

Some Applications of Boolean Valued Set Theory to Abstract Harmonic Analysis on Locally Compact Groups

By

Hirokazu NISHIMURA*

Abstract

The main purpose of this paper is to extend Takeuti's [23] Boolean valued treatment of abstract harmonic analysis on locally compact abelian groups to locally compact groups (neither abelian nor compact in general). The distinctive feature of our approach, compared with traditional treatments of the subject, is that we can establish many important theorems without resort to direct integrals or to the theory of Banach algebras. By way of illustration, we will give such a proof of renowned Bochner's theorem. This paper is not intended to be exhaustive at all but hopefully to be suggestive. How far we can proceed in this direction yet remains to be seen.

§ 1. Introduction

Abstract harmonic analysis has two origins. One is the classical Fourier analysis set forth, e.g., in Bochner [3] and Zygmund [31]. The other is the algebraic theory of finite groups and their representations, whose modern and comprehensive treatment can be seen, e.g., in Curtis and Reiner [5]. Indeed the spirit of abstract harmonic analysis is to do Fourier analysis on topological groups as general as possible, guided by the representation theory of finite groups while using the modern techniques of functional analysis.

The most central technique in the study of topological groups, which are usually assumed to be locally compact at least, is their unitary representations (on some appropriate Hilbert spaces) and, in particular, their irreducible unitary representations. As for compact groups, it is well known that their irreducible unitary representations are finite-dimensional and any unitary representation of such a

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* Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606, Japan.

group decomposes *discretely* into irreducible ones. As for locally compact abelian groups, their irreducible representations are all one-dimensional, which simplifies their study considerably. However, the decomposition of a unitary representation of a locally compact group is *continuous* in general and its irreducible representations may be infinite-dimensional. This is one of the main reasons why the study of general locally compact groups had been left almost untouched for fairly a long time even after we got to know compact groups and locally compact abelian ones pretty well.

One of the most familiar examples of continuous decomposition theories is celebrated von Neumann's reduction theory in operator algebras, where the notion of a direct integral plays a fundamental role. And it is quite natural that this notion should have been applied successfully to unitary representations of locally compact groups. See, e.g., Godement [7, 8], Mackey [12], Mautner [13], Segal [8, 9], Tomita [26], Tsuji [27] and Yoshizawa [28].

The principal deficiency of a direct integral is that this notion seems to be involved too much in separability conditions. Recently, Takeuti [25] proposed to replace this notion with a Boolean valued approach, which was then applied successfully to continuous geometries and the like by Eda [4] and Nishimura [15]. The main purpose of this paper is to show how to apply this new method to the study of locally compact groups.

After reviewing the rudiments of Boolean valued set theory in Section 2, we will see in Section 3 that any unitary representation \mathcal{V} of a locally compact group G is irreducible in a Boolean valued universe $V^{(\mathcal{B})}$, where \mathcal{B} is a Boolean algebra of projections that is maximal with respect to the property that every projection of \mathcal{B} commute with \mathcal{V}_x for all $x \in G$. We then apply this result, by way of illustration, to obtain a simple proof of Bochner's theorem in Section 4.

§ 2. The Rudiments of Boolean Valued Set Theory

Let \mathcal{B} be a complete Boolean algebra. We define $V_\alpha^{(\mathcal{B})}$ by transfinite induction on ordinal α as follows :

- (1) $V_0^{(\mathcal{B})} = \phi,$
- (2) $V_\alpha^{(\mathcal{B})} = \{u \mid u : \mathcal{D}(u) \rightarrow \mathcal{B} \text{ and } \mathcal{D}(u) \subseteq \bigcup_{\xi < \alpha} V_\xi^{(\mathcal{B})}\}.$

Then the Boolean valued universe $V^{(\mathcal{B})}$ of Scott-Solovay is defined as follows :

$V^{(\mathscr{A})} = \bigcup_{\alpha \in \text{On}} V_{\alpha}^{(\mathscr{A})}$, where On is the class of all ordinal numbers.

