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Extended Affine Root Systems I
(Coxeter transformations)

By

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Contents

Introduction

1. The present paper is the first part of the study on invariants for extended affine root systems with markings, which may be regarded as a development of the work of E. Looijenga [12] on Root systems and Elliptic curves, extending and strengthening the results by introducing the flat structure on the invariants.

We introduce in this paper the concept of *an extended affine root system with a marking* ((2.1) Definition).

The objective of this first part is the study of *the Coxeter transformations* for a

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marked extended affine root system. It is achieved in the paragraphs § 9, § 10 and § 11. The results are summarized in **Lemma A** (9.7), **Lemma B** (10.1) and **Lemma C** (11.3).

These **Lemmas** are used essentially in the second part of the study [20] for the construction of *flat θ-invariants for marked extended affine root systems*.

2. An extended affine root system R is, by definition, a root system belonging to a positive semi-definite quadratic form I, whose radical has *rank two*. A marking G for the root system R is a rank 1 subspace of the radical. (cf. (1.2) **Definition**, (2.1) **Definition**)

The main difficulty for the study of such extended affine root systems arise from the fact that there does not exist an analogous of a Weyl chamber, compared with the cases of finite or affine root systems, since the group W_R generated by the reflexions with respect to all roots of R: i.e. the extended affine Weyl group, does not act anywhere properly on the ambient real vector space*). Nevertheless we define *a Dynkin diagram* $\Gamma_{R,G}$ for (R, G) by the help of *the exponents* introduced in § 7. (See (8.2) for a definition of the diagram, and (8.6) for discussions on the diagram.) The diagrams are listed in the following **Table** 1.

Then a Coxeter transformation for (R, G) is defined as a product of reflexions of roots corresponding to the nodes of the diagram ((9.7) **Definition**).

*) W_R acts properly on a domain in the complexification of the real ambient space.

3. Let us explain briefly some geometric backgrounds, which helps but is not necessary for the understanding of this paper.

A rational double point of a complex analytic surface and its universal deformation are described by a Dynkin diagram and a simple Lie group (or algebra) of type A_l, D_l or E_l by E. Brieskorn [3,4]. (See also P. Slodowy [21].) Then there exists canonically a vector space Ω and a non-degenerate symmetric bilinear form J on Ω such that the base space S of the deformation (whose coordinate ring is the ring of invariant polynomials on the Cartan algebra by the action of the Weyl group) is canonically isomorph to Ω (see [16]). We call such structure, *the flat structure of* S or *the flat structure on the invariant ring*.**)

A simple elliptic singularity of a complex analytic surface is, by definition [28], obtained by blowing down a smooth elliptic curve in a smooth surface, whose deformation is studied by E. Looijenga [11][12], P. Slodowy [22] and others, where an affine root system and affine Lie algebra (i.e. Kac-Moody Lie algebra (or group) of Euclidean type) are used for the construction of the family.

The present paper gives a construction of a flat structure for the base space S of the universal deformation of a simple elliptic singularity or equivalently *a flat structure on the invariant ring of θ-functions*, which will be actually done in the second half of this paper [20].

For the purpose, it was necessary to introduce a new root system, which is an extension of an affine root system by one dimensional radical: *the extended affine root system with a marking*, as introduced in this paper.

Even the extended affine root system does not correspond to a Kac-Moody Lie algebra (cf. [33], [34]), one may naturally ask an existence of a Lie algebra corresponding to them which would describe the universal deformation of the simple elliptic singularity (cf. (8.5), P. Slodowy [23]).

**) The flat structure on the base space S of a universal unfolding of any hypersurface isolated singular point is introduced in [17][18], where Ω is a space of relative differential forms, J is a residue pairing on Ω and the embedding $S \to \Omega$ is defined by a flat connection ∇ depending on a choice of a primitive form $\zeta^{(0)}$.

