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Stability of Kähler Metrics in Deformations of Non-Compact Complex Manifolds of Dimension Two

Dedicated to Professor S. Nakano on his 60th birthday

By

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Introduction

In deformations of non-compact complex manifolds, not much is known about the stability theorem. As in deformations of compact complex manifolds (see [5]), it seems to be a fundamental problem to establish some stability theorem in the non-compact case. In this paper, we prove the following stability theorem.

Main Theorem. *Suppose π: $\mathcal{M}\to B$ is a differentiable family of non-compact complex manifolds of dimension two over a ball B centered at the origin of $\boldsymbol{R}^m$ and the fibre $M_0=\pi^{-1}(0)$, $t=0\in B$, is provided with a Kähler metric. Then for any relatively compact domain X_0 of M_0, any sufficiently small deformation $X_t\Subset M_t=\pi^{-1}(t)$, $t\in B$, of X_0 admits a Kähler metric. Moreover for given any Kähler metric on M_0, we can choose a Kähler metric on each domain X_t which depends differentiably on t and coincides on X_0 with the given Kähler metric.*

In the case π: $\mathcal{M}\to B$ being a differentiable family of compact complex manifolds of arbitrary dimension, the Main Theorem was proved by Kodaira and Spencer for M_t using an elliptic differential operator of fourth order ([5]). On the other hand, in the case $\dim_{\boldsymbol{C}} M_t=1$, the Main Theorem is more or less known as a special case of the pseudo-rigidity of Stein manifolds of arbitrary dimension ([1]). Our method of the proof is based on the theory of elliptic differential equations of second order on bounded domains with smooth bound-

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ary. But our method can not apply to the case $\dim_C M_t \geqq 3$.

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§ 1. Preliminaries

Let $\pi: \mathcal{M} \to B$ be a differentiable family of non-compact complex manifolds of dimension n over a ball B centered at the origin of $\boldsymbol{R}^m$ where π is a surjective differentiable map of maximal rank. We denote $M_t = \pi^{-1}(t)$ for $t \in B$. Let $\pi: \mathcal{X} \to B$ be a differentiable family of bounded domains with smooth boundary i.e. each domain $X_t = \pi^{-1}_{|\mathcal{X}}(t)$ is relatively compact in M_t and its boundary ∂X_t is defined by a real valued C^∞-function h_t on M_t depending differentiably on t in such a way that $X_t = \{h_t > 0\}$ and the gradient of h_t nowhere vanishes on ∂X_t. Then replacing each M_t by a relatively compact neighborhood of $\bar{X}_t = X_t \cup \partial X_t$ and shrinking B arbitrarily, we can assume that *there exists a diffeomorphism* $f: \mathcal{M} \to M_0 \times B$ *such that* i) $p \circ f = \pi$ $(p: M_0 \times B \to B)$ ii) *each restriction f_t of f onto M_t yields a diffeomorphism from M_t onto M_0 and $f_0 = $identity* iii) *each f_t maps ∂X_t diffeomorphically onto ∂X_0.* We set $f_{st} = f_s^{-1} \circ f_t$ for s and $t \in B$. From now on, we fix the above diffeomorphism f.

We denote by $\Theta \to \mathcal{M}$ the complex vector bundle of holomorphic tangent vectors along the fibres of M. We denote by Θ^*, $\bar{\Theta}$ and $\wedge^p \Theta$, the dual, the conjugate and p-tuple exterior product of Θ. Let $\Theta^*(p, q) = (\wedge^p \Theta^*) \wedge (\wedge^q \bar{\Theta}^*)$ and let $\Theta_t^*(p, q)$ be the restriction of $\Theta^*(p, q)$ to the fibre M_t. We denote by $C^{p,q}(M_t)$ (resp. $C^{p,q}(X_t)$) the space of sections of class C^∞ of $\Theta_t(p, q)$ over M_t (resp. X_t) and set $C^{p,q}(\bar{X}_t) = $Image $(C^{p,q}(M_t) \to C^{p,q}(X_t))$. We denote by $C^{p,q}(\mathcal{M})$ (resp. $C^{p,q}(\mathcal{X})$) the space of sections of class C^∞ of $\Theta^*(p, q)$ over $\mathcal{M}$ (resp. $\mathcal{X}$) and set $C^{p,q}(\bar{\mathcal{X}}) = $Image $(C^{p,q}(\mathcal{M}) \to C^{p,q}(\mathcal{X}))$.