$V^{(\mathscr{A})}$ can be considered to be a Boolean valued model of set theory by defining $\llbracket u \in v \rrbracket$ and $\llbracket u = v \rrbracket$ for $u, v \in V^{(\mathscr{A})}$ with the following properties

- (1) $\llbracket u \in v \rrbracket = \sup_{y \in \mathscr{D}(v)} (v(y) \wedge \llbracket u = y \rrbracket)$,
- (2) $\llbracket u = v \rrbracket = \inf_{x \in \mathscr{D}(u)} (u(x) \implies \llbracket x \in v \rrbracket) \wedge \inf_{y \in \mathscr{D}(v)} (v(y) \implies \llbracket y \in u \rrbracket)$,

and by assigning a Boolean value $\llbracket \varphi \rrbracket$ to each formula φ without free variables inductively as follows: ‘

- (1) $\llbracket \neg \varphi \rrbracket = \neg \llbracket \varphi \rrbracket = 1 - \llbracket \varphi \rrbracket$,
- (2) $\llbracket \varphi_1 \vee \varphi_2 \rrbracket = \llbracket \varphi_1 \rrbracket \vee \llbracket \varphi_2 \rrbracket$,
- (3) $\llbracket \varphi_1 \wedge \varphi_2 \rrbracket = \llbracket \varphi_1 \rrbracket \wedge \llbracket \varphi_2 \rrbracket$,
- (4) $\llbracket \forall x \varphi(x) \rrbracket = \inf_{u \in V^{(\mathscr{A})}} \llbracket \varphi(u) \rrbracket$,
- (5) $\llbracket \exists x \varphi(x) \rrbracket = \sup_{u \in V^{(\mathscr{A})}} \llbracket \varphi(u) \rrbracket$.

The following theorem is fundamental to Boolean valued analysis.

Theorem 2.1. *If φ is a theorem of ZFC, then so is $\llbracket \varphi \rrbracket = 1$.*

Now we present several elementary properties of $V^{(\mathscr{A})}$ without proofs.

Proposition 2.2.

- (1) $\llbracket \exists x \in u \varphi(x) \rrbracket = \sup_{x \in \mathscr{D}(u)} (u(x) \wedge \llbracket \varphi(x) \rrbracket)$,
- (2) $\llbracket \forall x \in u \varphi(x) \rrbracket = \inf_{x \in \mathscr{D}(u)} (u(x) \implies \llbracket \varphi(x) \rrbracket)$.

The class V of all sets can be embedded into $V^{(\mathscr{A})}$ by transfinite induction as follows.

$$\check{y} = \{ \langle \check{x}, 1 \rangle \mid x \in y \} \text{ for } y \in V.$$

Proposition 2.3. *For $x, y \in V$,*

- (1) $\llbracket \check{x} \in \check{y} \rrbracket = \begin{cases} 1 & \text{if } x \in y \\ 0 & \text{otherwise.} \end{cases}$

$$(2) \quad \llbracket \check{x} = \check{y} \rrbracket = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

A subset $\{b_\alpha\}$ of \mathcal{B} is called a *partition of unity* if $\sup_\alpha b_\alpha = 1$ and $b_\alpha \wedge b_\beta = 0$ for any $\alpha \neq \beta$. Given a partition of unity $\{b_\alpha\}$ and a subset $\{u_\alpha\}$ of $V^{(\mathcal{B})}$, it can be proved easily that :

Proposition 2.4. *There exists an element u of $V^{(\mathcal{B})}$ such that $\llbracket u = u_\alpha \rrbracket \geq b_\alpha$ for any α . Furthermore this u is determined uniquely in the sense that $\llbracket u = v \rrbracket = 1$ for any $v \in V^{(\mathcal{B})}$ with the above property.*

The above u is denoted by $\sum_\alpha u_\alpha b_\alpha$ or $u_{\alpha_1} b_{\alpha_1} + \cdots + u_{\alpha_n} b_{\alpha_n}$ if $\{b_\alpha\}$ is a finite set. Then we have

$$\text{Proposition 2.5.} \quad \llbracket \varphi(\sum u_\alpha b_\alpha) \rrbracket = \sup_\alpha (\llbracket \varphi(u_\alpha) \rrbracket \wedge b_\alpha).$$

The techniques of partitions of unity give the following two propositions.

