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The Root System of Sign (1, 0, 1)

Dedicated to Professor Shigeo Nakano on his 60th birthday

By

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Contents

§0. Introduction

Let F be a real vector space equipped with a quadratic form q, whose signature is (μ_+, μ_0, μ_-). (i.e. μ_+ and μ_- are maximal ranks of linear subspaces of F, on which q is positive or negative definite respectively. μ_0 is the rank of the radical of q.) A subset R of F will be called a root system of sign (μ_+, μ_0, μ_-) in this note, if it satisfies certain system of axioms similar to the classical one. (See for instance [1] Chap. VI. We reformulate it in §1 (1.2).)

In this note we give a classification of *root systems of sign* (1, 0, 1). The result is summarized in *Theorem* in §1 (1.6), which contains *a list of* 72 *types of root systems*. (Three types of them are reducible root systems.) Each of the types contains an infinite sequence of root systems, whose isomorphism classes are parametrized by a positive integer $m \in \mathbb{N}$, called the period of R, and some

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finite numerical datas $r_1, \cdots, r_k$, called the coefficients of the diagram of the type.

For a root system R, the group generated by reflexions of $\alpha \in R$, which one may call the Weyl group for R, is easily calculated, whose invariants are elementary hyperbolic functions. (See §5.)

The proof of the theorem is divided in §2–§5. A general view of the proof is given at the end of §1.

As an application of above study of root systems we classify *indefinite quadratic forms defined on a* $\mathbb{Z}$*-free module* L *of* rank 2 *from a view point of a maximal root system* belonging to the quadratic form (cf. Def. (7.1)). The result is summarized in (7.4), which contains 7 *types of maximal root systems* with parameter $(m, c) \in \mathbb{N} \times \mathbb{N}$. The number 7 of types is surprisingly small comparing to 72 types of root systems in (1.6). This fact is well explained from a duality property for maximal root systems in §8 (cf. (8.1) *Lemma*). The duality was first observed by an experimental study using the computer DEC System-2020 at RIMS in Kyoto University. The author is grateful to his colleagues T. Miwa, M. Jimbo and I. Naruki for their interest in the subject and for the helps in execution of the computer.

For the convenience of the reader, this note is written so that one can read the tables of the classification as quickly as possible. Therefore some readers may be suggested to go to (1.6) and (7.4) directly after reading some preparations of notations, definitions and some general programs in (1.1) – (1.5) and (7.1) – (7.3).

The set of units in a real quadratic field forms automatically a root system of sign (1, 0, 1) in the sense of this note, which we shall investigate in §6. This paragraph was added after a discussion with Prof. W. Borho, to whom the author is grateful.

It should be mentioned that this work has a motivation in a study of period domains for singularities. (See for instance [3], [4], [5].) However, there does not exist a singularity, which corresponds to a root system studied in this note, since the root systems in this note are too simple for the application. Therefore this note may be regarded to be rather preliminary computations. Nevertheless some complications concerning with arithmetics of quadratic forms and quadratic fields, which appeared in this note, seem to indicate already some possible complications which we shall meet in a further study of such generalized root systems.

The main part of this note was carried out during the stay of the author in Universite de Nancy I, in June 1983. He is grateful to the hospitality of the university and expresses his gratitude to Prof. D. Barlet.

§1. Statement of the Classification

We define a root system of sign (μ_+, μ_0, μ_-) in (1.2). Then in (1.6) Theorem we give a result of classification of root systems of sign (1, 0, 1), containing a table of 72 types of root systems. A general view of the proof is given in (1.7).

(1.1) As in the introduction, let F be a real vector space with a quadratic form q on it. The associated symmetric bilinear form given by $q(x+y)-q(x)-q(y)$ is denoted by $I(x, y)$. Let (μ_+, μ_0, μ_-) be the signature of I.

As usual, if an element $\alpha \in F$ has non zero length $I(\alpha, \alpha) \neq 0$, we define the dual $\alpha^\vee \in F$ and the reflexion $w_\alpha \in GL(F)$ as follows.

i) $\alpha^\vee := \dfrac{2}{I(\alpha, \alpha)} \alpha$

ii) $w_\alpha(u) := u - I(u, \alpha^\vee)\alpha$

so that $\alpha^{\vee\vee}=\alpha$ and $w_\alpha = w_{\alpha^\vee}$ and $w_\alpha^2 = \mathrm{id}$.

(1.2) **Definition 1.** *A subset R of F is called a root system of sign (μ_+, μ_0, μ_-), if it satisfies the following axioms* 1) – 5).

1) *Denote by* $Q(R)$ *the additive subgroup of F generated by the elements of R. Then* $Q(R)$ *is a full lattice of F.* (i.e. $\mathbb{R} \underset{\mathbb{Z}}{\otimes} Q(R) \simeq F$.)
2) *For any* $\alpha \in R$, $I(\alpha, \alpha) \neq 0$.
3) *For any* $\alpha \in R$, $w_\alpha R = R$.
4) *For any* $\alpha, \beta \in R$, $I(\alpha, \beta^\vee) \in \mathbb{Z}$.
5) *Irreducibility. If $R = R_1 \cup R_2$ such that $I(\alpha_1, \alpha_2) = 0$ for $\alpha_i \in R_i$, $i = 1, 2$, then $R_1 = \phi$ or $R_2 = \phi$.*

2. *Two root systems $R \subset F$ and $R' \subset F'$ are said to be isomorphic, if there exists a linear isomorphism $\varphi: F \to F'$, such that $\varphi(R) = R'$.*

Note 1. The only difference of the above definition of a root system from the conventional one (see for example [1] Chap. VI) is that we do not assume that I is positive (negative) definite and R is finite. R may contain positive length roots and negative length roots simultaneously as we shall see in this note.

Under this general setting, we have studied some general facts on the root systems in [5], which we shall use in this note.

Note 2. If a subset $R \subset F$ satisfies only axioms 1) – 4), we shall call R a root system, which may not be irreducible.

Note 3. If R and R' are irreducible root systems, then the above definition of the isomorphism φ: $R \simeq R'$ implies automatically an existence of a constant C such that

$$I = CI \circ \varphi .$$

(cf. §5 (5.1) Assertion, [5 §1 (1.4) Lemma]).

Therefore including reducible root systems, sometimes we use the following 2' as for a definition of isomorphism.

2'. *Two root systems $R \subset F$ and $R' \subset F'$ are said to be isomorphic, if there exists a linear isomorhism $\varphi: F \to F'$ and a constant C such that $\varphi(R)=R'$ and $I=CI\circ\varphi$.*

(1.3) Here after in this note, we assume that *F is a vector space of rank 2 over $\boldsymbol{R}$ and the signature of I is* (1, 0, 1).

To state our main theorem, we prepare some notations and fix some coordinates of F as follows.

(1.4) Due to the signature of I, the set of isotropic vectors$=\{f \in F: I(f,f)=0\}$ is consisting of two lines $\boldsymbol{R}e_1 \cup \boldsymbol{R}e_2$, where $e_1, e_2 \in F$ are some linearly independent isotropic vectors. Normalize their constant factors, so that, we have $I(e_1, e_1)=I(e_2, e_2)=0$ and $I(e_1, e_2)=1$. (There exists still an ambiguity of changing (e_1, e_2) to $(ce_1, c^{-1}e_2)$ for an non zero $c \in \boldsymbol{R}$.)

Using these coordinates, let us define an element of F,

(1.4.1) $$\alpha(t, r) := e^t e_1 + e^{-t} r e_2 \qquad \text{for } t, r \in \boldsymbol{R} .$$

Obviously any element of $F - \boldsymbol{R}e_1 \cup \boldsymbol{R}e_2$ is expressed uniquely either as $\alpha(t, r)$ or $-\alpha(t, r)$ for some $t \in \boldsymbol{R}$ and $r \in \boldsymbol{R} - \{0\}$.

(1.5) **Definition.** *For $m \in \boldsymbol{N}$ $m>2$, $q \in \boldsymbol{R}$ and $r \in \boldsymbol{R}-\{0\}$, let us define a subset of F,*

(1.5.1) $$R_{m,q,r} := \{\pm\alpha(np_m+q, r) : n \in \boldsymbol{Z}\}$$

where the number p_m is defined as follows.

(1.5.2) $$p_m := \log\left(\frac{m+\sqrt{m^2-4}}{2}\right) = \cosh^{-1}\left(\frac{m}{2}\right).$$

It is not hard to see that *$R_{m,q,r}$ is an irreducible root system in F* (cf. (2.3) Lemma). *We shall call m, q and r the period, the phase and the coefficient of $R_{m,q,r}$ respectively.*

By definition $R_{m,q,r}=R_{m',q',r'}$ iff $m=m'$, $r=r'$ and $q \equiv q' \bmod p_m$. Since the change of basis e_1, e_2 to $e^t e_1, e^{-t} e_2$ induces the translation of the phase q by t and the change of the bilinear form I to CI induces a multiplication by C on the coefficient r, $R_{m,q,r}$ and $R_{m',q',r'}$ are isomorphic as root system iff $m=m'$ (cf. (5.3) (5.4)).

For a non zero constant $d\in\mathbb{R}-\{0\}$, we define, $dR_{m,q,r}:=\{d\alpha\in F: \alpha\in R_{m,q,r}\}$. Thus $dR_{m,q,\varepsilon}=R_{m,q+\log|d|,\varepsilon d^2}$.

We need one more definition to state the classification.

Definition. *Let m be a positive integer. We define a set,*

$$M_m:=\{m/d^2: d\in\mathbb{N},\ d\,|\,m\}\ .$$

Direct from the definition, we have properties,

i) $M_m^{-1}=M_m$ *for* $m\in\mathbb{N}$,

ii) $M_{m_1}\circ M_{m_2}\subset M_{m_1\cdot m_2}$ *for* $m_1, m_2\in\mathbb{N}$.

(1.6) The classification of root systems R in F goes in the following way.

First we define a *diagram* ΓR associated to R and define the *type* of it as follows.

1) *Let R be a root system of sign* (1, 0, 1) *in F. Then it has a unique decomposition* (cf. (1.7) Lemma i) ii), (2.2) Lemma),

$$R=\bigcup_{i=1}^{k}R_i \tag{1.6.1}$$

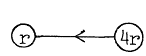
where

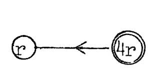
$$R_i:=d_iR_{m_i,q_i,\varepsilon_i}=R_{m_i,q_i+\log|d_i|,\varepsilon_i d_i^2}\ (i=1,\cdots,k)\ ,$$

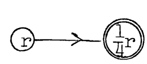
for some $k\in\mathbb{N}$ *and* $d_i\in\mathbb{R}_+=\{d>0\}$, $m_i\in\mathbb{N}$, $q_i\in\mathbb{R}/\mathbb{Z}p_{m_i}$, $\varepsilon_i\in\{\pm 1\}$ *for* $i=1,\cdots,k$, *s.t.* $d_i^2\varepsilon_i\neq d_j^2\varepsilon_j$ *for* $i\neq j$.

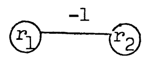
We put $r_i:=d_i^2\varepsilon_i$, $i=1,\cdots,k$ and call them the coefficients of R. $R_1,\cdots,R_k$ will be called the components of R.

2) *Let R be a root system with the decomposition* (1.6.1). *Then any pair of components* R_i, R_j *for* $1\leqq i, j\leqq k$, *falls into one of the following seven types of pairs, for suitable constants m, q, ε depending on the pair.* (Lemma in (3.3)) (On the left side of the list we associate a diagram, which will be explained in the next step 3).)

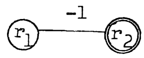
1)  $dR_{m,q,\varepsilon}$, $2dR_{m,q,\varepsilon}$ with $2|m$,

2)  $dR_{m,q,\varepsilon}$, $2dR_{m^2-2,q,\varepsilon}$ with $2|m$,

3)  $dR_{m,q,\varepsilon}$, $\frac{1}{2}dR_{m^2-2,q,\varepsilon}$ with $2|m$,

4) 

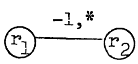
$d_1R_{m,q,\varepsilon}$, $d_2R_{m,q,-\varepsilon}$ with $r_2/r_1 \in M_{m^2-4}$,

5) 

$d_1R_{m,q,\varepsilon}$, $d_2R_{m^2-2,q,-\varepsilon}$ with $r_2/r_1 \in M_{m^2-4}$,

6) 

$d_1R_{m,q,\varepsilon}$, $d_2R_{m,q+\frac{1}{2}p_m,\varepsilon}$ with $r_2/r_1 \in M_{m+2}$,

7) 

$d_1R_{m,q,\varepsilon}$, $d_2R_{m,q+\frac{1}{2}p_m,-\varepsilon}$ with $r_2/r_1 \in M_{m-2}$.

As a consequence, we obtain:

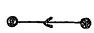
Put $m=\min\{m_i,\ i=1, \cdots, k\}$. *Then* $m_i=m$ *or* m^2-2 *for* $1 \leqq i \leqq k$. *We shall call this number* m *the period of* R.

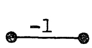
3) **Definition of the diagram** ΓR.

i) *For each component* R_i $i=1, \cdots, k$ *of* (1.6.1), *we associate a circle or a double circle according as* $m_i=m$ *or* m^2-2.

ii) *Inside the* (*double*) *circle of a component* R_i, *we put the coefficient* $r_i := d_i^2\varepsilon_i$ $(=I(\alpha, \alpha)/2$ *for* $\alpha \in R_i)$.

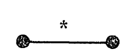
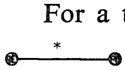
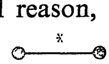
iii) *Two* (*double*) *circles of components* R_i *and* R_j *are combined by a segment with an additional symbols as follows*;

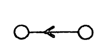
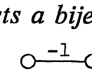
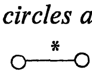
 *if there exists an element* $\alpha \in F$ *s.t.* $\alpha \in R_i$, $2\alpha \in R_j$.



if the signs ε_i, ε_j *of coefficients of* R_i *and* R_j *are different.*

if the difference $q_i - q_j$ *of phases is* $\not\equiv 0 \bmod \boldsymbol{Z}p_{m_i} + \boldsymbol{Z}p_{m_j}$.

For a typographical reason, we shall some times employ  or  instead of  in this note.

4) *Two diagrams are said to be of the same type, if there exists a bijection of circles and double circles which keeps the symmbols* ,  *and*  *on the segments.*

Two root systems are said to be of the same type, if their associated diagrams are of the same type.

5) *Let* R *be a root system in* F *of the decomposition* (1.6.1). *To the root system* R, *we associate the data,*

(1.6.2) $$d_i(m_i, q_i, \varepsilon_i) \quad i = 1, \cdots, k$$

where $d_i \in \boldsymbol{R}^+$, $m_i \in \boldsymbol{N}$ $(m_i > 2)$, $q_i \in \boldsymbol{R}/\boldsymbol{Z}p_{m_i}$, $\varepsilon_i \in \{\pm 1\}$.

Then we have:

i) *Two root systems in F are equal as subsets of F, iff their associated* (1.6.2) *data are equal (for a suitable permutation of* $\{1, \cdots, k\}$).

ii) *Two root systems in F are isomorphic, iff they are of the same type and the numerical invariants; the period* $m:=\inf(m_i)$ *and the proportion of the coefficients* $(r_1: \cdots : r_k) \in \boldsymbol{P}_{k-1}$ *(for* $r_i:=d_i^2\varepsilon_i$, $i=1, \cdots, k$) *are equal* (cf. (5.4)).

Theorem. *Root systems in F of sign* (1, 0, 1) *are classified into* 72 *types. Precisely, by the classification we mean the followings.*

i) *In the following table, we give* 72 *types of diagrams.*

ii) *To each type of the diagrams, we give a set of data,*

$$d_i(m_i, q_i, \varepsilon_i) \quad i=1, \cdots, k$$

where $d_i \in \boldsymbol{R}^+$, $m_i \in \boldsymbol{N}$ $(m_i > 2)$, $q_i \in \boldsymbol{R}/\boldsymbol{Z}p_{m_i}$, $\varepsilon_i \in \{\pm 1\}$ *for* $i=1, \cdots, k$, *which should satisfy the numerical conditions described at each table and the condition* $r_i \neq r_j$ *for* $1 \leqq i < j \leqq k$. *(Here we put* $r_i:=d_i^2\varepsilon_i$, $i=1, \cdots, k$.)

iii) *If R is a root system in F of sign* (1, 0, 1), *then the diagram ΓR is of one of the* 72 *types in the table and the data* (1.6.2) *associated to R belongs to the set of the data of the type in the table.*

Conversely any data of a type in the table is associated to a root system of the type in F.

iv) *We put numbers from* 1 *to* 72 *to each type of the diagrams. We shall refer for simplicity "the diagram Γ_n" or "type Γ_n" etc., instead of drawing the diagram itself.*

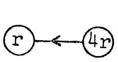
Table of Root Systems of Sign (1, 0, 1).

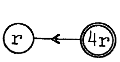
Numbering, Diagram, Datum $d_i(m_i, q_i, \varepsilon_i)$ $i=1, \cdots, k$ **and relations of** $(r_1, \cdots, r_k)$.

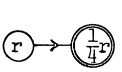
$k=1$ **case (one type)**

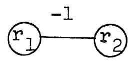
1.  $d(m, q, \varepsilon)$.

$k=2$ **case (7 types)**

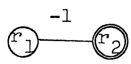
2.  $d(m, q, \varepsilon), 2d(m, q, \varepsilon)$ *s.t.* $2|m$.

3.  $d(m, q, \varepsilon), 2d(m^2-2, q, \varepsilon)$ *s.t.* $2|m$.

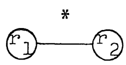
4.  $d(m, q, \varepsilon), \frac{1}{2}d(m^2-2, q, \varepsilon)$ *s.t.* $2|m$.

5. 

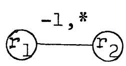
$d_1(m, q, \varepsilon), d_2(m, q, -\varepsilon)$ *s.t.* $r_1/r_2 \in M_{m^2-4}$.

6. 

$d_1(m, q, \varepsilon), d_2(m^2-2, q, -\varepsilon)$ *s.t.* $r_1/r_2 \in M_{m^2-4}$.

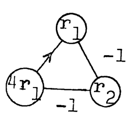
7. 

$d_1(m, q, \varepsilon), d_2(m, q+\frac{1}{2}p_m, \varepsilon)$ *s.t.* $r_1/r_2 \in M_{m+2}$.

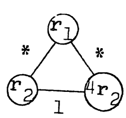
8. 

$d_1(m, q, \varepsilon), d_2(m, q+\frac{1}{2}p_m, -\varepsilon)$ *s.t.* $r_1/r_2 \in M_{m-2}$.

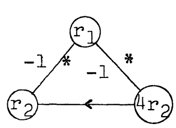
$k=3$ case (14 types)

9. 

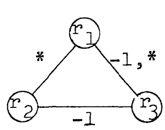
$d_1(m, q, \varepsilon), 2d_1(m, q, \varepsilon), d_2(m, q, -\varepsilon)$
s.t. $2|m$ *and*
$r_2/r_1, r_2/4r_1 \in M_{m^2-4}$.

10. 

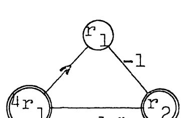
$d_1(m, q, \varepsilon), d_2(m, q+\frac{1}{2}p_m, \varepsilon), 2d_2(m, q+\frac{1}{2}p_m, \varepsilon)$
s.t. $2|m$ *and*
$r_2/r_1, 4r_2/r_1 \in M_{m+2}$.

11. 

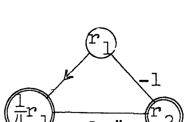
$d_1(m, q, \varepsilon), d_2(m, q+\frac{1}{2}p_m, -\varepsilon), 2d_2(m, q+\frac{1}{2}p_m, -\varepsilon)$
s.t. $2|m$ *and*
$r_2/r_1, 4r_2/r_1 \in M_{m-2}$.

12. 

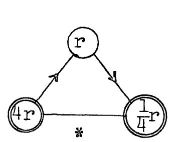
$d_1(m, q, \varepsilon), d_2(m, q+\frac{1}{2}p_m, \varepsilon), d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$,
s.t. $r_2/r_1 \in M_{m+2}$, $r_3/r_1 \in M_{m-2}$.

13. 

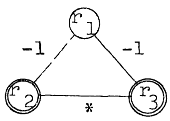
$d_1(m, q, \varepsilon), 2d_1(m^2-2, q+p_m, \varepsilon), d_2(m^2-2, q, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/r_1, r_2/4r_1 \in M_{m^2-4}$.

14. 

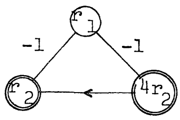
$d_1(m, q, \varepsilon), \frac{1}{2}d_1(m^2-2, q+p_m, \varepsilon), d_2(m^2-2, q, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/r_1, 4r_2/r_1 \in M_{m^2-4}$.

15. 

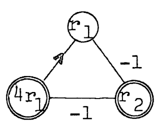
$d(m, q, \varepsilon), 2d(m^2-2, q+p_m, \varepsilon), \frac{1}{2}d(m^2-2, q, \varepsilon)$,
s.t. $4|m$.

16. 

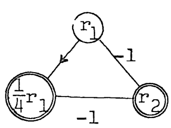
$d_1(m, q, \varepsilon)$, $d_2(m^2-2, q+p_m, -\varepsilon)$, $d_3(m^2-2, q, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/r_1, r_3/r_1 \in M_{m^2-4}$, $r_2/r_3 \in \{4^{\pm 1}, 16^{\pm 1}\}$.

17. 

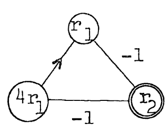
$d_1(m, q, \varepsilon)$, $d_2(m^2-2, q, -\varepsilon)$, $2d_2(m^2-2, q, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/r_1, 4r_2/r_1 \in M_{m^2-4}$.

18. 

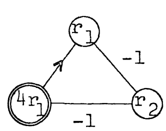
$d_1(m, q, \varepsilon)$, $2d_1(m^2-2, q, \varepsilon)$, $d_2(m^2-2, q, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/r_1 \in M_{m^2-4}$.

19. 

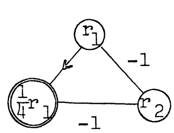
$d_1(m, q, \varepsilon)$, $\frac{1}{2}d_1(m^2-2, q, \varepsilon)$, $d_2(m^2-2, q, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/r_1 \in M_{m^2-4}$.

20. 

$d_1(m, q, \varepsilon)$, $2d_1(m, q, \varepsilon)$, $d_2(m^2-2, q, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/r_1, r_2/4r_1 \in M_{m^2-4}$.

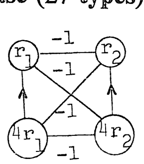
21. 