Now we introduce hermitian metrics $\{ds_t^2\}_{t \in B}$ on the fibres M_t which depend differentiably on t and from now on we fix this family of hermitian metrics. Then the inner product $(\ ,\ )_t$ on the space $C^{p,q}(\bar{X}_t)$ is defined by ds_t^2 as usual. Let $L^{p,q}(X_t)$ be the Hilbert space obtained by completing the space of sections in $C^{p,q}(X_t)$ with compact supports under the norm $\|\ \|_t = (\ ,\ )_t$. Let d_t, ∂_t and $\bar{\partial}_t$ be the d-operator, the ∂-operator and the $\bar{\partial}$-operator on M_t with $d_t = \partial_t + \bar{\partial}_t$. We denote by ϑ_t the formal adjoint operator of $\bar{\partial}_t$. We denote again by $\bar{\partial}_t$ the closed maximal extension of the original $\bar{\partial}_t$ and denote by $\bar{\partial}_t^*$ the adjoint operator of $\bar{\partial}_t$ in the Hilbert spaces $L^{p,q}(X_t)$ respectively. Moreover we consider the restricted Laplace-Beltrami operator $\boldsymbol{L}_t = \bar{\partial}_t \bar{\partial}_t^* + \bar{\partial}_t^* \bar{\partial}_t$ in the spaces $L^{p,q}(X_t)$. These operators are densely defined closed operators. For a given densely defined operator $\boldsymbol{T}$: $L^{p,q}(X_t) \rightarrow L^{r,s}(X_t)$, we denote the domains, ranges and nullities of $\boldsymbol{T}$ and its adjoint operator $\boldsymbol{T}^*$ by $D_T^{p,q}$, $D_{T^*}^{r,s}$, $R_T^{r,s}$, $R_{T^*}^{p,q}$, $N_T^{p,q}$ and $N_{T^*}^{r,s}$.

We define the subspace $B^{p,q}(\bar{X}_t)$ of $C^{p,q}(\bar{X}_t)$ by

$$B^{p,q}(\bar{X}_t) := \{\varphi_t \in C^{p,q}(\bar{X}_t) \mid \partial_t h_t \wedge *_t \varphi_t = 0 \text{ on } \partial X_t\}$$

where $*_t$ is the star operator with respect to ds_t^2. The space $B^{p,q}(\bar{X}_t)$ describes the boundary condition of the $\bar{\partial}$*-Neumann problem* as follows (see [2] (1.3.2) and (1.3.5) Propositions).

$$C^{p,q}(\bar{X}_t) \cap D_{L_t}^{p,q} = \{\varphi_t \in B^{p,q}(\bar{X}_t) \text{ and } \bar{\partial}_t \varphi_t \in B^{p,q+1}(\bar{X}_t)\} \tag{1.1}$$

and $\boldsymbol{L}_t = \square_t$ *on* $C^{p,q}(\bar{X}_t) \cap D_{L_t}^{p,q}$, *where* $\square_t = \bar{\partial}_t \vartheta_t + \vartheta_t \bar{\partial}_t$ *is the Laplace-Beltrami operator associated to* ds_t^2.

In particular, when $q = n$ ($n = \dim_C M_t$), by (1.1) we obtain

$$C^{p,n}(\bar{X}_t) \cap D_{L_t}^{p,n} = \{\varphi_t \in C^{p,n}(\bar{X}_t) \mid \varphi_t = 0 \text{ on } \partial X_t\} . \tag{1.2}$$

Namely in this case the $\bar{\partial}$*-Neumann boundary conditions* reduce to the *Dirichlet boundary condition*. Hence the $\bar{\partial}$*-Neumann problem* for (p, n) forms on X_t is *solvable* (see Sect. 2., Proposition 1 and Theorem 2). Moreover this boundary condition is invariant under the transformation f_t^*.