In the case of a simple elliptic singularity, a choice of a primitive form $\zeta^{(0)}$ is equivalent to a choice of an element a in the radical of the intersection form on the middle homology group of the Milnor fiber, which defines the marking $G := \mathbb{R}a$. (This was announced in [17]. Details will appear in [20].)

4. Let us give a brief view on the contents of the note.

i) The first three paragraphs § 1, § 2 and § 3 contain preliminaries: *definition of a root system R belonging to a quadratic form* (1.2) *and its generality.*

For the first reading, the readers are suggested to skip this part until § 4, after looking at some basic definitions and notations in (1.1), (1.2), (2.1), (2.2), (2.3), (3.1) and (3.4) without proofs, and to come back to § 1, § 2 or § 3 according to the necessity.

ii) The next three paragraphs § 4, § 5 and § 6 contain *the classification of marked extended affine root systems* (R, G). They are classified using numerical invariants $t_1(R, G)$, $t_2(R, G)$, called the first and the second tier numbers introduced in § 4 (cf. (1.11)), as follows.

Except for some exceptional cases, the isomorphism class of (R, G) is determined by the triple (P, t_1, t_2) where P is the type of the quotient finite root system $R/\mathrm{rad}I$ and $t_i := t_i(R, G)$ $(i=1,2)$ are tier numbers. We call $P^{(t_1,t_2)}$ as the type for (R, G). For the exceptional cases, we define types by a slite modification such as $A_1^{(1,1)*}$, $B_l^{(2,2)*}$, $C_l^{(1,1)*}$, $BC_l^{(2,2)}(1)$ and $BC_l^{(2,2)}(2)$, which are called exceptional types (cf. (5.1), (5.2)).

The result of classification is exposed in the table in (5.2). The proof is reduced in § 6 to classify weighted diagrams $(\Gamma,(k(\alpha))_{\alpha\in|\Gamma|})$ where Γ is a Dynkin diagram for an affine root system and $k(\alpha)$ $(\alpha\in|\Gamma|)$ are positive integers called the counting (6.1).

If one admits the result of the classification in § 5, one may skip these paragraphs till § 7.

iii) After these preparations, we arrive basically important concepts for a marked extended affine root system: *the exponents* m_i $(i=0,\ldots,l)$ in § 7 and *the Dynkin diagram* $\Gamma_{R,G}$ in § 8.

The following Table 1 is the complete list of the types and the Dynkin diagrams for marked extended affine root systems with an assumption that the quotient affine root system R/G is reduced.

The author is grateful to W. Ebeling, who noticed him that the figures $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ appear already in the representation theory of algebras [2], and the importance of the branching points of the diagrams.

iv) The Dynkin diagram $\Gamma_{R,G}$ gives a most intrinsic description of the marked extended affine root system (R, G) in the sense that (R, G) can be reconstructed only from the data of the diagram ((9.6) Theorem). In the reconstruction of (R, G), a (pre-) Coxeter transformation, which is a product of reflexions w.r.t. the nodes of the diagram (cf. (9.3)(9.7)), plays an essential role.

v) The Coxeter transformation c is unique up to (autor) conjugacy in W_R. Then we shall show in § 9, § 10 and § 11 that;

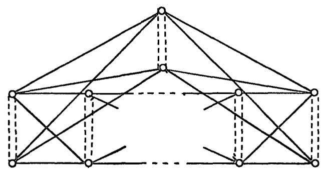
i) *c is a semi-simple of finite order $l_{\max}+1$, whose eigenvalues are described by the exponents* (**Lemma A** (9.7)).

ii) *No element of R can be expressed as an image of the transformation $c-1$* (**Lemma B** (10.1)).

