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Some Infinite Dimensional Groups and Bundles

By

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Abstract

The group $\mathcal{O}$ of Bogoliubov automorphisms of the infinite dimensional Clifford albegra, implementable in a Fock representation, the analogous group of automorphisms of the canonical commutation relations and various generalisations are discussed. Their homotopy type is determined in a topology naturally defined by the spin and metaplectic representations. A theorem of Araki and Evans on a $\mathbf{Z}_2$-index for certain projections is generalised using our "mod 2" index for $\mathcal{O}$. Connections with K_1 of certain Banach algebras are described.

§1. Introduction

This note is a continuation of [1]. There we considered the topological structure of certain infinite dimensional groups, the prototype of which is the group $\mathcal{O}$, of Bogoliubov automorphisms of the (infinite dimensional) Clifford algebra implementable in a given Fock representation. To describe $\mathcal{O}$, let E be an infinite dimensional real separable Hilbert space and let $\mathcal{C}(E)$ denote the Clifford algebra over E generated by $\{c(u) \mid u \in E\}$ with $c(u)^2 = ||u||^2 \circ \mathrm{I}$. Then if J is a complex structure on E, $\mathcal{O}$ is the group of orthogonal operators O on E such that $OJ-JO$ is Hilbert-Schmidt. Moreover J determines a Fock representation of $\mathcal{C}(E)$ [2] denoted by π_J, and each $O \in \mathcal{O}$ defines an automorphism of $\mathcal{C}(E)$ by $c(u) \to c(Ou)$, $u \in E$, which is implementable in π_J in the sense that there is a unitary operator $\Gamma(O)$ on the Hilbert space $\mathcal{F}$ of π_J such that

$$\Gamma(O)\, \pi_J(c(u))\, \Gamma(O)^{-1} = \pi_J(c(Ou)) .$$

The main object of Section 2 is to determine the coarsest topology on $\mathcal{O}$ such that the map $O \to \Gamma(O)$ from $\mathcal{O}$ into the projective unitary group of $\mathcal{F}$ is a

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continuous homomorphism. This topology on $\mathcal{O}$ is weaker than that of [1], however it turns out that the results of [1] go through in the new topology. In particular the map $j\colon \mathcal{O}\to \boldsymbol{Z}_2$ defined by

$$j(O) = \dim_C \ker(JOJ-O) \pmod 2$$

is a continuous homomorphism which separates the connected components of $\mathcal{O}$. (Note that $\ker(JOJ-O)$ is J-invariant and so the complex dimension, $\dim_C$, makes sense.) As a corollary of this, in Section 3, a generalisation of a theorem of Araki and Evans (which is used in their work on the Ising model [3]) is proved.

The remainder of Section 2 is taken up with some technicalities which are expanded on as the results are useful in another context [4]. In Section 3 various generalisations of $\mathcal{O}$ are considered, beginning with the replacement of the Hilbert-Schmidt operators by more general ideals of compact operators. Symplectic analogues of $\mathcal{O}$ are introduced and analogous results obtained for them. The bundles referred to in the title are those associated with the analysis of the topological structure of these groups, the principal $U(1)$-bundle determined by the map $O\to\Gamma(O)$ from $\mathcal{O}$ into the projective unitary group, some universal bundles (Section 4) for certain groups and a proof that infinite dimensional Clifford bundles are trivialisable (Section 2).

Section 4 contains some remarks which connect the groups studied here with the unitary groups of certain Banach algebras and hence interprets the index maps of [1] and [5] as determining K_1 of these algebras.

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§2. Topological Structure of $\mathcal{O}$

In this section I will show that $\mathcal{O}$ has a natural topology in which it becomes a Polish group. The homotopy groups of $\mathcal{O}$ are then determined following [1], and the significance of this topology is established.

Definition 2.1. A net $\{O_\alpha\}$ in $\mathcal{O}$ converges to O if $JO_\alpha J-O_\alpha\to JOJ-O$ in the strong operator topology and $JO_\alpha J+O_\alpha\to JOJ+O$ in Hilbert-Schmidt norm.

Lemma 2.2. *$\mathcal{O}$ is a topological group.*

Proof. For $O \in \mathcal{O}$ define

$$T_1 = \tfrac{1}{2}(O - JOJ), \quad T_2 = \tfrac{1}{2}(O + JOJ) \tag{2.1}$$

and note that $||T_1|| \leq 1$. Then continuity of multiplication in $\mathcal{O}$ follows easily if one notes three facts. Firstly that multiplication is continuous in Hilbert-Schmidt norm, secondly that it is continuous on the unit ball in the strong operator topology and thirdly Grümm's result [6] that if a sequence $\{R_n\}$ of operators with uniformly bounded norms converges strongly to R and S is Hilbert-Schmidt then $R_n S \to RS$ in Hilbert-Schmidt norm. Now, to verify continuity of taking inverses it is sufficient to show that if $O_\alpha \to I$ then $O_\alpha^{-1} \to I$. But if $O_\alpha \to I$ then $T_2^\alpha \to 0$ in Hilbert-Schmidt norm so the identity

$$T_1^* T_1 + T_2^* T_2 = I \tag{2.2}$$

shows that $(T_1^\alpha)^* T_1^\alpha \to I$ in norm (where T_1^α, T_2^α are defined in terms of O_α as in (2.1)). But now $T_2^{\alpha *} \to 0$ in Hilbert-Schmidt norm and from

$$||(T_1^\alpha)^* v - v||^2 = \langle T_1^\alpha (T_1^\alpha)^* v, v\rangle + \langle v, v\rangle - \langle v, T_1^\alpha v\rangle - \langle T_1^\alpha v, v\rangle$$

we deduce that $(T_1^\alpha)^* \to I$ strongly.

