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On the Algebraic K-Cohomology of Lens Spaces

Dedicated to Professor Nobuo Shimada on his 60th birthday

By

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§ 1. Introduction

Let F_q be a finite field of order $q=p^d$ and b_{F_q} be the 0-connected spectrum of algebraic K-theory for F_q. Then the homotopy groups of b_{F_q} are

$$\pi_{2k}(b_{F_q}) = 0\,,$$
$$\pi_{2k-1}(b_{F_q}) = \begin{cases} \boldsymbol{Z}/(q^k-1) & \text{if } k>0\,, \\ 0 & \text{if } k\leqq 0\,. \end{cases}$$

Let l be an odd prime number ($l\neq p$) and $L^n(l)$ the standard $2n+1$ dimensional lens space $S^{2n+1}/(\boldsymbol{Z}/l)$. We write $L^n_0(l)$ for its $2n$-skeleton.

The cohomology groups $b^*_{F_q}(L^n_0(l))$ were studied by G. Nishida [6] in a special case. The purpose of the paper is to determine the cohomology group $b^*_{F_q}(L^n_0(l))$. The main theorem is Theorem 5.2.

This paper is organized as follows:
In Section 2 we state the splitting of algebraic K-theory for a finite field. In Sections 3 and 4, we study the topological K-group of a lens space and its generators. In the last section we state the main theorem and prove it.

§ 2. Splitting of b_{F_q}

Denote by Λ the ring of l-adic integers $\boldsymbol{Z}_l$. By X_Λ we denote the l-adic completion of a spectrum X. Let K (resp. bu) be the periodic (resp. 1-connected) spectrum which represents topological K-theory.

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Definition 2.1. Let $\rho \in \Lambda$ be a primitive $(l-1)$-th root of unity. Then for $1 \leqq i \leqq l-1$ we define $\Phi_i: K_\Lambda \to K_\Lambda$ by

$$\Phi_i = \frac{1}{l-1} \sum_{m=1}^{l-1} \rho^{-mi} \psi^{\rho^m},$$

where ψ^k is the Adams operation.

Then splitting of topological K-theory is as follows (see [1]):

Theorem 2.2. *We have*

(i) $\sum_{i=1}^{l-1} \Phi_i = id$,

(ii) $\Phi_i \Phi_j = \begin{cases} \Phi_i & \text{if } i=j \\ 0 & \text{if } i \neq i, \end{cases}$

(iii) $K_\Lambda(-) \cong \bigoplus_{i=1}^{l-1} \Phi_i K_\Lambda(-)$ *and*

(iv) $bu_\Lambda(-) \cong \bigoplus_{i=1}^{l-1} \Phi_i bu_\Lambda(-)$.

Then $\Phi_i K_\Lambda(-)$ (resp. $\Phi_i bu_\Lambda(-)$) is a generalized cohomology theory and we write G_i (resp. g_i) for its spectrum. Theorem 2.2 (iii) and (iv) imply

(iii)′ $K_\Lambda \simeq \bigvee_{i=1}^{l-1} G_i$ and

(iv)′ $bu_\Lambda \simeq \bigvee_{i=1}^{l-1} g_i$.

Let $\iota_i: g_i \to bu_\Lambda$ and $\pi_i: bu_\Lambda \to g_i$ be the canonical inclusion and projection of the splitting (iv)′.

Let b_{F_q} be the 0-connected spectrum which represents the algebraic K-theory for a finite field F_q. By Fiedorowicz-Priddy [3], we know that $(b_{F_q})_\Lambda$ is the homotopy fibre of $1-\psi^q: bu_\Lambda \to bu_\Lambda$ where ψ^q is the Adams operation.

Definition 2.3. Let $(1-\psi^q)_i: g_i \to g_i$ be the composition $g_i \xrightarrow{\iota_i} bu_\Lambda \xrightarrow{1-\psi^q} bu_\Lambda \xrightarrow{\pi_i} g_i$ and $g_{F_q,i}$ the homotopy fibre of $(1-\psi^q)_i$. Let r be the least positive integer such that $q^r \equiv 1 \pmod{l}$.

Then we have the splitting of $(b_{F_q})_\Lambda$ as follows (see [4]):

Theorem 2.4.

(i) $(b_{F_q})_\Lambda \simeq \bigvee_{\substack{1 \leqq i \leqq l-1 \\ i \equiv 0 \ (\mathrm{mod}\ r)}} g_{F_q,i}$,

(ii) $\pi_{2k}(g_{F_q,i}) \simeq 0$,

$$\pi_{2k-1}(g_{F_q,i}) \simeq \begin{cases} \boldsymbol{Z}/(q^k-1)\otimes \Lambda & \textit{if } k>0 \textit{ and } k\equiv i \pmod{l-1} \\ 0 & \textit{otherwise.} \end{cases}$$

§ 3. Topological K-Group of Lens Spaces

Let η be the canonical complex line bundle of $L^n(l)$ and put $x=\eta-1$. By $[t]$ we denote the greatest integer which is less than or equal to t.