Proposition 2.6 (The Maximum Principle). *Let φ be a formula. Then there exists a $u \in V^{(\mathcal{B})}$ such that $\llbracket \varphi(u) \rrbracket = \llbracket \exists x \varphi(x) \rrbracket$.*

Proposition 2.7. *Let $\varphi(x)$ be a formula with only x as a free variable and $\llbracket \varphi(u) \rrbracket = 1$ for some $u \in V^{(\mathcal{B})}$. Then*

$$(1) \quad \llbracket \forall x (\varphi(x) \implies \psi(x)) \rrbracket = \inf_{\llbracket \varphi(u) \rrbracket = 1} \llbracket \psi(u) \rrbracket,$$

$$(2) \quad \llbracket \exists x (\varphi(x) \wedge \psi(x)) \rrbracket = \sup_{\llbracket \varphi(u) \rrbracket = 1} \llbracket \psi(u) \rrbracket.$$

We define the interpretation $X^{(\mathcal{B})}$ of $X = \{x \mid \varphi(x)\}$ with respect to $V^{(\mathcal{B})}$ to be $\{u \in V^{(\mathcal{B})} \mid \llbracket \varphi(u) \rrbracket = 1\}$, assuming that it is not empty. For technical convenience, if X is a set, then $X^{(\mathcal{B})}$ is usually considered to be a set by choosing a representative from an equivalence class $\{v \in V^{(\mathcal{B})} \mid \llbracket u = v \rrbracket = 1\}$. Then, by Proposition 2.7, we have $X^{(\mathcal{B})} \times \{1\} \in V^{(\mathcal{B})}$ and $\llbracket X = X^{(\mathcal{B})} \times \{1\} \rrbracket = 1$.

Let $D \subseteq V^{(\mathcal{B})}$. A function $g : D \rightarrow V^{(\mathcal{B})}$ is called *extensional* if $\llbracket d = d' \rrbracket \leq \llbracket g(d) = g(d') \rrbracket$ for any $d, d' \in D$. A \mathcal{B} -valued set $u \in V^{(\mathcal{B})}$ is said to be *definite* if $u(d) = 1$ for any $d \in \mathcal{D}(u)$. Then we have the following characterization theorem of

extensional maps.

Theorem 2.8. *Let $u, v \in V^{(\mathcal{B})}$ be definite and $D = \mathcal{D}(u)$. Then there is a bijective correspondence between $f \in V^{(\mathcal{B})}$ satisfying $\llbracket f : u \rightarrow v \rrbracket = 1$ and extensional maps $\varphi : D \rightarrow v^{(\mathcal{B})}$, where $v^{(\mathcal{B})} = \{u \mid \llbracket u \in v \rrbracket = 1\}$. The correspondence is given by the relation $\llbracket f(d) = \varphi(d) \rrbracket = 1$ for any $d \in D$.*

Now we restrict our consideration to more concrete complete Boolean algebras. Let H be a Hilbert space. A set \mathcal{B} of projections of H is called a *Boolean algebra of projections* if it satisfies the following conditions :

- (i) *both the identity and zero operators are members of \mathcal{B} and members of \mathcal{B} are pairwise commutable ;*
- (ii) *if P_1 and P_2 are members of \mathcal{B} , so are $P_1 \vee P_2 (= P_1 + P_2 - P_1 P_2)$ and $\neg P_1 = I - P_1$.*

A Boolean algebra \mathcal{B} of projections is said to be *complete* if \mathcal{B} is not only complete as a Boolean algebra but also whenever $P = \sup_{\alpha} P_{\alpha}$, the range of P is the closure of the linear space spanned by all the ranges of P_{α} 's.

From now on, let \mathcal{B} be a complete Boolean algebra of projections. A self-adjoint operator T whose spectral resolution is $\int \lambda dE_{\lambda}$ is said to be *in* (\mathcal{B}) if every E_{λ} belongs to \mathcal{B} . A normal operator T which can be written as $T_1 + iT_2$ for self-adjoint T_1, T_2 is said to be *in* (\mathcal{B}) if both T_1 and T_2 are in (\mathcal{B}). The real numbers in $V^{(\mathcal{B})}$ correspond to self-adjoint operators in (\mathcal{B}), the complex numbers of absolute value 1 correspond to unitary operators in (\mathcal{B}) and the complex numbers correspond to normal operators in (\mathcal{B}).

Recently Ozawa [16] succeeded in showing that the Hilbert space H can be embedded in $V^{(\mathcal{B})}$ as a Hilbert space \tilde{H} simply by changing the truth value of the equality between vectors in such a way as

$$\llbracket \tilde{x} = \tilde{y} \rrbracket = \sup \{P \in \mathcal{B} : Px = Py\} \quad \text{for any } x, y \in H.$$

We say that the complete Boolean algebra \mathcal{B} *reduces* a bounded operator T if $PT = TP$ for any $P \in \mathcal{B}$. The characterization of bounded operators on \tilde{H} is a bit cumbersome, but the partial isometry operators on \tilde{H} have a simple characterization as follows :

Theorem 2.9. *Partial isometry operators on \tilde{H} correspond to partial isometry*

operators on H that \mathcal{B} reduces.