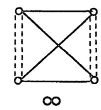
iii) The hyperbolic extension $\tilde{W}_{R,G}$ is an extension of W_R *by an infinite cyclic group, which is generated by the power $\tilde{c}^{l_{\max}+1}$ of a hyperbolic Coxeter transformation $\tilde{c}$*

Table 1. Dynkin Diagrams for Extended Affine Root Systems

$A_l^{(1,1)}$

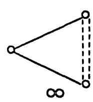


$(l \geq 2)$

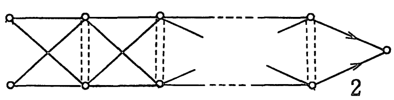


$(l=1)$

$A_1^{(1,1)*}$

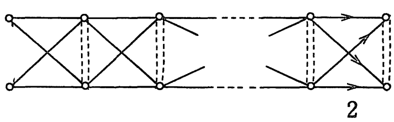


$B_l^{(1,1)}$



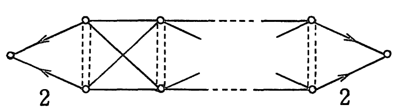
$(l \geq 3)$

$B_l^{(1,2)}$



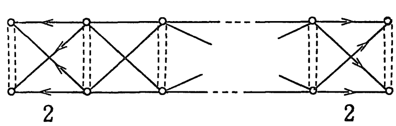
$(l \geq 3)$

$B_l^{(2,1)}$



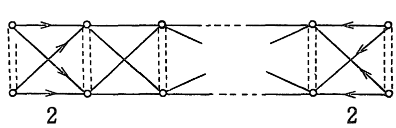
$(l \geq 2)$

$B_l^{(2,2)}$



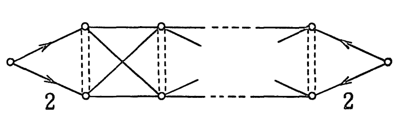
$(l \geq 2)$

$C_l^{(1,1)}$



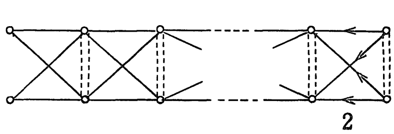
$(l \geq 2)$

$C_l^{(1,2)}$



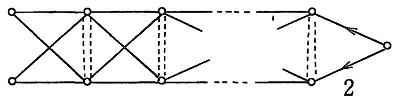
$(l \geq 2)$

$C_l^{(2,1)}$



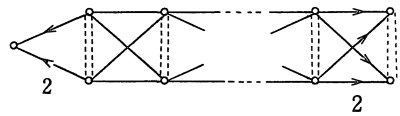
$(l \geq 3)$

$C_l^{(2,2)}$



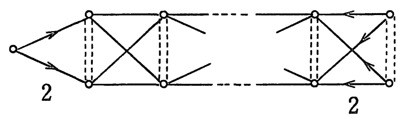
$(l \geq 3)$

$B_l^{(2,2)*}$



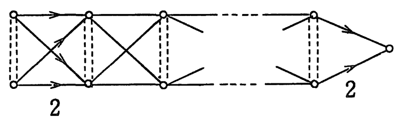
$(l \geq 2)$

$C_l^{(1,1)*}$

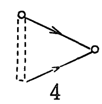


$(l \geq 2)$

$BC_l^{(2,1)}$

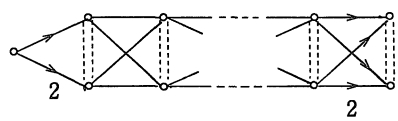


$(l \geq 2)$,

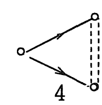


$(l = 1)$

$BC_l^{(2,4)}$

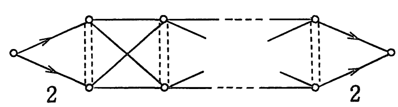


$(l \geq 2)$,



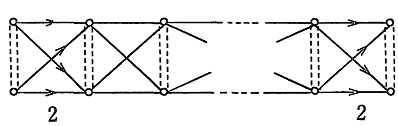
$(l = 1)$

$BC_l^{(2,2)}(1)$

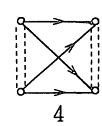


$(l \geq 2)$

$BC_l^{(2,2)}(2)$

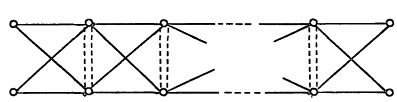


$(l \geq 2)$,



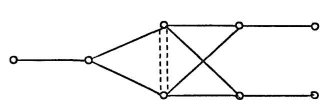
$(l = 1)$

$D_l^{(1,1)}$

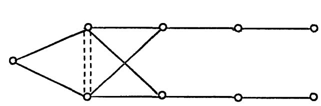


$(l \geq 4)$

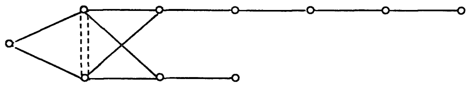
$E_6^{(1,1)}$



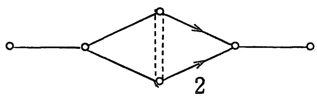
$E_7^{(1,1)}$



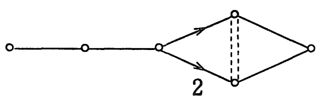
$E_8^{(1,1)}$



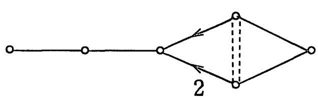
$F_4^{(1,1)}$



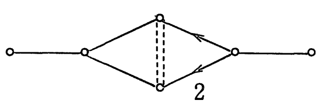
$F_4^{(1,2)}$



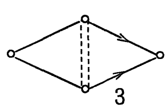
$F_4^{(2,1)}$



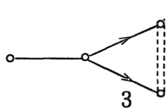
$F_4^{(2,2)}$



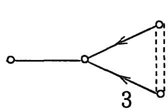
$G_2^{(1,1)}$



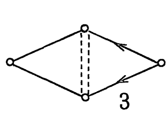
$G_2^{(1,3)}$



$G_2^{(3,1)}$



$G_2^{(3,3)}$



(Lemma C (11.3)).

