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On the Domain of Existence for P-pluriharmonic Functions

By

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Introduction

The domain of existence for a holomorphic function is pseudoconvex in the sense of Oka (théorème de la continuite (C) in [5]). In the present note we prove an analogous result for a certain class of harmonic maps:

Theorem. *Let X be a Riemann domain and let h be a pluriharmonic map into the unit disc whose metric is the Poincaré metric. Suppose X is the domain of existence for h. Then X is pseudoconvex in the sense of Oka.*

Definitions

Let $u=u(z)$ be a function defined on a domain $D\subset \boldsymbol{C}$ into the unit disc $\varDelta\subset \boldsymbol{C}$. We call u P-harmonic if u is of class C^2 and satisfies the following nonlinear differential equation of second order:

$$u_{z\bar{z}}+\frac{2\bar{u}}{1-|u|^2}u_z u_{\bar{z}}=0\,.$$

This is the Euler-Lagrange equation for the energy functional with respect to a hermitian metric on D and the Poincaré metric $|dw|/(1-|w|^2)$ on $\varDelta$ (cf. [2]).

Let X be a Riemann domain of dimension n and let f: $X\to\varDelta$ be a function of class C^2. We call f P-pluriharmonic if for any holomorphic map g: $D\to X$, $D\subset \boldsymbol{C}$, $f\circ g$ is harmonic. Given a P-pluriharmonic function f on X, X is called the domain of existence for f if for any Riemann domain $\tilde{X}$ containing X and a P-pluriharmonic function $\tilde{f}$ which extends f, we have $\tilde{X}=X$. A Riemann domain X is called pseudoconvex (in the sense of Oka) if $n=1$ or

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every holomorphic imbedding of

$$\Delta^n \setminus \{z_1;\ |z_1| \geq 1/2\} \times \{(z_2, \cdots, z_n);\ \max_{2\leq i\leq n} |z_i| \leq 1-\varepsilon\}$$

into X is extended to a holomorphic imbedding of Δ^n into X, where $(z_1, \cdots, z_n)$ denotes the coordinate of Δ^n and $0<\varepsilon<1$.

Preliminaries

We recall basic facts concerning P-(pluri) harmonic functions.

Lemma 1. *Let u: $D\to\Delta$ be a P-harmonic function and let K be a compact subset of D. Then,*

$$\sup_K |u| = \sup_{\partial K} |u|.$$

Proof is immediate from the fact that $-\log(1-|u|^2)$ is a subharmonic function.

Proposition 2. *Let $x\in D$ and let u be a P-harmonic function defined on $D\setminus\{x\}$. Suppose $u(D\setminus\{x\})$ is relatively compact. Then, there exists a P-harmonic function on D which extends u.*

Proof. See [1].

Proposition 3. *Let u_1 and u_2 be two P-harmonic functions defined on D. Suppose $u_1=u_2$ on a nonempty open subset of D. Then, $u_1=u_2$ on D.*

Proof is immediate from the following proposition. See also [6].

Proposition 4. *Every P-pluriharmonic function is real analytic.*

Proof. See [4].

Proof of Theorem

We may assume $X\subset\Delta^n$. Let $G\subset\Delta^n$ be a non-pseudoconvex domain and $\check{h}$ a P-pluriharmonic function defined on G. Then, for some $\varepsilon>0$ we can choose a holomorphic imbedding ι: $\bar{\Delta}^n\to\Delta^n$ such that the image of $\Delta^n\setminus\{z_1;$ $|z_1|\leqslant 1-\varepsilon\}\times\{(z_2, \cdots, z_n);\ \max|z_i|\geq 1/2\}$ is contained in G and that $\iota(\{(z_1, 1/2, 0, \cdots, 0);\ |z_1|\leq 1\})\setminus G=\iota((0, 1/2, 0, \cdots, 0))$, where $\bar{\Delta}=\{z;\ |z|\leq 1\}$. Since $\check{h}\circ\iota$ is real analytic, we can expand $\check{h}\circ\iota$ into a power series in $(z_2-1/2, \bar{z}_2-1/2, \cdots, z_n, \bar{z}_n)$ on a neighbourhood of $\{|z_1|=\delta\}\times(1/2, 0, \cdots, 0)$ for $0<\delta\leqq 1$. Namely we put