In this paper we are interested almost exclusively in projection operators and unitary operators, both of which are subsumed under partial isometry operators. Thus this theorem is sufficient for our purpose.

§ 3. Unitary Representations and Boolean Valued Analysis

Let G be an arbitrary locally compact group. A unitary representation of G is a homomorphism from G into the group of unitary operators of a Hilbert space H . An unitary representation \mathcal{V} of G assigning a unitary operator \mathcal{V}_x to each $x \in G$, is called *continuous*¹⁾ if for any $\xi \in H$ and any $\varepsilon > 0$, there exists an open neighborhood U of e (=the identity element of G) such that

$$\|\mathcal{V}_{x\xi} - \mathcal{V}_y\xi\| < \varepsilon \text{ provided } x^{-1}y \in U.$$

In the sequel, every unitary representation is assumed to be continuous unless stated to the contrary. A unitary representation \mathcal{V} of G on a Hilbert space H is called *irreducible* if $\{0\}$ and H are the only closed subspaces of H that are invariant under all \mathcal{V}_x . Otherwise \mathcal{V} is called *reducible*. It is well-known that the following three conditions on \mathcal{V} are equivalent :

- (i) \mathcal{V} is irreducible ;
- (ii) every nonzero vector in H is a cyclic vector for \mathcal{V} (i.e., the closed linear span of $\{\mathcal{V}_x\xi : x \in G\}$ for each nonzero vector $\xi \in H$ is H itself) ;
- (iii) the only bounded operators on H commuting with all \mathcal{V}_x are of the form αI , where α is a complex number and I is the identity operator.

Let \mathcal{B} be a complete Boolean algebra of projections on a Hilbert space H and let G be a locally compact group with \mathcal{O} as its family of open sets and \mathcal{Z} as an open basis at e . It is easy to see that \check{G} is a group in $V^{(\mathcal{B})}$. $\check{\mathcal{O}}$ is not a topology of \check{G} in general, but fortunately $\check{\mathcal{Z}}$ can be an open basis of e making \check{G} a topological group in $V^{(\mathcal{B})}$. We denote this topological group also by \check{G} . As Takeuti did in [23], we consider the completion of \check{G} with respect to the two-sided uniformity, denoted by

¹⁾ See Hewitt and Ross [9, (22.20)] for the equivalence of various continuities of \mathcal{V} .

\tilde{G} , to obtain a locally compact group (cf. Kelley [11, p. 210]).

The following two theorems follow readily from Theorem 2.9 and the fact that m -convergence implies strong convergence for unitary operators (cf. Takeuti [21, 23, 24]).

Theorem 3.1. Every unitary representation \mathcal{V} of G which \mathcal{B} reduces¹⁾ corresponds to a unitary representation of \tilde{G} in $V^{(\mathcal{B})}$.

Conversely,

Theorem 3.2. Every unitary representation of \tilde{G} in $V^{(\mathcal{B})}$ corresponds to a unitary representation of G which \mathcal{B} reduces.

Theorems 3.1 and 3.2 show that unitary representations of G which \mathcal{B} reduces and unitary representations of \tilde{G} in $V^{(\mathcal{B})}$ are the same things from different viewpoints. Our applications of Boolean valued set theory to abstract harmonic analysis are based almost solely on this simple but overwhelmingly powerful principle as well as

Theorem 3.3. Let \mathcal{V} be a unitary representation of G and let \mathcal{B} be a maximal Boolean algebra of projections contained in $\{\mathcal{V}_x : x \in G\}$ ²⁾. Then \mathcal{V} corresponds to an irreducible unitary representation in $V^{(\mathcal{B})}$.

Proof. Any projection operator which reduces \mathcal{V} in $V^{(\mathcal{B})}$ must correspond to a projection operator in the standard universe V which reduces \mathcal{V} and which \mathcal{B} reduces. But the maximality of \mathcal{B} implies readily that such a projection operator belongs to \mathcal{B} .