Then the topological K-group of lens spaces is as follows:

Theorem 3.1. (Kambe [5]) *Let M_i $(1\leqq i\leqq l-1)$ be a cyclic group generated by x^i of order $a_i^{(n)}=l^{[(n+l-i-1)/(l-1)]}$. Then*

(i) $K^0(L_0^n(l)) \simeq \boldsymbol{Z}[x]/((1+x)^l-1,\, x^{n+1})$

$\simeq \boldsymbol{Z}\oplus M_1\oplus M_2\oplus\cdots\oplus M_{l-1}$,

(ii) $K^1(L_0^n(l))=0$.

We define the filtration $F^k\tilde{K}^0(L_0^n(l))$ of $\tilde{K}^0(L_0^n(l))$ as follows:

Definition 3.2. Let $L_0^k(l)\to L_0^n(l)$ $(0\leqq k\leqq n)$ be the canonical inclusion, then we put

$$F^k\tilde{K}^0(L_0^n(l)) = \begin{cases} \tilde{K}^0(L_0^n(l)) & \text{if } k<0\,, \\ \operatorname{Ker}(\tilde{K}^0(L_0^n(l))\to\tilde{K}^0(L_0^k(l))) & \text{if } 0\leqq k<n\,, \\ 0 & \text{if } k\geqq n\,. \end{cases}$$

Since Atiyah-Hirzeburch spectral sequence of $K^*(L_0^n(l))$ collapses, we have:

Corollary 3.3. *The bu-cohomology group of lens space is*

(i) $\widetilde{bu}^{2k}(L_0^n(l))=F^k\tilde{K}^0(L_0^n(l))$,

(ii) $\widetilde{bu}^{2k+1}(L_0^n(l))=0$.

By the cohomology exact sequence of the fibration $(b_{F_q})_\Lambda\to bu_\Lambda \xrightarrow{1-\psi^q} bu_\Lambda$ and $\tilde{b}_{F_q}^*(L_0^n(l))\simeq(\tilde{b}_{F_q})_\Lambda^*(L_0^n(l))$, we have the following exact sequence:

$$0\to \tilde{b}_{F_q}^{2k}(L_0^n(l))\to \widetilde{bu}_\Lambda^{2k}(L_0^n(l)) \xrightarrow{(1-\psi^q)^*} \widetilde{bu}_\Lambda^{2k}(L_0^n(l))\to \tilde{b}_{F_q}^{2k+1}(L_0^n(l))\to 0\,.$$

The Bott periodicity and (3.3) imply the diagram

$$\begin{array}{ccc} \widetilde{bu}_\Lambda^{2k}(L_0^n(l)) & \xrightarrow{(1-\psi^q)^*} & \widetilde{bu}_\Lambda^{2k}(L_0^n(l)) \\ \wr\| & \left(1-\dfrac{1}{q^k}\psi^q\right)^* & \wr\| \\ F^k\tilde{K}^0(L_0^n(l)) & \longrightarrow & F^k\tilde{K}^0(L_0^n(l)) \end{array}$$

commutes. Then we have

Proposition 3.4. *The b_{F_q}-cohomology group of lens space is*

(i) $\tilde{b}^{2k}_{F_q}(L^n_0(l)) \cong \mathrm{Ker}\left(\left(1-\frac{1}{q^k}\psi^q\right)^*: F^k\tilde{K}^0(L^n_0(l)) \to F^k\tilde{K}^0(L^n_0(l))\right)$,

(ii) $\tilde{b}^{2k+1}_{F_q}(L^n_0(l)) \cong \mathrm{Coker}\left(\left(1-\frac{1}{q^k}\psi^q\right)^*: F^k\tilde{K}^0(L^n_0(l)) \to F^k\tilde{K}^0(L^n_0(l))\right)$.

Recall that $K^0(L^n_0(l)) = \boldsymbol{Z}[x]/((1+x)^l-1, x^{n+1})$ and $(\psi^q)^*x = (1+x)^q-1$. To the generators $x, x^2, \cdots$, and x^{l-1} of $\tilde{K}^0(L^n_0(l))$, the action of $\left(1-\frac{1}{q^k}\psi^q\right)^*$ is

$$\left(1-\frac{1}{q^k}\psi^q\right)^* x^i = x^i - \frac{1}{q^k}\{(x+1)^q-1\}^i.$$

§ 4. On Generators of $\tilde{K}^0(L^n_0(l))$

To compute the kernel and cokernel of $\left(1-\frac{1}{q^k}\psi^q\right)^*$, we define new generators of $\tilde{K}^0(L^n_0(l))$.