Lastly we define the hermitian form $\boldsymbol{Q}_t$ on the space $B^{p,q}(\bar{X}_t)$ by

$$\boldsymbol{Q}_t(\varphi_t, \psi_t) = (\bar{\partial}_t \varphi_t, \bar{\partial}_t \psi_t)_t + (\vartheta_t \varphi_t, \vartheta_t \psi_t)_t + (\varphi_t, \psi_t)_t$$

for φ_t, $\psi_t \in B^{p,q}(\bar{X}_t)$ and $t \in B$. Here we remark that $B^{p,q}(\bar{X}_t) = C^{p,q}(\bar{X}_t) \cap D_{\bar{\partial}_t^*}^{p,q}$ and $\bar{\partial}_t^* = \vartheta_t$ on $B^{p,q}(\bar{X}_t)$ (see [2] (1.3.2) Proposition).

§ 2. The Parametrized $\bar{\partial}$-Neumann Problem

In this section, using the solvability of the $\bar{\partial}$-Neumann problem for (p, n) forms on X_t and the vanishing theorem for L^2-harmonic (p, n) forms on X_t, we show that the *Neumann operator* $\boldsymbol{N}_t$ depends differentiably on t. We begin with the following estimate (see [2] (3.2.13) Corollary).

Proposition 1 (*a priori estimate for* (p, n) *forms*). *There exists a continuous function* $C(t) > 0$ *such that*

$$\|\varphi_t\|_{1,t}^2 \leqq C(t)\, \boldsymbol{Q}_t(\varphi_t, \varphi_t), \quad \varphi_t \in B^{p,n}(\bar{X}_t),\ t \in B . \tag{2.1}$$

Here we denote by $\|\ \|_{\nu,t}$ the Sobolev ν-norm on the spaces $C^{p,q}(\bar{X}_t)$, $\nu=1, 2, \cdots$. The following theorem is deduced from the above proposition (see [2]).

Theorem 2 (*Solvability of the $\bar\partial$-Neumann problem for (p, n) forms*). *For each $t\in B$, the nullity $N^{p,n}_{L_t}$ of $\boldsymbol{L}_t=\bar\partial_t\,\bar\partial^*_t+\bar\partial^*_t\,\bar\partial_t$ is a finite dimensional subspace of $C^{p,n}(\bar{X}_t)$ and there exists a unique bounded self-adjoint operator $\boldsymbol{N}_t$: $L^{p,n}(X_t)\to L^{p,n}(X_t)$, which is called the Neumann operator associated to the operator $\boldsymbol{L}_t$, such that*

1) $D^{p,n}_{N_t}=L^{p,n}(X_t)$, $R^{p,n}_{N_t}\hookrightarrow D^{p,n}_{L_t}$, $R^{p,n}_{N_t}\perp N^{p,n}_{L_t}$ *and* $N^{p,n}_{N_t}=N^{p,n}_{L_t}$

2) *for any* $\alpha_t\in L^{p,n}(X_t)$, $\alpha_t=\bar\partial_t\,\bar\partial^*_t\,\boldsymbol{N}_t\,\alpha_t+\boldsymbol{H}_t\,\alpha_t$, *where $\boldsymbol{H}_t$ is the orthogonal projection onto the space $N^{p,n}_{L_t}$.*

3) $\boldsymbol{N}_t\boldsymbol{L}_t=\boldsymbol{L}_t\boldsymbol{N}_t=I-\boldsymbol{H}_t$ *on* $D^{p,n}_{L_t}$ *and* $\boldsymbol{P}_t=I-\bar\partial^*_t\boldsymbol{N}_t\bar\partial_t$ *on* $D^{p,n-1}_{\bar\partial_t}$, *where $\boldsymbol{P}_t$ is the orthogonal projection onto the space $N^{p,n-1}_{\bar\partial_t}$*

4) $\boldsymbol{N}_t$ *maps* $C^{p,n}(\bar{X}_t)$ *into* $C^{p,n}(\bar{X}_t)$

5) *for any non-negative integer ν, there exists a continuous function $C_\nu(t)>0$ such that*

$$\|\varphi_t\|_{\nu+2,t}\leqq C_\nu(t)\,(\|\Box_t\,\varphi_t\|_{\nu,t}+\|\varphi_t\|_t) \qquad (2.2)$$
$$\varphi_t\in C^{p,n}(\bar{X}_t)\cap D^{p,n}_{L_t}\ \textit{and}\ t\in B\,.$$

We first prove the following.

