psi$. q.e.d.

§7. Proof of Theorem

(7.1) First we consider the problem of separation.

Proposition 10. *Let* $\mathfrak{R}=\{(X, \omega)\in\mathfrak{M}; X$ *is ruled*$\}$. *Then* $\mathfrak{M}-\mathfrak{R}$ *is separated with respect to the canonical topology of* $\mathfrak{M}$.

Proof. Let $\Delta\subseteq(\mathfrak{M}-\mathfrak{R})\times(\mathfrak{M}-\mathfrak{R})$ be the diagonal. Let $(m_1, m_2)\in\Delta^-$, where Δ^- is the closure of Δ. Take a sequence (m_{1k}, m_{2k}), $k=1, 2, \cdots$, of points of Δ (so that $m_{1k}=m_{2k}$), converging to (m_1, m_2). Let $m_i=(X_i, \omega_i)$. Let $(f_i\colon \mathcal{X}_i\to S_i, \tilde{\omega}_i)$ be the local modular family of (X_i, ω_i) with the base point $s_{i0}\in S_i$. Then by Lemma 8 there exists a sequence $\{s_{ik}\}$ of points of S_i converging to s_{i0} and with $\rho_i(s_{ik})=m_{ik}$. Let $S=S_1\times S_2$ and $p_i\colon S\to S_i$ be the natural projection. Let $(\hat{f}_i\colon \hat{\mathcal{X}}_i\to S, \hat{\omega}_i)$ be the pull-back of $(f_i, \tilde{\omega}_i)$ to S by p_i. Then the set $S_1=\{(s_1, s_2)\in S; (X_{1s_1}, \tilde{\omega}_{1s_1})\cong(X_{2s_2}, \tilde{\omega}_{2s_2})\}$ is a closed analytic subset of S by Corollary to Proposition 4. Hence $(s_{10}, s_{20})\in S_1$ since $(s_{1k}, s_{2k})\in S_1$. Namely $m_1=\rho_1(s_{10})=\rho_2(s_{20})=m_2$. Thus $\Delta^-=\Delta$ since $\mathfrak{M}\times\mathfrak{M}$ satisfies the first countability axiom as follows from Lemma 8. q.e.d.

The following is a refinement of the main result of [8].

Proposition 11. *Let* $(f\colon \mathcal{X}\to S, \tilde{\omega})$ *be a polarized family of compact Kähler manifolds with* S *connected. If* X_o *is uniruled for some* $o\in S$, *then* X_s *is uniruled for any* $s\in S$.

Proof. By [8] the proposition is true if each point $s\in S$ admits a neighborhood $s\in U$ such that $f^{-1}(U)$ is Kähler. We shall prove the proposition by reducing to this result. Let $D=\{t\in\boldsymbol{C}; |t|<1\}$ be the unit disc. Fix an arbitrary point $\xi\in D-\{0\}$. Then, since S is connected, for any $s\in S$ we can find a finite number of morphisms $h_i\colon D\to S$, $i=1, \cdots, m$, such that $h_1(0)=o$, $h_i(\xi)=h_{i+1}(0)$, $1\leqq i\leqq m-1$, and $h_m(\xi)=s$. Then it suffices to show inductively that $X_{h_i(\xi)}$ is uniruled for any i. Then the pull-back by h_i reduces the proof to the case where $S=D$. In this case, however, f satisfies the condition above by virtue of Proposition 5. q.e.d.

Proof of Theorem. By Proposition 11 $\mathfrak{A}$ is a union of connected components of $\mathfrak{M}$. In particular it is open in $\mathfrak{M}$. Since $\mathfrak{A}\subseteqq\mathfrak{M}-\mathfrak{R}$ by the definition of $\mathfrak{A}$ and $\mathfrak{R}$, by Proposition 10 $\mathfrak{A}$ is separated. Finally from Proposition 4 it follows that $\mathfrak{A}\subseteqq\mathfrak{B}$. Thus Theorem follows from Theorem 2.

(7.2) We shall give an immediate application of Theorem to local isomorphy problem of polarized families of compact Kähler manifolds.