Let $\mathcal{O}_0$ denote the set of $O \in \mathcal{O}$ with $\ker(O - JOJ) = (0)$, then if $O \in \mathcal{O}_0$ it follows from (2.2) that one is not an eigenvalue of $T_2^* T_2$ (T_2 being defined by (2.1)). But if O' is sufficiently close to $O \in \mathcal{O}_0$ in this topology on $\mathcal{O}$ then $(T_2')^* T_2'$ cannot have one as an eigenvalue either. Thus $\mathcal{O}_0$ is open in $\mathcal{O}$. By translating $\mathcal{O}_0$ around we obtain an open cover for $\mathcal{O}$.

Lemma 2.3. *Every $O \in \mathcal{O}_0$ may be written uniquely as a product, $O = U (I + X)$ where U is J-linear (i.e., unitary on E) and X is Hilbert-Schmidt with $J(I+X)J - (I+X)$ positive. Moreover U and $I+X$ depend continuously on O in $\mathcal{O}_0$.*

Proof. This result appears in part in [1]. Here U is the unitary in the polar decomposition of $T_1 = \tfrac{1}{2}(O - JOJ)$ from which it follows easily that $U^{-1} O = I + X$ for some Hilbert-Schmidt X with $J(I+X)J - (I+X)$ positive. Uniqueness is straightforward to verify. If $O_\alpha \to O$ in $\mathcal{O}_0$ then we need to show that the unitary U_α in the polar decomposition of $T_1^\alpha = \tfrac{1}{2}(O_\alpha - JO_\alpha J)$ converges strongly to U. But (cf. proof of Lemma 2.2) $U_\alpha = T_1^\alpha (1 - (T_2^\alpha)^* T_2^\alpha)^{-1/2}$ and so the convergence is clear. It remains to show that $I + X_\alpha = U_\alpha^{-1} O_\alpha$ satisfies $||X_\alpha - X||_2 \to 0$. Using (2.2) and the definition of X_α one may write $I + X_\alpha = (I - Y_\alpha^* Y_\alpha)^{1/2} + Y_\alpha$ where

$Y_\alpha = JX_\alpha J + X_\alpha$ and similarly $I+X=(I-Y^*Y)^{1/2}+Y$. As $O_\alpha \to O$ in $\mathcal{O}$ so $\|Y_\alpha - Y\|_2$ converges to zero and hence it is sufficient to show that $(I-Y_\alpha^* Y_\alpha)^{1/2}-I$ converges to $(I-Y^*Y)^{1/2}-I$ in Hilbert-Schmidt norm. We already have convergence of $(I-Y_\alpha^* Y_\alpha)^{1/2}$ to $(I-Y^*Y)^{1/2}$ in the strong operator topology so Theorem 2.21 of [7] will give the result, provided we can show that

$$\|(I-Y_\alpha^* Y_\alpha)^{1/2}-I\|_2 \to \|(I-Y^*Y)^{1/2}-I\|_2 .$$

But

$$\|(I-Y_\alpha^* Y_\alpha)^{1/2}-I\|_2^2 = 2\operatorname{tr}[I-(I-Y_\alpha^* Y_\alpha)^{1/2}]-\operatorname{tr}(Y_\alpha^* Y_\alpha)$$

so it is sufficient to prove that $\operatorname{tr}[(I-Y_\alpha^* Y_\alpha)^{1/2}-(I-Y^*Y)^{1/2}]$ converges to zero. If the eigenvalues of $Y_\alpha^* Y_\alpha$ and Y^*Y are $\{\lambda_j(\alpha)\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty$ respectively then

$$\textstyle\sum_j (1-\lambda_j(\alpha))^{1/2}(1-\lambda_j)^{1/2} = \sum_j (\lambda_j - \lambda_j(\alpha))/[(1-\lambda_j(\alpha))^{1/2}+(1-\lambda_j)^{1/2}]$$

and as $I+X_\alpha \in \mathcal{O}_0$ for all α there is a constant K such that

$$[(1-\lambda_j(\alpha))^{1/2}+(1-\lambda_j)^{1/2}] \geq K$$

for all j (and α sufficiently large). Thus the result follows from the fact that $\|Y_\alpha^* Y_\alpha\|_2 \to \|Y^*Y\|_2$.

Remark 2.4. Let $\mathcal{O}_2$ denote the group of orthogonal operators on E of the form $I+X$ with X Hilbert-Schmidt. Then it follows from [1] that every element of $\mathcal{O}$ may be written as a product of a unitary operator on E and an element of $\mathcal{O}_2$. Hence the map from the product $\mathcal{U}(E)\times\mathcal{O}_2$ (where $\mathcal{U}(E)$ is the unitary group of E) onto $\mathcal{O}$ given by $(U, I+X)\to U(I+X)$, is onto and continuous if $\mathcal{U}(E)$ is given the strong operator topology and $\mathcal{O}_2$ its natural topology.