§ 4. A Boolean Valued Approach to Bochner's Theorem

Let G be a locally compact group. A continuous complex-valued function φ on G is said to be *positive definite* if the inequality

$$\sum_{j=1}^m \sum_{k=1}^m \bar{a}_j a_k \varphi(x_j^{-1} x_k) \geq 0$$

¹⁾ i. e., $\mathcal{V}_x P = P \mathcal{V}_x$ for any $x \in G$ and any $P \in \mathcal{B}$.

²⁾ For a set S of bounded operators, S' denotes the commutant of S .

holds for all finite sequences x_1, \dots, x_m of distinct elements of G and complex numbers $\alpha_1, \dots, \alpha_m$. We remark that a positive definite function φ is always left uniformly continuous, since $|\varphi(x) - \varphi(y)|^2 \leq 2\varphi(e)[\varphi(e) - \operatorname{Re}[\varphi(x^{-1}y)]]$ for $x, y \in G$, as is well-known.

A positive definite function φ on G is called *normalized* if $\varphi(e) = 1$. Given two positive definite functions φ_1, φ_2 on G , the function φ_1 is said to be *dominated* by the function φ_2 if $\varphi_2 - \varphi_1$ is a positive definite function. A positive definite function φ is called *elementary* if every positive definite function dominated by the function φ is a multiple of φ . We denote by Γ the set of all normalized elementary positive definite functions on G , which is called the *dual space* of G .

It is easy to see that any cyclic unitary representation \mathcal{V} of G on H with a cyclic vector ξ yields a positive definite function $\varphi(x) = \langle \mathcal{V}_x \xi, \xi \rangle$, which gives a one-to-one correspondence between positive definite functions on G and cyclic unitary representations of G up to unitary equivalence. Under this correspondence, elementary positive definite functions on G correspond to irreducible unitary representations of G . Since $|\varphi(x)| \leq \varphi(e)$ for any $x \in G$ and any positive definite function φ on G , the set Γ can be considered to be a subset of the conjugate space $L_\infty(G)$ of $L_1(G)$ and so the weak closure $\bar{\Gamma}$ of Γ is weakly compact.

If μ is a bounded complex regular measure on Γ , a function $\hat{\mu}$ on G defined by

$$\hat{\mu}(x) = \int_{\Gamma} \varphi(x) d\mu(\varphi)$$

is called the *Fourier transform* of μ . It is easy to see that $\hat{\mu}$ is a positive definite function provided μ is non-negative. The converse of this is celebrated Bochner's theorem, for which we shall give a simple proof, using Boolean valued analysis.

Theorem 4.1. *Every positive definite function on G is the Fourier transform of a suitable non-negative regular Borel measure on Γ .*

Proof. Let φ be an arbitrary positive definite function on G . We can assume without loss of generality that φ is normalized. As we have remarked before, φ can be expressed as $\varphi(x) = \langle \mathcal{V}_x \xi, \xi \rangle$ for a suitable unitary representation \mathcal{V} of G on a Hilbert space H and a vector $\xi \in H$. Let \mathcal{B} be a maximal Boolean algebra of projections contained in $\{\mathcal{V}_x : x \in G\}'$. Then \mathcal{V} becomes irreducible unitary representation in $V^{(\mathcal{B})}$, as Theorem 3.3 shows, and so φ becomes elementary in $V^{(\mathcal{B})}$. Therefore φ belongs to $\bar{\Gamma}$ in $V^{(\mathcal{B})}$. By the way, $\bar{\Gamma}$ is a compact Hausdorff

space and so it is a uniform space in the unique way (cf. Kelley [11, pp. 197-200]). Therefore there must exist a \mathcal{B} -valued regular Borel measure E such that $E(N) = \llbracket \varphi \in \check{N} \rrbracket$ for any Borel subset N of $\bar{\Gamma}$, as Takeuti [24, § 1] showed. Let $\mu(N) = \langle E(N)\xi, \xi \rangle$. Then μ is a non-negative regular Borel measure on $\bar{\Gamma}$ and $\mu(\bar{\Gamma} - \Gamma) = 0$. A similar discussion of Takeuti [24, Theorem 2 of § 1] establishes easily that $\varphi = \hat{\mu}$.

This proof suggests typically how Boolean valued analysis can supersede not only direct integrals but also the theory of Banach algebras in some important theorems of abstract harmonic analysis. To establish a theorem on G , we have often had to make a detour through $L_1(G)$, to which we can apply the theory of Banach algebras.

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