Proposition 3. *For each $t\in B$, $\boldsymbol{L}_t$: $L^{p,n}(X_t)\to L^{p,n}(X_t)$ is surjective and there exist a neighborhood $W_t\Subset B$ of t and a positive constant C_t such that*

$$\|\varphi_s\|_s\leqq C_t\|\boldsymbol{L}_s\,\varphi_s\|_s,\quad \varphi_s\in D^{p,n}_{L_s}\ \textit{and}\ s\in W_t\,. \qquad (2.3)$$

Proof. First we prove the former assertion. By Theorem 2, 2), we have only to show

$$N^{p,n}_{L_t}=\{0\}\,,\ t\in B\,.$$

We fix a point $t\in B$. If $\delta_0>0$ is sufficiently small, then we can apply Proposition 1 and Theorem 2 to each domain $X_{t,\delta}=\{h_t<\delta\}$, $0<\delta<\delta_0$. We define the homomorphism $\rho_{t,\delta}$: $N^{p,n}_{L_{t,\delta}}\to N^{p,n}_{L_t}$ by $\rho_{t,\delta}:=\boldsymbol{H}_t\circ r_{t,\delta}$ where $r_{t,\delta}$: $L^{p,n}(X_{t,\delta})\to L^{p,n}(X_t)$ is the restriction homomorphism. Then *there exists a positive constant δ_1 such that $\rho_{t,\delta}$: $N^{p,n}_{L_{t,\delta}}\to N^{p,n}_{L_t}$ is an isomorphism*, $0<\delta<\delta_1$. The *existence* of δ_1 and the *injectivity* of $\rho_{t,\delta}$ are proved as follows. Suppose that there were sequences $\{\delta_k\}$ and $\{\theta_{t,k}\}$ such that $\delta_k\to 0$ as $k\to\infty$, $\theta_{t,k}\in N^{p,n}_{L_{t,\delta_k}}$, $\|\theta_{t,k}\|_{t,\delta_k}=1$ and $\rho_{t,\delta_k}(\theta_{t,k})=0$. Since the value $C(t)$ of the inequality (2.1) can be taken independently of δ if δ is sufficiently small, combining Theorem 2 with the inequality (2.1), $||\theta_{t,k}||_{1,t}$ bounded. By Rellich's lemma (see [2] (A.2.3) Proposition), taking a subsequence, $\theta_{t,k}$ converges to some $\theta_t \in L^{p,n}(X_t)$ such that $\boldsymbol{H}_t\,\theta_t=0$ and $||\theta_t||_t=1$ since the inner product $(\ ,\ )_t$ depends continuously on t. On the other hand, we recall that $\varphi_t \in D^{p,n}_{\bar\partial_t^*}$ if and only if for some positive constant C, $|(\varphi_t,\ \bar\partial_t \psi_t)_t| \leqq C||\psi_t||_t$, $\psi_t \in C^{p,n-1}(\bar X_t)$. Since $\bar\partial^*_{t,\delta_k}\theta_{t,k}=0$, $(\theta_t,\ \bar\partial_t\psi_t)_t=0$, $\psi_t \in C^{p,n-1}(\bar X_t)$. Hence $\theta_t \in N^{p,n}_{L_t} \cap N^{p,n\perp}_{L_t}=\{0\}$. This is a contradiction. The *surjectivity* of $\rho_{t,\delta}$ is proved as follows. If $\rho_{t,\delta}$ were not surjective $(0<\delta<\delta_1)$, then there would be a non-zero element ω_t of $N^{p,n}_{L_t}$ with $\omega_t \perp \rho_{t,\delta}(N^{p,n}_{L_{t,\delta}})$. Extending the definition of ω_t by setting $\omega_t=0$ on $X_{t,\delta}\backslash X_t$, we denote it by ω'_t. Then since $\omega'_t \in L^{p,n}(X_{t,\delta})$ and $\omega'_t \perp N^{p,n}_{L_{t,\delta}}$, from Theorem 2, 2), $\omega'_t=\bar\partial_{t,\delta}\,\bar\partial^*_{t,\delta}\,N_{t,\delta}\,\omega'_t$. Setting $\psi'_t=\bar\partial^*_{t,\delta}\,N_{t,\delta}\,\omega'_t$, $(\omega_t,\ \theta_t)_t=(\bar\partial_t\, r_{t,\delta}(\psi'_t),\ \theta_t)_t=(r_{t,\delta}(\psi'_t),\ \bar\partial^*_t\,\theta_t)_t=0$ for any $\theta_t \in N^{p,n}_{L_t}$. Hence $\omega_t=0$. This is a contradiction. Therefore our assertion holds:

To show $N^{p,n}_{L_t}=\{0\}$, we have only to prove that $\rho_{t,\delta}$: $N^{p,n}_{L_{t,\delta}} \rightarrow N^{p,n}_{L_t}$ is the zero map for $0<\delta<\delta_1$. For any element $\theta_{t,\delta}$ of $N^{p,n}_{L_{t,\delta}}$, combining the fact $H^n(X_{t,\delta},\ \Omega^p)=0$ ([6] Theorem) with the Dolbeault theorem, there exists $\eta_{t,\delta} \in C^{p,n-1}(X_{t,\delta})$ with $\theta_{t,\delta}=\bar\partial_{t,\delta}\,\eta_{t,\delta}$. Hence $\rho_{t,\delta}(\theta_{t,\delta})=0$. Therefore we have $N^{p,n}_{L_t}=\{0\}$.

Next we prove the latter assertion. We may put $t=0$. Suppose that there were sequences $\{t_k\}$ and $\{\varphi_k\}$ such that $t_k \rightarrow 0$ as $k \rightarrow \infty$, $\varphi_k \in D^{p,n}_{L_{t_k}}$, $||\varphi_k||_{t_k}=1$ and $||\boldsymbol{L}_{t_k}\,\varphi_k||_{t_k} \rightarrow 0$ as $k \rightarrow \infty$. Since $B^{p,n}(\bar X_t)$ is dense in $D^{p,n}_{\bar\partial_t^*}$ with respect to the form $\boldsymbol{Q}_t$ (see [3] Proposition 1.2.3 and 1.2.4), the inequality (2.1) holds for any form contained in $D^{p,n}_{L_t}$. Hence we may assume that $||\varphi_k||_{1,t_k}$ is bounded. Fiom the local invariance of the Sobolev spaces under coordinate transformations and Rellich's lemma, there exists $\varphi_0 \in L^{p,n}(X_0)$ such that $||f^*_{t_k 0}\,\varphi_k - \varphi_0||_0 \rightarrow 0$ as $k \rightarrow \infty$ in $\sum_{r+s=p+n} L^{r,s}(X_0)$. Then $||\varphi_0||_0>0$. On the other hand, using $||\boldsymbol{L}_{t_k}\varphi_k||_{t_k} \rightarrow 0$ as $k \rightarrow \infty$, $(\varphi_0,\ \bar\partial_0\psi_0)_0=0$ for any $\psi_0 \in C^{p,n-1}(\bar X_0)$. Hence $\varphi_0 \in N^{p,n}_{L_0}=\{0\}$. This is a contradiction. q.e.d.

Any section $\varphi \in C^{p,q}(\bar{\mathfrak{X}})$ determines a family $\{r_t(\varphi)\,|\,t \in B\}$ of sections $r_t(\varphi) \in C^{p,q}(\bar X_t)$, where r_t: $C^{p,q}(\bar{\mathfrak{X}}) \rightarrow C^{p,q}(\bar X_t)$ is the restriction homomorphism. Given, for each $t \in B$, a form $\varphi_t \in C^{p,q}(\bar X_t)$, we say that φ_t depends *differentiably* on t if and only if there exists $\varphi \in C^{p,q}(\bar{\mathfrak{X}})$ such that

$$\varphi_t = r_t(\varphi),\ t \in B\,.$$

Given, for each $t \in B$, a linear operator $\mathbb{A}_t$ of $L^{p,q}(X_t)$ into $L^{r,s}(X_t)$, we say that A_t depends *differentiably* on t if and only if $A_t r_t(\varphi)$ depends differentiably on t for any section $\varphi \in C^{p,q}(\overline{\mathscr{X}})$. Next we prove the following proposition.