vi) The last paragraph § 12 treats *the folding relations among Dynkin diagrams* for marked extended affine root systems. The folding relation induces a hierarchy relations among the diagrams so that all diagrams are divided into 4 groups, illustrating the classification in § 5. Particularly the exceptional types form one group.

The study of foldings also illustrate *the importance of rank two-ness of the radical of the quadratic form for an extended affine root system*, since the two extensions correspond to the two types of foldings defined in (12.3).

5. Some part of the result, including the classification of marked extended affine root systems, is published in [19].

Part of this work was carried out in Sept. '82 - Dec. '82, when the author was a visitor of the University of Nijmegen under the support of Z.W.O, in Jan. '83 - Feb. '83, when he was a visitor of the Universioy Bonn and the Max Planck Institut in Bonn and March '83, when he was a visitor of the Ecole Polytechniques in Paris. He expresses his gratitude for the organizations and the mathematicians for the hospitality.

Particularly thanks goes to Professors T. Springer and E. Looijenga in the Netherlands, Professors E. Brieskorn, P. Slodowy, F. Knörrer and W. Ebeling in Bonn and Professors M. Demazur and B. Tits in Paris for several valuable discussions.

§ 1. Root Systems Belonging to a Quadratic Form

This paragraph is devoted for generalities on root systems belonging to quadratic forms. We prepare terminologies and concepts for the uses in later paragraphs.

A rough view of the paragraph is the following.

i) *The axioms for a root system R belonging to a quadratic form I and its examples are given in* (1.2), (1.3). *The isomorphism class of the root systems determines the quadratic form I up to a constant factor* (1.4).

ii) *In* (1.5)-(1.13) *one is concerned with the rational structure* (1.7) *and the finiteness of the set of length of roots* (1.9) *and their direct consequences.*

One obtains the concepts of *dual root system* $R^{\vee}$ (1.5), *quotient root system* R/G (1.8), *marking* G, *tier numbers* $t(R)$, $t(R, G)$ (1.10) (1.12), and *even lattice structure* I_R (1.11).

iii) *In* (1.14)-(1.20) *the group* W_R *generated by reflexions of roots is investigated*. The main tool is the use of Eichler-Siegel map and its inverse (1.14.1)(1.14.5).