Noting that $\mathcal{U}(E)\times\mathcal{O}_2$ is separable and metrisable we have

Proposition 2.5. *$\mathcal{O}$ is a Polish group.*

Proof. We need only check completeness. If $\{O_n\}$ is a Cauchy sequence in $\mathcal{O}$ then $\{O_n\}$ is a Cauchy sequence in the orthogonal operators on E equipped with the strong operator topology and hence converges to some orthogonal operator O. But then JO_nJ-O_n and JO_nJ+O_n must converge strongly to $JOJ-O$ and to $JOJ+O$ in Hilbert-Schmidt norm respectively as JO_nJ+O_n is a Cauchy sequence in the Hilbert-Schmidt operators. So $\mathcal{O}$ is complete.

Lemma 2.6. *$\mathcal{O}_0$ contains a contractible, open, symmetric neighbourhood of* *the identity.*

Proof. Observe that if $O \in \mathcal{O}_0$ and $O = U(I+X)$ is the decomposition of Lemma 2.3 then -1 does not lie in the spectrum of $I+X$, because with T_1 and T_2 as in (2.1), $\mathcal{O} < U^{-1}T_1 \leq 1$ and

$$||X|| = ||U^{-1}T_1 + U^{-1}T_2 - 1|| \leq ||U^{-1}T_1 - 1|| + ||T_2|| < 2 .$$

Thus there is a skew-adjoint real linear Hilbert-Schmidt operator A on E with $I+X = \exp A$. On restricting exp to the set of real skew-adjoint A such that $||\exp A - I|| < 1$ we find that log is a continuous inverse for exp. So we can choose $\mathcal{O}_{00} \subseteq \mathcal{O}_0$ to be the set of all $O = U(I+X)$ with $||X|| < 1$. If $\exp A \in \mathcal{O}_{00}$ let $A = J_0|A|$ be the polar decomposition of A so that

$$\exp A = \cos|A| + J_0 \sin|A| .$$

Let g_s, $0 \leq s \leq 1$, be a homotopy contracting $\mathcal{U}(E)$ to I and define

$$h_s \colon \mathcal{O}_{00} \to \mathcal{O}_{00}\,, \quad 0 \leq s \leq 1$$

by

$$h_s(O) = g_s(U) \exp sA$$

where $O = U(I+X) = U \exp A$ is the decomposition of Lemma 2.3. Note that because $\cos s|A| \geq 0$, $g_s(U)$ is the unitary in the polar decomposition of $Jh_s(O)J - h_s(O)$ for all $s \in [0, 1]$ so that h_s is well-defined. Then h_s is a homotopy contracting $\mathcal{O}_{00}$ to the identity as required.

With these preliminaries out of the way the spin representation of $\mathcal{O}$ may be analysed. Let

$$\tilde{\mathcal{U}} = \{e^{i\theta}\Gamma(O) \,|\, 0 \leq \theta \leq 2\pi\,, \quad O \in \mathcal{O}\}$$

and note that $\tilde{\mathcal{U}}$ is a subgroup of the unitary group $\mathcal{U}(\mathcal{F})$ of $\mathcal{F}$ equipped with the strong operator topology. $\tilde{\mathcal{U}}$ is a topological group and we may define p: $\tilde{\mathcal{U}} \to \mathcal{O}$ by $p(e^{i\theta}\Gamma(O)) = O$. Assuming that $p \colon \tilde{\mathcal{U}} \to \mathcal{O}$ is continuous then it follows from a result of Gleason [8] that p is a locally trivial fibration (i.e., there exist local cross-sections for p) with fibre $p^{-1}(O) = \ker p = \{e^{i\theta} \,|\, 0 \leq \theta \leq 2\pi\}$. It then follows that $p \colon p^{-1}(\mathcal{O}_{00}) \to \mathcal{O}_{00}$ is a locally trivial fibration with contractible base and so is trivial. That is we can find a continuous cross-section from $\mathcal{O}_{00}$ into $\tilde{\mathcal{U}}$. Thus except for the continuity of p we have established

Proposition 2.7. *There is a countable open cover $\{\mathcal{O}_i\}_{i \in I}$ of $\mathcal{O}$ obtained by* *translating $\mathcal{O}_{00}$, such that for each $\mathcal{O}_i$ there is a continuous map Γ_i: $\mathcal{O}_i \rightarrow \mathcal{U}(\mathcal{F})$ with*

$$\mathrm{Ad}\ \Gamma_i(O)\ [\pi_J(c(u))] = \pi_J(c(Ou)) .$$

(*Note that* (Ad U) $(A) = UAU^{-1}$).

Corollary 2.8. *The map associating to $O \in \mathcal{O}$, an implementing unitary $\Gamma(O)$, defines a continuous projective representation of $\mathcal{O}$.*

Remark 2.9. Let Ω be the G.N.S. cyclic vector in the Hilbert space $\mathcal{F}$ of π_J and fix the phase of $\Gamma(O)$ for $O \in \mathcal{O}$ by requiring

$$\langle \Omega, \Gamma(O)\ \Omega \rangle \geq 0 , \quad O \in \mathcal{O} . \tag{2.3}$$

Note that if $O \notin \mathcal{O}_0$ this is no restriction at all (cf. [9]). With this choice of phase the map $O \rightarrow \Gamma(O)$, $O \in \mathcal{O}_0$ can be shown to be continuous.