Proposition 12. *Let* $(f_i\colon \mathscr{X}_i\to T, \tilde{\omega}_i)$, $i=1, 2$, *be polarized families of compact Kähler manifolds. Suppose that* $X_{i,t}$ *is not ruled for any* $t\in T$, *or* $X_{i,t}$ *is not uniruled for some* $t\in T$ *and* T *is reduced and connected. Suppose further that there exists a dense subset* $V\subseteqq T$ *such that* $(X_{1t}, \tilde{\omega}_{1t})\cong(X_{2t}, \tilde{\omega}_{2t})$ *for all* $t\in V$. *Then* $(X_{1t}, \tilde{\omega}_{1t})\cong(X_{2t}, \tilde{\omega}_{2t})$ *for all* $t\in T$. *Further if* T *is locally irreducible, then the two families are locally isomorphic to each other.*

Proof. By our assumption and by Proposition 11 $\rho(f_i, \tilde{\omega}_i)(|T|)\subseteqq\mathfrak{B}-\mathfrak{R}$. Since $\mathfrak{B}-\mathfrak{R}$ is separated by Proposition 10 and since $\rho(f_1, \tilde{\omega}_1)(t)=\rho(f_2, \tilde{\omega}_2)(t)$ for all $t\in V$, $\rho(f_1, \tilde{\omega}_1)=\rho(f_2, \tilde{\omega}_2)$ on the whole T. This proves the first assertion. Next, suppose that T is locally irreducible. Let $t\in T$ be an arbitrary point. Let $(f\colon \mathscr{X}\to S, \tilde{\omega})$, $(X_o, \tilde{\omega}_o)=(X_{1t}, \tilde{\omega}_{1t})$, $o\in S$, be the local modular family of $(X_{1t}, \tilde{\omega}_{1t})$. We have the associated universal map $\tau_i\colon U\to S$, $\tau_i(t)=o$, $i=1, 2$ (with respect to some isomorphism $(X_{2t}, \tilde{\omega}_{2t})\cong(X_{1t}, \tilde{\omega}_{1t})$ for τ_2 (cf. (5.3)), where U is a small irreducible neighborhood of t. Let $\rho=\rho(f, \tilde{\omega})\colon S\to\bar{S}\cong S/H$ be the canonical map, where $H:=H(X_{1t}, \tilde{\omega}_{1t})$. By the first assertion we have $\rho\tau_1=\rho\tau_2$. This implies that $(\tau_1\times\tau_2)(U)\subseteqq R=\bigcup_{h\in H}R^h\subseteqq S\times S$ in the notations of (6.2). Since U is irreducible, $(\tau_1\times\tau_2)(U)\subseteqq R^h$ for some h. Since $I^h\to R^h$ is smooth by Lemma 10, $\tau_1\times\tau_2$ lifts to a morphism $\lambda\colon U\to I^h$. Hence the two families are isomorphic over U.

Remark 8. The final assertion is clearly false if T is not locally irreducible (cf. [15, Cor. 7.3]). For a related result see [30].

We also prove the next result concerning a jumping phenomenon of complex structures, which is actually an application of Theorem 1.

Proposition 13. *Let* $D=\{t\in \boldsymbol{C}; |t|<1\}$ *be the unit disc. Let* $f\colon \mathscr{X}\to D$ *be a proper smooth morphism with connected fibers. Suppose that* X_0 *is Kähler and* X_t *are isomorphic to each other for all* $t\neq 0$. *Then either* $X_0\ncong X_t$, $t\neq 0$, *or* X_0 *is ruled.*

Proof. Suppose that $X_0\ncong X_t$. Then we have $\dim H^0(X_0, \Theta_{X_0})>\dim H^0$

(X_t, Θ_{X_t}) for $t \neq 0$ (cf. [14, Prop. 5.6]). Then by Theorem 1 $\mathrm{Aut}_0 X_0$ is not a complex torus. Then by [6, Prop. 5.10] we see that X_0 is ruled. q.e.d.

Note that the above proof gives more information than the proposition itself.

§8. The Case of Complex Tori and K3 Surfaces

We are interested in the coarse moduli space for non-algebraic manifolds. Typical examples of such are complex tori and K3 surfaces. In these case the coarse moduli space can be obtained directly by considering the period maps. Let $\mathfrak{M}_T = \{(X, \omega); X \text{ is a complex torus}\}$ and $\mathfrak{M}_K = \{(X, \omega); X \text{ is a K3 surface}\}$. Then $\mathfrak{M}_T$ and $\mathfrak{M}_K$ are union of connected components of $\mathfrak{A}$ in Theorem. In this section we shall give a rather explicit description of the structure of the corresponding moduli spaces $\mathfrak{M}_T$ and $\mathfrak{M}_K$ by summarizing the known results on these spaces.