For the name of the transformation, the author is indebted to W. Ebeling, who pointed out him that such transformations (1.14.1)(1.14.5) are used implicitly in the works of M. Eichler and C. L. Siegel.

A criterium for W_R *to split* into a semi-direct product of another $W_{R'}$ and a free abelian group is given in (1.15).

We introduce a hyperbolic extension $\tilde{F}_G$ (1.17) *of the space* F *w.r.t. a marking* G, *which induces a central extension* $\tilde{W}_{R,G}$ *of* W_R *in* (1.18). *Some structual study of* $\tilde{W}_{R,G}$ *is done in* (1.19) (1.20) *using a map* $r: \tilde{W}_{R,G} \to M_G$.

(1.1) **Reflexion w_α of α.** Let F be a real vector space equipped with a quadratic form q, which induces a symmetric bilinear form,

(1.1.1)
$$I: F\times F \to \mathbb{R}$$
$$I(x, y) := q(x+y)-q(x)-q(y).$$

If an element $\alpha \in F$ is non-isotropic (i. e. $q(\alpha)\neq 0$), then we define the dual $\alpha^{\vee}\in F$ and the reflexion $w_\alpha \in GL(F)$ as follows.

(1.1.2)
$$\alpha^{\vee} := \frac{1}{q(\alpha)}\alpha = \frac{2}{I(\alpha, \alpha)}\alpha,$$

(1.1.3)
$$w_\alpha(u) := u - I(u, \alpha^{\vee})\alpha \quad \text{for} \quad u\in F.$$

By the definition we have,

(1.1.4)
$$\alpha^{\vee\vee}=\alpha, \ \frac{I(\alpha,\alpha)}{2}\circ\frac{I(\alpha^{\vee},\alpha^{\vee})}{2}=1,$$

(1.1.5)
$$w_\alpha = w_{\alpha^{\vee}}, \ w_\alpha^2 = id_F,$$

(1.1.6)
$$w_\alpha \in O(F, I),$$

where

$$O(F,I) := \{g\in GL(F) : I\circ g = I\}$$

is the orthogonal group of the metric I.

For a non isotropic subset B of F (i.e. $q(\alpha)\neq 0$ for any $\alpha\in B$), we define a reflexion group by,

(1.1.7) $\quad W_B :=$ *the subgroup of* $O(F, I)$ *generated by* w_α *for* $\alpha\in B$.

Note. To avoid a confusion on the word "dual", we shall denote by F^* the dual vector space $\mathrm{Hom}_{\mathbb{R}}(F, \mathbb{R})$ of F. Note that the element $\alpha^\vee$ belongs to F but not to F^*.

(1.2) **The axioms for a root system R belonging to I.**

As in (1.1) let F be a real vector space of finite rank with a metric I, whose signature is (μ_+, μ_0, μ_-). (I.e. μ_+, μ_0, or μ_- are number of positive, zero or negative eigenvalues of I respectively.)

Definition 1. *A subset R of F is called a root system belonging to I or a root system of sign (μ_+, μ_0, μ_-), if it satisfies the following* **Axioms** 1), ..., 5).

1) *Let $Q(R)$ be the additive subgroup of F generated by R. Then $Q(R)$ is a full lattice of F. (i.e. the natural map induces an isomorphism $\mathbb{R}\otimes_{\mathbb{Z}} Q(R)\simeq F$.)*
2) *For any* $\alpha\in R$, $I(\alpha, \alpha)\neq 0$.
3) *For any* $\alpha\in R$, $w_\alpha R=R$.
4) *For any* $\alpha, \beta\in R$, $I(\alpha, \beta^\vee)\in\mathbb{Z}$.
5) *Irreducibility. If $R=R_1\amalg R_2$ and $R_1\perp R_2$ with respect to I for subsets R_i of R, then either $R_1=\phi$ or $R_2=\phi$.*

2. *Two root systems R in F and R' in F' are isomorphic if there exists a linear isomorphism $\varphi: F\simeq F'$ which induces a bijection $\varphi: R\simeq R'$* (cf. (1.4) **Lemma**).