Lemma 2.10. *The map p: $\widetilde{\mathcal{U}} \rightarrow \mathcal{O}$ is continuous.*

Proof. As p is a homomorphism we need only prove continuity at the identity. If $U_n \rightarrow I$ strongly in $\widetilde{\mathcal{U}}$ then there exists a sequence $\{O_n\}_{n=1}^{\infty}$ in $\mathcal{O}$ such that $U_n \pi_J(c(u))\ U_n^{-1} = \pi_J(c(O_n u))$ and O_n converges strongly to I. In order to prove that O_n converges to I in the topology on $\mathcal{O}$ we need some results of [9]. As U_n implements the automorphism defined by O_n [9] gives an expression for $U_n \Omega$. Introduce the notation Λ for the operator

$$\Lambda v = \overline{T_2 T_1^{-1} v}, \quad v \in E \tag{2.4}$$

where T_1 and T_2 are defined in terms of O by (2.1) and the bar denotes a complex conjugation on E so that Λ is J-linear. Similarly define Λ_n in terms of O_n.

In order to use [9] we need to interpret the notation of that paper. To that end note that [9] defines $H = H_+ \oplus H_-$ where $H_\pm$ are copies of E (regarded as a complex Hilbert space with complex structure J) and lets C: $H \rightarrow H$ be defined by $(Cv)_\varepsilon = \bar{v}_-$, $\varepsilon = \pm$, $v_\varepsilon \in H_\varepsilon$. Identify the Clifford algebra over E with the CAR algebra over H_+ and orthogonal operators on E with unitary operators on H via equations (3.21), (3.22) of [9]. Then Λ_n defined above (cf. equation (2.4)) is written Λ_{+-} in equation (4.2) of [9]. Inspection of equations (4.41) to (4.45) of [9] shows that $U_n \Omega$ is a sum of terms which involve powers of the operator Λ_n. The term linear in Λ_n we call $K_n \Omega$ and from [9] p. 123 we deduce

$$||K_n \Omega|| = ||\Lambda_n||_2 \det [1 + \Lambda_n^* \Lambda_n]^{-1/4}$$

(here $||\Omega|| = 1$ is assumed and Λ_n is defined in the obvious way in terms of O_n

using (2.1) and (2.4)). Moreover from [9] (equ. 4.45)

$$|\langle\Omega, U_n\Omega\rangle| = \det[1+A_n^*A_n]^{-1/4}.$$

Since $K_n\Omega$ converges to the zero vector we have $||A_n||_2\to 0$ as $n\to\infty$. But

$$||A_n||_2^2 \geq ||A_n^*A_n||_1 = ||((T_n)_1^*(T_n)_1)^{-1}-1||$$

so that $\{||(T_n)_1^{-1}||\}_{n=1}^{\infty}$ is bounded abvoe. Hence $\{||(T_n)_1||\}_{n=1}^{\infty}$ is bounded away from zero. From this and the fact that $O_n\to I$ strongly it follows that A_n and A_n^* converge strongly to zero. By a result of Grümm [6] this means that A_n converges to zero in Hilbert-Schmidt norm. But as $(T_n)_1\to I$ strongly and $||(T_n)_1||=1$ another result of [6] gives $(T_n)_2=(T_n)_2(T_n)_1^{-1}(T_n)_1$ converging to zero in Hilbert-Sehmidt norm. This proves that $O_n\to I$ in $\mathcal{O}$ as required.

The next task of this section is to determine the homotopy groups of $\mathcal{O}$. The argument is essentially that of [1] (although there are small differences as the topology on $\mathcal{O}$ given there is not the same as that of this paper) and so the following is just a sketch.

Let $\mathcal{X}$ denote the space of all complex structures on E which differ from J by a Hilbert-Schmidt operator. Then $\mathcal{X}$ is the homogeneous space $\mathcal{O}/\mathcal{U}(E)$. On the other hand by Lemma 2.3 the normal subgroup $\mathcal{O}_2$ of $\mathcal{O}$ also acts transitively on $\mathcal{X}$ so that

$$\mathcal{O}/\mathcal{U}(E) \cong \mathcal{O}_2/\mathcal{U}(E)\cap\mathcal{O}_2.$$

This latter space has known homotopy type, namely that of the space $O(\infty)/U(\infty)$ where $O(\infty)$ (resp $U(\infty)$) denotes the stable orthogonal (resp unitary) group (cf. [10]). We prove below that the fibration $\mathcal{O}\to\mathcal{X}$ is locally trivial (this follows also from Lemma 2.3). So $\mathcal{O}$ is a locally trivial fibre bundle with contractible fibre $\mathcal{U}(E)$ over a paracompact base $\mathcal{X}$ and hence has the homotopy type of the base. From [11] the non-zero homotopy groups of $\mathcal{O}$ are thus: $\pi_0(\mathcal{O})\cong \boldsymbol{Z}_2\cong\pi_1(\mathcal{O})$, $\pi_2(\mathcal{O})\cong\pi_6(\mathcal{O})\cong\boldsymbol{Z}$, $\pi_{i+8}(\mathcal{O})\cong\pi_i(\mathcal{O})$ $i\geq 0$.

The following result exhibits a local cross-section for the fibration $\mathcal{O}\to\mathcal{X}$ and we include it for use elsewhere.