$d_1(m, q, \varepsilon)$, $2d_1(m^2-2, q, \varepsilon)$, $d_2(m, q, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/4r_1, r_2/r_1 \in M_{m^2-4}$.

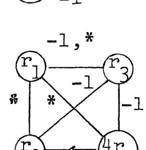
22. 

$d_1(m, q, \varepsilon)$, $\frac{1}{2}d_1(m^2-2, q, \varepsilon)$, $d_2(m, q, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/r_1, 4r_2/r_1 \in M_{m^2-4}$.

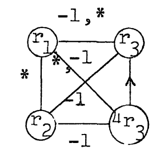
$k=4$ case (27 types)

23. 

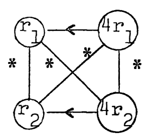
$d_1(m, q, \varepsilon)$, $2d_1(m, q, \varepsilon)$, $d_2(m, q, -\varepsilon)$, $2d_2(m, q, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/4r_1, r_2/r_1, 4r_2/r_1 \in M_{m^2-4}$.

24. 

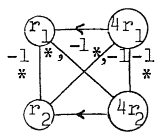
$d_1(m, q, \varepsilon)$, $d_2(m, q+\frac{1}{2}p_m, \varepsilon)$, $2d_2(m, q+\frac{1}{2}p_m, \varepsilon)$,
$d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/r_1, 4r_2/r_1 \in M_{m+2}$, $r_3/r_1 \in M_{m-2}$.

25. 

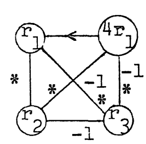
$d_1(m, q, \varepsilon)$, $d_2(m, q+\frac{1}{2}p_m, \varepsilon)$, $d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$,
$2d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/r_1 \in M_{m+2}$, r_3/r_1, $4r_3/r_1 \in M_{m-2}$.

26. 

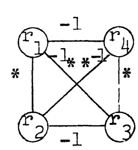
$d_1(m, q, \varepsilon)$, $2d_1(m, q, \varepsilon)$, $d_2(m, q+\frac{1}{2}p_m, \varepsilon)$,
$2d_2(m, q+\frac{1}{2}p_m, \varepsilon)$,
s.t. $4|m+2$ *and*
$r_2/4r_1$, r_2/r_1, $4r_2/r_1 \in M_{m+2}$.

27. 

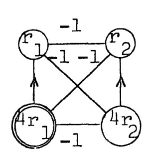
$d_1(m, q, \varepsilon)$, $2d_1(m, q, \varepsilon)$, $d_2(m, q+\frac{1}{2}p_m, -\varepsilon)$,
$2d_2(m, q+\frac{1}{2}p_m, -\varepsilon)$,
s.t. $4|m-2$ *and*
$r_2/4r_1$, r_2/r_1, $4r_2/r_1 \in M_{m-2}$.

28. 

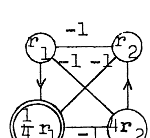
$d_1(m, q, \varepsilon)$, $2d_1(m, q, \varepsilon)$, $d_2(m, q+\frac{1}{2}p_m, \varepsilon)$,
$d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/4r_1$, $r_2/r_1 \in M_{m+2}$, $r_3/4r_1$, $r_3/r_1 \in M_{m-2}$.

29. 

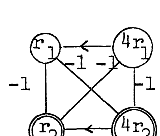
$d_1(m, q, \varepsilon)$, $d_2(m, q+\frac{1}{2}p_m, \varepsilon)$,
$d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$, $d_4(m, q, -\varepsilon)$,
s.t.
r_2/r_1, $r_4/r_3 \in M_{m+2}$, r_3/r_1, $r_4/r_2 \in M_{m-2}$.

30. 

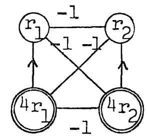
$d_1(m, q, \varepsilon)$, $d_2(m, q, -\varepsilon)$, $2d_2(m, q, -\varepsilon)$,
$2d_1(m^2-2, q, \varepsilon)$,
s.t. $2|m$ *and*
$r_2/4r_1$, r_2/r_1, $4r_2/r_1 \in M_{m^2-4}$.

31. 

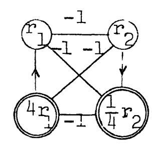
$d_1(m, q, \varepsilon)$, $d_2(m, q, -\varepsilon)$, $2d_2(m, q, -\varepsilon)$,
$\frac{1}{2}d_1(m^2-2, q, \varepsilon)$,
s.t. $2|m$ *and*
r_2/r_1, $4r_2/r_1$, $16r_2/r_1 \in M_{m^2-4}$.

32. 

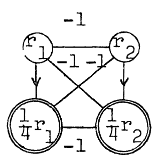
$d_1(m, q, \varepsilon)$, $2d_1(m, q, \varepsilon)$, $d_2(m^2-2, q, -\varepsilon)$,
$2d_2(m^2-2, q, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/4r_1$, r_2/r_1, $4r_2/r_1 \in M_{m^2-4}$.

33. 

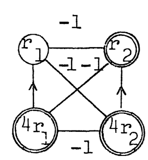
$d_1(m, q, \varepsilon)$, $d_2(m, q, -\varepsilon)$, $2d_1(m^2-2, q, \varepsilon)$, $2d_2(m^2-2, q, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/4r_1, r_2/r_1, 4r_2/r_1 \in M_{m^2-4}$.

34. 

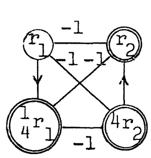
$d_1(m, q, \varepsilon)$, $d_2(m, q, -\varepsilon)$, $2d_1(m^2-2, q, \varepsilon)$, $\frac{1}{2}d_2(m^2-2, q, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/4r_1, r_2/r_1 \in M_{m^2-4}$.

35. 

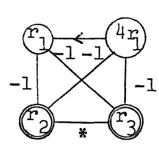
$d_1(m, q, \varepsilon)$, $d_2(m, q, -\varepsilon)$, $\frac{1}{2}d_1(m^2-2, q, \varepsilon)$, $\frac{1}{2}d_2(m^2-2, q, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/4r_1, r_2/r_1, 4r_2/r_1 \in M_{m^2-4}$.

36. 

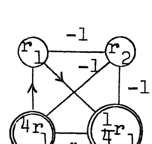
$d_1(m, q, \varepsilon)$, $2d_1(m^2-2, q, \varepsilon)$, $d_2(m^2-2, q, -\varepsilon)$, $2d_2(m^2-2, q, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/4r_1, r_2/r_1, 4r_2/r_1 \in M_{m^2-4}$.

37. 

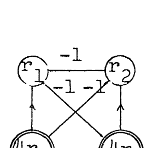
$d_1(m, q, \varepsilon)$, $\frac{1}{2}d_1(m^2-2, q, \varepsilon)$, $d_2(m^2-2, q, -\varepsilon)$, $2d_2(m^2-2, q, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/r_1, 4r_2/r_1, 16r_2/r_1 \in M_{m^2-4}$.

38. 

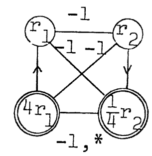
$d_1(m, q, \varepsilon)$, $2d_1(m, q, \varepsilon)$, $d_2(m^2-2, q+p_m, -\varepsilon)$, $d_3(m^2-2, q, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/r_1, r_2/4r_1, r_3/r_1, r_3/4r_1 \in M_{m^2-4}$, $r_2/r_3 \in \{4^{\pm 1}\}$.

39. 

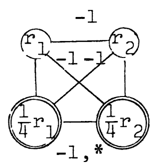
$d_1(m, q, \varepsilon)$, $d_2(m, q, -\varepsilon)$, $2d_1(m^2-2, q+p_m, \varepsilon)$, $\frac{1}{2}d_1(m^2-2, q, \varepsilon)$,
s.t. $4|m$ *and*
$r_2/4r_1, r_2/r_1, 4r_2/r_1 \in M_{m^2-4}$.

40. 

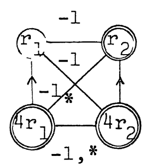
$d_1(m, q, \varepsilon)$, $d_2(m, q, -\varepsilon)$, $2d_1(m^2-2, q+p_m, \varepsilon)$, $2d_2(m^2-2, q, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/4r_1, r_2/r_1, 4r_2/r_1 \in M_{m^2-4}$.

41. 

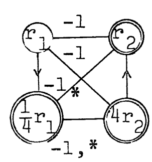
$d_1(m, q, \varepsilon)$, $d_2(m, q, -\varepsilon)$,
$2d_1(m^2-2, q+p_m, \varepsilon)$, $\frac{1}{2}d_2(m^2-2, q, -\varepsilon)$,
s.t. $2|m$ *and*
r_2/r_1, $r_2/4r_1$, $r_2/16r_1 \in M_{m^2-4}$.

42. 

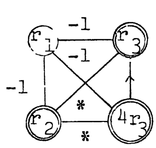
$d_1(m, q, \varepsilon)$, $d_2(m, q, -\varepsilon)$,
$\frac{1}{2}d_1(m^2-2, q+p_m, \varepsilon)$, $\frac{1}{2}d_2(m^2-2, q, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/4r_1$, r_2/r_1, $4r_2/r_1 \in M_{m^2-4}$.

43. 

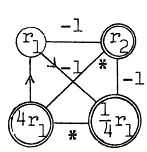
$d_1(m, q, \varepsilon)$, $2d_1(m^2-2, q+p_m, \varepsilon)$,
$d_2(m^2-2, q, -\varepsilon)$, $2d_2(m^2-2, q, -\varepsilon)$.
s.t. $2|m$ *and*
$r_2/4r_1$, r_2/r_1, $4r_2/r_1 \in M_{m^2-4}$.

44. 

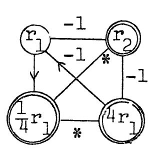
$d_1(m, q, \varepsilon)$, $\frac{1}{2}d_1(m^2-2, q+p_m, \varepsilon)$,
$d_2(m^2-2, q, -\varepsilon)$, $2d_2(m^2-2, q, -\varepsilon)$,
s.t. $2|m$ *and*
r_2/r_1, $4r_2/r_1$, $16r_2/r_1 \in M_{m^2-4}$.

45. 

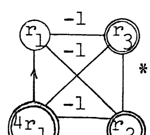
$d_1(m, q, \varepsilon)$, $d_2(m^2-2, q+p_m, -\varepsilon)$, $d_3(m^2-2, q, -\varepsilon)$,
$2d_3(m^2-2, q, -\varepsilon)$,
s.t. $4|m$ *and*
r_2/r_1, r_3/r_1, $4r_3/r_1 \in M_{m^2-4}$, $r_3/r_2 \in \{4^{\pm 1}, 16^{\pm 1}\}$.

46. 

$d_1(m, q, \varepsilon)$, $2d_1(m^2-2, q+p_m, \varepsilon)$,
$\frac{1}{2}d_1(m^2-2, q, \varepsilon)$, $d_2(m^2-2, q, -\varepsilon)$,
s.t. $4|m$ *and*
$r_2/4r_1$, $r_2/r_1 \in M_{m^2-4}$.

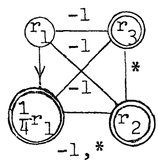
47. 

$d_1(m, q, \varepsilon)$, $\frac{1}{2}d_1(m^2-2, q+p_m, \varepsilon)$,
$2d_1(m^2-2, q, \varepsilon)$, $d_2(m^2-2, q, -\varepsilon)$,
s.t. $4|m$ *and*
r_2/r_1, $4r_2/r_1 \in M_{m^2-4}$.

48. 

$d_1(m, q, \varepsilon)$, $d_2(m^2-2, q+p_m, -\varepsilon)$,
$d_3(m^2-2, q, -\varepsilon)$, $2d_1(m^2-2, q, \varepsilon)$,
s.t. $2|m$ *and*
r_2/r_1, $r_2/4r_1$, $r_3/r_1 \in M_{m^2-4}$, $r_2/r_3 \in \{4^{\pm 1}, 16^{\pm 1}\}$.

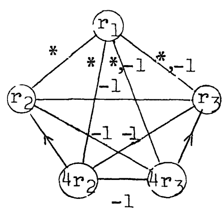
49.



$d_1(m, q, \varepsilon)$, $d_2(m^2-2, q+p_m, -\varepsilon)$,
$d_3(m^2-2, q, -\varepsilon)$, $\frac{1}{2}d_1(m^2-2, q, \varepsilon)$,
s.t. $2|m$ *and*
r_2/r_1, $4r_2/r_1$, $r_3/r_1 \in M_{m^2-4}$, $r_2/r_3 \in \{4^{\pm 1}, 16^{\pm 1}\}$.

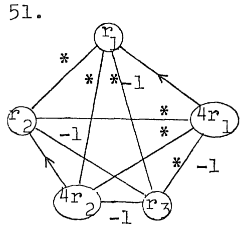
$k=5$ **case (11 types)**

50.



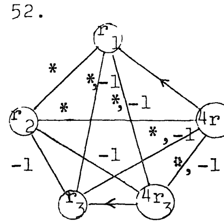
$d_1(m, q, \varepsilon)$, $d_2(m, q+\frac{1}{2}p_m, \varepsilon)$, $2d_2(m, q+\frac{1}{2}p_m, \varepsilon)$,
$d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$, $2d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$,
s.t. $2|m$ *and*
r_2/r_1, $4r_2/r_1 \in M_{m+2}$, r_3/r_1, $4r_3/r_1 \in M_{m-2}$.

51.



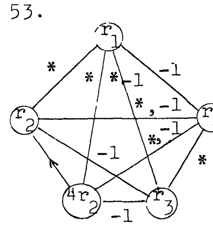
$d_1(m, q, \varepsilon)$, $2d_1(m, q, \varepsilon)$, $d_2(m, q+\frac{1}{2}p_m, \varepsilon)$,
$2d_2(m, q+\frac{1}{2}p_m, \varepsilon)$, $d_3(m, q, -\varepsilon)$,
s.t. $4|m+2$ *and*
$r_2/4r_1$, r_2/r_1, $4r_2/r_1 \in M_{m+2}$, $r_3/4r_2$, $r_3/r_2 \in M_{m-2}$.

52.



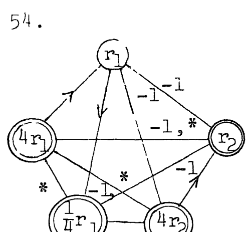
$d_1(m, q, \varepsilon)$, $2d_1(m, q, \varepsilon)$, $d_2(m, q+\frac{1}{2}p_m, \varepsilon)$,
$d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$, $2d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$,
s.t. $4|m+2$ *and*
$r_2/4r_1$, $r_2/r_1 \in M_{m+2}$, $r_3/4r_1$, r_3/r_1, $4r_3/r_1 \in M_{m-2}$.

53.



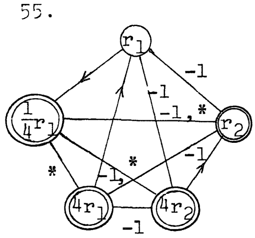
$d_1(m, q, \varepsilon)$, $d_4(m, q, -\varepsilon)$, $d_2(m, q+\frac{1}{2}p_m, \varepsilon)$,
$2d_2(m, q+\frac{1}{2}p_m, \varepsilon)$, $d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$,
s.t. $2|m$ *and*
r_3/r_1, r_4/r_2, $r_4/4r_2 \in M_{m-2}$, r_4/r_3, r_2/r_1, $4r_2/r_1 \in M_{m+2}$.

54.



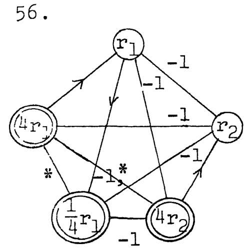
$d_1(m, q, \varepsilon)$, $2d_1(m^2-2, q+p_m, \varepsilon)$, $d_2(m^2-2, q, -\varepsilon)$,
$\frac{1}{2}d_1(m^2-2, q, \varepsilon)$, $2d_2(m^2-2, q, -\varepsilon)$,
s.t. $4|m$ *and*
$r_2/4r_1$, r_2/r_1, $4r_2/r_1 \in M_{m^2-4}$.

55.



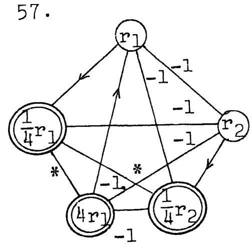
$d_1(m, q, \varepsilon)$, $2d_1(m^2-2, q, \varepsilon)$, $d_2(m^2-2, q, -\varepsilon)$,
$\frac{1}{2}d_1(m^2-2, q+p_m, \varepsilon)$, $2d_2(m^2-2, q, -\varepsilon)$,
s.t. $4|m$ *and*
$r_2/r_1, 4r_2/r_1, 16r_2/r_1 \in M_{m^2-4}$.

56.



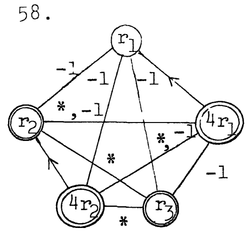
$d_1(m, q, \varepsilon)$, $d_2(m, q, -\varepsilon)$, $2d_2(m^2-2, q, -\varepsilon)$,
$2d_1(m^2-2, q+p_m, \varepsilon)$, $\frac{1}{2}d_1(m^2-2, q, \varepsilon)$,
s.t. $4|m$ *and*
$r_2/4r_1, r_2/r_1, 4r_2/r_1 \in M_{m^2-4}$.

57.



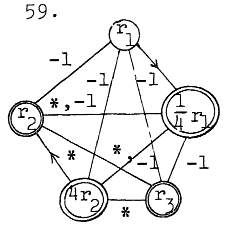
$d_1(m, q, \varepsilon)$, $\frac{1}{2}d_1(m^2-2, q+p_m, \varepsilon)$, $2d_1(m^2-2, q, \varepsilon)$,
$d_2(m, q, -\varepsilon)$, $\frac{1}{2}d_2(m^2-2, q, -\varepsilon)$,
s.t. $4|m$ *and*
$r_2/4r_1, r_2/r_1, 4r_2/r_1 \in M_{m^2-4}$.

58.



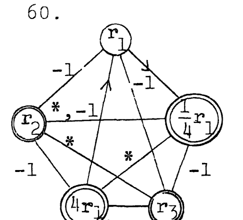
$d_1(m, q, \varepsilon)$, $d_2(m^2-2, q+p_m, -\varepsilon)$, $2d_2(m^2-2, q+p_m, -\varepsilon)$,
$d_3(m^2-2, q, -\varepsilon)$, $2d_1(m^2-2, q, \varepsilon)$,
s.t. $4|m$ *and*
$r_2/4r_1, r_2/r_1, 4r_2/r_1, r_3/r_1 \in M_{m^2-4}$, $r_3/r_2 = 4^{-1}$.

59.



$d_1(m, q, \varepsilon)$, $d_2(m^2-2, q+p_m, -\varepsilon)$, $2d_2(m^2-2, q+p_m, -\varepsilon)$,
$d_3(m^2-2, q, -\varepsilon)$, $\frac{1}{2}d_1(m^2-2, q, \varepsilon)$,
s.t. $4|m$ *and*
$r_2/r_1, 4r_2/r_1, 16r_2/r_1, r_3/r_1 \in M_{m^2-4}$, $r_3/r_2 = 16$.

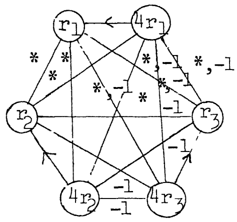
60.



$d_1(m, q, \varepsilon)$, $2d_1(m^2-2, q+p_m, \varepsilon)$, $\frac{1}{2}d_1(m^2-2, q, \varepsilon)$,
$d_2(m^2-2, q+p_m, -\varepsilon)$, $d_3(m^2-2, q, -\varepsilon)$,
s.t. $4|m$ *and*
$r_2/r_1, 4r_2/r_1, r_3/r_1, r_3/4r_1 \in M_{m^2-4}$, $r_3/r_2 \in \{4, 16\}$.

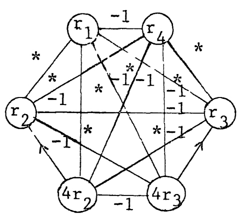
$k=6$ **case (7 types)**

61.



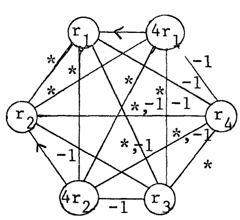
$d_1(m, q, \varepsilon)$, $2d_1(m, q, \varepsilon)$, $d_2(m, q+\frac{1}{2}p_m, \varepsilon)$,
$2d_2(m, q+\frac{1}{2}p_m, \varepsilon)$, $d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$,
$2d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$,
s.t. $4|m+2$ *and*
$r_2/4r_1,\ r_2/r_1,\ 4r_2/r_1 \in M_{m+2},\quad r_3/4r_1,\ r_3/r_1,\ 4r_3/r_1 \in M_{m-2}$.

62.



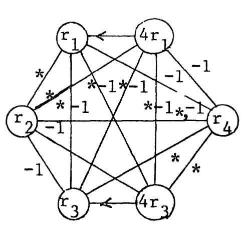
$d_1(m, q, \varepsilon)$, $d_2(m, q+\frac{1}{2}p_m, \varepsilon)$, $2d_2(m, q+\frac{1}{2}p_m, \varepsilon)$,
$d_4(m, q, -\varepsilon)$, $d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$, $2d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$,
s.t. $2|m$ *and*
$r_2/r_1,\ 4r_2/r_1,\ r_4/r_3,\ r_4/4r_3 \in M_{m+2}$,
$r_3/r_1,\ 4r_3/r_1,\ r_4/r_2,\ r_4/4r_2 \in M_{m-2}$.

63.



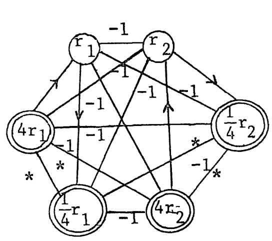
$d_1(m, q, \varepsilon)$, $d_2(m, q+\frac{1}{2}p_m, \varepsilon)$, $2d_2(m, q+\frac{1}{2}p_m, \varepsilon)$,
$2d_1(m, q, \varepsilon)$, $d_4(m, q, -\varepsilon)$, $d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$,
s.t. $4|m+2$ *and*
$r_2/4r_1,\ r_2/r_1, 4r_2/r_1,\ r_4/r_3 \in M_{m+2}$,
$r_3/4r_1,\ r_3/r_1,\ r_4/r_2,\ r_4/4r_2 \in M_{m-2}$.

64.