Note 1. i) *If $\alpha\in R$, then $-\alpha\in R$.* $(\because\ w_\alpha\alpha=-\alpha)$

ii) *If α, $c\alpha\in R$ for a constant $c\in\mathbb{R}$, then*

$$c\in\{\pm 2, \pm 1, \pm 1/2\}.$$

$$(\because \text{ Axiom 4) implies } I(c\alpha, \alpha^\vee)=2c,\ I(\alpha, (c\alpha)^\vee)=2/c\in\mathbb{Z})$$

iii) *A root system R is called reduced if $\alpha, c\alpha\in R$ implies $c\in\{\pm 1\}$.*

Note 2. *Let us call a subset R of F to be a root system belonging to I, which may not be irreducible, if R satisfies the axioms* 1), ..., 4) *of* Def. 1. *Then we have the following.*

Assertion. *Let R be a root system belonging to I, which may not be irreducible. Then there exists a unique disjoint decomposition $R=\coprod_{j=1}^{N} R_j$, $R_j \neq \phi$ $(j=1, \dots, N)$ such that*

i) *R_j is an irreducible root system belonging to $I \mid \mathbb{R}R_j$ for $j=1, \dots, N$. (Here $\mathbb{R}R_j$ is the linear span of R_j).*

ii) *$R_i \perp R_j$ for $i \neq j$. (I.e. $I(\alpha_i, \alpha_j)=0$ for $^\forall \alpha_i \in R_i$, $^\forall \alpha_j \in R_j$.)*

Proof. If $R=\coprod_{j=1}^{N} R_j$, $R_j \neq \phi$ for $j=1, \dots, N(\geq 2)$ is a decomposition of R satisfying only ii) of the assertion, then one check easily that automatically each R_j is a root system which may not be irreducible. Furthermore $\mathbb{R}R_i \subsetneq F$ for all i, since otherwise ii) implies $R_j \subset \operatorname{rad} I$ for $j \neq i$, which contradicts to the fact that $R_j \subset R$ is non-isotropic. Thus the existence of the decomposition with i) and ii) can be shown by induction on rank F.

If $\coprod_j R_j$ and $\coprod_i S_i$ are two irreducible decompositions. Then $R_j = \coprod_i (R_j \cap S_i)$ is a decomposition of R_j. Since R_j is irreducible $R_j \cap S_i = \phi$ or R_j. The same argument for S_i shows that up to a permutation of the indexes i, j, the two decompositions coincide. This is the uniqueness of the decomposition. q.e.d.

Note 3. In the above situation *we have* $F=\bigoplus_j \mathbb{R}R_j$ *if I is non-degenerate.* ($\because$ $\mathbb{R}R_j \cap \sum_{i \neq j} \mathbb{R}R_i \subset \operatorname{rad} I$ for all j)

Note 4. By definition W_R preserve the metric I. Conversely *up to a constant factor, I is the unique symmetric bilinear form on F which is invariant by W_R.*

Proof. Let J be another symmetric invariant form on F. The equality $J(w_\alpha u, w_\alpha \alpha)=J(u, \alpha)$ for $\alpha \in R$, $u \in F$ implies $J(u, \alpha)=c_\alpha I(u, \alpha)$ where $c_\alpha := J(\alpha, \alpha)/I(\alpha, \alpha)$. If $I(\alpha, \beta) \neq 0$ for $\alpha, \beta \in R$, the symmetry of J implies $c_\alpha = c_\beta$. Thus the irreducibility of R implies $J=cI$ for $c=c_\alpha$ for any $\alpha \in R$.