$$\check{h}\circ\iota = \sum u^{\delta}_{IJ}(z_1)\, z'^{I}\, \bar{z}'^{J}\ ,$$

where $z'=(z_2', z_3, \cdots, z_n)$, $z_2'=z_2-1/2$, and $z'^{I}\,\bar{z}'^{J}=z_2'^{i_2}\cdots z_n^{i_n}\,\bar{z}_2'^{j_2}\cdots\bar{z}_n^{j_n}$. Combining Lemma 1 and Proposition 2, we have a P-harmonic function h_0: $\bar{\Delta}\to\Delta$ such that $h_0=u^1_{00}$ on $\partial\Delta$. Now we are going to extend $\check{h}\circ\iota$ on a neighbourhood of $(0, 1/2, 0, \cdots, 0)$. After an isometric coordinate change of Δ, we may assume that $h_0(0)=0$. First we consider a formal power series

$$h = h_0(z_1)+\sum_{|I|+|J|\geq 1} h_{IJ}(z_1)\, z'^{I}\, \bar{z}'^{J}$$

which satisfy

$$(1)\qquad \begin{cases} h_{z_1\bar{z}_1}+\dfrac{2\bar{h}}{1-|h|^2}\, h_{z_1} h_{\bar{z}_1} = 0 \\ h_{IJ}=u^{\delta}_{IJ} \quad \text{on } \{|z_1|=\delta\}\ . \end{cases}$$

Note that for sufficiently small δ, $1/(1-|h|^2)$ is well-defined as a formal power series. (1) is equivalent to the following equations;

$$(2)\qquad \begin{cases} L\, h_{IJ}+c\, \bar{h}_{JI} = g_{IJ} \quad \text{on } \{|z_1|<\delta\} \\ h_{IJ} = u_{IJ} \quad \text{on } \{|z_1|=\delta\}\ , \end{cases}$$

where L is a linear elliptic operator of second order, c is a real analytic function on $\{|z_1|\leq\delta\}$ and g_{IJ} is a convergent power series with respect to $h_{I'J'}$, $h_{I'J'z_1}$, $h_{I'J'\bar{z}_1}$ and their conjugates, for the indices I', J' satisfying $|I'|+|J'|<|I|+|J|$. Thus we can determine h_{IJ} inductively by solving (2) (cf. [4]). Let $A(z_2', \bar{z}_2', \cdots, z_n, \bar{z}_n)$ be Kodaira's power series, i.e., $A=A(z_2', \bar{z}_2', \cdots, z_n, \bar{z}_n):=(a/16b)\sum_{k=0}^{\infty} b^k(z_2'+\bar{z}_2'+\cdots+z_n+\bar{z}_n)^k/k^2$. Then, for any $\varepsilon>0$ and $a>0$, we can choose b so that

$$\frac{A^3}{1-A^2}\ll \varepsilon\, A\ .$$

Hence, applying a priori estimate for the equation (2) (cf. [4] p.148 Th. 5.4.2), we see that for a sufficiently small δ we can choose a and b so that $h\ll A$, $h_{z_1}\ll A$, and $h_{\bar{z}_1}\ll A$. Thus we obtain a real analytic function h on a neighbourhood U of $(0, 1/2, 0, \cdots, 0)$. Since P-harmonic functions are uniquely determined by their boundary values (cf. [3]), $h=\check{h}\circ\iota$ on a connected component of $U\cap\iota^{-1}(G)$. Therefore, by the real analyticity of h, h is P-pluriharmonic on U. Hence $\check{h}$ is extendable across a boundary point of G as a P-pluriharmonic function. Therefore, if X is the domain of existence for $\check{h}$, then X should be pseudoconvex.

q.e.d.

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