Lemma 2.11. *Let $\mathcal{X}_0=\{J_1 \mid ||J_1-J||<1\}$ and define*

$$s(J_1) = \tfrac{1}{2}(2-JJ_1-J_1J)^{1/2}-\tfrac{1}{2}\, J_0(JJ_1+J_1J+2)^{1/2} \quad (J_1\neq J)$$
$$s(J) = I;$$

where J_0 is the isometric part of the polar decomposition of JJ_1-J_1J then $J_1\to S(J_1)$ is a locally continuous cross section for $\mathcal{O}\to\mathcal{X}$

Proof. Let $s_1(J_1)$ and $s_2(J_1)$ stand for the first and second terms respectively in the definition of $s(J_1)$. Notice that s is well-defined since with $||J_1-J||<1$ both $(2-JJ_1-J_1J)$ and J_1J+J_1J+2 are positive and moreover $J_0^2=-I$, $J_0^*=-J_0$ since $\ker(JJ_1-J_1J)=0$. Now $s(J_1)\in\mathcal{O}$ since $s_2(J_1)^*s_2(J_1)=\frac{1}{4}(1+J_1J)^2JJ_1$ is trace class and by noting that J_0 commutes with both $s_1(J_1)$ and $s_2(J_1)$ some elementary algebra proves that $s(J_1)$ is orthogonal. It is straightforward to check that s is indeed a section so that only continuity remains to be established. Let $J_n\to J_1$ and note that $s_1(J_n)s_2(J_n)=JJ_n-J_nJ$ converges to JJ_1-J_1J in Hilbert-Schmidt norm. As $s_1(J_n)\to s_1(J_1)$ strongly we conclude that

$$s_1(J_n)^{-1}\, s_1(J_n)\, s_2(J_n)\to s_1(J_1)^{-1}\, s_1(J_1)\, s_2(J_1)$$

in Hilbert-Schmidt norm. This proves that s is continuous.

The original motivation for the results of this section was the observation that the usual construction of Clifford bundles (see for example [12]) does not immediately go through in the infinite dimensional case. An alternative approach is to start with a locally trivial principal $\mathcal{O}$-bundle P over a paracompact space $\mathcal{X}$. This may be specified in terms of a locally finite cover $\{X_\alpha\}_{\alpha\in I}$ of $\mathcal{X}$ and transition functions $g_{\alpha\beta}\colon X_\alpha\cap X_\beta\to\mathcal{O}$. Then one may define functions

$$\tilde{g}_{\alpha\beta}\colon X_\alpha\cap X_\beta\to \text{Aut}\, \mathcal{C}(E);\quad \alpha,\ \beta\in I$$

by

$$\tilde{g}_{\alpha\beta}(x)\ (c(u)) = c(g_{\alpha\beta}(x)u)\,,\quad x\in X_\alpha\cap X_\beta\,.$$

Then the functions $\tilde{g}_{\alpha\beta}$ may be used to patch together $\{X_\alpha\times\mathcal{C}(E)\}_{\alpha\in I}$ to form a bundle of C^*-algebras over $\mathcal{X}$ with fibre $\mathcal{C}(E)$ just as in Dixmier [13] (para. 10.1.3).

Now while non-trivial principal $\mathcal{O}$-bundles exist (as $\mathcal{O}$ has non-trivial topology) the associated Clifford bundle constructed above is always trivialisable whenever $\mathcal{X}$ has finite topological dimension. To see this construct first the associated Hilbert bundle $\mathcal{P}\times_{\mathcal{O}}E$. Then, as a bundle with structure group the orthogonal operators on E equipped with the strong operator topology this bundle is trivialisable. (This is Lemma 10.8.7 of [13] which goes through in the case of real Hilbert spaces as well.) So there exist maps r_α from X_α into the orthogonal group on E such that

$$g_{\alpha\beta}(x) = r_\alpha(x)\ r_\beta(x)^{-1}\,,\quad x\in X_\alpha\cap X_\beta\,.$$

By defining $\tilde{r}_\alpha\colon X_\alpha\to\text{Aut}\ C(E)$ by $\tilde{r}_\alpha(x)\ (c(u))=c(r_\alpha(x)u)$ we see that the Clifford bundle is trivialisable. This argument fails of course if E is finite dimensional (cf. [12]).

§3. Generalisations

Notice that many of the results of the previous section still hold when we replace the Hilbert-Schmidt ideal by any separable symmetrically normed ideal $\mathfrak{S}$ of compact operators on E (see [7], [14] for a discussion of symmetric norms). Thus $\mathcal{O}$ is the group of orthogonal O such that $OJ-JO\in\mathfrak{S}$. Specifically Lemmas 2.2 and 2.3 go through although now we must define $\mathcal{O}_{\mathfrak{S}}$ (in place of $\mathcal{O}_2$) to be the group of orthogonal operators on E which differ from the identity by an element of $\mathfrak{S}$, equipped with the topology inherited as a subgroup of $\mathcal{O}$. With this reinterpretation Remark 2.4, Proposition 2.5 and Lemma 2.6 still hold. Similarly the determination of the homotopy type of $\mathcal{O}$ (discussion preceding and including Lemma 2.11) goes through.

The main consequence of the preceding paragraph which will be used here is that for every $O\in\mathcal{O}$, $O-JOJ$ is Fredholm (use equation (2.2)) and that the map j: $\mathcal{O}\to\boldsymbol{Z}_2$ given by

$$j(O)=\dim_C \ker(O-JOJ) \pmod 2$$

is a continuous homomorphism which separates the connected components of $\mathcal{O}$. (In view of the remarks of the previous paragraph the only assertion which needs checking is the continuity of j and this depends only on the continuity of $O\to O+JOJ$ in the uniform topology by [1]. As the $\mathfrak{S}$-topology is, in general, stronger than the uniform topology, continuity of j follows.)

We will now deduce from the preceding remarks a generalisation of a theorem of Araki and Evans (Theorem 3 of [3]).