Definition 4.1. We define the element ξ_i $(1 \leqq i \leqq l-1)$ of $\tilde{K}^0(L^n_0(l))$ by $\xi_i = \Phi_i(x)$ and put $N_i = \mathrm{Im}\,(\Phi_i\colon \tilde{K}^0(L^n_0(l)) \to \tilde{K}^0(L^n_0(l)))$.

Then we have

Theorem 4.2. *Let $a^{(n)}_i$ be the integer defined in Theorem* 3.1, *then*

(i) $K^0_\Lambda(L^n_0(l)) \cong \Lambda \oplus N_1 \oplus N_2 \oplus \cdots \oplus N_{l-1}$,

(ii) *N_i is a cyclic group generated by ξ_i of order $a^{(n)}_i$.*

Proof of (i) is clear by Theorem 2.2. To prove (ii) we need following two lemmas.

By y_i $(i \in \boldsymbol{Z}/l)$ we denote the element of $\tilde{K}^0(L^n_0(l))$ such that $y_i = (1+x)^i - 1$. This notation is well defined since $(1+x)^l = 1$.

When k is an element of $\boldsymbol{Z}$ or Λ we write $\bar{k}$ for the mod l reduction of k. Then $(\psi^k)^*x = y_{\bar{k}}$. Thus we can regard that the Adams operation $(\psi^k)^*\colon \tilde{K}^0(L^n_0(l)) \to \tilde{K}^0(L^n_0(l))$ is defined for $k \in \boldsymbol{Z}/l$.

Let k be an element of Λ such that $k \not\equiv 0 \pmod{l}$. Then there exists one and only one element m of Λ such that $m^{l-1} = 1$ and $k \equiv m \pmod{l}$. We write $\tilde{k}$ for the element m. Then we have

Lemma 4.3. *$\Phi_i(y_k) = \tilde{k}^i \xi_i$ for $1 \leqq k \leqq l-1$.*

Proof. By definition

$$\Phi_i(y_k) = \frac{1}{l-1}\sum_{m=1}^{l-1}\rho^{-mi}\psi^{\rho^m}\psi^k(x) = \frac{1}{l-1}\sum_{m=1}^{l-1}\rho^{-mi}\psi^{\rho^m \tilde{k}}(x)\,.$$

Since $\{\rho\tilde{k}, \rho^2\tilde{k}, \cdots, \rho^{l-1}\tilde{k}\} = \{\rho, \rho^2, \cdots, \rho^{l-1}\}$, we have

$$\Phi_i(y_k) = \frac{1}{l-1}\sum_{m=1}^{l-1}\tilde{k}^i\rho^{-mi}\psi^{\rho^m}(x) = \tilde{k}^i\Phi_i(x) = \tilde{k}^i\xi_i\,.$$

Since ψ^k commutes with Φ_i we have the following corollary:

Corollary 4.4. $\psi^k(\xi_i) = \tilde{k}^i\xi_i$ *for* $1 \leqq i \leqq l-1$.

Lemma 4.5. *Let* $\overline{\Phi_i(x)}$ *be the* mod l *reduction of* $\Phi_i(x)$ *and put* $\overline{\Phi_i(x)} = c_1x + c_2x^2 + \cdots + c_{l-1}x^{l-1}$ $(c_k \in \boldsymbol{Z}/l)$. *Then*

(i) $c_k = 0$ *for* $1 \leqq k < i$, *and*

(ii) $c_i \neq 0$.

Proof. Since $\{\overline{\rho^1}, \overline{\rho^2}, \cdots, \overline{\rho^{l-1}}\} = \{1, 2, \cdots, l-1\}$ we have

$$\overline{\Phi_i(x)} = \overline{\frac{1}{l-1}\sum_{m=1}^{l-1}\rho^{-mi}\psi^{\rho^m}(x)} = \frac{1}{l-1}\sum_{m=1}^{l-1}m^{-i}\{(1+x)^m - 1\}\,.$$

Inductively we define $f_k(x)$ $(0 \leqq k \leqq i)$ by $f_0(x) = \overline{\Phi_i(x)}$ and $f_k(x) = (1+x)\frac{d}{dx}f_{k-1}(x)$. Then, for $1 \leqq k \leqq i$

$$f_k(x) = \frac{1}{l-1}\sum_{m=1}^{l-1}m^{-i+k}(1+x)^m\,.$$

Therefore

$$k!\,c_k = f_k(0) = \frac{1}{l-1}\sum_{m=1}^{l-1}m^{-i+k}$$
$$= \begin{cases} 0 & \text{if } 1 \leqq k < i \\ 1 & \text{if } k = i\,. \end{cases}$$