Proposition 4. *The Neumann operator N_t: $L^{p,n}(X_t) \to L^{p,n}(X_t)$ defined in Theorem 2 depends differentiably on t.*

Proof. By the inequalities (2.2) and (2.3), we get

$$||\eta_s||_{\nu+2,s} \leqq C_\nu ||\Box_s \eta_s||_{\nu,s}$$

if $\eta_s \in C^{p,n}(\bar{X}_s) \cap D^{p,n}_{L_s^*}$, $\nu \geqq 0$ and $s \in W_t$, where C_ν is a positive constant not depending on $s \in W_t$. By Sobolev's lemma (see [2] (A.2.3) Proposition), we get

$$|\eta_s|_{l,s} \leqq C_l ||\Box_s \eta_s||_{\nu,s} \tag{2.4}$$

if $\eta_s \in C^{p,n}(\bar{X}_s) \cap D^{p,n}_{L_s^*}$, $\nu \geqq n+l-1$ and $s \in W_t$, where $|\ \ |_{l,s}$ is the supremum norm up to the l-th derivatives on $C^{p,n}(\bar{X}_s)$ and C_l is a positive constant not depending on $s \in W_t$.

Consider a family $\{\varphi_t | t \in B\}$ of sections $\varphi_t \in C^{p,n}(\bar{X}_t)$. We say that φ_t *is of class C^κ in t* if and only if all derivatives of φ_t with respect to the fibre coordinates z_t^α and $z_t^{\bar{\beta}}$ are of class C^κ in t. We denote by $\prod_t^{p,q}$ the projection of $\sum_{0 \leqq i+j \leqq 2n} C^{i,j}(\bar{X}_t)$ onto $C^{p,q}(\bar{X}_t)$. Since $\prod_t^{p,n}(f^{-1})^*$ induces a differentiable isomorphism of $\Theta^*(p, n)_{|W_t}$ onto $\Theta_t^*(p, n) \times W_t$, φ_t *is of class C^κ* in t if and only if for any $t \in B$, all derivatives of $\prod_t^{p,n} f_{st}^* \varphi_s$ with respect to the fibre coordinates z_t^α and $z_t^{\bar{\beta}}$ are of class C^κ in s. If φ_t is of class C^κ in t for every $0 \leqq \kappa < \infty$, then clearly φ_t depends differentiably on t.

By Theorem 2, 4), we have only to prove that $\varphi_t := N_t \psi_t$ depends differentiably on t for any family $\{\psi_t | t \in B\}$ of sections $\psi_t \in C^{p,n}(\bar{X}_t)$ depending differentiably on t. We prove this statement in several steps. We first prove the following assertion.

i) *If ψ_t is continuous* (*i.e. of class C^0*) *in t, then φ_t is continuous in t.*

Proof. We set $\varphi_s^* = \prod_t^{p,n} f_{st}^* \varphi_s$. By (1.2), we get

$$\varphi_s^* \in C^{p,n}(\bar{X}_t) \cap D^{p,n}_{L_t^*}. \tag{2.5}$$

Replacing η_s by φ_s and $\varphi_s^* - \varphi_t^*$ ($\varphi_t^* = \varphi_t$), we have from (2.4) and (2.5) the following inequalities

$$|\varphi_s|_{l,s} \leqq C_l ||\psi_s||_{\nu,s} \tag{2.6}$$

(2.7) $$|\varphi_s^* - \varphi_t^*|_{l,t} \leqq C_l(\|f_{st}^* \psi_s - \psi_t\|_{\nu,t} + \|(\Box_t f_{st}^* - f_{st}^* \Box_s)\varphi_s\|_{\nu,t})$$

if $\nu > n+l-1$. Since ψ_t is continuous in t and the coefficients of the operator $\Box_t$ depend differentiably on t, from (2.6) and (2.7), we have

$$|\varphi_s^* - \varphi_t^*|_{l,t} \to 0 \text{ as } s \to t, \quad l \geqq 0\,.$$

Hence φ_t is continuous in t.