First of all we need to reformulate our discussion in terms of Araki's self-dual CAR algebra formalism [15]. Let H be a complex separable infinite dimensional Hilbert space, Γ be an anti-unitary involution, P_+ be an orthogonal projection on H with $\Gamma P_+\Gamma=1-P_+\equiv P_-$ and $\mathfrak{S}$ a separable symmetrically normed ideal of operators on H. Define

$$\mathcal{U}_\Gamma=\{U\mid U \text{ unitary on } H,\ P_+UP_-+P_-UP_+\in\mathfrak{S}\}\,.$$

Then $\mathcal{U}_\Gamma$ and $\mathcal{O}$ are isomorphic [simply let $H=E\oplus E$ with complex structure $J\oplus -J$, $\Gamma=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and map $\mathcal{O}\ni O\to\begin{pmatrix}T_1&T_2\\T_2&T_1\end{pmatrix}\in\mathcal{U}_\Gamma$ for $P_+=\begin{pmatrix}1&0\\0&0\end{pmatrix}$]. The analogue of a complex structure on E is a *basis projection* on H, that is, an orthogonal projection P on H with $\Gamma P\Gamma=1-P$. Hence let

$$\mathscr{X}_{P_+} = \{P \mid P \text{ basis projection, } P-P_+\in\mathfrak{S}\}$$

and equip $\mathscr{X}_{P_+}$ with the obvious metric topology: $\|P_1-P_2\|_{\mathfrak{S}}$ where $\|\ \ \|_{\mathfrak{S}}$ is the norm on $\mathfrak{S}$. Then $\mathscr{X}_{P_+}$ is homeomorphic to $\mathscr{X}$ as $\mathcal{U}_\Gamma$ acts transitively on $\mathscr{X}_{P_+}$ (so that both are homeomorphic to $\mathcal{O}/\mathcal{U}(E)$ with P_+ corresponding to J). The index map j becomes $\tilde{j}$: $\mathcal{U}_\Gamma\to\boldsymbol{Z}_2$ where

$$\tilde{j}(U) = \dim\ker(P_+UP_+) \pmod 2\,, \quad U\in\mathcal{U}_\Gamma$$

(here P_+UP_+ is regarded as an operator on P_+H). This gives the commutative diagram

$$\begin{array}{ccc} \mathcal{U}_\Gamma & \longrightarrow & \mathscr{X}_{P_+} \\ & \tilde{j}\searrow \quad \swarrow j_+ & \\ & \boldsymbol{Z}_2 & \end{array}$$

where $\mathcal{U}_\Gamma\to\mathscr{X}_{P_+}$ is the quotient map given by the transitive action of $\mathcal{U}_\Gamma$ on $\mathscr{X}_{P_+}$ and the dotted arrow is defined by

$$j_+(P) = \dim\ker P_+P \pmod 2, \quad P\in\mathscr{X}_{P_+}\,,$$

where P_+P: $PH\to P_+H$.

There is another way of defining j_+. Notice that

$$\dim\ker P_+UP_+ = \dim\ker P_+U^*P_+$$

as P_+UP_+ has Fredholm index zero. Moreover if $P_+U^*P_+v=0$, $v\in P_+H$ then $Pv=UP_+U^*P_+\ v=0$. Thus $\ker P_+U^*P_+$ is contained in the eigenspace of P_+-P corresponding to the eigenvalue 1. Conversely if $(P_+-P)v=v$ then one has

$$P_+Pv = 0 \text{ and } PP_+v = 2Pv \text{ so that } P_+PP_+v = 0\,.$$

But $P_+PP_+=P_+UP_+U^*P_+=|P_+U^*P_+|^2$ and thus $\ker P_+PP_+=\ker P_+U^*P_+$. Let $P_+\wedge(1-P)$ denote the projection onto the eigenspace of P_+-P of eigenvalue 1. Then we have (cf. [3], Theorem 3):

Theorem 3.1. *The group $\mathcal{U}_\Gamma$ acts continuously and transitively on $\mathscr{X}_{P_+}$, the spaces $\mathcal{U}_\Gamma$, $\mathscr{X}_{P_+}$ and $O(\infty)/U(\infty)$ have the same homotopy type and the connected components of $\mathscr{X}_{P_+}$ are separated by the continuous map j_+: $\mathscr{X}_{P_+}\to\boldsymbol{Z}_2$ given by*

$$j_+(P) = \text{rank } P_+ \wedge (1-P) \pmod 2, \quad P \in \mathcal{X}_{P_+}.$$

In [5] a subgroup of $\mathcal{O}$ was discussed although again the topology given there was different from that of this paper. If we fix a complex structure J_0 on E which commutes with J then (E, J_0) forms a complex Hilbert space and the subgroup $\mathcal{U}$ of $\mathcal{O}$ considered in [5] consists of those elements of $\mathcal{O}$ which commute with J_0. When $\mathfrak{S}=\mathfrak{S}_2$, the Hilbert-Schmidt ideal, $\mathcal{U}$ is the group of unitary Bogoliubov automorphisms implementable in the representation π_J of the CAR over E (regarded as a complex Hilbert space with complex structure J_0). In the context of this paper the natural topology on $\mathcal{U}$ is that inherited from $\mathcal{O}$. The appropriate index map for $\mathcal{U}$ is defined in [5] as $i(U)=$Fredholm index P_+UP_+ where $U\in\mathcal{U}$, $J=J_0(P_+-P_-)$. The continuity of i in the topology described here is proved in [16] in the case $\mathfrak{S}=\mathfrak{S}_2$ and the proof given there goes over to the general case. $\mathcal{U}$ turns out to be relevant to the discussion of representations of the group of unitaries which differ from the identity by a Hilbert-Schmidt operator and I will discuss its properties in more detail in [4].