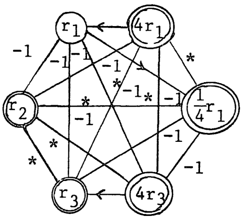
$d_1(m, q, \varepsilon)$, $2d_1(m, q, \varepsilon)$, $d_2(m, q+\frac{1}{2}p_m, \varepsilon)$,
$d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$, $2d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$, $d_4(m, q, -\varepsilon)$,
s.t. $4|m-2$ *and*
$r_2/4r_1,\ r_2/r_1,\ r_4/r_3,\ r_4/4r_3 \in M_{m+2}$,
$r_3/4r_1,\ r_3/r_1,\ 4r_3/r_1,\ r_4/r_2 \in M_{m-2}$.

65.



$d_1(m, q, \varepsilon)$, $d_2(m, q, -\varepsilon)$, $2d_1(m^2-2, q+p_m, \varepsilon)$,
$\frac{1}{2}d_2(m^2-2, q+p_m, -\varepsilon)$,
$\frac{1}{2}d_1(m^2-2, q, \varepsilon)$, $2d_2(m^2-2, q, -\varepsilon)$,
s.t. $4|m$ *and*
$r_2/4r_1,\ r_2/r_1,\ 4r_2/r_1 \in M_{m^2-4}$.

66.

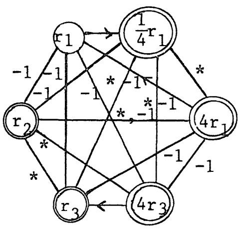


$d_1(m, q, \varepsilon)$, $d_2(m^2-2, q+p_m, -\varepsilon)$, $2d_1(m^2-2, q+p_m, \varepsilon)$, $d_3(m^2-2, q, -\varepsilon)$, $\frac{1}{2}d_1(m^2-2, q, \varepsilon)$, $2d_3(m^2-2, q, -\varepsilon)$,

s.t. $4|m$ *and*

$r_2/r_1, 4r_2/r_1, r_3/4r_1, r_3/r_1, 4r_3/r_1 \in M_{m^2-4}$, $r_3/r_2=4$.

67.



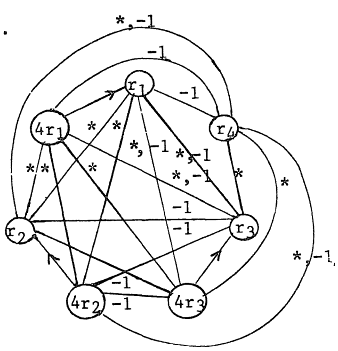
$d_1(m, q, \varepsilon)$, $d_2(m^2-2, q+p_m, -\varepsilon)$, $\frac{1}{2}d_1(m^2-2, q+p_m, \varepsilon)$, $d_3(m^2-2, q, -\varepsilon)$, $2d_1(m^2-2, q, \varepsilon)$, $2d_3(m^2-2, q, -\varepsilon)$,

s.t. $4|m$ *and*

$r_2/4r_1, r_2/r_1, r_3/r_1, 4r_3/r_1, 16r_3/r_1 \in M_{m^2-4}$, $r_2/r_3=16$.

$k=7$ case (1 type)

68.



$d_1(m, q, \varepsilon)$, $2d_1(m, q, \varepsilon)$, $d_4(m, q, -\varepsilon)$,

$d_2(m, q+\frac{1}{2}p_m, \varepsilon)$, $2d_2(m, q+\frac{1}{2}p_m, \varepsilon)$,

$d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$, $2d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$,

s.t. $4|m+2$ *and*

$r_2/4r_1, r_2/r_1, 4r_2/r_1, r_4/r_3, r_4/4r_3 \in M_{m+2}$,

$r_3/4r_1, r_3/r_1, 4r_3/r_1, r_4/r_2, r_4/4r_2 \in M_{m-2}$.

$k=8$ case (1 type)

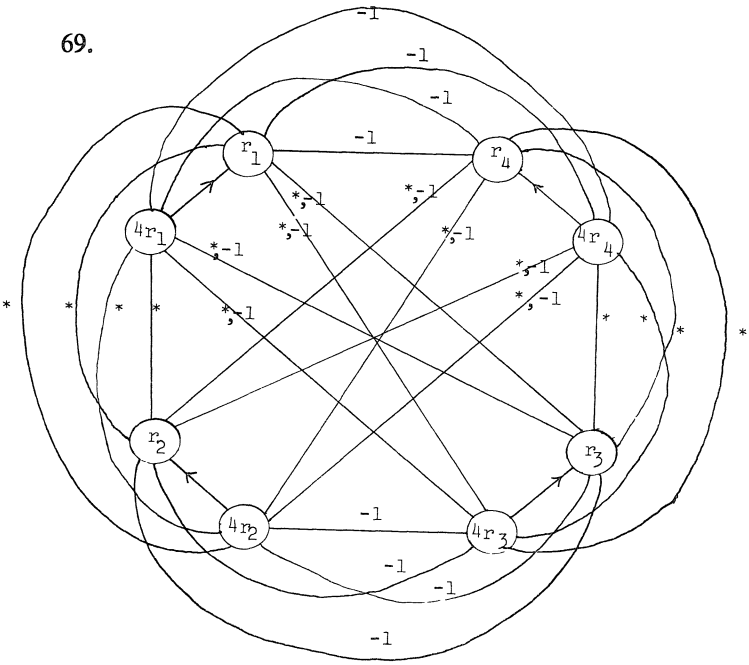
69. $d_1(m, q, \varepsilon)$, $2d_1(m, q, \varepsilon)$, $d_2(m, q+\frac{1}{2}p_m, \varepsilon)$, $2d_2(m, q+\frac{1}{2}p_m, \varepsilon)$, $d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$, $2d_3(m, q+\frac{1}{2}p_m, -\varepsilon)$, $d_4(m, q, -\varepsilon)$, $2d_4(m, q, -\varepsilon)$,

s.t. $4|m+2$ *and*

$r_2/4r_1, r_2/r_1, 4r_2/r_1, 4r_4/r_3, r_4/r_3, r_4/4r_3 \in M_{m+2}$,

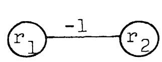
$r_3/4r_1, r_3/r_1, 4r_3/r_1, r_4/4r_2, r_4/r_2, 4r_4/r_2 \in M_{m-2}$.

69.



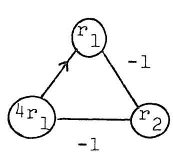
There are three more diagrams for *reducible root systems*,*) where we understand them as root systems whose periods are ∞.

70.



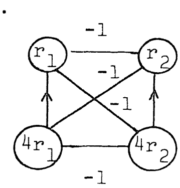
$$d_1(\infty,\ q,\ \varepsilon) := \{\pm d_1(e^q e_1 + \varepsilon e^{-q} e_2)\}\ ,$$
$$d_2(\infty,\ q,\ -\varepsilon) := \{\pm d_2(e^q e_1 - \varepsilon e^{-q} e_2)\}\ .$$

71.



$$d_1(\infty,\ q,\ \varepsilon) := \{\pm d_1(e^q e_1 + \varepsilon e^{-q} e_2)\}\ ,$$
$$2d_1(\infty,\ q,\ \varepsilon) := \{\pm 2d_1(e^q e_1 + \varepsilon e^{-q} e_2)\}\ ,$$
$$d_2(\infty,\ q,\ -\varepsilon) := \{\pm d_2(e^q e_1 - \varepsilon e^{-q} e_2)\}\ .$$

72.



$$d_1(\infty,\ q,\ \varepsilon) := \{\pm d_1(e^q e_1 + \varepsilon e^{-q} e_2)\}\ ,$$
$$2d_1(\infty,\ q,\ \varepsilon) := \{\pm 2d_1(e^q e_1 + \varepsilon e^{-q} e_2)\}\ ,$$
$$d_2(\infty,\ q,\ -\varepsilon) := \{\pm d_2(e^q e_1 - \varepsilon e^{-q} e_2)\}\ ,$$
$$2d_2(\infty,\ q,\ -\varepsilon) := \{\pm 2d_2(e^q e_1 - \varepsilon e^{-q} e_2)\}\ .$$

Note 1) *The automorphism group of a root system R and the Weyl group W_R generated by the reflexions w_α for $\alpha \in R$, are explicitely given in* §5 (5.5) (5.7).
2) *Let us define the dual of R as*

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*) For the last three types, we employ (1.2) *Note* 2' as for the definition of the isomorphism.

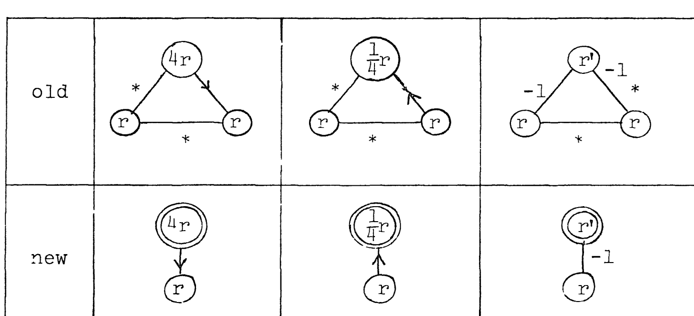
$$R^{\vee}:=\{\alpha^{\vee}\in F\colon \alpha\in R\}$$

which is also a root system in F. (Cf. [5] §1). *If R belongs to a datum* $d_i(m_i, q_i, \varepsilon_i)$ $i=1, \cdots, k$, *the dual* $R^{\vee}$ *belongs to the datum* $d_i^{-1}(m_i, q_i, \varepsilon_i)$ $i=1, \cdots, k$. *Therefore the diagram for* $R^{\vee}$ *is obtained from that of R by replacing the coefficients* r_i *by* r_i^{-1}, $i=1, \cdots, k$, *and changing the directions of the arrows of* ○—←—○, ○—←—◎ *and* ○—→—◎ .

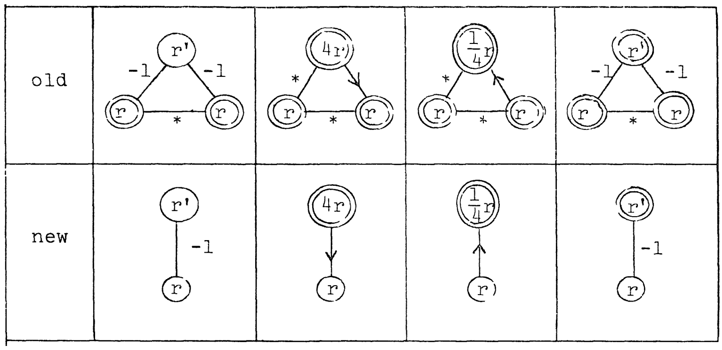
3) *The assumption of Theorem* 2), *that coefficients* r_i, $i=1, \cdots, k$, *are pairwisely different, was made to avoid the overlapping of the classification. Without this assumption, the union* (1.6.1) *associated to a datum* (1.6.2) *in the table still form a root system in F, whose diagram* Γ' *is obtained by "collapsing" the vertexes of the diagram* Γ *of the table as follows.*

First notice that two coefficients can be equal only among the component combined by the segments ⓡ—*—ⓡ or ◎ⓡ—*—◎ⓡ. Then new diagram Γ' is obtained by replacing this part by one circle ⓡ and changing the other part of Γ as follows.

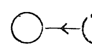
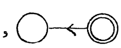
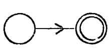
i) **case of** ⓡ—*—ⓡ. There exist an integer $l>2$ such that $m=l^2-2$ and the new period for the new diagram Γ' is equal to l. Hence all circles in the old diagram Γ (except the part ⓡ—*—ⓡ) should be changed to double circles. The numbering of the segments between the new circle ⓡ and the others is given by the following rule.



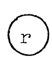
ii) **case of** ◎ⓡ—*—◎ⓡ . The period m is unchanged. The numbering of the segment between the new circle ⓡ and the others are given by the following rule.

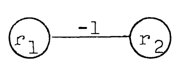


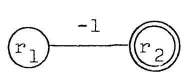
4) *A root system R is called to be reduced, if $\alpha=c\beta$ for $\alpha, \beta\in R$ and $c\in\mathbb{R}$ implies $c=\pm 1$.*

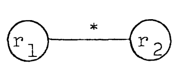
One sees easily from the definition that *a root system of sign* (1, 0, 1) *is reduced iff its associated diagram does not contain a segment*  ,  *or*  . *As a consequence, there are 9 types of reduced root systems as follows.*

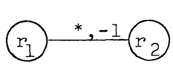
Numbering Diagram

1. 

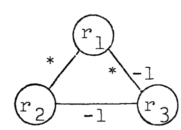
5. 

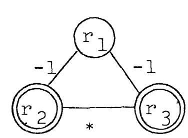
6. 

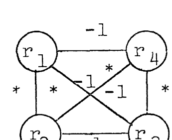
7. 

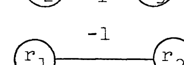
8. 

Numbering Diagram

12. 

16. 

29. 

70. 

5) For each diagram in the table, for each fixed period m, the possible set $(r_1:\cdots:r_k)\in\boldsymbol{P}_{k-1}$ is a finite set, since M_n for $n\in\boldsymbol{N}$ is a finite set. This implies the following,

The number of isomorphism classes of root system of sign (1, 0, 1), *whose period is m, is finite.*

6) In [5] §1, we introduced a concept of accumulating set A_R and proper dimension and codimension of R.

For a root system of sign (1, 0, 1), we get,

i) $A_R = \boldsymbol{R}e_1 \cup \boldsymbol{R}e_2$

ii) p-dim(R)=1, p-cod(R)=1.

iii) *W_R acts properly on $F \backslash A_R$*

(1.7) **A general view of the proof of the theorem.**

Let R be any root system in the sense of (1.2) Definition. In [5] §1 (1.9), (1.13), the followings are proven as generality.

Lemma. i) *The set of lengths of roots* $= \{I(\alpha, \alpha) : \alpha \in R\}$ *is a finite set.* ii) *For each length* $2r \in \{I(\alpha, \alpha) : \alpha \in R\}$, *put* $R_r := \{\alpha \in R : I(\alpha, \alpha) = 2r\}$. *Then* R_r *is a root system in F which may not be irreducible.* (*In particular* $Q(R_r)$ *is a full lattice of F.*)

Assuming this lemma, the proof of the theorem is divided into parts.

i) *Determination of each* R_r *for* $2r \in \{I(\alpha, \alpha) : \alpha \in R\}$.

ii) *Study of "interaction" among two* R_r *and* $R_{r'}$.

iii) *Construction of R as a union of* R_r's.

In §2 we shall show that every R_r is equal to $R_{m,q,r}$ for some $m \in N$ and $q \in \boldsymbol{R}$. (§2 (2.2) Lemma) In §3 we determine the cases when a union $R_{m,q,r} \cup R_{m',q',r'}$ can form a root system in F. All cases are classified into seven types in Lemma of §3 (3.3).

In §4 we determine all possible unions $\bigcup_{i=1}^{k} R_{m_i, q_i, \varepsilon_i}$, which form a root system in F. We do not go any details of such classification in this note, since it asks too many studies of cases. Instead of that we explain a general principle along which the classification will be done.

Apriori there is no reason that all root systems are classified into finite types as in the theorem. This finiteness is due to the strong limitation of the interactions among two components, stated in the previous lemma in (3.3).

The paragraph §5 is devoted for the study of isomorphisms, automorphisms and Weyl groups of these root systems.

The units of a real quadratic field is a root system of type $\textcircled{1}$ or $\textcircled{1}\overset{*,-1}{\text{———}}\textcircled{-1}$ as studied in §6.

In §7, for all isomorphism classes of binary quadratic forms, we shall calculate root systems belonging to them.

§2. A Single $R_{m,q,r}$ Case

This paragraph treats a root system R, which has only a fixed length roots. We shall show that such root system is equal to $R_{m,q,r}$ for some m, q, r ((2.2) Lemma).

(2.1) Recall notations and definitions from paragraph 1.

i) We fix isotropic vectors $e_1, e_2 \in F$ with $I(e_1, e_2)=1$.

ii) Define

$$\alpha(t, r) := e^t e_1 + e^{-t} r e_2 \quad \text{for} \quad t \in \mathbb{R},\ r \in \mathbb{R}$$

iii) From the definition,

$$I(\alpha(t, r), \alpha(s, \rho)) = \rho e^{t-s} + r e^{s-t}$$

In particular

$$I(\alpha(t, r), \alpha(t, r)) = 2r$$

iv) For $r \neq 0$, define the dual as,

$$\alpha(t, r)^\vee = r^{-1}\alpha(t, r) = r^{-1}e^t e_1 + e^{-t} e_2 = \operatorname{sgn}(r)\alpha(t - \log|r|, r^{-1}) .$$

Therefore,

$$I(\alpha(t, r)^\vee, \alpha(s, \rho)) = (\rho/r)e^{t-s} + e^{s-t}$$

v) $w_{\alpha(t,r)}(u) := u - r^{-1} I(u, \alpha(t, r))\alpha(t, r)$

In particular,

$$\begin{aligned} w_{\alpha(t,r)}(\alpha(s, \rho)) &= -(\rho/r)e^{2t-s}e_1 + re^{s-2t}e_2 \\ &= -(\rho/r)\alpha(2t-s, r^2\rho^{-1}) \\ &= -\operatorname{sgn}(\rho/r)\alpha(2t-s+\log|\rho/r|, \rho) \end{aligned}$$

vi) Recall the definition (1.5),

$$R_{m,q,r} := \{\pm\alpha(np_m + q, r) : n \in \mathbb{Z}\}$$

for $m \in \mathbb{N}$ with $m > 2$, $q \in \mathbb{R}$ and $r \in \mathbb{R} - \{0\}$

where $p_m = \cosh^{-1}\left(\frac{m}{2}\right)$.

(2.2) **Lemma.** *Let R be a root system in F, which may not be irreducible. Suppose that the set $\{I(\alpha, \alpha) : \alpha \in R\}$ consists of a single element $2r \in \mathbb{R} - \{0\}$. Then there exists some $m \in \mathbb{N}$ with $m > 2$ and $q \in \mathbb{R}$, such that*

$$R = R_{m,q,r} .$$

Proof. Any element of R is expressed as $\alpha(t, r)$ or $-\alpha(t, r)$ for some $t \in \mathbb{R}$. Define a set,

$$A := \{t \in \boldsymbol{R} : \alpha(t, r) \in R\}$$

Due to the *axiom* 1) of (1.2), A is a discrete subset of $\boldsymbol{R}$, which contains at least two elements.

If $t, s \in A$, by definition, $\alpha(t, r), \alpha(s, r) \in R$ and therefore $w_{\alpha(t,r)}(\alpha(s, r)) = -\alpha(2t-s, r) \in R$ by 3) of (1.2). Hence $\alpha(2t-s, r) \in R$ and therefore $2t-s \in A$. This implies that A is closed under the reflexions with the center of each point $t \in A$. Thus there exists $p, q \in \boldsymbol{R}, p > 0$, such that

$$A = \{pn+q : n \in \boldsymbol{Z}\} ,$$

and therefore

$$R = \{\pm\alpha(pn+q, r) : n \in \boldsymbol{Z}\} .$$

For two roots $\alpha(q, r)$ and $\alpha(q+p, r) \in R$, the *axiom* 4) of (2.1) demands that

$$I(\alpha(q, r), \alpha(q+p, r)^{\vee}) = e^{p}+e^{-p} \text{ (cf. (2.1) iv))}$$

to be an integer, say $m \in \boldsymbol{N}$

$$e^{p}+e^{-p} = m. \quad \text{(Since } p>0 \text{, we have } m>2.)$$

This relation is easily solved as,

$$p = \log\left(\frac{m+\sqrt{m^2-4}}{2}\right) = \cosh^{-1}\left(\frac{m}{2}\right).$$

Let us denote this number by p_m and we have proven the lemma.

Note. The proof of this lemma shows also the following fact.

If a root system R of sign (1, 0, 1) *contains two roots* $\alpha, \beta \in R$ *s.t.* $\alpha \neq \pm\beta$ *and* $I(\alpha, \alpha) = I(\beta, \beta)$, *then R is irreducible. In particular if there are two roots* $\alpha, \beta \in R$ *s.t.* $\alpha \neq \pm\beta$ *and* $I(\alpha, \beta) \neq 0$, *then R is irreducible.*

(2.3) To complete this paragraph, we show the following.

Lemma. *For any* $m \in \boldsymbol{N}$ $m>2$, $q \in \boldsymbol{R}$ *and* $r \in \boldsymbol{R}-\{0\}$, *the set* $R_{m,q,r}$ *is an irreducible root system in F.*

Proof. Except the axiom 1), all other axioms are directly verified as follows.

2) $I(\alpha(np_m+q, r), \alpha(np_m+q, r)) = 2r$,

3) $w_{\alpha(np_m+q,r)}(\alpha(n'p_m+q, r)) = -\alpha((2n-n')p_m+q, r)$,

4) $I(\alpha(np_m+q, r)^{\vee}, \alpha(n'p_m+q, r)) = e^{(n-n')p_m}+e^{-(n-n')p_m} \in \boldsymbol{Z}$,

5) $I(\alpha(np_m+q, r), \alpha(n'p_m+q, r)) \neq 0$.

Let us show the axiom 1) in a slitely stronger form:

1)′ $Q(R_{m,q,r}) = \boldsymbol{Z}\alpha(q, r) + \boldsymbol{Z}\alpha(p_m+q, r)$.

Proof. For short, let us denote by α_n the element $\alpha(np_m+q,r)$. Then the above 3) and 4) implies $\alpha_{2n-n'}=-w_{\alpha_n}(\alpha_{n'})$ is contained in $\mathbb{Z}\alpha_n+\mathbb{Z}\alpha_{n'}$. In particular, $\alpha_{n+2}, \alpha_{n-1}\in\mathbb{Z}\alpha_n+\mathbb{Z}\alpha_{n+1}$. Thus by induction, in positive and negative directions, the module $\mathbb{Z}\alpha_0+\mathbb{Z}\alpha_1$ contains all roots of $R_{m,q,r}$. This completes the proof of the lemma.

§3. Interaction of Two Components

In this paragraph we determine the cases when a union of $R_{m,q,r}$ and $R_{m',q',r'}$ become a root system in F. The result is summarized in (3.3) Lemma, where all such pairs are classified into seven types.

(3.1) **Lemma.** *Take $m_1, m_2\in\mathbb{N}$ with $m_1, m_2>2, q_1, q_2\in\mathbb{R}, r_1, r_2\in\mathbb{R}-\{0\}$ such that $r_1\neq r_2$. If the union $R_{m_1,q_1,r_1}\cup R_{m_2,q_2,r_2}$ is a root system in F, one of the following three cases happens.*

i) $m_1=m_2$

ii) $m_1^2=m_2+2$ $(\Leftrightarrow 2p_{m_1}=p_{m_2})$

iii) $m_2^2=m_1+2$ $(\Leftrightarrow 2p_{m_2}=p_{m_1})$

Proof. Due to the axiom 3), the following element

$$w_{\alpha(np_{m_1}+q_1,r_1)}(\alpha(n'p_{m_2}+q_2,r_2)) = -\alpha(2(np_{m_1}+q_1)-n'p_{m_2}-q_2+\log|r_2/r_1|, r_2)$$

should belong again to R_{m_2,q_2,r_2}. This implies a condition,

$$2(np_{m_1}+q_1)-n'p_{m_2}-q_2+\log|r_2/r_1|\in\mathbb{Z}p_{m_2}+q_2,$$

for $n, n'\in\mathbb{Z}$. Thus one gets relations,

i) $p_{m_2}|2p_{m_1}$

ii) $p_{m_2}|2(q_1-q_2)+\log|r_2/r_1|$.