(8.1) *Case of complex tori* (cf. [15]). Fix once and for all a real vector space W of even dimension $2n$. Let Q be a non-degenerate skew-symmetric bilinear form on W. Let $W_C = W \otimes_R C$. Q extends naturally to W_C. Let $Gr(n, W_C)$ be the Grassmann manifold of n-dimensional complex linear subspaces of W_C. We identify a point $S \in Gr(n, W_C)$ with the corresponding subspace $S \subseteq W_C$. We define a submanifold D_Q of $Gr(n, W_C)$ by

$$D_Q = \{S \in Gr(n, W_C); Q(S, S) = 0, Q(S, \bar{S}) > 0\} .$$

Here $Q(S, S) = 0$ means that $Q(w_1, w_2) = 0$ for any $w_1, w_2 \in S$ and $Q(S, \bar{S}) > 0$ means that the Hermitian form $\sqrt{-1}Q(w_1, \bar{w}_2)$, $w_1, w_2 \in S$, is positive definite on S. Let G_Q be the orthogonal group of Q; $G_Q = \{g \in GL(W); Q(gw, gw) = Q(w, w), w \in W\}$. Then G_Q acts naturally on D_Q. Since Q takes the form $Q = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ with respect to a suitable base of W, the pair (D_Q, G_Q) is isomorphic as a transformation space to the pair $(H_n, Sp(n, \mathbb{R}))$, where H_n is the Siegel upper half space of degree n on which the real symplectic group $Sp(n, \mathbb{R})$ acts in the usual manner. Fix now a lattice Λ in W. Let

$$\Gamma = \{g \in GL(W); g\Lambda = \Lambda\} \text{ and } \Gamma_Q = \Gamma \cap G_Q .$$

Then Γ_Q is a discrete subgroup of G_Q so that its action on D_Q is properly discontinuous. Hence $M_Q := D_Q / \Gamma_Q$ is naturally a normal analytic space.

Let $\mathfrak{Q}$ be the set of all nondegenerate skew-symmetric bilinear forms on

W. $\mathfrak{Q}$ is a homogeneous space under the natural action of $GL(W)$; the stabilizer at $Q \in \mathfrak{Q}$ is just G_Q.

Proposition 14. *For any $Q \in \mathfrak{Q}$, M_Q is isomorphic to a connected component of $\mathfrak{M}_T$. Conversely, any connected component of $\mathfrak{M}_T$ is isomorphic to M_Q for some $Q \in \mathfrak{Q}$ which is determined uniquely up to the action of Γ on Q. Thus $M_T = \coprod_Q M_Q$, where the sum is taken over a complete set of representatives of $\mathfrak{Q}/\Gamma$.*

We omit the proof, only indicating how one associates to each point of M_Q a polarized complex torus. For each $S \in D_Q$ we have the direct sum decomposition $W_C = S \oplus \bar{S}$. Let Λ_S be the projection of Λ to S which is a lattice in S. Then $T := S/\Lambda_S$ is a complex torus with the natural isomorphism $W_C \equiv \Lambda \otimes \boldsymbol{R} \cong H_1(T_S, \boldsymbol{R})$, and then Q, as a skew-symmetric form on W, defines an element $\omega_S \in H^2(T_S, \boldsymbol{R}) \cong \wedge^2 H^1(T_S, \boldsymbol{R})$. Moreover the condition that $S \in D_Q$ implies that ω_S is a Kähler class. Then we associate to the point $\pi(S) \in M_Q$ the polarized complex torus (T_S, ω_S) whose isomorphism class is uniquely determined by $\pi(S)$, where $\pi: D_Q \to M_Q$ is the natural projection.

(8.2) *Case of K3 surfaces.* For general facts on K3 surfaces we refer the reader to [2] and [27]. Let $(L, \langle\ ,\ \rangle)$ be an even unimodular Euclidian lattice of signature (3,19), i.e., L is a free abelian group of rank 22 and $\langle\ ,\ \rangle$ is an even unimodular symmetric bilinear form on L such that its extension to $L \otimes \boldsymbol{R}$ has signature (3,19). Such an $(L, \langle\ ,\ \rangle)$ is up to isomorphisms unique. Let $\boldsymbol{P}(L \otimes \boldsymbol{C})$ be the projective space of lines of $L \otimes \boldsymbol{C}$. Let $M = \{x \in \boldsymbol{P}(L \otimes \boldsymbol{C});$ $\langle x, x\rangle = 0, \langle x, \bar{x}\rangle > 0\}$, where the natural extension of $\langle\ ,\ \rangle$ to $L \otimes \boldsymbol{C}$ is still denoted by $\langle\ ,\ \rangle$. M is a locally closed submanifold of $\boldsymbol{P}(L \otimes \boldsymbol{C})$ of codimension 1.