Finally one may also consider the symplectic analogues of the groups discussed above. Let $\sigma\colon E\times E\to\mathbb{R}$ be the imaginary part of the complex inner product

$$\langle u, v\rangle = (u, v)+i(Ju, v); \quad u, v\in E.$$

Then the group of bounded operators R on E with

$$\sigma(Ru, Rv) = \sigma(u, v); \quad u, v\in E$$

is the symplectic group of E and we define $\mathcal{S}p$ to be the subgroup consisting of those R with $RJ-JR\in\mathfrak{S}$. Let $\mathcal{S}p_{\mathfrak{S}}$ denote the subgroup of $\mathcal{S}p$ consisting of operators which differ from the identity by an element of $\mathfrak{S}$, equipped with the metric topology: $\|R_1-R_2\|_{\mathfrak{S}}$, R_1, $R_2\in\mathcal{S}p_{\mathfrak{S}}$. Following [17], the polar decomposition allows us to write every element of $\mathcal{S}p$ as the product of a unitary operator on E (that is, an orthogonal operator commuting with J) and a positive element of $\mathcal{S}p$. So we may equip $\mathcal{S}p$ with the product topology (the unitaries having the strong operator topology) in which it becomes a topological group [17].

Lemma 3.2. (*Araki* [18]). *The group $\mathcal{S}p$ acts transitively on the space $\mathcal{X}_\sigma$ consisting of symplectic operators J_1 on E which differ from J by an element of $\mathfrak{S}$ and satisfy $J_1^2=-1$ and*

$$(Jv, J_1v)\geq(v, v) \quad \textit{for all} \quad v\in E.$$

Proof. This result could be proved using [18] however it is simpler to just exhibit the element R of $\mathcal{S}p$ such that $J_1=RJR^{-1}$. In fact it is sufficient to show there is a symplectic operator R with $J_1=RJR^{-1}$ for then $RJ-JR=(J_1-J)R\in\mathfrak{S}$. So define

$$R_{J_1} = \tfrac{1}{2}(2I-J_1J-JJ_1)^{1/2}+\tfrac{1}{2}K_0(-2I-J_1J-JJ_1)^{1/2}$$

where K_0 is the isometry in the polar decomposition of J_1J-J_1J. R_{J_1} is well-defined as the conditions on J_1 give $J_1J+JJ_1\leq-2\cdot I$ so that the square roots exist. Moreover the kernel of J_1J-JJ_1 is the joint eigenspace of JJ_1 and J_1J corresponding to the eigenvalue -1 and hence coincides with the kernel of $-2I-J_1J-JJ_1$. Thus as J anticommutes with JJ_1-J_1J it anticommutes with K_0. This, together with the fact that K_0 commutes with JJ_1+J_1J, implies that R_{J_1} is symplectic and it is then elementary algebra to verify that $R_{J_1}JR_{J_1}^{-1}=J_1$ as required.

Equip $\mathcal{X}_\sigma$ with the metric topology:

$$\|J_1-J_2\|_{\mathfrak{S}} \quad (J_1, J_2\in\mathcal{X}_\sigma)\,.$$

Theorem 3.3. *The group $\mathcal{S}p$ has the homotopy type of the space $\mathcal{X}_\sigma$. $\mathcal{X}_\sigma$ is homeomorphic to $\mathcal{S}p/\mathcal{U}(E)\cap\mathcal{S}p_{\mathfrak{S}}$ which has the homotopy type of $\mathcal{S}p(\infty)/\mathcal{U}(\infty)$. In particular $\mathcal{S}p$ is connected.*

Proof. That $\mathcal{S}p$ has the homotopy type of $\mathcal{X}_\sigma$ is immediate from the definition of the topology on $\mathcal{S}p$. Lemma 3.2 and the fact that $\mathcal{S}p=\mathcal{U}(E)\cdot\mathcal{S}p_{\mathfrak{S}}$ give the homeomorphism $\mathcal{X}_\sigma\simeq\mathcal{S}p_{\mathfrak{S}}/\mathcal{S}p_{\mathfrak{S}}\cap\mathcal{U}(E)$ while the homotopy type of the latter is known [10] to be that of $\mathcal{S}p(\infty)/U(\infty)$. The homotopy groups are listed in [11].

In the case where $\mathfrak{S}=\mathfrak{S}_2$, the Hilbert-Schmidt operators, the analogue of the spin representation of $\mathcal{O}$ is the metaplectic representation of $\mathcal{S}p$. The latter arises from the fact that in this case $\mathcal{S}p$ is the group of Bogoliubov automorphisms of the CCR over E (with symplectic form σ) implementable in the Fock representation determined by the complex structure J. Continuity of the metaplectic representation is proved in [17] (see also [19] for a clearer argument) and the fact that the topology on $\mathcal{S}p$ is the weakest for which the metaplectic representation is continuous is proved in [20].

§4. Remarks on K-Theory

Some of the groups of the preceding sections are related to the groups of invertible elements of certain Banach algebras. In this section H denotes E regarded as a complex Hilbert space with complex structure J_0 such that J_0 and J commute. Define algebras $\mathcal{B}_{\mathfrak{S}}(E)$, $\mathcal{B}_{\mathfrak{S}}(H)$ to consist of those bounded operators A on E, H respectively such that $AJ-JA\in\mathfrak{S}$. The following result is straightforward.