By changing 1 and 2, we obtain also relations,

iii) $p_{m_1}|2p_{m_2}$

iv) $p_{m_1}|2(q_1-q_2)+\log|r_2/r_1|$.

In particular, the relations i) and iii) implies that p_{m_2}/p_{m_1} is 1/2, 1 or 2. Noting $e^{2p_m}+e^{-2p_m}=m^2-2$, this proves the lemma.

(3.2) Recall from paragraph 1 (1.5) the definition,

$$M_m:=\{\pm m/d^2: d\in\mathbb{N}, d|m\}.$$

Assertion. *The following statements for positive integers $m, u\in\mathbb{N}$ are equivalent.*

i) $u|m$

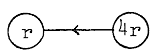
ii) *There exists an element* $r\in M_m$ *s.t.* $v^2r\in M_m$ *for any* $v\in \mathbf{N}$ *with* $v\,|\,u$.

iii) *There exists an element* $r\in M_m$ *s.t.* $u^2r\in M_m$.

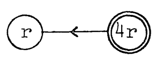
(3.3) Since there is a relation $dR_{m,q,r}=R_{m,q+\log|d|,d^2r}$, any $R_{m,q,r}$ is uniquely expressed as $dR_{m,q',\varepsilon}$ for some $d>0$, $q'\in\mathbf{R}$ mod $\mathbf{Z}p_m$ and $\varepsilon\in\{\pm1\}$.

Now we prove a lemma which was stated in (1.6) 2).

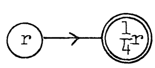
Lemma. *The following is the list of pair* $d_iR_{m_i,q_i,\varepsilon_i}$, $i=1,2$ s.t. *the union in* F *forms a root system.* (For the diagram cf. (1.6) 3).) *Put* $r_i:=d_i^2\varepsilon_i$.

1) 

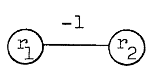
$dR_{m,q,\varepsilon}$, $2dR_{m,q,\varepsilon}$ *with* $2\,|\,m$,

2) 

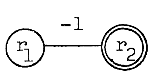
$dR_{m,q,\varepsilon}$, $2dR_{m^2-2,q,\varepsilon}$ *with* $2\,|\,m$,

3) 

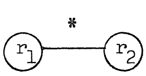
$dR_{m,q,\varepsilon}$, $\frac{1}{2}dR_{m^2-2,q,\varepsilon}$ *with* $2\,|\,m$,

4) 

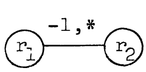
$d_1R_{m,q,\varepsilon}$, $d_2R_{m,q,-\varepsilon}$ *such that* $r_2/r_1\in M_{m^2-4}$,

5) 

$d_1R_{m,q,\varepsilon}$, $d_2R_{m^2-2,q,-\varepsilon}$ *such that* $r_2/r_1\in M_{m^2-4}$,

6) 

$d_1R_{m,q,\varepsilon}$, $d_2R_{m,q+\frac{1}{2}p_m,\varepsilon}$ *such that* $r_2/r_1\in M_{m+2}$,

7) 

$d_1R_{m,q,\varepsilon}$, $d_2R_{m,q+\frac{1}{2}p_m,-\varepsilon}$ *such that* $r_2/r_1\in M_{m-2}$.

Here in the above table, d_1, $d_2\in\mathbf{R}^+$, $m\in\mathbf{N}$ $m>2$, $q\in\mathbf{R}$ *and* $\varepsilon\in\{\pm1\}$.

Proof. Put $R_i=d_iR_{m_i,q_i,\varepsilon_i}$ $i=1,2$ and assume $R=R_1\cup R_2$ is a root system in F. The Lemma in (3.1) implies the following.

C 1) $$p_{m_1}=p_{m_2} \text{ or } 2p_{m_1}=p_{m_2} \text{ or } p_{m_1}=2p_{m_2}\,.$$

The relations ii) and iv) in the proof of Lemma in (3.1) is expressed now as follows.

C 2) $$p_{m_1}\,|\,2(q_1-q_2) \quad\text{and}\quad p_{m_2}\,|\,2(q_1-q_2)\,.$$

Put $d:=d_1/d_2$, $\varepsilon:=\varepsilon_1/\varepsilon_2$ and $m:=\inf\{m_1,m_2\}$. The axiom 4) for the union $R_1\cup R_2$ to be a root system demands that the numbers $I(d_1\alpha(np_{m_1}+q_1,\varepsilon_1)^\vee, d_2\alpha(n'p_{m_2}+q_2,\varepsilon_2))=d(\varepsilon e^N+e^{-N})$, where $N=np_{m_1}-n'p_{m_2}+q_1-q_2$, are integers for any n, $n'\in\mathbf{Z}$. Therefore one gets another condition:

C 3) $\quad d(\varepsilon e^{N}+e^{-N}),\ d^{-1}(\varepsilon e^{N}+e^{-N})\in\mathbb{Z}$ where $N=q_1-q_2+np_m$ for $n\in\mathbb{Z}$.

Assertion. *If C 3) holds for $n=n_1$, n_1+1 for some $n_1\in\mathbb{Z}$, then C 3) holds for all integers $n\in\mathbb{Z}$.*

Proof. The following equality holds for any n, $n'\in\mathbb{Z}$.

$$\begin{aligned} &d(\varepsilon e^{(q_1-q_2+(n+n')p_m)}+e^{-(p_1-q_2+(n+n')p_m)}) \\ &+d(\varepsilon e^{(q_1-q_2+(n-n')p_m)}+e^{-(q_1-q_2+(n-n')p_m)}) \\ &=d(\varepsilon e^{(q_1-q_2+np_m)}+e^{-(q_1-q_2+np_m)})\,(e^{n'p_m}+e^{-n'p_m}) \end{aligned}$$

where $e^{n'p_m}+e^{-n'p_m}$ is an integer. Using the equality the assertion is shown by induction on positive and negative directions. Q.E.D.

Now let us show that if the pair R_1, R_2 satisfies the above three conditions C 1), 2), 3), then the pair is one of the types listed in the lemma.

If $m_1\neq m_2$, due to C 1), $2p_{m_1}=p_{m_2}$ or $p_{m_1}=2p_{m_2}$. Therefore using C 2) one gets $\min\{p_{m_1}, p_{m_2}\}\mid(q_1-q_2)$. Thus we have only to study the following three cases.

Case 1. $q_1-q_2\equiv 0 \bmod \mathbb{Z}p_{m_1}+\mathbb{Z}p_{m_2}$ and $\varepsilon=+1$.

Case 2. $q_1-q_2\equiv 0 \bmod \mathbb{Z}p_{m_1}+\mathbb{Z}p_{m_2}$ and $\varepsilon=-1$.

Case 3. $m_1=m_2$ and $q_1-q_2=\frac{1}{2}p_{m_1} \bmod \mathbb{Z}p_{m_1}+\mathbb{Z}p_{m_2}$.

Notify that this distinction into three cases is symmetric by the exchange of the roll of R_1 and R_2.

Case 1. Since $q_1-q_2\equiv 0 \bmod \mathbb{Z}p_m$, for a suitable choice of $n\in\mathbb{Z}$, N of C 3) is 0. Thus the condition C 3) implies,

$$\begin{aligned} d(e^0+e^{-0}) &= 2d, & d^{-1}(e^0+e^{-0}) &= 2d^{-1}, \\ d(e^{p_m}+e^{-p_m}) &= dm, & d^{-1}(e^{p_m}+e^{-p_m}) &= d^{-1}m, \end{aligned}$$

are integers. Since $d^2\varepsilon\neq 1$, d is either 2 or 1/2 and $2\mid m$.

This corresponds to the cases 1), 2), and 3) of the Lemma.

Case 2. Since $q_1-q_2\equiv 0 \bmod \mathbb{Z}p_m$, for a suitable choice of $n\in\mathbb{Z}$, N of C 3) is 0. Thus the condition C 3) implies,

$$\begin{aligned} &d(-e^0+e^{-0}) = 0,\ d^{-1}(-e^0+e^{-0}) = 0 \\ &d(-e^{-p_m}+e^{p_m}) = d\sqrt{m^2-4},\ d^{-1}(-e^{-p_m}+e^{p_m}) = d^{-1}\sqrt{m^2-4}, \end{aligned}$$

are integers. Hence $d^2\in M_{m^2-4}$.

This corresponds to the cases 4) and 5) of the Lemma.

Case 3. Since $q_1-q_2\equiv\frac{1}{2}p_m \bmod \mathbf{Z}_{p_m}$, by choosing n suitably, $N=-\frac{1}{2}p_m$ in C 3). Thus the condition C 3) implies,

$$d(\varepsilon e^{-\frac{1}{2}p_m}+e^{\frac{1}{2}p_m}) = d\sqrt{m+2\varepsilon}\,,\quad d^{-1}(\varepsilon e^{-\frac{1}{2}p_m}+e^{\frac{1}{2}p_m}) = d^{-1}\sqrt{m+2\varepsilon}\,,$$
$$d(\varepsilon e^{\frac{1}{2}p_m}+e^{-\frac{1}{2}p_m}) = \varepsilon d\sqrt{m+2\varepsilon}\,,\quad d^{-1}(\varepsilon e^{\frac{1}{2}p_m}+e^{-\frac{1}{2}p_m}) = \varepsilon d^{-1}\sqrt{m+2\varepsilon}\,,$$

are integers. Hence $d^2\in M_{m+2\varepsilon}$.

This corresponds to the cases 6) and 7) of the Lemma.

It remains to show that the unions $R_1\cup R_2$ of the pairs in the Lemma form root systems in F. The axioms 2)–5) are already checked in the above calculations.

To show the axiom 1), it is enough to show that the linear transformation matrix from a basis α_0, α_1 of $Q(R_1)$ to a basis β_0, β_1 of $Q(R_2)$ is of rational coefficients.

Put

$$\alpha_n := d_1\alpha(np_{m_1}+q_1, \varepsilon_1),\ \beta_n := d_2\alpha(np_{m_2}+q_2, \varepsilon_2) \quad \text{for } n = 0, 1\,,$$

so that we have

$$\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} = d\begin{pmatrix}1 & 1\\ e^{p_{m_1}} & e^{-p_{m_1}}\end{pmatrix}\begin{pmatrix}e^{q_1-q_2} & 0\\ 0 & \varepsilon e^{-(q_1-q_2)}\end{pmatrix}\begin{pmatrix}1 & 1\\ e^{p_{m_2}} & e^{-p_{m_2}}\end{pmatrix}^{-1}\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}.$$

Thus the each entry of the transformation matrix has a form

$$d(e^N-\varepsilon e^{-N})\,(e^{p_{m_2}}-e^{-p_{m_2}})^{-1},\ \text{where } N = q_1-q_2+np_m$$

for suitable $n\in\mathbf{Z}$. We may assume $m_2=m=\inf\{m_1, m_2\}$.

Case 1. Assume $q_1-q_2=0$ and $\varepsilon=1$. Then

$$d(e^{np_m}-e^{-np_m})\,(e^{p_m}-e^{-p_m})^{-1} = d(e^{(n-1)p_m}+\cdots+e^{-(n-1)p_m})\in\mathbf{Z}\,.$$

Case 2. Assume $q_1-q_2=0$ and $\varepsilon=-1$. Then

$$d(e^{np_m}+e^{-np_m})\,(e^{p_m}-e^{-p_m})^{-1} = (e^{np_m}+e^{-np_m})\,(d^{-1}(e^{p_m}-e^{-p_m}))^{-1}\in\mathbf{Q}\,.$$

Case 3. Assume $q_1-q_2=\frac{1}{2}p_m$. Then

$$d(e^{(n+1/2)p_m}-\varepsilon e^{-(n+1/2)p_m})\,(e^{p_m}-e^{-p_m})^{-1}$$
$$= (e^{np_m}+\cdots+e^{-np_m})\,(d^{-1}(e^{1/2p_m}+\varepsilon e^{-1/2p_m}))^{-1}\in\mathbf{Q}\,.$$

This completes the proof of the Lemma.

§4. Decomposition of a Root System

In this paragraph we classify root systems of sign (1, 0, 1).

(4.1) By classification of a root system of sign (1, 0, 1), we mean the following.

Due to the Lemma i), ii) of (1.7) and Lemma of (2.2), any root system R in F has a decomposition,

$$R = \bigcup_{i=1}^{k} d_i R_{m_i, q_i, \varepsilon_i} \tag{4.1.1}$$

where k is an integer and $d_i \in \mathbb{R}^+$, $m_i \in \mathbb{N}$ with $m_i > 2$, $q_i \in \mathbb{R}$ and $\varepsilon_i \in \{\pm 1\}$ for $i=1, \cdots, k$. Furthermore the coefficients $d_i^2 \varepsilon_i$ $(:= r_i)$ $i=1, \cdots, k$ are pairwisely different. Such datas d_i, m_i, q_i, ε_i $i=1, \cdots, k$ coming from a root system R are strongly limited and arbitrarily given numbers do not form a root system. Thus by a classification of R, we mean the determination and the classification of all such pair of datas d_i, m_i, q_i, ε_i $i=1, \cdots, k$, for which (4.1.1) form a root system in F.

We do not present in this note any details of such classification work, since it is involved in study of many cases, which is rather cumbersome and long. Instead of that, we are going to explain some general principles for the classification, which altogether shows that there are only finite number of types of root systems as stated in Theorem in paragraph 1. (For a definition of the word "type", see (4.2) 6).)

(4.2) Let R be a union of the form (4.1.1). For R to become a root system it is necessary and sufficient that for any pair i, j $1 \leq i, j \leq k$, the union $d_i R_{m_i, q_i, \varepsilon_i} \cup d_j R_{m_j, q_j, \varepsilon_j}$ form a root system so that they are one of the seven types of the pair listed in Lemma of (3.3). (The proof is trivial and omitted.)

By posing this condition on R we are going to give a description of R as follows.

1) *There exists an integer $m \in \mathbb{N}$ with $m > 2$, which we shall call the period of R, such that*

i) *Any integer m_i is equal to either m or m^2-2.*

ii) *There exists at least one $1 \leq i \leq k$ such that $m_i = m$.* (*Proof.* (3.1) Lemma.)

By changing of the ordering, let us assume,

$$\begin{aligned} m_i &= m && \text{for } 1 \leq i \leq k_1 \\ m_i &= m^2-2 && \text{for } k_1 < i \leq k \quad \text{for some } 1 \leq k_1 \leq k\,. \end{aligned}$$

Put

$$R(m):=\bigcup_{i=1}^{k_1} d_i R_{m_i,q_i,\varepsilon_i}\,,$$

$$R(m^2-2):=\bigcup_{i=k_1+1}^{k} d_i R_{m_i,q_i,\varepsilon_i}\,.$$

2) *There exists a real number $q\in\boldsymbol{R}$ which we call the phase of R such that*

i) *For i, $1\leqq i\leqq k_1$,*

$$q_i\equiv\begin{cases} q & \text{mod. } p_m\\ or & \\ q+\frac{1}{2}p_m & \text{mod. } p_m\,.\end{cases}$$

Furthermore, if $R(m^2-2)\neq\phi$, the second case does not occur.

ii) *For i, $k_1<i\leqq k$,*

$$q_i\equiv\begin{cases} q & \text{mod. } 2p_m\\ or & \\ q+p_m & \text{mod. } 2p_m\,.\end{cases}$$

(The proof follows from (3.3) Lemma or iv) in the proof of (3.1) Lemma.)

We may change q_i to $q, q+\frac{1}{2}p_m$ or $q+p_m$ according as the above congruence relations.

3) *Put, for $\varepsilon\in\{\pm 1\}$*

$$R(m,q,\varepsilon):=\bigcup_{\substack{1\leqq i\leqq k_1\\ q_1=q\\ \varepsilon_i=\varepsilon}} d_i R_{m_i,q_i,\varepsilon_i}\,,$$

$$R(m,q+\frac{1}{2}p_m,\varepsilon):=\bigcup_{\substack{1\leqq i\leqq k_1\\ q_i=q+(1/2)p_m\\ \varepsilon_i=\varepsilon}} d_i R_{m_i,q_i,\varepsilon_i}\,,$$

$$R(m^2-2,q,\varepsilon):=\bigcup_{\substack{k_1<i\leqq k\\ q_i=q\\ \varepsilon_i=\varepsilon}} d_i R_{m_i,q_i,\varepsilon_i}\,,$$

$$R(m^2-2,q+p_m,\varepsilon):=\bigcup_{\substack{k_1<i\leqq k\\ q_i=q+p_m\\ \varepsilon_i=\varepsilon}} d_i R_{m_i,q_i,\varepsilon_i}\,.$$

Thus R is decomposed into 8 parts as follows.

$$(*)\qquad R=R(m,q,+1)\coprod R(m,q,-1)\coprod R(m,q+\frac{1}{2}p_m,+1)$$
$$\coprod R(m,q+\frac{1}{2}p_m,-1)\coprod R(m^2-2,q,+1)\coprod R(m^2-2,q,-1)$$
$$\coprod R(m^2-2,q+p_m,+1)\coprod R(m^2-2,q+p_m,-1)\,.$$

4) *Let $R(M, Q, \varepsilon)$ be a component in the decomposition* (∗) *above. Then it is either one of the following.*

i) *void,*

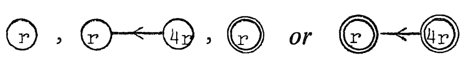
ii) $dR_{M,Q,\varepsilon}$ *for some* $d\in \boldsymbol{R}^+$,

iii) $dR_{M,Q,\varepsilon}\cup 2dR_{M,Q,\varepsilon}$ *for some* $d\in \boldsymbol{R}^+$.

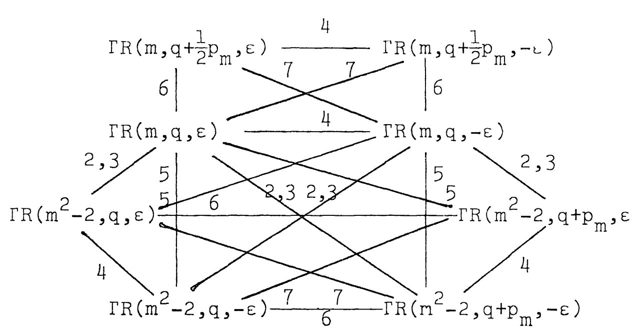
Proof. $R(M, Q, \varepsilon)$ consists of all such components $d_iR_{m_i,q_i,\varepsilon_i}$ with $m_i=M$, $q_i=Q$, $\varepsilon_i=\varepsilon$. Thus if there are more than two such components, then according to (3.3) Lemma, the only possible interaction among them is that of type 1) in the lemma. Hence one coefficient d_i should be a double or a half of the other coefficient d_i'. Therefore $R(M, Q, \varepsilon)$ can not contain more than three components.

5) *For a given root system R, we associate a diagram ΓR, which is already explained in* (1.6) 3). (Note that this is not a graph, since any two vertexes are always combined by a segment.)

Using the decomposition (∗) *in* 3), *the diagram ΓR decomposes into the following figure, where each component $\Gamma R(M, Q, \varepsilon)$ is either void or one of the following diagrams,*

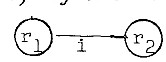


for some r.



6) *Two root systems are called to be the same type if associated diagrams are the same, forgetting the coefficients $r_1, \cdots, r_k$.*

A rough approximation of the number of types is given by $3^8=6561$ which is already finite.

7) *If two vertexes in the diagram are combined by a segment i, for instance*


, *then the ratio of the coefficients r_1/r_2 should take only finite possible values depending on m and i, which is described in* (3.3) Lemma.

Thus if we demand to the coefficients $r_1, \cdots, r_k$ to be consistent to all such conditions, it disclude many possibility of diagrams.

8) In the above 1) – 7), we stated conditions for a union R of the form (4.1.1), so that any subunions of two components form root systems. These are also sufficient for R to be a root system. Thus *if we classify all diagrams Γ with coefficients $r_1, \cdots, r_k$ satisfying* 7), *we have done the classification of the root systems of* sign (1, 0, 1).

The result of classification, whose details are omitted in this note, says that there are only 72 types of the root system including reducible cases, as stated in the theorem in paragraph 1.

§5. Automorphism of Root Systems and the Weyl Group W_R

We determine all isomorphisms among two root systems and the automorphism group of a root system. The Weyl group is a subgroup of the automorphism group of index either 1, 2 or 4.

(5.1) **Assertion.** *Let $R \subset F$ and $R' \subset F'$ be root systems of type* (1, 0, 1) *associated to the bilinear forms I and I' respectively.*

If a linear isomorphism φ: $F \to F'$ induces an isomorphism φ: $R \to R'$ of the root systems, then there exists a non-zero constant $C \in \mathbf{R} - \{0\}$ such that $I' \circ \varphi = CI$.

Proof. φ induces a bijection between two projective lines, $(F - \{0\})/(\mathbf{R} - \{0\})$ and $(F' - \{0\})/(\mathbf{R} - \{0\})$. Two points corresponding to two isotropic vectors e_1, e_2 of I, are mapped to that of I', since *they are characterized as accumulating points of the sets* $R/(\mathbf{R} - \{0\})$ and $R'/(\mathbf{R} - \{0\})$, which are bijective by assumption. One sees easily, that this bijectivity of isotropic vectors implies the assertion.

(5.2) The group of automorphisms g of the vector space F such that $I \circ g$ is a constant multiple of I is generated by the following four types of transformations.

i) **homothety**: H_d: $e \mapsto de$ for a $d \in \mathbf{R} - \{0\}$,

ii) **hyperbolic translation**: g_t: $e_1 \mapsto e^t e_1$
$e_2 \mapsto e^{-t} e_2$ for a $t \in \mathbf{R}$,

iii) **change of basis**: T: $e_1 \mapsto e_2$
$e_2 \mapsto e_1$,

iv) **change of sign**: E: $e_1 \mapsto e_1$, $e_2 \mapsto -e_2$.