Proof of Theorem 4.2 (ii). It is clear that $\tilde{K}^0(L_0^n(l))$ is generated by $y_1, y_2, \cdots$, and y_{l-1}. Then $N_i = \Phi_i(\tilde{K}^0(L_0^n(l)))$ is generated by $\Phi_i(x) = \Phi_i(y_1)$, $\Phi_i(y_2), \cdots$, and $\Phi_i(y_{l-1})$. By Lemma 4.3, N_i is a cyclic group generated by $\xi_i = \Phi_i(x)$. By Lemma 4.5, the mod l reduction of ξ_i is

$$\bar{\xi}_i = c_ix^i + c_{i+1}x^{i+1} + \cdots + c_{l-1}x^{l-1} \quad (c_i \neq 0)\,.$$

Theorem 3.1 implies that the order $a_k^{(n)}$ of x^k $(1 \leqq k \leqq l-1)$ has the following two properties:

(i) $a_1^{(n)} \geqq a_2^{(n)} \geqq \cdots \geqq a_{l-1}^{(n)}$ and

(ii) $a_j^{(n)}/a_k^{(n)} = l$ or 1 if $j < k$.

Therefore the order of ξ_i is that of x^i. This completes the proof of Theorem 4.2 (ii).

§ 5. b_{F_q}-Cohomology of Lens Spaces

Let $\nu_l: \Lambda \to \boldsymbol{Z} \cup \{\infty\}$ be the l-adic valuation, that is $\nu_l(\lambda)$ is the largest integer ν such that l^ν divides λ where λ is an element of Λ.

Lemma 5.1. *Let ξ_i be the element of $\tilde{K}^0(L_0^n(l))$ defined in* (4.1), *then*

(i) $\left(1-\frac{1}{q^k}\psi^q\right)^* \xi_i = \left(1-\frac{\tilde{q}^i}{q^k}\right)\xi_i$,

(ii) $1-\frac{\tilde{q}^i}{q^k}$ *is a unit of Λ if $i \not\equiv k$* (mod r), *and*

(iii) $\nu_l\left(1-\frac{\tilde{q}^i}{q^k}\right) = \nu_l(q^r-1)+\nu_l(k)$ *if* $i \equiv k$ (mod r).

Proof. By Corollary 4.4, (i) is clear. (ii) holds since $q^k \not\equiv q^i \equiv \tilde{q}^i$ (mod l). To prove (iii), assume $i \equiv k$ (mod r). Since $\tilde{q}^i = \tilde{q}^k$ and $\nu_l\left(1-\frac{\tilde{q}}{q}\right) \geqq 1$, we have

$$\nu_l\left(1-\frac{\tilde{q}^i}{q^k}\right) = \nu_l\left(1-\left(\frac{\tilde{q}}{q}\right)^k\right) = \nu_l\left(1-\frac{\tilde{q}}{q}\right)+\nu_l(k) .$$

On the other hand

$$q^r-1 = q^r-\tilde{q}^r = (q-\tilde{q})(q^{r-1}+q^{r-2}\tilde{q}+\cdots+\tilde{q}^{r-1}) ,$$

where

$$q^{r-1}+q^{r-2}\tilde{q}+\cdots+\tilde{q}^{r-1} \equiv q^{r-1}+q^{r-1}+\cdots+q^{r-1} = rq^{r-1} \not\equiv 0 \pmod{l} .$$

Therefore

$$\nu_l\left(1-\frac{\tilde{q}}{q}\right) = \nu_l(q-\tilde{q}) = \nu_l(q^r-1) .$$

This completes the proof of the lemma.

Theorem 5.2. *The b_{F_q}-cohomology of lens space is*

$$\tilde{b}_{F_q}^{2k}(L_0^n(l)) \cong \tilde{b}_{F_q}^{2k+1}(L_0^n(l)) \cong \bigoplus_{\substack{1 \leq i \leq l-1 \\ i \equiv k \ (\mathrm{mod}\ r)}} \boldsymbol{Z}/(l^{m_i}\boldsymbol{Z})$$

where m_i (for $i \equiv k$ (mod r)) *is the integer defined as follows:*

$$m_i = \begin{cases} \min\{\nu_l(a_i^{(n)}),\ \nu_l(q^r-1)+\nu_l(k)\} & \text{if } k \leqq 0 , \\ \min\{\nu_l(a_i^{(n)})-\nu_l(a_i^{(k)}),\ \nu_l(q^r-1)+\nu_l(k)\} & \text{if } 0<k<n , \\ 0 & \text{if } k \geqq n . \end{cases}$$

Proof. Since $\{a_i^{(k)}\xi_i\}_{1\leqq i\leqq l-1}$ is a basis of $F^k\tilde{K}^0(L_0^n(l))$, it is easy from Proposition 3.4, Theorem 4.2 and Lemma 5.1.

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