Lemma 4.1. *$\mathcal{B}_{\mathfrak{S}}(E)$ and $\mathcal{B}_{\mathfrak{S}}(H)$ are Banach algebras in the norm*

$$||A|| = ||A||_\infty + ||JAJ+A||_{\mathfrak{S}} \tag{4.1}$$

where $||\ \ ||_\infty$ is the uniform norm.

Denote by $\mathcal{GL}_{\mathfrak{S}}(E)$ and $\mathcal{GL}_{\mathfrak{S}}(H)$ the respective groups of invertible elements of these Banach algebras.

Proposition 4.2. *$\mathcal{GL}_{\mathfrak{S}}(E)$ (resp $\mathcal{GL}_{\mathfrak{S}}(H)$) has the homotopy type of $\mathcal{O}$ (resp $\mathcal{U}$).*

Proof. We equip $\mathcal{O}$ and $\mathcal{U}$ with the topologies they inherit from the metric defined by the norm (4.1). The group $\mathcal{GL}_{\mathfrak{S}}(H)$ is denoted G in [1] where we showed, using the polar decomposition, that $\mathcal{U}$ is a deformation retract of G. The same argument shows that $\mathcal{O}$ is a deformation retract of $\mathcal{GL}_{\mathfrak{S}}(E)$. Hence the result.

Proposition 4.3. $K_1(\mathcal{B}_{\mathfrak{S}}(E))\simeq\mathbb{Z}_2$, $\quad K_1(\mathcal{B}_{\mathfrak{S}}(H))\simeq\mathbb{Z}$.

Proof. We consider the case of $\mathcal{B}_{\mathfrak{S}}(E)$ only, the other being similar using the results of [5]. By definition of K_1 we need to consider the group of invertible elements of the algebra of $n\times n$ matrices over $\mathcal{B}_{\mathfrak{S}}(E)$ for every n. This latter algebra is isomorphic to the algebra of operators A on $E\otimes\mathbb{R}^n$ which satisfy $A\tilde{J}-\tilde{J}A\in\mathfrak{S}$ where $\tilde{J}=J\otimes I_n$ (I_n being the $n\times n$ identity), that is, isomorphic to the algebra $\mathcal{B}_{\mathfrak{S}}(E\otimes\mathbb{R}^n)$. But the group of invertible elements of the latter has the homotopy type of $\mathcal{O}$ by the preceding proposition and hence the group of invertible elements of $\mathcal{B}_{\mathfrak{S}}(E\otimes\mathbb{R}^n)$ factored by its connected component of the identity is isomorphic to $\mathbb{Z}_2$ independently of n. So $K_1(\mathcal{B}_{\mathfrak{S}}(E))\simeq\mathbb{Z}_2$.

Remark 4.4. There is another way in which the algebras $\mathcal{B}_{\mathfrak{S}}(E)$ and $\mathcal{B}_{\mathfrak{S}}(H)$ arise. If $\mathcal{A}$ is a complex C^*-algebra then given an element $\tau\in\mathrm{Ext}(\mathcal{A})$ it follows as in [20] that there is a representation π of $\mathcal{A}$ on a Hilbert space H and projections $P_\pm$ with $P_++P_-=I$ such that

$$\tau(A) = P_+\pi(A)P_+ + \mathcal{K}(H), \quad A\in\mathcal{A}$$

where $\mathcal{K}(H)$=compact operators on H. Thus $\mathcal{A}$ imbeds in $\mathcal{B}_{\mathfrak{S}}(H)$ via π. It is not difficult moreover to see that an embedding of $\mathcal{A}$ into $\mathcal{B}_{\mathfrak{S}_p}(H)$ (where $\mathfrak{S}_p$ is the ideal of compact operators S with $\sum_i s_i^p < \infty$, $\{s_i\}$ being the eigenvalues of $(S^*S)^{1/2}$) defines a p-summable Fredholm module in the sense of [22].

As a final observation we note that when $\mathfrak{S}=\mathcal{K}$, the ideal of compact operators, the groups $\mathcal{GL}(E)$ and $\mathcal{GL}(H)$ of invertible operators on E and H respectively, equipped with the uniform topology, are universal principal bundles for $\mathcal{GL}_{\mathcal{K}}(E)$ and $\mathcal{GL}_{\mathcal{K}}(H)$ respectively under the obvious quotient maps

$$\mathcal{GL}(E)\to\mathcal{GL}(E)/\mathcal{GL}_{\mathcal{K}}(E);\ \mathcal{GL}(H)\to\mathcal{GL}(H)/\mathcal{GL}_{\mathcal{K}}(H)\,.$$

This follows from the fact that $\mathcal{GL}(E)$ and $\mathcal{GL}(H)$ are contractible (Kuiper's theorem [23]) and the observation that the preceding quotient maps define locally trivial fibrations because $\mathcal{GL}_{\mathcal{K}}(E)$ and $\mathcal{GL}_{\mathcal{K}}(H)$ are closed Lie subgroups of $\mathcal{GL}(E)$ and $\mathcal{GL}(H)$ respectively. Similarly the orthogonal and unitary groups provide universal principal bundles for the retracts $\mathcal{O}$ and $\mathcal{U}$ respectively (with $\mathfrak{S}=\mathcal{K}$).

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