These are satisfying the following relations.

$$[H_d, g_t] = 1, [H_d, T] = 1, T^2 = 1, Tg_tT = g_{-t}, E^2 = 1 ,$$
$$H_dH_{d'} = H_{dd'}, g_tg_{t'} = g_{t+t'}, [H_d, E] = 1, [g_t, E] = 1, [T, E] = H_{-1} .$$

(5.3) The above four transformations transform a system $R_{m,q,r}$ as follows.

i) $H_dR_{m,q,r} = |d|R_{m,q,r} (=R_{m,q+\log|d|,d^2r})$.

ii) $g_tR_{m,q,r} = R_{m,q+t,r}$.

iii) $TR_{m,q,r} = R_{m,\log|r|-q,r}$.

iv) $ER_{m,q,r} = R_{m,q,-r}$.

(5.4) Obviously all these four transformations induces the isomorphisms among root systems. Notify that any of these transformations, if it applies to a root system (1.6.1), it does not change the type of the root system, since it does not change the type of interaction of two components. Thus *the isomorphism appears only among the root systems of the same type.* The transformations do not change the periods m_i, but changes the coefficients r_i by homothety d^2r_i or by the sign $-r_i$, and the phases q_i by a translation q_i+t or sign $-q_i$.

By summerizing, one obtain an assertion, proving (1.6) Theorem 5).

Assertion. 1) *If two root systems are isomorphic, they are of the same type.* 2) *Two root systems associated to two datas,*

$$d_i(m_i, q_i, \varepsilon_i)\ i = 1, \cdots, k, \text{ and } d_i'(m_i', q_i', \varepsilon_i')\ i = 1, \cdots, k ,$$

belonging to the same type in the table of Theorem (1.6), *are isomorphic, iff* $m=m'$ *and* $(r_1: \cdots r_k)=(r_1':\cdots r_k')$ in $\mathbb{P}_{k-1}$.

(5.5) **Assertion.** *The automorphism group of a root system R is given by:*

$$\langle H_{-1}, g_{2q}T, g_{p_m}\rangle \quad \text{if } R(m^2-2) = \phi ,$$
$$\langle H_{-1}, g_{2q}T, g_{2p_m}\rangle \quad \text{if } R(m^2-2) \neq \phi ,$$

or an extension of it by $\mathbb{Z}_2$, *where the generator of* $\mathbb{Z}_2$ *induces an automorphism of the diagram of R, which reverses the sign of the coefficients.*
Here m and q are the periods and the phase defined in (4.2) 1) *and* 2) *for R.*

(The proof is easy and omitted.)

(5.6) Let us compute the group W_R, which is generated by reflexions w_α for $\alpha\in R$.

For an element $d\alpha(t, \varepsilon)=d(e^te_1+e^{-t}\varepsilon e_2)$, the reflexion $w=w_{d\alpha(t,\varepsilon)}$ is calculated as

$$w\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = -\varepsilon \begin{bmatrix} 0 & e^{-2t} \\ e^{2t} & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Hence $w = H_{-\varepsilon} g_{2t} T$ in the notation of (5.1). Therefore we see

$$\begin{aligned} W_{dR_{m,q,\varepsilon}} &= \langle H_{-\varepsilon} g_{2(np_m+q)} T;\ n \in \boldsymbol{Z} \rangle \\ &= \langle g_{2p_m},\ H_{-\varepsilon} g_{2q} T \rangle \end{aligned}$$

(5.7) **Assertion.** *For a root system R, the group W_R is calculated as follows.*

i) $W_R = \langle g_{2p_m},\ H_{-\varepsilon} g_{2q} T \rangle$

For all R except the following cases ii)$\sim$v).

ii) $W_R = \langle g_{2p_m},\ H_{-1},\ g_{2q} T \rangle$

For all R whose diagram contains a segment $\bigcirc\overset{-1}{\text{—}}\bigcirc$, $\circledcirc\overset{-1}{\text{—}}\circledcirc$ *or* $\bigcirc\overset{-1}{\text{—}}\circledcirc$ *except the case* v)

iii) $W_R = \langle g_{p_m},\ H_{-\varepsilon} g_{2q} T \rangle$

For all R whose diagram contains a segment $\bigcirc\overset{*}{\text{—}}\bigcirc$ *except the case* v).

iv) $W_R = \langle H_{-1} g_{p_m},\ g_{2p_m},\ H_{-\varepsilon} g_{2q} T \rangle$

For all R whose diagram contains a segment $\bigcirc\overset{-1,*}{\text{—}}\bigcirc$ *except the case* v).

v) $W_R = \langle g_{pm},\ H_{-1},\ g_{2q} T \rangle$

For all R whose diagram contains segments either a pair $\bigcirc\overset{-1}{\text{—}}\bigcirc$ *and* $\bigcirc\overset{*}{\text{—}}\bigcirc$ *or a pair* $\bigcirc\overset{-1}{\text{—}}\bigcirc$ *and* $\bigcirc\overset{-1,*}{\text{—}}\bigcirc$.

(5.8) Let $z_1 e_1 + z_2 e_2$ for $z_1, z_2 \in \boldsymbol{C}$ be coordinates for $F_C := \boldsymbol{C} \otimes F$. Then obviously the function;

(5.8.1) $$z_1 z_2$$

is invariant on F_C under the actions of H_{-1}, g_t for $t \in \boldsymbol{R}$ and T.

The domain $\{\mathrm{Im}(z_1/z_2) > 0\}$ in F_C is invariant under the actions of H_{-1}, g_t for $t \in \boldsymbol{R}$ and T. A suitable choice of a branch of the function;

(5.8.2) $$\left(e^{2q}\frac{z_1}{z_2}\right)^{\pi i/p_m} + \left(e^{-2q}\frac{z_2}{z_1}\right)^{\pi i/p_m}$$

on the domain is a univalent function which is invariant by the action of $W_{R_{m,q,r}}$.

Similarly one can determine the invariants for the other types of the root systems.

§ 6. Units of a Real Quadratic Field

We shall see in this paragraph that the unit group E of a real quadratic field K is a root system of sign (1, 0, 1).

The type of the root system is either $\textcircled{1}$ or $\textcircled{1}\overset{-1,*}{\text{---}}\textcircled{-1}$ according as the sign of the norm of the fundamental unit ε_1 is $+1$ or -1. (cf. (6.3) Assertion 2.) A comparison of E and W_E is given in (6.5) Assertion 3.

(6.1) Let $K:=\boldsymbol{Q}(\sqrt{D})$ be a real quadratic field, where D is a square-free positive integer. The norm and spur are defined for $a \in K$ as,

$$\mathrm{N}(a) = a\bar{a}\,, \quad \mathrm{S}(a) = a+\bar{a}$$

where $\bar{a}$ denotes the conjugate of a by the action of the non trivial element of the Galois group of K. Since $\mathrm{N}(u+v\sqrt{D})=u^2-v^2D$ for $u, v \in \boldsymbol{Q}$, N defines a quadratic form of signature (1, 0, 1) on K as a vector space of rank 2 over $\boldsymbol{Q}$.

Denote by O the ring of algebraic integers in K and denote by E the unit group of K. Namely E is the set of element ε of O, satisfying the equation

$$\mathrm{N}(\varepsilon) = \pm 1\,.$$

We first show the following.

(6.2) **Assertion 1.** *Let K be a real quadratic field and E be the unit group. Then E is a root system of sign* (1, 0, 1) (in the sense of (1.2) **Definition**) *as a sub set of sign* (1, 0, 1) *vector space K* (*over $\boldsymbol{Q}$*).

Proof. We verify the axioms easily as follows.

i) The additive group $\mathrm{Q}(E)$ generated by E is contained in the ring O of integers, which is a lattice in K. On the other hand, since we know the existence of a unit $\varepsilon \in E$ such that $\varepsilon \neq \pm 1$, $\mathrm{rank}_{\boldsymbol{Z}}(\mathrm{Q}(E))>1$, which altogether shows that $\mathrm{Q}(E)$ is a full lattice in K.

ii) Obviously $\mathrm{N}(\varepsilon)=\pm 1 \neq 0$ for $\varepsilon \in E$.

iii) Define a bilinear form $\tilde{\mathrm{N}}$ by

$$\tilde{\mathrm{N}}(a, b):= \mathrm{N}(a+b)-\mathrm{N}(a)-\mathrm{N}(b) = (a\bar{b}+\bar{a}b)\,.$$

Therefore one gets a formula,

$$\text{(6.2.1)} \qquad w_\varepsilon(a) = a-\frac{\tilde{\mathrm{N}}(a, \varepsilon)}{\mathrm{N}(\varepsilon)}\varepsilon = a-\frac{1}{\varepsilon\bar{\varepsilon}}(a\bar{\varepsilon}+\bar{a}\varepsilon)\varepsilon = -\varepsilon\bar{\varepsilon}^{-1}\bar{a}\,.$$

In particular if ε, ε' are units, $w_\varepsilon(\varepsilon')=-\varepsilon\bar{\varepsilon}^{-1}\varepsilon'$ is a unit.

iv) $\tilde{N}(\varepsilon, \varepsilon')=\pm(N(\varepsilon+\varepsilon')-N(\varepsilon)-N(\varepsilon'))\in\boldsymbol{Z}$.

v) See (2.2) *Note*. Q.E.D.

(6.3) Let us recall some well known facts on units of a real quadratic field. (see [2] Chap. 28.)

Any unit ε of K has the unique basis representation

$$\varepsilon = (-1)^{\mu}\varepsilon_1^{\mu_1} \quad (\mu \bmod 2,\ \mu_1 \text{ a rational integer})$$

where ε_1 is the fundamental unit. The fundamental unit ε_1 with the normalization $\varepsilon_1>1$, can be calculated by solving Pell's equation as follows.

Let u_1, v_1 be the uniquely determined rational integral solution of the equation

$$(P^-) \qquad u^2-dv^2 = -4$$

or in case this equation does not have a solution in rational integers, of the equation

$$(P^+) \qquad u^2-dv^2 = 4$$

for which u_1, v_1 both are positive and minimal values. Here $d=D$ or $4D$ according as $D\equiv 1$ or 2, 3 mod 4. Then

$$\varepsilon_1 = \frac{u_1+v_1\sqrt{d}}{2}.$$

(6.4) To obtain a description of E in terms of $R_{m,q,r}$ as in paragraph 1 (1.4), (1.5), we fix basis e_1, e_2 of $F:=\boldsymbol{R}\underset{\boldsymbol{Q}}{\otimes}K$ as follows.

$$e_1 = \frac{1}{2}(1\otimes 1+\sqrt{d}^{-1}\otimes\sqrt{d})$$

$$e_2 = \frac{1}{2}(1\otimes 1-\sqrt{d}^{-1}\otimes\sqrt{d}).$$

The quadratic form N on K extends to a real quadratic form on F uniquely, denoted by the same N.

One computes easily

$$N(e_1) = N(e_2) = \frac{1}{4}(1\otimes 1+\sqrt{d}^{-1}\otimes\sqrt{d})(1\otimes 1-\sqrt{d}^{-1}\otimes\sqrt{d})$$

$$= \frac{1}{4}(1\otimes 1+\sqrt{d}^{-1}\otimes\sqrt{d}-\sqrt{d}^{-1}\otimes\sqrt{d}-d^{-1}\otimes d) = 0$$

$$N(e_1+e_2) = N(1\otimes 1) = 1$$

and therefore

$$\tilde{\mathrm{N}}(e_1, e_2) = 1 \, .$$

Using these basis e_1, e_2 and the terms of (1.5), we get the following description of E.

Assertion 2. *Let K be a real quadratic field and E be its unit group. We regard it to be embedded in $\boldsymbol{R}\underset{\boldsymbol{Q}}{\otimes}K$.*

i) *If* $\mathrm{N}(\varepsilon_1)=1$ (i.e. u_1, v_1 *is a solution of* (P^+)), *then*

$$E = R_{u_1,0,1} \, .$$

(i.e. *E is of type* ① *with the period*$=u_1$.)

ii) *If* $\mathrm{N}(\varepsilon_1)=-1$ (i.e. u_1, v_1 *is a solution of* (P^-)), *then*

$$E = R_{u_1^2-2,0,1} \cup R_{u_1^2-2,\frac{1}{2}p_{u^2-2},-1} \, .$$

(i.e. *E is of type* ①$\overset{-1,\,*}{\text{——}}$① *with period*$=u_1^2-2$.)

Proof. Using the basis e_1, e_2 of F, any element $\varepsilon=\dfrac{u+v\sqrt{d}}{2}>0$ of $K\subset F$ is described as

$$\begin{aligned}
\varepsilon &= \frac{u+v\sqrt{d}}{2} = \frac{u+v\sqrt{d}}{2}e_1+\frac{u-v\sqrt{d}}{2}e_2 \\
&= e^a e_1+e^{-a}\mathrm{N}(\varepsilon)e_2 = \alpha(a, \mathrm{N}(\varepsilon))
\end{aligned}$$

where $a=\log\left(\dfrac{u+v\sqrt{d}}{2}\right)$.

If $\mathrm{N}(\varepsilon)=1$, we have a relation $u^2-v^2d=4$ and therefore $v\sqrt{d}=\sqrt{u^2-4}$ and $a=\log\left(\dfrac{u+\sqrt{u^2-4}}{2}\right)=\cosh^{-1}\left(\dfrac{u}{2}\right)=p_u$. Thus we get a representation

$$\begin{aligned}
\varepsilon &= \alpha(p_u, 1) \, , \\
\varepsilon^\mu &= \alpha(\mu p_u, 1) \, .
\end{aligned}$$

If $\mathrm{N}(\varepsilon)=-1$, we have a relation $u^2-v^2d=-4$. Then $\left(\dfrac{u+v\sqrt{d}}{2}\right)^2=\dfrac{1}{4}(u^2+v^2d+2uv\sqrt{d})=\dfrac{1}{2}(u^2-2+\sqrt{(u^2-2)^2-4})$. Therefore $a=\dfrac{1}{2}\log\times\left(\left(\dfrac{u+v\sqrt{d}}{2}\right)^2\right)=\dfrac{1}{2}\cosh^{-1}\left(\dfrac{u^2-2}{2}\right)=\dfrac{1}{2}p_{u^2-2}$. Thus we get a presentation

$$\varepsilon = \alpha\left(\frac{1}{2}p_{u^2-2}, -1\right)$$

$$\varepsilon^{\mu}=\alpha\left(\frac{\mu}{2}p_{u^2-2},(-1)^{\mu}\right).$$ Q.E.D.

(6.5) Define a map,

$$\rho:\varepsilon\in E\mapsto w_{\varepsilon}w_1\in W_E.$$

Assertion 3. i) *ρ is a group homomorphism.*
ii) *The following sequence is exact.*

$$1\to \boldsymbol{Z}_2\to E\overset{\rho}{\to} W_E\to \boldsymbol{Z}_2\to 1.$$

Here $\boldsymbol{Z}_2=\ker\rho$ is the torsion $\{\pm 1\}$ of the unit group E and $\boldsymbol{Z}_2=\operatorname{coker}\rho$ is isomorphic to the cyclic subgroup in W_E generated by a reflexion w_1.

Proof. Recall a formula (6.2.1)

$$w_{\varepsilon}(a)=-\varepsilon\bar{\varepsilon}^{-1}\bar{a}\quad\text{for a unit}\quad \varepsilon\in E.$$

In particular

$$w_1(a)=-\bar{a}$$

which implies relations

$$w_1w_{\varepsilon}w_1=w_{\bar{\varepsilon}}\quad\text{and}\quad \rho(\varepsilon)(a)=\varepsilon\bar{\varepsilon}^{-1}a.$$

This means that ρ is a group homomorphism. Also one verifies easily that any element in W_E which is a product of even number reflexions lies in the image of ρ. Q.E.D.

§ 7. Maximal Root Systems

Let L be a $\boldsymbol{Z}$-free module of finite rank and I be a rational valued symmetric bilinear form on L. Put

$$R(L,I):=\{\alpha\in L: I(\alpha,\alpha)\neq 0,\quad 2I(\alpha,\beta)/I(\alpha,\alpha)\in\boldsymbol{Z}\quad\text{for}\quad {}^{\forall}\beta\in L\}$$

Then $R(L,I)$ is a root system of sign (1, 0, 1) which may not be irreducible, if rank $L=2$ and I is indefinite. We call $R(L,I)$ the maximal root system belonging to (L,I).

The purpose of this paragraph is to classify isomorphism classes of maximal root systems of sign (1, 0, 1). The result is summarized in (7.4). Among 72 types of root systems in the table (1.6), seven types: 29, 34, 62, 65, 69, 70, 72, appear as types of maximal root systems. We give a parameter presentation of maximal root systems for each type. The proofs are given in §8, §9.

(7.1) Let L be a free $\boldsymbol{Z}$-module and I be a rational valued symmetric bilinear form on L. We shall denote by $q(x)$ the associated quadratic form $\frac{1}{2}I(x, x)$ and by $O(L, I)$, the group of orthogonal linear transformations which preserves I (i.e. $\{g \in GL(L): I \circ g = I\}$).

Definition 1. *An element $\alpha \in L \otimes \mathbf{Q}$ is said to belong to (L, I) if $\alpha \in L$, $q(\alpha) \neq 0$ and $I(\alpha^\vee, \beta) = \frac{1}{q(\alpha)} I(\alpha, \beta) \in \mathbf{Z}$ for all $\beta \in L$.*

Put

$$R(L, I) := \{\alpha \in L: \alpha \text{ belongs to } (L, I)\} .$$

Assertion. *If $R(L, I)$ spans $L \otimes \mathbf{Q}$ over $\mathbf{Q}$, then $R(L, I)$ is a root system in the sense of* (1.2), *which may not be irreducible.*

Proof. Almost obvious, since

i) for any $\alpha \in R(L, I)$, $w_\alpha \in O(L, I)$ and

ii) for any $g \in O(L, I)$, $gR(L, I) = R(L, I)$.

Note 1. Does the assumption automatically follows, so that one can delete it? (This is true for an indefinite form I on L with rank $L=2$. cf. (9.4) *Note.*)

Definition 2. i) *A root system R associated to I is said to belong to (L, I), if $R \subset R(L, I)$. In particular, if $R = R(L, I)$, we call R to be a maximal root system belonging to (L, I).*

ii) *If R is maximal w.r.t.* $(\mathrm{Q}(R), I \mid \mathrm{Q}(R))$, *we call R to be a complete root system.*

Note 2. Two concepts to be maximal or to be complete are different. The former depends on the choice of the lattice L containing R, whereas the latter depends only on the isomorphism class of R.

Definition 3. *Two maximal root systems R and R' belonging to (L, I) and (L', I') respectively, are said to be isomorph as maximal root systems, if there exists an isomorphism $\varphi: L \simeq L'$ of abelian groups and a constant $C \neq 0$ s.t. $I' \circ \varphi = CI$ and $\varphi(R) = R'$.*

Note 3. In the above definition of the isomorphism, the condition $\varphi R = R'$ follows automatically from the others, since $R(L, I) = R(L, CI)$ for any constant $C \neq 0$.

(7.2) Format on the classification of maximal root systems

In (7.4) we shall give a list of all isomorphism classes of maximal root systems. As a consequence of the classification, there are 7 types of diagrams $\Gamma R(L, I)$ associated to maximal root systems $R(L, I)$ (see (1.6) 3) 4) for the definition of a diagram and a type of a root system). Therefore we divide the list into 7 tables according to the seven types Γ_{29}, Γ_{34}, Γ_{62}, Γ_{65}, Γ_{69}, Γ_{70}, Γ_{72}. The first five tables classify irreducible cases. The remaining two tables classify reducible cases, when the discriminant of I are square of rational numbers.

Each table of a type Γ contains infinite number of isomorphism classes of maximal root systems, parametrized by a set P_Γ.

Instead of giving exactly one representative for each isomorphism class of a maximal root system, the representation in this note has finite overlapping, which is described by a finite group $\mathrm{Aut}(\Gamma)$ as follows. (cf. the following v) vi)).

Definition. *For a type Γ of a diagram of a root system, define,*

$$\mathrm{Aut}(\Gamma) := \{\textit{automorphism } S \textit{ of } \Gamma\}\ . \tag{7.2.1}$$

*Here S is an automorphism of Γ if it is a bijection of vertexes of Γ, which maps double circles to double circles and which preserves the direction of the arrow, the sign -1 and the symbol * on the segment* (cf. (1.6) 2), 3)).

In fact directly from the table of diagrams in (1.6), we see that $\mathrm{Aut}(\Gamma)$ is isomorphic to either $\{0\}$, $\mathbf{Z}_2$ or $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.

Now the description of the classification of maximal root systems in (7.4) is in the following form.

i) *In each table of type Γ, we define a set P_Γ of parameters. The set P_Γ is either a set of pair of integers $(m, c) \in \mathbf{N} \times \mathbf{N}$ satisfying certain elementary number theoretic relations for the case when Γ is irreducible, or equal to $\mathbf{Q}^+ := \{c \in \mathbf{Q} : c > 0\}$ for the case when Γ is reducible.*

ii) *For each $p \in P_\Gamma$, we associate a quadratic form*

$$q(u) := q_p(u) \quad \textit{for} \quad u \in L := \mathbf{Z}\gamma_0 + \mathbf{Z}\gamma_1\,. \tag{7.2.2}$$

Thus one gets a family of lattices,

$$L_p := (L, I_p) \quad \textit{for} \quad p \in P_\Gamma\,, \tag{7.2.3}$$

where $I_p(u, v) := q_p(u+v) - q_p(u) - q_p(v)$, $u, v \in L$.

iii) *For each $p \in P_\Gamma$, the maximal root system R_p belonging to L_p and its diagram ΓR_p are explicitly calculated in the table.*

The type of ΓR_p is equal to Γ so that we fix an identification of ΓR_p

with the type Γ in the following way.

Let $v_1, \cdots, v_k$ ($k:=\#\Gamma$) be the vertexes of Γ. Then the coefficient r_i of the diagram ΓR_p at v_i is given inside the vertex v_i of Γ in the table as a function of $p\in P_\Gamma$.

$$r_i = r_i(p) \quad \text{for} \quad p\in P_\Gamma\,, \quad i=1,\cdots,\#\Gamma\,. \tag{7.2.4}$$

iv) *Let us denote by $m(p)$ the period* (cf. (1.6) 2)) *of R_p. Then the parametrization by P_Γ has the following unicity: The map*

$$P_\Gamma \to \mathbb{N}\times\mathbb{P}_{k-1}\,, \quad p\mapsto (m(p), (r_1(p):\cdots:r_k(p)))\,, \quad k:=\#|\Gamma|\,, \tag{7.2.5}$$

is injective (cf. *Note* 2 *of this section*).

v) *There exist an action of* Aut(Γ) *on the parameter set P_Γ, denoted*

$$*: \text{Aut}(\Gamma) \to \text{Bijection}(P_\Gamma)\,, \tag{7.2.6}$$

a cycle C with the relation,

$$C: \text{Aut}(\Gamma)\times P_\Gamma \to \mathbb{Q}^\times \tag{7.2.7}$$

$$C_{ST}(p) = C_S(T^*p)C_T(p) \quad \text{for} \quad S,\, T\in \text{Aut}(\Gamma)\,,\ p\in P_\Gamma\,, \tag{7.2.8}$$

and an isomorphism φ_S of L depending on $S\in$ Aut(Γ) *and $p\in P_\Gamma$,*

$$\varphi_S: L\simeq L\,, \tag{7.2.9}$$

inducing a relation of quadratic forms,

$$q_p(u) = C_S(p)q_{S^*(p)}(\varphi_S(u)) \tag{7.2.10}$$

such that φ_S defines an isomorphism of root systems (cf. (7.1) *Note* 3)

$$\varphi_S: R_p\simeq R_{S^*(p)} \tag{7.2.11}$$

which induces the automorphism

$$S: \Gamma \xrightarrow{\sim} \Gamma$$

using the identifications of Γ with ΓR_p and $\Gamma R_{S^*(p)}$ in iii).