ii) *If ψ_t is of class C^1 in t, then φ_t is of class C^1 in t.*

Proof. We define the linear operator $\boldsymbol{F}_s$: $C^r(\bar{X}_t) \to C^r(\bar{X}_t)$ by $\boldsymbol{F}_s := f_{st}^* \Box_s \Pi_s^{p,n} f_{ts}^*$ where $C^r(\bar{X}_t) = \sum_{p+q=r} C^{p,q}(\bar{X}_t)$. We denote by $\dfrac{\partial \boldsymbol{F}_t}{\partial t_\mu}$ the operator obtained from $\boldsymbol{F}_t$ by applying $\dfrac{\partial}{\partial t_\mu}$ to each coefficient of $\boldsymbol{F}_t$ $(1 \leqq \mu \leqq m)$ i.e. $\dfrac{\partial \boldsymbol{F}_t}{\partial t_\mu} = \left(\dfrac{\partial}{\partial s_\mu} \boldsymbol{F}_s\right)_{s=t}$. We define $\dfrac{\partial \psi_t}{\partial t_\mu}$ by $\left(\dfrac{\partial}{\partial s_\mu}(f_{st}^* \psi_s)\right)_{s=t}$. We set

$$\eta_t = \boldsymbol{N}_t\left(\Pi_t^{p,n}\left(\frac{\partial \psi_t}{\partial t_\mu} - \frac{\partial \boldsymbol{F}_t}{\partial t_\mu}\varphi_t\right)\right)$$

and

$$\chi_t(h) = \frac{1}{h}(\varphi_{t+h}^* - \varphi_t^*) - \eta_t$$

for $t+h = (t_1, \cdots, t_{\mu-1}, t_\mu + h, t_{\mu+1}, \cdots, t_m)$. Then $\chi_t(h)$ is written as follows using (2.5) and Theorem 2, 3).

(2.8) $$\chi_t(h) = \boldsymbol{N}_t\left(\Pi_t^{p,n}\left(\frac{1}{h}(f_{t+h,t}^* \psi_{t+h} - \psi_t) - \frac{\partial \psi_t}{\partial t_\mu}\right)\right) - \boldsymbol{N}_t\left(\Pi_t^{p,n}\left(\frac{1}{h}(\boldsymbol{F}_{t+h} - \boldsymbol{F}_t) f_{t+h,t}^* \varphi_{t+h} - \frac{\partial \boldsymbol{F}_t}{\partial t_\mu}\varphi_t\right)\right).$$

Since $\chi_t(h) \in D_{L_t}^{p,n}$ for any $h \neq 0$ by (2.8), applying $\chi_t(h)$ to (2.4), we have

(2.9) $$|\chi_t(h)|_{l,t} \to 0 \text{ as } h \to 0,\ l \geqq 0\,.$$

This means that φ_t is of class C^1 in t.

iii) *If ψ_t is of class C^κ in t, then φ_t is of class C^κ in t.*

Proof. By induction on κ. Assume that the statement is true for $\kappa - 1$ $(\kappa \geqq 2)$. Let U and V be local coordinate neighborhoods on M_t such that

$U \Subset V$ and $U \cap \bar{X}_t \neq \phi$ and let $(x_1, \cdots, x_{2n})$ be a system of real coordinates on V. Then from (2.9) we have

$$(2.10) \qquad \frac{\partial}{\partial t_\mu} D^\sigma(\varphi_{t,A_p\bar{B}_n}) = D^\sigma(N_t\left(\Pi_t^{p,n}\left(\frac{\partial \psi_t}{\partial t_\mu} - \frac{\partial F_t}{\partial t_\mu}\varphi_t\right)\right)_{A_p\bar{B}_n}) \text{ on } U$$

where $1 \leqq \mu \leqq m$, $\sigma = (\sigma_1, \cdots, \sigma_{2n})$, $D^\sigma = \prod_{i=1}^{2n}\left(\frac{\partial}{\partial x_i}\right)^{\sigma_i}$ and $\varphi_t = \frac{1}{p!n!} \sum_{A_pB_n} \varphi_{t,A_p\bar{B}_n}$ $dz^{A_p} \wedge dz^{\bar{B}_n}$ and so on. By inductive hypothesis, $N_t\left(\Pi_t^{p,n}\left(\frac{\partial \psi_t}{\partial t_\mu} - \frac{\partial F_t}{\partial t_\mu}\varphi_t\right)\right)$ is of class $C^{\kappa-1}$ in t. Hence the equality (2.10) implies that φ_t is of class C^κ in t. This completes the proof of Proposition 4.