Let $V = \{\omega \in L \otimes \boldsymbol{R}; \langle\omega, \omega\rangle > 0\}$. For any $a \in L \otimes \boldsymbol{R}$ we set $D_a = \{x \in M;$ $\langle a, x\rangle = 0\}$. D_a is an analytic submanifold of codimension 1 in M if $a \neq 0$. Let $G_0 = \{g \in GL(L \otimes \boldsymbol{R}); \langle gx, gx\rangle = \langle x, x\rangle, x \in L \otimes \boldsymbol{R}\}$. Let $G_a = \{g \in G_0;\ ga = a\}$. Then G_a acts naturally on D_a. Further if $a = \omega \in V$ we have an isomorphism $(D_\omega, G_\omega) \cong (D_{\mathrm{IV}}, O(2, 19))$ as transformation spaces, where D_{IV} is the bounded symmetric domain of type IV (cf. [14]). Let $\Gamma = GL(L) \subseteq GL(L \otimes \boldsymbol{R})$, $\Gamma_0 = \Gamma \cap G_0$ and $\Gamma_\omega = G_\omega \cap \Gamma$. Then Γ_ω is a discrete subgroup of G_ω and hence acts properly discontinuously on D_ω. Let $B_\omega = \{a \in L;\ \langle a, a\rangle = -2, \langle\omega, a\rangle = 0\}$ and $U_\omega = D_\omega - \bigcup_{a \in B_\omega} D_a$.

Proposition 15. *For any $\omega \in V$, U_ω is a G_ω-invariant Zariski open subset* *of D_ω such that U_ω/Γ_ω is naturally isomorphic to a connected component of $\mathfrak{M}_K$. Conversely, any connected component of $\mathfrak{M}_K$ is of this form for some $\omega \in V$ and ω is determined uniquely up to the natural action of Γ_0 on V; thus $\mathfrak{M}_K = \coprod_\omega U_\omega/\Gamma_\omega$, where the summation is over a complete set of representatives of V/Γ_0.*

a. First we show that U_ω is Zariski open in D_ω. Let $D_{\omega,a} = D_\omega \cap D_a$. Then it suffices to show that $\{D_{\omega,a}\}_{a \in B_\omega}$ is locally finite, i.e., for any $x \in D_\omega$ there exists a neighborhood $x \in W$ such that $\{a \in B_\omega; D_a \cap W \neq \phi\}$ is a finite set. For any $x \in D_\omega$ let x' be any nonzero point on the line in $L \otimes \boldsymbol{C}$ corresponding to x. Then Re x', Im x' and ω span a 3-dimensional subspace F_x of $L \otimes \boldsymbol{R}$ on which $\langle\ ,\ \rangle$ is positive definite, where Re, Im denote the real and imaginary parts respectively (cf. [27]). Let $E_x = F_x^\perp$ be the orthogonal of F_x with respect to $\langle\ ,\ \rangle$. Then $\{E_x\}_{x \in D_\omega}$ is a C^∞ family of subspaces of $L \otimes \boldsymbol{R}$ on which $\langle\ ,\ \rangle$ is negative definite. Hence for any $x_0 \in D_\omega$ and any relatively compact neighborhood W of x_0 in D_ω, $B_\omega \cap C_W$ is a finite set, where $C_W = \bigcup_{x \in W} E_x \subseteq L \otimes \boldsymbol{R}$. Thus the assertion follows since, for $x \in W$ and $a \in B_\omega$, $x \in D_{\omega,a}$ if and only if $a \in E_x$.

b. *A marked K3 surface* is a pair (X, ψ) consisting of a K3 surface and an isomorphism $\psi: H^2(X, \boldsymbol{Z}) \to L$ of Euclidian lattices, where the bilinear form $\langle\ ,\ \rangle_X$ on $H^2(X, \boldsymbol{Z})$ is given via the cup product. Define a functor F: An $\to$ Sets by $F(S)$ = the set of isomorphism classes (in the obvious sense) of the pairs $(f: \mathcal{X} \to S, \varphi)$, where f is a proper smooth morphism of complex spaces with each fiber X_s a K3 surface and $\varphi: R^2 f_* \boldsymbol{Z} \cong L \times S$ is a trivialization of the local system $R^2 f_* \boldsymbol{Z}$. For any $(f, \varphi) \in F(S)$, let $\varphi_{\boldsymbol{R}}: R^2 f_* \boldsymbol{R} \xrightarrow{\sim} (L \otimes \boldsymbol{R}) \times S$ be the induced isomorphism. For any $\omega \in V$ let $\tilde{\omega} = \varphi_{\boldsymbol{R}}^{-1}(\omega \times S) \in \Gamma(S, R^2 f_* \boldsymbol{R})$. Let $\tilde{\omega}_s \in H^2(X_s, \boldsymbol{R})$ be induced by $\tilde{\omega}$.