Therefore (7.2.10) *implies relations of coefficients,*

$$r_i(p) = C_S(p)r_{S(i)}(S^*(p)) \quad \text{for} \quad i=1,\cdots,\#|\Gamma|\,. \tag{7.2.12}$$

(Here $S(i)$ is defined by the relation $S(v_i)=v_{S(i)}$.)

The choice of φ_S is not unique, but is ambiguous up to multiplications of the elements of $O(L_p)$ from the right and $O(L_{S^*(p)})$ from the left.

vi) *The above process induces the natural bijection,*

$$(7.2.13) \qquad P_\Gamma/(\mathrm{Aut}\,(\Gamma))^* \simeq \left\{\begin{matrix}\textit{isomorphism classes of maximal root}\\ \textit{systems, whose type is } \Gamma.\end{matrix}\right\}.$$

(Cf. (7.4) *Corollary*.)

vii) *The action of* Aut(Γ) *on* P_Γ *may have fixed points. We shall calculate, the fixed point set, the isotropy subgroup* $I_{\Gamma,p}$ *of* Aut(Γ) *and the quadratic form* $q_p(u)$ *at the fixed point p in the tables of* (7.4) (cf. *Note* 1).

Note 1. Put

$$(7.2.14) \quad \tilde{O}(L, I) := \{g \in GL(L)\colon \textit{there exists a } C \in \mathbf{Q}\setminus\{0\} \quad \textit{s.t.} \quad q = Cq\cdot g.\}$$

Then using above Aut(Γ), one can determine $\tilde{O}(L, I)$ as follows.

i) *There exists a natural isomorphism,*

$$(7.2.15) \qquad \tilde{O}(L_p)/O(L_p) \simeq I_{\Gamma,p} \quad \textit{for} \quad p \in P_\Gamma\,,$$

where $I_{\Gamma,p}$ *is the isotropy subgroup of* Aut(Γ) *by the action* * *at* $p \in P_\Gamma$.

ii) *In fact* $I_{\Gamma,p}$ *is either* 0 *or* $\mathbf{Z}_2$.

If $S \neq 0$ *is a generator of* $I_{\Gamma,p}$, *then*

$$(7.2.16) \qquad \tilde{O}(L_p) = \langle O(L_p), \varphi_S\rangle$$

where φ_S *is the isomorphism* (7.2.9), *which satisfies,*

$$(7.2.17) \qquad q_p(u) = -q_p\circ\varphi_S(u) \quad \textit{and} \quad \varphi_S^2 \in O(L_p)\,.$$

Proof. Let φ be an element of $\tilde{O}(L_{p_0})$ *s.t.* $q_{p_0} = Cq_{p_0}\circ\varphi$. Then φ defines an automorphism $\varphi\colon R_{p_0} \simeq R_{p_0}$, which induces an automorphism $S_\varphi\colon \Gamma \simeq \Gamma$ with relations of coefficients:

$$*) \qquad r_i(p_0) = Cr_{S_\varphi(i)}(p_0) \quad \text{for} \quad i = 1, \cdots, \#|\Gamma|\,.$$

Then by definition, $\varphi \in O(L_{p_0}) \Leftrightarrow C = 1 \Leftrightarrow S_\varphi \in \mathrm{Aut}(\Gamma)$ is an identity. (Remember that the coefficients r_i are pairwisely different (1.6) 1)).

This implies the natural embedding,

$$\tilde{O}(L_{p_0})/O(L_{p_0}) \subsetneq \mathrm{Aut}\,(\Gamma)\,, \quad \varphi \mapsto S_\varphi\,.$$

Thus if $C \neq 1$, $S_\varphi^2 = 1$ and $C = -1$.

On the other hand, comparing *) with (7.2.12) one gets relations,

$$**) \qquad r_i(p_0) = C^{-1}C_{S_\varphi(p_0)}r_i(S_\varphi^*(p_0)) \quad \text{for} \quad i = 1, \cdots, \#|\Gamma|\,.$$

Since the period is an invariant of isomorphism class of a root system, we have,

$$***) \qquad m(p_0) = m(S_\varphi^*(p_0))\,.$$

Thus the unicity of iv) together with **) and ***) implies $p_0 = S_\varphi^*(p_0)$ and hence $S_\varphi \in I_{\Gamma, p_0}$.

Conversely for $S \in I_{\Gamma, p_0}$, take $\varphi_S \in GL(L)$ of (7.2.9). Then (7.2.10) implies $q_{p_0} = C_S(p_0) q_{p_0} \circ \varphi_S$ and hence $\varphi_S \in \tilde{O}(L_{p_0})$ such that $S = S_{\varphi_S}$. Q.E.D.

Note 2. Even the image set of (7.2.5) has an invariant meaning as a parameter set for maximal root systems of type Γ, it does not imply that the family $q_p(u)$, $p \in P_\Gamma$ of quadratic forms is uniquely determined, since there remains an ambiguity of a constant factor C in front of q_p.

In the tables in (7.4), we have the following normalization.

vii) *In the table of type Γ of* (7.4), *one vertex, say v_1, of Γ is fixed, so that the coefficient at the point is normalized to* 1. *i.e.*

(7.2.18) $$r_1(p) \equiv 1 \quad \textit{for all} \quad p \in P_\Gamma .$$

Note 3. The choice of the basis γ_0, γ_1 of L for the quadratic forms q_p, $p \in P_\Gamma$ in the tables of (7.4) is done from a view point of the duality of maximal root systems in §8, which seems relatively simple and natural.

(7.3) For the description of the tables in (7.4), we introduce two lattices $L_1(D)$ and $L_2(D)$.

1. **The lattice** $L_1(D)$.

1.1. *By $L_1(D)$ for a $D \in \mathbb{Q}^+$, we mean the lattice (L, I) defined as follows.*

i) $L := \mathbb{Z}\gamma_0 + \mathbb{Z}\gamma_1$, $L' := \mathbb{Z}\gamma_0 + \mathbb{Z}2\gamma_1$.

ii) $q(x\gamma_0 + y\gamma_1) := x^2 - \dfrac{D}{4} y^2$ *for* $x, y \in \mathbb{Z}$.

iii) $I(u, v) := q(u+v) - q(u) - q(v)$, $u, v \in L$.

iv) *The descriminant of q is given as,* $D = -\det(I(\gamma_i, \gamma_j))_{i,j}$.

v) *Let us define an element $\hat{T} \in O(L, I)$ by,*

$$\hat{T}\begin{bmatrix}\gamma_0 \\ \gamma_1\end{bmatrix} := \begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix}\begin{bmatrix}\gamma_0 \\ \gamma_1\end{bmatrix}.$$

1.2. i) *By $L_1(m, c)$ for $m, c \in \mathbb{N}$ $m > 2$, we mean the lattice $L_1\left(\dfrac{m^2-4}{c^2}\right)$. The descriminant is $D := \dfrac{m^2-4}{c^2}$.*

ii) *Put*

$$P_1 := \{(m, c) \in \mathbb{N} \times \mathbb{N} \colon m > 2,\ 2 \mid m,\ 2c \mid m^2-4\}$$

$$\backslash \{(m, c) \in \mathbb{N} \times \mathbb{N} \colon \textit{There exists an integer } n \textit{ s.t.}$$

$$2 < n < m \textit{ and } \frac{n}{2},\ c\sqrt{\frac{n^2-4}{m^2-4}},\ \frac{\sqrt{(m^2-4)(n^2-4)}}{2c} \in \mathbb{Z}\}$$

$$\backslash\{(m, c)\in \boldsymbol{N}\times \boldsymbol{N}:\ \sqrt{m+2}\in \boldsymbol{N} \text{ s.t. } \sqrt{m+2}\,|\,2c,\ 4c\,|\,\sqrt{m+2}(m-2)\}\,.$$

Then the maximal root system $R(L_1(m, c))$ belonging to $L_1(m, c)$ for $(m, c)\in P_1$ has the period m (cf. (8.4), (9.2)).

iii) *Define an element $F\in GL(L\underset{\boldsymbol{Z}}{\otimes}\boldsymbol{Q})$ by*

$$F\begin{bmatrix} \gamma_0 \\ \gamma_1 \end{bmatrix} := \begin{bmatrix} \dfrac{m}{2} & c \\ \dfrac{m^2-4}{4c} & \dfrac{m}{2} \end{bmatrix}\begin{bmatrix} \gamma_0 \\ \gamma_1 \end{bmatrix}.$$

Then $F^2\in O(L, I)$, $F\in O(L, I)$ if $\dfrac{m^2-4}{4c}\in \boldsymbol{Z}$ and $F\in O(L', I)$ if $c\in 2\boldsymbol{Z}$.

2. **The lattice $L_2(D)$.**

2.1. *By $L_2(D)$ for a $D\in \boldsymbol{Q}^+$, we mean the lattice (L, I) defined as follows.*

i) $L:=\boldsymbol{Z}\gamma_0+\boldsymbol{Z}\gamma_1$, $L':=\boldsymbol{Z}\gamma_0+\boldsymbol{Z}2\gamma_1$.

ii) $q(x\gamma_0+y\gamma_1) := x^2-xy+\frac{1}{4}(1-D)y^2$ *for* $x, y\in \mathbb{Z}$.

iii) $I(u, v) := q(u+v)-q(u)-q(v)$, $u, v\in L$.

iv) *The descriminant of q is given as,*

$$D = -\det\,(I(\gamma_i, \gamma_j))_{i,j}\,.$$

v) *Define an element $\hat{T}\in O(L, I)$ by*

$$\hat{T}\begin{bmatrix} \gamma_0 \\ \gamma_1 \end{bmatrix} := \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} \gamma_0 \\ \gamma_1 \end{bmatrix}.$$

2.2. i) *By $L_2(m, c)$ for $m, c\in \boldsymbol{N}$ $m>2$, we mean the lattice $L_2\left(\dfrac{m^2-4}{c^2}\right)$. The descriminant is $D:=\dfrac{m^2-4}{c^2}$.*

ii) *Put*

$$P_2 := \{(m, c)\in \boldsymbol{N}\times \boldsymbol{N}:\ m>2,\ 2\,|\,m+c,\ 2c\,|\,m^2-4+mc\}$$
$$\backslash\{(m, c)\in \boldsymbol{N}\times \boldsymbol{N}: \text{There exists an integer } n \text{ s.t. } 2<n<m \text{ and}$$
$$\frac{n}{2}+\frac{c}{2}\sqrt{\frac{n^2-4}{m^2-4}},\ c\sqrt{\frac{n^2-4}{m^2-4}},\ \frac{n}{2}-\frac{\sqrt{(m^2-4)(n^2-4)}}{2c}\in \boldsymbol{Z}\}\,.$$

Then the maximal root system $R(L_2(m, c))$ belonging to $L_2(m, c)$ for $(m, c)\in P_2$ has the period m. (cf. (8.4)).

iii) *Define an element $F\in GL(L\underset{\boldsymbol{Z}}{\otimes}\boldsymbol{Q})$ by*

$$F\begin{bmatrix} r_0 \\ r_1 \end{bmatrix} := \begin{bmatrix} \frac{m+c}{2} & c \\ \frac{m^2-4-c^2}{4c} & \frac{m-c}{2} \end{bmatrix}\begin{bmatrix} r_0 \\ r_1 \end{bmatrix}.$$

Then $F^2 \in O(L, I)$, $F \in O(L, I)$ *if* $\frac{m^2-4-c^2}{4c} \in \mathbf{Z}$ *and* $F \in O(L', I)$ *if* $c \in 2\mathbf{Z}$.

(7.4) **Classification of maximal root systems of sign** (1, 0, 1).

In the following, we list all isomorphism classes of maximal root systems of sign (1, 0, 1) (cf. (7.1) Def. 1, 2). There are seven tables according to the seven types of the maximal root systems. We use the types as titles of the tables.

In the each table of type Γ, we shall exhibit:

1) Description of a family $L_p := (L, I_p)$, $p \in P_\Gamma$ of lattices, where P_Γ is the parameter set. (We use here the notations of (7.3).)
2) Description of the set of the maximal root system $R_p := R(L_p)$ belonging to the lattice L_p as a union of components for $p \in P_\Gamma$.
3) Description of the diagram $\Gamma R(L_p)$ for $p \in P_\Gamma$. The coefficients $r_1(p), \cdots, r_k(p)$ of the diagram are given at the vertexes of the diagram as functions of $p \in P_\Gamma$.
4) Description of $Q(R_p)$ for $p \in P_\Gamma$. (In fact it is either L or L' (cf. (7.5).)
5) Generators of the orthogonal group $O(L, I_p) = W_{R_p}$ (cf. (8.3) Corollary).
6) Description of the action of $\mathrm{Aut}(\Gamma)$ on P_Γ (cf. (7.2) v)). Namely: for each $S \in \mathrm{Aut}(\Gamma)$, we present the representation $S^* \in \mathrm{Bijection}\,(P_\Gamma)$, the constant $C_S(p) \in \mathbf{Q}$ for $p \in P_\Gamma$, and an isomorphism $\varphi_S: L \to L$ for $p \in P_\Gamma$.
7) Fixed Points set of the action $\mathrm{Aut}(\Gamma)$ on P_Γ, the Isotropy group at a fixed point and the Quadratic form $q(xr_0+yr_1)$ at a fixed point. (Cf. (7.2) Note 1.)

For a detailed explanation of the meaning of the above data in the tables, one is refered to (7.2).

Table 1. Type Γ_{34}.

1) $(L, I) := L_1(m, c)$ for $(m, c) \in P_{34}$, i.e.

$L := \mathbf{Z}r_0 + \mathbf{Z}r_1$, $q(xr_0+yr_1) := x^2 - \frac{D}{4}y^2$ for $x, y \in \mathbf{Z}$,

where

$D := \frac{m^2-4}{c^2}$,

$P_{34} := \{(m, c) \in P_1 \colon 2|m,\ 2|c,\ \dfrac{m^2-4}{2c} \text{ is odd}\}.$

2) $R(m, c)$ is a union of the following components:

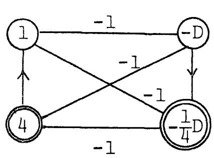
(1) $:= \{\pm F^n \gamma_0 \colon n \in \boldsymbol{Z}\}$,

((4)) $:= \{\pm F^{2n} 2\gamma_0 \colon n \in \boldsymbol{Z}\}$,

(-D) $:= \{\pm F^n 2\gamma_1 \colon n \in \boldsymbol{Z}\}$,

((-D/4)) $:= \{\pm F^{2n} \gamma_1 \colon n \in \boldsymbol{Z}\}$.

3) $R(m, c)$ is a root system of the diagram 34,



period $= m$.

4) $Q(R) = L$.
5) $W = O(L, I) = \langle F^2, \hat{T}, -1 \rangle$.
6) $\mathrm{Aut}(\Gamma_{34}) = \{0\}$.
7) Fixed Point of $\mathrm{Aut}(\Gamma_{34}) = \phi$.

Table 2. Type Γ_{62}.

1) $(L, I) := L_1(m, c)$ for $(m, c) \in P_{62}$, i.e.

$L := \boldsymbol{Z}\gamma_0 + \boldsymbol{Z}\gamma_1,\ q(x\gamma_0 + y\gamma_1) := x^2 - \dfrac{D}{4} y^2$ for $x, y \in \boldsymbol{Z}$,

where

$D := \dfrac{m^2-4}{c^2}$,

$P_{62} := \{(m, c) \in P_1 \colon \dfrac{m}{2}$ is odd, either c or $(m^2-4)/4c$ is odd$\}$.

2) $R(m, c)$ is a union of the following components:

(1) $:= \{\pm F^n \gamma_0 \colon n \in \boldsymbol{Z}\}$,

(4) $:= \{\pm F^n 2\gamma_0 \colon n \in \boldsymbol{Z}\}$,

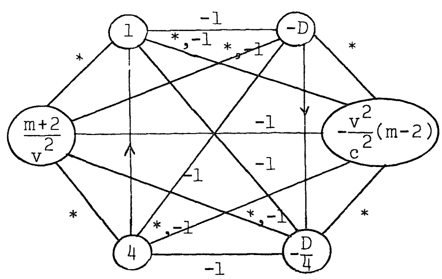
$\left(\frac{m+2}{v^2}\right)$ $:= \left\{\pm F^n \dfrac{1}{\nu}\left(\dfrac{m+2}{2}\gamma_0 + c\gamma_1\right) \colon n \in \boldsymbol{Z}\right\}$,

(-D) $:= \{\pm F^n 2\gamma_1 : n \in \mathbf{Z}\}$,

($-\frac{1}{4}$D) $:= \{\pm F^n \gamma_1 : n \in \mathbf{Z}\}$,

($-\frac{v^2}{c^2}(m-2)$) $:= \left\{\pm F^n v\left(\frac{1}{2c}(m-2)\gamma_0 + \gamma_1\right) : n \in \mathbf{Z}\right\}$.

3) $R(m, c)$ is a root system of diagram 62,



period $= m$,

where $v := \left(c, \dfrac{m+2}{2}\right)$.

4) $Q(R) = L$.

5) $W = O(L, I) = \langle F, \hat{T}, -1 \rangle$.

6) $\mathrm{Aut}(\Gamma_{62}) = \langle S \rangle \simeq \mathbf{Z}_2$, where S is the rotation π of the diagram.

$S^*(m, c) = \left(m, \dfrac{m^2-4}{4c}\right)$, $\quad C_S = -\dfrac{m^2-4}{4c^2}$,

$$\varphi_S \begin{bmatrix} \gamma_0 \\ \gamma_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \end{bmatrix}.$$

7) Fixed Point of $\mathrm{Aut}(\Gamma_{62}) = \phi$.

Table 3. Type Γ_{65}.

1) $(L, I) := L_1(m, c)$ for $(m, c) \in P_{65}$, i.e.

$L := \mathbf{Z}\gamma_0 + \mathbf{Z}\gamma_1$, $q(x\gamma_0 + y\gamma_1) := x^2 - \dfrac{D}{4} y^2$ for $x, y \in \mathbf{Z}$,

where

$$D := \frac{m^2-4}{c^2},$$

$P_{65} := \{(m, c) \in P_1 : 4 \mid m,\ 2 \mid c\}$.

2) $R(m, c)$ is a union of the following components:

(1) $:= \{\pm F^n \gamma_0 : n \in \mathbf{Z}\}$,

((4)) $:= \{\pm F^{2n}(2\gamma_0) : n \in \mathbf{Z}\}$,

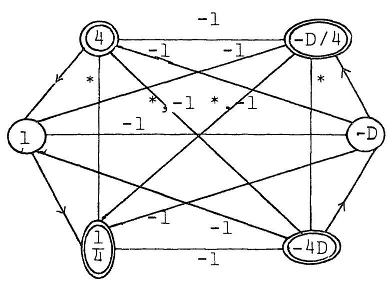
(($\frac{1}{4}$)) $:= \left\{\pm F^{2n+1}\left(\frac{1}{2}\gamma_0\right) : n \in \mathbf{Z}\right\}$,

(-D) $:= \{\pm F^n 2\gamma_1 : n\in\mathbf{Z}\}$,

(-D/4) $:= \{\pm F^{2n}\gamma_1 : n\in\mathbf{Z}\}$,

(-4D) $:= \{\pm F^{2n+1}4\gamma_1 : n\in\mathbf{Z}\}$.

3) $R(m, c)$ is a root system of diagram 65,



period $= m$.

4) $Q(R)=L$.

5) $W=O(L, I)=\langle F^2, \hat{T}, -1\rangle$.

6) $\mathrm{Aut}(\Gamma_{65})=\langle S\rangle\simeq\mathbf{Z}_2$,
where S is the rotation π of the diagram.

$$S^*(m, c)=\left(m, \frac{m^2-4}{c}\right), \quad C_S=-\frac{m^2-4}{c^2},$$

$$\varphi_S\begin{bmatrix}\gamma_0\\ \gamma_1\end{bmatrix}=\begin{bmatrix}\frac{c}{2} & m\\ \frac{m}{4} & \frac{m^2-4}{2c}\end{bmatrix}\begin{bmatrix}\gamma_0\\ \gamma_1\end{bmatrix}.$$

7) Fixed Point of $\mathrm{Aut}(\Gamma_{65})=\phi$.

Table 4. Type Γ_{69}.

1) $(L, I):=L_1(m, c)$ for $(m, c)\in P_{69}$, i.e.

$$L:=\mathbf{Z}\gamma_0+\mathbf{Z}\gamma_1,\ q(x\gamma_0+y\gamma_1):=x^2-\frac{D}{4}y^2 \quad \text{for} \quad x, y\in\mathbf{Z},$$

where

$$D:=\frac{m^2-4}{c^2},$$

$$P_{69}:=\{(m, c)\in P_1;\ \frac{m}{2} \text{ is odd, } c \text{ and } \frac{m^2-4}{4c} \text{ are even}\}.$$

2) $R(m, c)$ is a union of the following components:

(1) $:= \{\pm F^n\gamma_0 : n\in\mathbf{Z}\}$,

(4) $:= \{\pm F^n 2r_0 \colon n \in \mathbf{Z}\}$,

$\left(\frac{m+2}{v^2}\right)$ $:= \left\{\pm F^n \frac{1}{v}\left(\frac{m+2}{2} r_0 + c r_1\right) \colon n \in \mathbf{Z}\right\}$,

$\left(\frac{4(m+2)}{v^2}\right)$ $:= \left\{\pm F^n \frac{2}{v}\left(\frac{m+2}{2} r_0 + c r_1\right) \colon n \in \mathbf{Z}\right\}$,

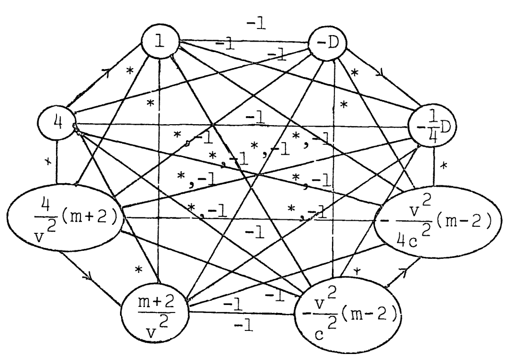
(-D) $:= \{\pm F^n 2r_1 \colon n \in \mathbf{Z}\}$,

$\left(-\frac{D}{4}\right)$ $:= \{\pm F^n r_1 \colon n \in \mathbf{Z}\}$,

$\left(-\frac{v^2}{c^2}(m-2)\right)$ $:= \left\{\pm F^n v\left(\frac{m-2}{2c} r_0 + r_1\right) \colon n \in \mathbf{Z}\right\}$,

$\left(-\frac{v^2}{4c^2}(m-2)\right)$ $:= \left\{\pm F^n \frac{v}{2}\left(\frac{m-c}{2c} r_0 + r_1\right) \colon n \in \mathbf{Z}\right\}$.