We summarize the results obtained in this section.

Theorem 5. 1) *For any* $\alpha_t \in L^{p,n}(X_t)$ *and* $t \in B$, $\alpha_t = \bar{\partial}_t \bar{\partial}_t^* N_t \alpha_t$.
2) *The operators* N_t: $L^{p,n}(X_t) \to L^{p,n}(X_t)$ *and* P_t: $L^{p,n-1}(X_t) \to L^{p,n-1}(X_t)$ *depend differentiably on* t.

§ 3. Proof of Main Theorem

We take an exhaustion function Φ_0 of class C^∞ on M_0. Since any relatively compact domain on M_0 is contained in a sublevel set $\{\Phi_0 < c\}$, $c \in \mathbf{R}$, we may replace its domain by such a sublevel set. We take a non-critical value c of Φ_0 and set $X_0 = \{\Phi_0 < c\}$. Extending Φ_0 to a neighborhood of X_0 in $\mathcal{M}$ and shrinking B arbitrarily, we obtain a differentiable family π: $\mathcal{X} \to B$ of bounded domains with smooth boundary such that $\pi_{|\mathcal{X}}^{-1}(0) = X_0$. We fix the diffeomorphism f as taken in Sect. 1. To prove the theorem, we have only to prove the following assertion.

Assertion. *For given any Kähler metric* $d\sigma^2$ *on* M_0, *there exist a neighborhood* W *of* $t=0$ *in* B *and a family* $\{ds_t^2 = \sum g_{i,\alpha\bar{\beta}}\, dz_{t,i}^\alpha\, dz_{t,i}^{\bar{\beta}}\}_{t \in W}$ *of hermitian metrics such that* 1) ds_t^2 *is a Kähler metric on* X_t, $t \in W$ *and* $ds_0^2 = d\sigma^2$ *on* X_0
2) *the functions* $g_{i,\alpha\bar{\beta}}(z_i, \bar{z}_i, t)$ *are of class* C^∞ *in* z_i, $\bar{z}_i$ *and* t.

Proof. Let ω_0 be the Kähler form associated to $d\sigma^2$ on M_0. We set

$$\omega_t' = f_{0t}^* \, \omega_0, \; t \in B\,.$$

Then ω_t' is a d_t-closed real two form on $\bar{X}_t$ depending differentiably on t. Since $\dim_C M_t = 2$, we can apply Theorem 5 to $(p, 2)$ forms of class C^∞ on $\bar{X}_t$. Combining the d_t-closedness of ω_t' with Theorem 5, each ω_t' satisfies the following differential equations.

$$\bar{\partial}_t(\Pi_t^{1,1}\omega_t' - \partial_t\vartheta_t N_t \Pi_t^{0,2}\omega_t') = 0$$

$*)_t$

$$\partial_t(\Pi_t^{1,1}\omega_t' - \overline{\partial_t\vartheta_t N_t \Pi_t^{0,2}\omega_t'}) = 0$$

Hence we set

$$\omega_t = \Pi_t^{1,1}\omega_t' - \partial_t\vartheta_t N_t \Pi_t^{0,2}\omega_t' - \overline{\partial_t\vartheta_t N_t \Pi_t^{0,2}\omega_t'}, \quad t \in B .$$

From the equation $*)_t$, ω_t is a d_t-closed real (1,1) differential form on $\bar{X}_t$ such that $\omega_0 = \omega$ on $\bar{X}_0$. Since ω_t depends differentiably on t by Theorem 5, 2), ω_t gives a Kähler form on X_t for each $t \in W$ if W is a sufficiently small neighborhood of $t=0$ in B. Hence the assertion is now clear. This completes the proof of Main Theorem.

Remark. Using Theorem 5, we can prove the following theorem too.

Theorem. *Let π: $\mathcal{M} \to B$ be a differentiable family of non-compact complex manifolds of dimension two. Then for any relatively compact domain in M_0, any holomorphic line bundle on M_0 can be extended to a differentiable family of holomorphic line bundles on a neighborhood of its domain in $\mathcal{M}$ whose restriction to its domain coincides with the given holomorphic line bundle.*

Since the idea of the proof is quite similar to the idea of Sect. 13 in [4], the detail is left to the reader.

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