Define then the subfunctor F_ω of F by $F_\omega(S) = \{(f, \varphi) \in F(S); (f, \tilde{\omega})$ is a polarized family of Kähler K3 surfaces$\}$. A complex space which represents the functor F_ω will be called the fine moduli space of polarized marked K3 surfaces with polarization ω.

Proposition 16. *For any $\omega \in V$ the fine moduli space for polarized marked K3 surfaces of polarization type ω exists and is naturally isomorphic to U_ω above.*

Proof. By Burns-Rappoport [2] F is represented by a 20-dimensional complex manifold T which is not separated. Let $(f: \mathcal{X} \to T, \varphi)$ be the universal family. Then define the subspace T_ω of T by $\eta(\tilde{\omega})(t) = 0$, where $\eta: R^2 f_* \boldsymbol{R} \to R^2 f_* \mathcal{O}_{\mathcal{X}}$ is the natural homomorphism. Let $T_\omega^\circ = \{t \in T_\omega;\ \tilde{\omega}_t$ is a Kähler class$\}$. Then T_ω° is open in T_ω by Lemma 1 and it is immediate to see that T_ω°

represents F_ω, where the universal family is the restriction of (f, φ) to T_ω°. Let $p: T\to M$ be the period map associated to (f, φ). Then for the second assertion it suffices to show that $p(T_\omega^\circ)\subseteq U_\omega$ and $p|_{T_\omega^\circ}: T_\omega^\circ\to U_\omega$ is isomorphic.

First, if $p(t)\in D_a$ for some $a\in B_\omega$, then $a_t:=\varphi_t^{-1}(a)$ is of type (1,1) in $H^2(X_t, \boldsymbol{Z})$ and then by Riemann-Roch either a_t or $-a_t$ is represented by an effective curve. Then $\langle\tilde{\omega}_t, a_t\rangle_{X_t}\neq 0$ because $\tilde{\omega}_t$ is a Kähler class. This contradicts the fact that $a\in B_\omega$. Thus $p(T_\omega^\circ)\subseteq U_\omega$. Next, p is locally an embedding by the local Torelli theorem (cf. [2]). Finally, the surjectivity can be seen as follows. For any $m\in U_\omega$, by Todorov [27] there exists a marked Kähler K3 surface (X, ψ) with its period m. Since $m\in U_\omega$, by the proof of [27, 3.5], by changing the marking if necessary we may assume that $\psi^{-1}(\omega)$ belongs to the Kähler cone $V_X:=\{x\in H^2(X, \boldsymbol{R}); \langle x, x\rangle_X>0, \langle x, c\rangle_X>0$ for any $c\in H^2(X, \boldsymbol{Z})$ with $\langle c, c\rangle_X=-2$ which is represented by an effective curve on $X\}$ of X. Then by Theorem 3 of [27] (cf. [20] for the proof) $\psi^{-1}(\omega)$ is a Kähler class, i.e., the point $t\in T$ corresponding to (X, ψ) belongs to T_ω° and we have $p(t)=m$. q.e.d.

c. From the above proposition and the global Torelli theorem the first and the second assertions of Proposition 15 follow. The last assertion follows from the next remark. Let $\mathfrak{M}_{K,\alpha}$ be any connected component of $\mathfrak{M}_K$. Let $\Omega_\alpha=\{\omega\in V; \mathfrak{M}_{K,\alpha}\cong U_\omega/\Gamma_\omega\}$. Fix any $(X, \omega_X)\in\mathfrak{M}_{K,\alpha}$. Then $\omega\in\Omega_\alpha$ if and only if there exists a marking $\psi: H^2(X, \boldsymbol{Z})\to L$ such that $\psi_{\boldsymbol{R}}(\omega_X)=\omega$.

Remark 9. In Propositions 14 and 15 Γ is countable so that for 'general' $Q\in\mathfrak{Q}$ (resp. $\omega\in V$), the corresponding Γ_Q (resp. Γ_ω) reduces to the identity and hence $M_Q\cong H_n$ (resp. $M_\omega\cong D_{\mathrm{IV}}$). In particular there is no analytic compactification of M_Q or M_ω in contrast to the algebraic case where the Baily-Borel compactification is available.

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