3) $R(m, c)$ is a root system of the diagram 69,



period $= m$,

where $v := \left(c, \frac{m+2}{2}\right)$.

4) $Q(R) = L$.

5) $W = O(L, I) = \langle F, \hat{T}, -1 \rangle$.

6) $\mathrm{Aut}(\Gamma_{69}) = \langle S_1, S_2 \rangle \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$, where

S_1 is the symmetry (1) $\longleftrightarrow$ $\left(-\frac{D}{4}\right)$, $\left(\frac{m+2}{v^2}\right)$ $\longleftrightarrow$ $\left(-\frac{v^2(m-2)}{4c^2}\right)$,

S_2 is the reflexion *w.r.t.* the horizontal center line,

$$S_1^*(m, c) = \left(m, \frac{m^2-4}{4c}\right), \quad C_{S_1} = -\frac{m^2-4}{4c^2},$$

$$S_2^*(m, c) = \left(m, \frac{c(m+2)}{v^2}\right), \quad C_{S_2} = \frac{m+2}{v^2},$$

$$(S_1S_2)^*(m, c) = \left(m, \frac{v^2(m-2)}{4c}\right), \quad C_{S_1S_2} = -\frac{v^2(m-2)}{4c^2}.$$

$$\varphi_{S_1}\begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} r_0 \\ r_1 \end{bmatrix},$$

$$\varphi_{S_2}\begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} \frac{v}{2} & \frac{c}{v} \\ \frac{v(m-2)}{4c} & \frac{m+2}{2v} \end{bmatrix}\begin{bmatrix} r_0 \\ r_1 \end{bmatrix},$$

$$\varphi_{S_1S_2}\begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} \frac{c}{v} & \frac{v}{2} \\ \frac{m+2}{2v} & \frac{v(m-2)}{4c} \end{bmatrix}\begin{bmatrix} r_0 \\ r_1 \end{bmatrix}.$$

7) Fixed Point of $\mathrm{Aut}(\Gamma_{69}) = \{(m, c)\in P_{69}: m=4n^2+2,\ c=2nw$ for $(n, w)\in \boldsymbol{N}\times\boldsymbol{N}$ *s.t.* $w\,|\,n^2+1\}$,

$I_{\Gamma,p} = \langle S_1S_2\rangle \simeq \boldsymbol{Z}_2$ for $p\in$ Fixed Point,

$$q_p(xr_0+yr_1) = x^2 - \frac{n^2+1}{w^2}y^2, \quad D = 4\frac{n^2+1}{w^2}.$$

Table 5. Type Γ_{29}.

1) $(L, I) := L_2(m, c)$ for $(m, c)\in P_{29}$, i.e.

$$L := \boldsymbol{Z}r_0 + \boldsymbol{Z}r_1, \quad q(xr_0+yr_1) := x^2 - xy + \frac{1-D}{4}y^2 \quad \text{for} \quad x, y\in\boldsymbol{Z},$$

where

$$D := \frac{m^2-4}{c^2},$$

$P_{29} := P_{29,1}\cup P_{29,2}$ and

$P_{29,1} := \{(m, c)\in P_2: m$ is odd$\}$,

$P_{29,2} := \{(m, c)\in P_2: m$ is even, $\frac{m+c}{2}$ and $\frac{(m+c)^2-4}{4c}$ are odd$\}$.

2) $R(m, c)$ is a union of the following components:

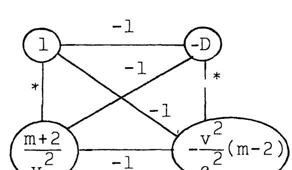
(1) $:= \{\pm F^n r_0: n\in\boldsymbol{Z}\}$,

$\left(\frac{m+2}{v^2}\right)$ $:= \left\{\pm F^n \frac{1}{v}\left(\frac{m+c+2}{2}r_0 + cr_1\right): n\in\boldsymbol{Z}\right\}$,

(-D) $:= \{\pm F^n(r_0+2r_1): n\in\boldsymbol{Z}\}$,

$\left(-\frac{v^2}{c^2}(m-2)\right)$ $:= \left\{\pm F^n v\left(\frac{m-2+c}{2c}r_0 + r_1\right): n\in\boldsymbol{Z}\right\}$.

3) $R(m, c)$ is a root system of the diagram 29,



period $= m$,

where $v = \left(c, \frac{m+c+2}{2}\right)$.

4) $Q(R)=L'$ if $8|c$ and $8|\bar{c}$ where $\bar{c}=\dfrac{m^2-4}{c}$.
$Q(R)=L$ otherwise.

5) $W=O(L, I)=\langle F, \hat{T}, -1\rangle$.

6) $\mathrm{Aut}(\Gamma_{29})=\langle S_1, S_2\rangle\simeq \mathbf{Z}_2\oplus\mathbf{Z}_2$
where S_1 is the reflexion *w.r.t.* the vertical center line, S_2 is the reflexion *w.r.t.* the horizontal center line,

$$S_1^*(m, c)=\left(m, \frac{m^2-4}{c}\right), \quad C_{S_1}=-\frac{m^2-4}{c^2},$$

$$S_2^*(m, c)=\left(m, \frac{c(m+2)}{\nu^2}\right), \quad C_{S_2}=\frac{m+2}{\nu^2},$$

$$(S_1S_2)^*(m, c)=\left(m, \frac{\nu^2(m-2)}{c}\right), \quad C_{S_1S_2}=-\frac{\nu^2(m-2)}{c^2},$$

$$\varphi_{S_1}\begin{bmatrix} r_0 \\ r_1 \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} r_0 \\ r_1 \end{bmatrix},$$

$$\varphi_{S_2}\begin{bmatrix} r_0 \\ r_1 \end{bmatrix}=\begin{bmatrix} \frac{1}{2}\left(\nu+\frac{c}{\nu}\right) & \frac{c}{\nu} \\ \frac{1}{4}\left(\frac{\nu(m-2-c)}{c}+\frac{m+2-c}{\nu}\right) & \frac{m+2-c}{2\nu} \end{bmatrix}\begin{bmatrix} r_0 \\ r_1 \end{bmatrix},$$

$$\varphi_{S_1S_2}\begin{bmatrix} r_0 \\ r_1 \end{bmatrix}=\begin{bmatrix} \frac{1}{c}\left(\nu+\frac{c}{2}\right) & \nu \\ \frac{1}{4}\left(\frac{\nu(m-2-c)}{c}+\frac{m+2-c}{\nu}\right) & \frac{\nu(m-2-c)}{2c} \end{bmatrix}\begin{bmatrix} r_0 \\ r_1 \end{bmatrix}.$$

7) Fixed Points of $\mathrm{Aut}(\Gamma_{29})=F_1\cup F_2$ where

$$F_1:=\left\{\begin{matrix} (m, c)\in P_{29,1}\colon m=n^2+2 & s.t. & n \text{ odd}\in\mathbf{N}, & \nu\in\mathbf{N} \\ c=n\nu & & \nu\,|\,n^2+4 & \end{matrix}\right\},$$

$$F_2:=\left\{\begin{matrix} (m, c)\in P_{29,2}\colon m=n^2+2 & s.t. & n \text{ even}\in\mathbf{N}, & \nu\in\mathbf{N} \\ c=n\nu & & 2\nu\,|\,n^2+4, & 2\nmid \frac{n}{2}+\frac{n^2+4}{2\nu} \end{matrix}\right\}.$$

$$I_{(m,c)}=\langle S_1S_2\rangle\simeq\mathbf{Z}_2,$$

$$q_{(m,c)}(xr_0+yr_1)=x^2-\frac{n^2+4}{4\nu^2}y^2.$$

There are two more tables of maximal root systems which are reducible.

Table 6. Type Γ_{72}.

1) $(L, I):=L_1(c^2)$ for $c\in P_{72}:=\mathbf{Q}^+$, i.e.

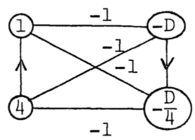
$$L:=\mathbf{Z}r_0+\mathbf{Z}r_1, \quad q(xr_0+yr_1):=x^2-\frac{c^2}{4}y^2 \quad \text{for} \quad x, y\in\mathbf{Z}.$$

2) $R(L, I)$ is a union of the following components:

① $:= \{\pm r_0\}$, (-D) $:= \{\pm 2r_1\}$,

④ $:= \{\pm 2r_0\}$, $(-\frac{D}{4})$ $:= \{\pm r_1\}$.

3) $R(L, I)$ is a root system of the diagram 72,



period $= \infty$.

4) $Q(R)=L$.

5) $W=O(L, I)=\langle \hat{T}, -1\rangle \simeq \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$.

6) $\mathrm{Aut}(\Gamma_{72})=\langle S\rangle \simeq \boldsymbol{Z}_2$, where S is the rotation π of the diagram.

$$S^*(c)=\frac{4}{c}, \quad C_S=-\frac{D}{4}=-\frac{c^2}{4}, \quad \varphi_S\begin{bmatrix} r_0 \\ r_1 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} r_0 \\ r_1 \end{bmatrix}.$$

7) Fixed Point of $\mathrm{Aut}(\Gamma_{72})=\{c=2\}$

$I_{\Gamma,\{c=2\}}=\langle S\rangle \simeq \boldsymbol{Z}_2$,

$q(xr_0+yr_1):=x^2-y^2$.

Table 7. Type Γ_{70}.

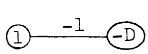
1) $(L, I):=L_2(c^2)$ for $c \in P_{70}:=\boldsymbol{Q}^+$, i.e.

$$L:=\boldsymbol{Z}r_0+\boldsymbol{Z}r_1, \quad q(xr_0+yr_1):=x^2-xy+\frac{1-c^2}{4}y^2 \quad \text{for} \quad x, y \in \boldsymbol{Z}.$$

2) $R(L, I)$ is a union of the following components:

① $:= \{\pm r_0\}$, (-D) $:=\{\pm(r_0+2r_1)\}$.

3) $R(L, I)$ is a root system of the diagram 70,



period $= \infty$.

4) $Q(R)=L$.

5) $W=O(L, I)=\langle \hat{T}, -1\rangle \simeq \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$.

6) $\mathrm{Aut}(\Gamma_{70})=\langle S\rangle \simeq \boldsymbol{Z}_2$, where S is the exchange of two vertexes;

$$S^*(c)=\frac{1}{c}, \quad C_S=-D=-c^2, \quad \varphi_S\begin{bmatrix} r_0 \\ r_1 \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} r_0 \\ r_1 \end{bmatrix}.$$

7) Fixed Point of $\mathrm{Aut}(\Gamma_{70})=\{c=1\}$.

$I_{\Gamma,\{c=1\}}=\langle S\rangle \simeq \boldsymbol{Z}_2$,

$q(xr_0+yr_1)=x(x-y)$.

(7.5) **Corollary 1.** *Let (L_i, I_i) be rank two free $\boldsymbol{Z}$-module L_i with indefinite symmetric bilinear form I_i on it for $i=1, 2$.*

Then followings are equivalent.

i) *There exists an isomorphism* φ: $L_1 \simeq L_2$ *of the additive groups and a constant* $C \neq 0$ *s.t.* $I_2 \circ \varphi = CI_1$.

ii) *Put* $R_i := R(L_i, I_i)$, $i=1, 2$. *Then* R_1 *and* R_2 *are isomorphic as root systems* (cf. (1.2)).

iii) *The periods of* R_1 *and* R_2 *are equal and there exists an isomorphism of the diagrams* ΓR_1 *and* ΓR_2 *so that the coefficients of corresponding vertexes are proportional.* (i.e. $^{\exists}C \neq 0$ *s.t.* $r_{2i} = Cr_{1i}$, $i=1, \cdots, \sharp|\Gamma R_1|$.)

Proof. The conditions ii) and iii) are already equivalent, due to (1.6) Theorem 4). The condition i) implies ii), as noted in (7.1) Note 3.

Let us show that ii) and iii) implies i). The isomorphism $R_1 \simeq R_2$ implies the existence of a linear isomorphism φ: $Q(R_1) \simeq Q(R_2)$ such that $I_1 = CI_2 \circ \varphi$ for a $C \neq 0$ (cf. (5.1) Assertion). Therefore all what we need to show is that this φ induces an isomorphism $L_1 \simeq L_2$. Due to the classification (7.4), in all cases $Q(R)$ is either equal to L or L'. Thus for instance the case when $L_1 = Q(R_1)$ and $L_2 = Q(R_2)$, the statement is obviously true.

In general we have only to show the followings.

Assertion. *Under the assumption* iii) *of the corollary,*

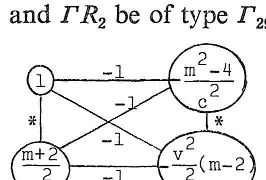
1) $L_1 = Q(R_1)$ *iff* $L_2 = Q(R_2)$.

2) *If* $L_1 \neq Q(R_1)$ *and* $L_2 \neq Q(R_2)$, *the isomorphism* $\varphi Q(R_1) \simeq Q(R_2)$ *of the root systems induces* $L_1 \simeq L_2$.

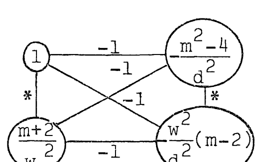
Proof of the assertion.

1) If the diagram ΓR_1 $(=\Gamma R_2)$ is not the type Γ_{29}, then due to 4) of the tables we have $L_1 = Q(R_1)$, $L_2 = Q(R_2)$.

Let ΓR_1 and ΓR_2 be of type Γ_{29} as follows.

$\Gamma R_1 =$ 

with $v = \left(c, \dfrac{m+c+2}{2}\right)$.

$\Gamma R_2 =$ 

with $w = \left(d, \dfrac{m+d+2}{2}\right)$.

Furthermore by assumption there exists an isomorphism of the diagram such that the proportions of the coefficients of corresponding vertexes are constant C. In this case the isomorphism group of the diagram is $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ generated by S_1 and S_2 so that $S_1^*(m, c) = \left(m, \frac{m^2-4}{c}\right)$, $S_2^*(m, c) = \left(m, \frac{c(m+2)}{v^2}\right)$, $(S_1S_2)^*(m, c) = \left(m, \frac{v^2(m-2)}{c}\right)$ as in Table 5.6).

Then an elementary number theoretic calculations show that

$$
\begin{aligned}
&Q(R_{(m,c)}) = L' \\
&\quad \Leftrightarrow 8 \mid c, \quad 8 \mid \frac{m^2-4}{c} \\
&\quad \Leftrightarrow 8 \mid \frac{c(m+2)}{v^2}, \quad 8 \mid \frac{v^2(m-2)}{c} \quad \text{with} \quad v := \left(c, \frac{m+c+2}{2}\right) \\
&\quad \Leftrightarrow Q(R_{S(m,c)}) = L' \quad \text{for any} \quad S \in \mathrm{Aut}(\Gamma_{29}).
\end{aligned}
$$

Furthermore since φ_{S_1}, φ_{S_2}, $\varphi_{S_1S_2}$ are integral matrixes, they preserve the group L. This proves the assertion, and hence Corollary 1.

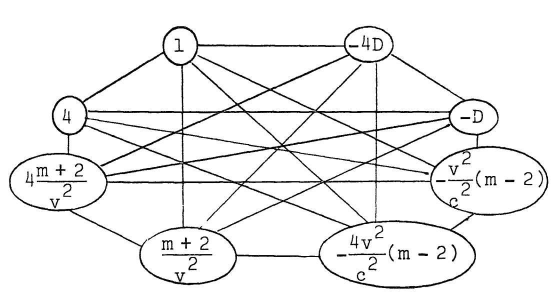
(7.6) **Classification of complete root systems of sign** (1, 0, 1).

Corollary 2. *The tables* $1\sim 7$ *of* (7.4), *deleted the case of table* 5, $8 \mid c$, $8 \mid \bar{c}$ *when* $Q(R)=L'$, *give a classification of complete root systems of signature* (1, 0, 1).

Two complete root systems corresponding to $p_1, p_2 \in P_\Gamma$ *are isomorphic iff they belong to the same orbit of* $\mathrm{Aut}(\Gamma)$.

Note. The deleted case $8 \mid c$, $8 \mid \bar{c}$ of the table 5 when $Q(R)=L'$ can even not be a maximal root system in L' as follows.

In this case the lattice $(L', I \mid L') \subset L_2(m, c)$ is isomorphic to $L_1\left(m, \frac{c}{2}\right)$ by a transformation $\gamma_0 \mapsto G_0$, $\gamma_0 + 2\gamma_1 \mapsto G_1$. Hence the maximal root system belonging to $(L', I \mid L')$ is isomorphic to $R\left(L_1\left(m, \frac{c}{2}\right)\right)$, which is the type of table 4 as follows.



(7.7) **A program for a proof of** (7.4).

To obtain the tables of (7.4), we shall proceed the following steps

i) In §8, we show that any lattice (L, I) is equal either to $L_1(m, c)$ or to $L_2(m, c)$, by a suitable choice of basis γ_0, γ_1 of L and by a suitable constant multiplication on the form I, by assuming that some $dR_{m,q,\varepsilon}$ belongs to (L, I).

This is shown by a use of a duality of maximal root systems (see (8.1) Lemma).

ii) In §9, we list up all components $d'R_{m',q',\varepsilon'}$, which belongs to $L_1(m, c)$ or $L_2(m, c)$ and we draw diagrams for the maximal root systems belonging to them.

Here we need to stratify the set of parameters (m, c) into several components P_Γ, to distinguish the type Γ of the diagram. By ordering them suitably, we obtain seven diagrams.

iii) We need to determine the action of $\mathrm{Aut}(\Gamma)$ on the component P_Γ and to determine the isomorphisms φ_S: $L \to L$ for $S \in \mathrm{Aut}(\Gamma)$.

This can be done by using the data in the above i) and ii), so that we omit the details.

iv) The case when $R(L, I)$ is reducible, is treated in (9.4).

§8. A Duality for Maximal Root Systems

Let us show that if a root system of a diagram (r) belongs to (L, I), then there exists another root system of the diagram (-D/r) belonging to $R(L, I)$ so that they form a diagram (r) $\overset{-1}{—}$ (-D/r). Here

(8.0.1) $D := -\det(I(\gamma_i, \gamma_j))_{i,j}$ for $\mathbb{Z}$-basis γ_0, γ_1 of L.

(8.1) More precisely, we show the following.

Lemma (Duality). *As before let I be an indefinite form on a $\mathbb{Z}$-module L of rank 2. If a root system $dR_{m,q,\varepsilon}$ with respect to basis $e_1, e_2 \in L \otimes \mathbb{R}$* (cf. (1.4), (1.5)) *belongs to (L, I), then another root system $(\sqrt{D}/d)R_{m,q,-\varepsilon}$ belongs to (L, I). Here D is given as in* (8.0.1).

Proof. For a suitable choice of e_1, e_2, one may assume that $q=0$.

Put
$$\alpha_i := d(e^{ip_m}e_1 + \varepsilon e^{-ip_m}e_2) \qquad i = 0, 1,$$
$$\beta_i := (\sqrt{D}/d)(e^{ip_m}e_1 - \varepsilon e^{-ip_m}e_2) \qquad i = 0, 1.$$

Since $Q(dR_{m,q,\varepsilon}) = \mathbb{Z}\alpha_0 + \mathbb{Z}\alpha_1$ and $Q((\sqrt{D}/d)R_{m,q,-\varepsilon}) = \mathbb{Z}\beta_0 + \mathbb{Z}\beta_1$ (cf. (2.3) 1)'), $dR_{m,q,\varepsilon}$ (resp. $(\sqrt{D}/d)R_{m,q,-\varepsilon}$) belongs to (L, I), iff α_0, α_1(resp. β_0, β_1) belong to

(L, I). Since $\alpha_0, \alpha_1 \in L$, there exists an integral matrix $M \in M(\mathbf{Z}, 2)$ *s.t.*

1) $$\begin{bmatrix}\alpha_0\\ \alpha_1\end{bmatrix} = M\begin{bmatrix}\gamma_0\\ \gamma_1\end{bmatrix}.$$

The relation between the intersections $I(\gamma_i, \gamma_j)$ and $I(\alpha_i, \alpha_j)$ is given by,

2) $$M(I(\gamma_i, \gamma_j))_{i,j}{}^tM = \varepsilon d^2\begin{bmatrix}2 & m\\ m & 2\end{bmatrix}.$$

Since α_0, α_1 belong to (L, I), $I(\gamma_i, \alpha_j^\vee) \in M(\mathbf{Z}, 2)$ and

3) $$\varepsilon M^{-1}\begin{bmatrix}2 & m\\ m & 2\end{bmatrix} \in M(\mathbf{Z}, 2).$$

By taking the determinant of 2), one obtains,

4) $$D(\det M)^2 = d^4(m^2-4).$$

By definition, linear relation between α_i's and β_i's is given by,

$$\begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix} = \frac{\sqrt{D}}{d}\begin{bmatrix}1 & -\varepsilon\\ e^{pm} & -\varepsilon e^{-pm}\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix} = \frac{\sqrt{D}}{d}\begin{bmatrix}1 & -\varepsilon\\ e^{pm} & -\varepsilon e^{-pm}\end{bmatrix}$$
$$\times\left[d\begin{bmatrix}1 & \varepsilon\\ e^{pm} & \varepsilon e^{-pm}\end{bmatrix}\right]^{-1}\begin{bmatrix}\alpha_0\\ \alpha_1\end{bmatrix}$$
$$= \frac{\sqrt{D}}{d^2\sqrt{m^2-4}}\begin{bmatrix}-m & 2\\ -2 & m\end{bmatrix}\begin{bmatrix}\alpha_0\\ \alpha_1\end{bmatrix}.$$

Therefore applying 4) and 1), one gets,

5) $$\begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix} = |\det M|^{-1}\begin{bmatrix}-m & 2\\ -2 & m\end{bmatrix}M\begin{bmatrix}\gamma_0\\ \gamma_1\end{bmatrix}.$$

A direct calculation shows that the entries of the matrixes of 5) and 3) coincides. This implies that β_0, β_1 are integral linear combinations of $\gamma_0, \gamma_1 \in L$.

Using 5), one computes,

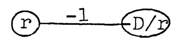
$$(I(\gamma_i, \beta_j^\vee))_{i,j} = \left[|\det M|^{-2}\begin{bmatrix}-m & 2\\ -2 & m\end{bmatrix}M\right]^{-1}\begin{bmatrix}-\varepsilon 2 & -\varepsilon m\\ -\varepsilon m & -\varepsilon 2\end{bmatrix}$$
$$= \varepsilon|\det M|M^{-1}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} \in M(\mathbf{Z}, 2).$$

Thus $I(\beta_0^\vee)$, $I(\beta_1^\vee) \in L^*$ and therefore β_0, β_1 belong to (L, I). Q.E.D.

(8.2) **Corollary 1.** *Due to this duality lemma, possible diagrams for maximal root systems can be subtracted from the table of* (1.6) *as follows.*

numbering **diagram**

5.



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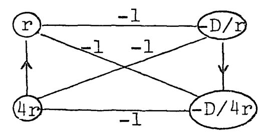

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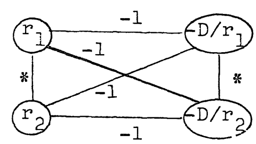
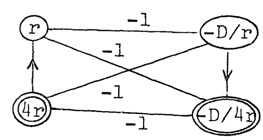
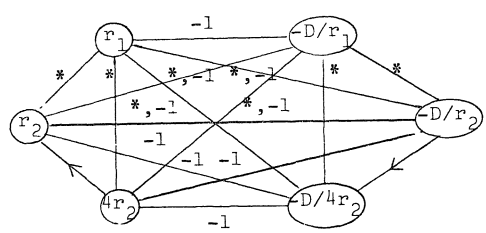


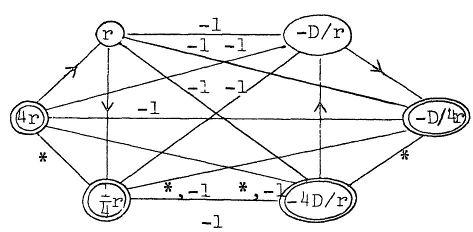
34.



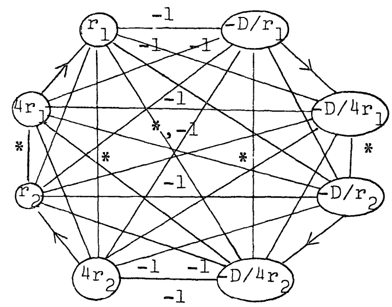
62.



65.



69.



In fact we shall see that among the above diagrams, the first two types 5 and 23 do not appear (cf. (9.2)). (To see this we need to give a more precise description of the relationships between the lattice (L, I) and the root system belonging to it.) Assuming this fact, let us show that the orthogonal group $O(L, I)$ is generated by reflexions.

(8.3) **Corollary 2.** *Let us denote by W the group generated by reflexions w_α for $\alpha \in R(L, I)$.*

Then

(8.3.1) $W = O(L, I)$, $\tilde{O}(L, I) =$ *Automorphism group of* $R(L, I)$.

Proof. The statement is true including the case $R(L, I)$ is reducible, when the proof is immediate from the description of $\alpha \in R(L, I)$.

Here we prove the case when $R(L, I)$ is irreducible.

Since we have a natural inclusion $W \subset O(L, I)$, we have only to show the opposite inclusion relation.

Since we may disclude the first two diagrams in Cor. 1., all the remaining diagram belong to one of the following two cases.

Case 1. The diagram contains edges $\bigcirc\overset{-1}{\text{———}}\bigcirc$ and $\bigcirc\overset{*}{\text{———}}\bigcirc$.

Case 2. The diagram contains edges $\bigcirc\overset{-1}{\text{———}}\bigcirc$ and vertexes $\circledcirc$.

Due to Assertion (5.7), in both cases, one computes W as

Case 1. $W = \langle g_{p_m}, H_{-1}, g_{2q}T \rangle$,

Case 2. $W = \langle g_{2p_m}, H_{-1}, g_{2q}T \rangle$.

Put $G := \{g_t : t \in \mathbb{R}\} \simeq \mathbb{R}$ (cf. (5.1) ii)). Let $g_p \in G$ be a generator of $G \cap O(L, I)$ (which is an infinite cyclic group.) Then since $g_p R(L, I) = R(L, I)$, $p_m | p$ for the case 1 and $2p_m | p$ for the case 2. Since $W \subset O(L, I)$, we get the equality $p = p_m$ or $2p_m$ according to the case 1 or 2. i.e. $G \cap O(L, I) = G \cap W$. By noticing the fact $-1 = H_{-1}$, $g_{2q}T \in W$ one concludes $W = O(L, I)$.

The last isomorphism follows from the description of the automorphism group of a root system given in (5.5) Assertion. Q.E.D.

Note. In 5) of the tables of (7.4), we use the notations $F = g_{p_m}$, $\hat{T} = \varepsilon T$ and $-1 = H_{-1}$.

(8.4) **A description of (L, I).** Let (L, I) be as before a pair of rank 2 $\mathbb{Z}$-module and an indefinite form on it. As in (8.1) let $R_{m,0,\varepsilon}$ and $\sqrt{D}R_{m,0,-\varepsilon}$ belong to (L, I), where D is the descriminant given in (8.0.1). Put also, $\alpha_i := e^{ip_m} + \varepsilon e^{-ip_m}$,

$\beta_i := \sqrt{D}(e^{ip_m} - \varepsilon e^{-ip_m})$ for $i \in \mathbf{Z}$. Note that we have relations

$$I(\alpha_i, \beta_i) = 0, \quad q(\alpha_i)q(\beta_i) = -D \quad (i \in \mathbf{Z}), \quad \text{(cf. (9.4) Lemma).}$$

The aim here is to find basis r_0, r_1 of L as far as "near" to α_0, β_0 so that they give a simple description of (L, I).

By choosing r_0 to be a constant multiple of α_0, we get

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \end{pmatrix}, \quad \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \frac{1}{ac} \begin{pmatrix} -ma+2b & 2c \\ -2a+mb & mc \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \end{pmatrix}$$

for some integers $a, b, c \in \mathbf{Z}$ with $a, c > 0$ (cf. 5) of (8.1)). Therefore $a=1$ or 2. By replacing r_1 by $r_1 + \left[\frac{-ma+2b}{2c}\right] r_0$ we have three cases $\beta_0 = r_1$, $2r_1$ or $r_0 + 2r_1$. On each case, one computes easily as follows:

1)
$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ m & c \\ 0 & 1 \\ \frac{m^2-4}{2c} & \frac{m}{2} \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \end{pmatrix} \quad \text{where } 2 \mid m,\ 2c \mid m^2-4,$$

$$q(xr_0 + yr_1) = \frac{1}{4}x^2 - Dy^2, \quad D = \frac{m^2-4}{c^2},$$

$$F\begin{pmatrix} r_0 \\ r_1 \end{pmatrix} = \begin{pmatrix} \frac{m}{2} & \frac{c}{2} \\ \frac{m^2-4}{2c} & \frac{m}{2} \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \end{pmatrix}.$$

2)
$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{m}{2} & c \\ 0 & 2 \\ \frac{m^2-4}{2c} & m \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \end{pmatrix} \quad \text{where } 2 \mid m,\ 2c \mid m^2-4,$$

$$q(xr_0 + yr_1) = x^2 - Dy^2, \quad D = \frac{m^2-4}{c^2}$$

$$F\begin{pmatrix} r_0 \\ r_1 \end{pmatrix} = \begin{pmatrix} \frac{m}{2} & c \\ \frac{m^2-4}{4c} & \frac{m}{2} \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \end{pmatrix}.$$

3)
$$\begin{pmatrix}\alpha_0\\ \alpha_1\\ \beta_0\\ \beta_1\end{pmatrix}=\begin{pmatrix}1 & 0\\ \frac{m+c}{2} & c\\ 1 & 2\\ \frac{m^2-4}{2c}+\frac{m}{2} & m\end{pmatrix}\begin{pmatrix}\gamma_0\\ \gamma_1\end{pmatrix} \quad \text{where } 2\,|\,m+c,\ 2c\,|\,m^2-4+mc\,,$$

$$q(x\gamma_0+y\gamma_1)=x^2-xy+\frac{1}{4}(1-D^4)y^2,\quad D=\frac{m^2-4}{c^2}\,,$$

$$F\begin{pmatrix}\gamma_0\\ \gamma_1\end{pmatrix}=\begin{pmatrix}\frac{m+c}{2} & c\\ \frac{m^2-4-c^2}{4c} & \frac{m-c}{2}\end{pmatrix}\begin{pmatrix}\gamma_0\\ \gamma_1\end{pmatrix}.$$

Here in the above tables F means a linear transformation *s.t.* $F(\alpha_i)=\alpha_{i+1}$, $F(\beta_i)=\beta_{i+1}$ for $i\in\mathbb{Z}$ and $F(e_1)=e^{p_m}e_1$, $F(e_2)=e^{-p_m}e_2$.

We delete the first case from our consideration, since the first and the second lattices above are isomorph by the transformation $\varphi(x\gamma_0+y\gamma_1)=y\gamma_0+x\gamma_1$, $m=m$, $c=(m^2-4)/c$ and $\bar{q}(\varphi(u))=(-\sqrt{D})q(u)$.

The second and the third lattices are named as $L_1(m, c)$ and $L_2(m, c)$ respectively in (7.3). By definition the root system $R_{m.0,\varepsilon}=\{\pm F^n\alpha_0\colon n\in\mathbb{Z}\}$ belongs to $L_1(m, c)$ and to $L_2(m, c)$.

If another root system $R_{n,0,\varepsilon}$ for an $n>2$ belongs to $L_i(m, c)$, then $L_i(,m\,c)$ is isomorphic to $L_i(n, d)$ for $d=c\sqrt{\frac{n^2-4}{m^2-4}}$ since we have followings.

$R_{n,0,\varepsilon}$ belongs to $L_1(m, c)$ (resp. $L_2(m, c)$).

$\Leftrightarrow$ $\alpha:=e^{p_m}e_1+e^{-p_m}\varepsilon e_2$ belongs to $L_1(m, c)$ (resp. $L_2(m, c)$).

$$\Leftrightarrow \frac{n}{2},\ c\sqrt{\frac{n^2-4}{m^2-4}},\ \sqrt{\frac{(m^2-4)\,(n^2-4)}{2c}}\in\mathbb{Z}\,,$$

$$\left(\text{resp. } \frac{n}{2}+\frac{c}{2}\sqrt{\frac{n^2-4}{m^2-4}},\ c\sqrt{\frac{n^2-4}{m^2-4}},\ \frac{n}{2}-\sqrt{\frac{(m^2-4)\,(n^2-4)}{2c}}\in\mathbb{Z}\right).$$

To avoid overlappings and complications in the classification, we disclude the cases when a root system $R_{n,0,\varepsilon}$ for $2<n<m$ belongs to $L_i(m, c)$.

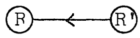
Therefore in the definitions of P_1 and P_2 in §7 (7.3), 1.2 ii) and 2.2 ii), we discluded the parameters (m, c) of these cases.

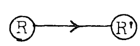
§9. A Criterium for Components to Belong to $(\mathbb{L}, \mathbb{I})$

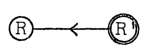
(7.9) Let us give a criterium for a pair of root systems ○←○, ○←◎, ○→◎, ○—(−1)—◎, ○—(*)—○ or ○—(−1, *)—○ to belong to (L, I).

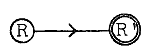
(9.1) **Criterium.** *Let a root system* $R:=dR_{m,q,\varepsilon}$ *belong to* (L, I). *Put* $\alpha_i:=d(e^{q+ip_m}e_1+\varepsilon e^{-q-ip_m}e_2)\in R\ (i=0, 1)$.

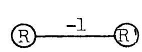
In the following, we give necessary and sufficient conditions for an existence of a root system $R':=d'R_{m',q',\varepsilon'}$ *belonging to* (L, I) *so that* R *and* R' *interacts in one of the seven types described in* (3.3) **Lemma.**

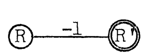
1)  *iff* $2\alpha_0$, $2\alpha_1$ *belong to* (L, I),

 *iff* $\frac{1}{2}\alpha_0$, $\frac{1}{2}\alpha_1$ *belong to* (L, I).

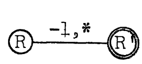
2)  *iff either* $2\alpha_0$ *belongs to* (L, I),
or $2\alpha_1$ *belongs to* (L, I).

3)  *iff either* $\frac{1}{2}\alpha_0$ *belongs to* (L, I),
or $\frac{1}{2}\alpha_1$ *belongs to* (L, I).

4)  *iff there exist* β_0, β_1 *belonging to* (L, I)
s.t. $I(\alpha_0, \beta_0)=I(\alpha_1, \beta_1)=0$, $I(\beta_0, \beta_0)=I(\beta_1, \beta_1)$.

5)  *iff either there exists* β_0 *belonging to* (L, I)
s.t. $I(\alpha_0, \beta_0)=0$,
or there exists β_1 *belonging to* (L, I)
s.t. $I(\alpha_1, \beta_1)=0$.

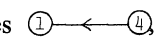
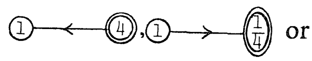
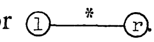
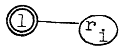
6)  *iff there exists* $\nu\in\mathbb{Z}\setminus\{0\}$
s.t. $\frac{1}{\nu}(\alpha_0+\alpha_1)$ *belongs to* (L, I).

7)  *iff there exists* $\nu\in\mathbb{Z}\setminus\{0\}$
s.t. $\frac{1}{\nu}(\alpha_0-\alpha_1)$ *belongs to* (L, I).

Proof. Let β_0 (or β_0, β_1) be the elements of L described in the above conditions and let W_R be the group generated by reflexions w_α for $\alpha\in R$. Then $R':=\pm W_R\beta_0$ or $=\pm W_R\beta_0\cup\pm W_R\beta_1$ is a root system belonging to (L, I), interacting with R according to the 7 types described.

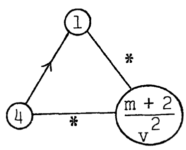
Conversely if $d'R_{m',q',\varepsilon'}$ belongs to (L, I), where $m'=m$ or m^2-2, $q'=q$ or $q+\frac{1}{2}p_m$, then the elements $\beta_i:=d'(e^{q'+ip_m}e_1+\varepsilon' e^{-q'-ip_m}e_2)\ (i=0, 1)$, satisfy the conditions described above. Detailed verifications are omitted. Q.E.D.

(9.2) Using the criterium in (9.1), we are now able to determine all components belonging to $L_1(m, c)$ and $L_2(m, c)$, so that one can draw the diagram of the maximal root systems. Due to the symmetry of the diagram in (8.1), it is enough to investigate a half of the diagram, namely the components $R_i:=d_iR_{m_i,q_i,\varepsilon_i}$ such that $\varepsilon_i=\varepsilon$, i.e. the components which are combined with the com-

ponent $R_{m,0,\varepsilon}$ by the edges   or .
By discluding the case, when there is a component R_i such that $m=m_i^2-2$ and , (cf. the definition of P_1 in (7.3) 1.2 ii)), one may assume that $m_i=m$ or $m_i=m^2-2$ for all $1\leqq i\leqq k$.

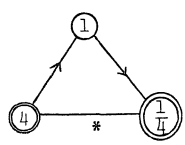
Followings are the list of the half diagrams.

Case 1. $L_1(m, c)$ *s.t.* m even, c odd,

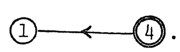


$v:=\left(\frac{m+2}{2}, c\right)$.

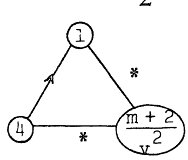
Case 2. $L_1(m, c)$ *s.t.* $4|m$, $2|c$,



Case 3. $L_1(m, c)$ *s.t.* $\frac{m}{2}$ odd, $\frac{m^2-4}{2c}$ odd,

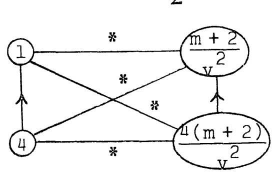


Case 4. $L_1(m, c)$ *s.t.* $\frac{m}{2}$ odd, $\frac{m^2-4}{4c}$ odd,



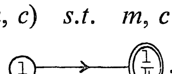
$v:=\left(\frac{m+2}{2}, c\right)$.

Case 5. $L_1(m, c)$ *s.t.* $\frac{m}{2}$ odd, $\frac{m^2-4}{4c}$ even,

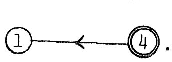


$v:=\left(\frac{m+2}{2}, c\right)$.

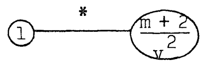
Case 6. $L_2(m, c)$ *s.t.* m, c even,



Case 7. $L_2(m, c)$ *s.t.* $\frac{m}{2}$ odd, $4|c$, $\frac{m^2-4}{2c}$ odd,

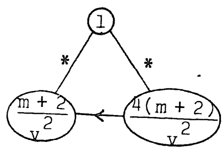


Case 8. $L_2(m, c)$ *s.t.* m even, $\dfrac{m+c}{2}$ odd, $\dfrac{(m+c)^2-4}{4c}$ odd,



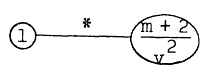
$$v := \left(c, \frac{m+c+2}{2}\right).$$

Case 9. $L_2(m, c)$ *s.t.* m even, $\dfrac{m+c}{2}$ odd, $\dfrac{(m+c)^2-4}{4c}$ even,



$$v := \left(c, \frac{m+c+2}{2}\right).$$

Case 10. $L_2(m, c)$ *s.t.* m, c odd,



$$v := \left(c, \frac{m+c+2}{2}\right).$$

It is not hard to check that the **cases** 6, 7 are isomorphic to the **case** 3 and the **case** 9 is isomorphic to the **case** 1 or to the **case** 4, where the **case** 1 and **case** 4 are isomorphic by a use of φ of (7.2) 1. vi).

(9.3) Using the discussions in §8, §9, it is no more hard to reconstruct the full diagram ΓR for each case in (9.2). Also the datas in the table of (7.4) is now possible to calculate. We omitt the calculations. These complete the classification of maximal root systems associated to an indefinite form on a $\boldsymbol{Z}$-module of rank 2.

(9.4) *A proof of* (7.4) *Table* 6, 7.

Let us show a slite modification of (8.1) Lemma (Duality).

Lemma. *Let I be an indefinite form on a $\boldsymbol{Z}$-module L of rank 2. If an element $\alpha \in L$ belongs to (L, I), then the element $\beta \in L \otimes \boldsymbol{R}$, characterized by the equations $I(\alpha, \beta)=0$ $q(\alpha)\, q(\beta)=-D$ up to a sign, belongs to (L, I).*

Proof. Let γ_0, γ_1 be $\boldsymbol{Z}$-basis of L such that α is an integral multiple of γ_0. Put,

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \end{bmatrix} \qquad \text{for } a \in \boldsymbol{N},\ b, c \in \boldsymbol{R},\ c > 0\,.$$

We have relations

1) $I(\alpha, \beta) = a(bI(\gamma_0, \gamma_0) + cI(\gamma_0, \gamma_1)) = 0$

2) $\det \begin{bmatrix} I(\alpha, \alpha) & 0 \\ 0 & I(\beta, \beta) \end{bmatrix} = \det \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \det(I(\gamma_i, \gamma_j)) \det \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$. i.e.

2)′ $ac=2$

Since α belongs to (L, I), using 1) and 2)′ one computes, that

$$\frac{2I(\alpha,\gamma_i)}{I(\alpha,\alpha)}=\begin{cases}\dfrac{2}{a}=c & i=0\\ \dfrac{2I(\gamma_0,\gamma_1)}{aI(\gamma_0,\gamma_0)}=-b & i=1\end{cases}$$

are integers.

On the other hand, again using 1) and 2)′ one computes that,

$$\frac{2I(\beta,\gamma_i)}{I(\beta,\beta)}=\begin{cases}0 & i=0\\ -a & i=1\end{cases}$$

are integers. Q.E.D.

Note. This lemma asserts the existence of linearly independent elements belonging to (L, I). Hence it gives a positive answer to the question Note 1 in (7.1), for the case of indefinite form on rank two module.

Now the same type argument as in (8.2) shows that one can choose basis γ_0, γ_1 of L in one of the following form.

Case 1. $\alpha=\gamma_0,\quad \beta=2\gamma_1,$

$$q(x\gamma_0+y\gamma_1):=x^2-\frac{D}{4}y^2,\quad \text{for}\quad x,y\in\boldsymbol{Z}.$$

Case 2. $\alpha=\gamma_0,\quad \beta=\gamma_0+2\gamma_1$

$$q(x\gamma_0+y\gamma_1):=x^2-xy+\frac{1}{4}(1-D)y^2,\quad \text{for}\quad x,y\in\boldsymbol{Z}.$$

Case 3. $\alpha=2\gamma_0,\quad \beta=\gamma_1$

$$q(x\gamma_0+y\gamma_1)=\frac{1}{4}x^2-Dy^2\quad \text{for}\quad x,y\in\boldsymbol{Z}.$$

We omit the case 3, which is isomorphic to case 1.

In the above cases it is not hard to see the equivalence:

$R(L, I)$ is irreducible.

$\Leftrightarrow$ There exists $\delta\in L$ *s.t.* δ belongs to (L, I), δ is not a constant multiple of α or β. (cf. (2.2) Note)

$\Leftrightarrow$ There exists $\delta\in L$ *s.t.* δ belongs to (L, I), $I(\delta,\delta)=2$, δ is not a constant multiple of α or β.

$\Leftrightarrow$ $D\notin\boldsymbol{Q}^2:=\{c^2;\ c\in\boldsymbol{Q}\}$.

(The verification of the last equivalence is reduced to the existence of integral solution of Pell's equation.)

Finally in the case $D \in \boldsymbol{Q}^2$, one computes directly

Case 1. $R(L_1(c^2)) = \{\pm r_0, \pm 2r_0\} \cup \{\pm r_1, \pm 2r_1\}$.

Case 2. $R(L_2(c^2)) = \{\pm r_0\} \cup \{\pm r_1\}$.

In both cases $O(L, I) = \langle T, -1 \rangle = W \simeq \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$.

This completes the proof of (7.4) for reducible cases.

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