(1.3) Example 1. *Let R be a root system belonging to I, where I is positive semi definite of sign $(l_+, l_0, 0)$.*

Then

$$I(\alpha, \beta^\vee) \in \mathbb{Z} \cap [-4, 4] \quad \textit{for any} \quad \alpha, \beta \in R.$$

($\because$ Semi-definiteness of $q \mid \mathbb{R}\alpha + \mathbb{R}\beta$ implies that $I(\alpha^\vee, \beta) I(\alpha, \beta^\vee) \leq 4$. Then apply **Axiom** 4).)

i) *If* $l_0 = 0$ (i.e. *I is positive definite*), *then R is finite and is a root system in the classical sense* (see [1] Ch. VI).

Proof. We have only to show that $\# R < \infty$. Let $\alpha_1, \ldots, \alpha_{l_+} \in R$ be a $\mathbb{R}$ basis for F. Since I is non-degenerate, the correspondence $\alpha \in R \mapsto (I(\alpha, \alpha_i^\vee))_{i=1,\cdots,l_+} \in (\mathbb{Z} \cap [-4, 4])^{l_+}$ is injective.

Note 1. We shall generalize this finiteness property for general I in (1.21) **Lemma**.

Note 2. *If a root system R which belongs to I is finite, then I is definite*, due to the uniqueness of I in (1.2) *Note* 3.

ii) *If* $l_0 = 1$, *then R is an affine root system*. (*For a definition, see* [14, 2], [7]).

Proof. All axioms for an affine root system stated in [14] are direct consequences of the axioms in (1.2), except an axiom (AR4), which asks that W_R acts properly on an affine hyperplane $E := \{b = 1\} \subset F^*$, where b is a generator of rad $I \simeq \mathbb{R}b$. This is an direct consequence of an exact sequence $0 \to T_{\mathrm{rad} I} \to W_R \to W_{R/\mathrm{rad} I} \to 0$ in (1.15) **Assertion**, where $T_{\mathrm{rad} I}$ is a lattice of the translation group of E and $W_{R/\mathrm{rad} I}$ is a finite Weyl group. (cf. (1.3) **Ex** 1. i), (1.7) **Corollary**, (1.8) **Assertion**).

iii) *In general if* $l_0 = k \geq 2$, *let us call R a k-extended affine root system. Particularly we shall investigate the case* $k = 2$ *in this paper, when we call R an extended affine root system simply*.

Example 2. *Let L be a free abelian group of finite rank and I be a symmetric bilinear form on it. Put,*

$$R(L, I) := \{\alpha \in L : I(\alpha, \alpha) \neq 0,\ I(\alpha^\vee, \beta) \in \mathbb{Z} \text{ for all } \beta \in L\}.$$

If $R(L, I)$ *spans* $\mathbb{R} \otimes L$ *over* $\mathbb{R}$, $R(L, I)$ *is a root system belonging to* I, *which may not be irreducible. We shall call* $R(L, I)$ *the maximal root system belonging* *to* (L, I).

In [29, § 7 (7.4)], all maximal root systems belonging to an indefinite form of sign $(1, 0, 1)$ are classified into seven types.

Example 3. All *root systems of sign* $(1, 0, 1)$ are classified into 72 types in [29, § 1 (1.6)].

Example 4. Let B be a (finite) subset of a vector space F with a form I. If B satisfies the axioms for a root system except axiom 3, then $R := W_B B$ is a root system belonging to I. Sometimes we shall call B a basis for R. We have relations, $Q(R) = Q(B)$, $W_R = W_B$.

(1.4) **Isomorphisms of Root Systems.**

Lemma. *Let φ be an isomorphism between two root systems R and R' which are belonging to I and I' respectively. Then there exists a non-zero constant $c \in \mathbb{R}$ such that $I = cI' \circ \varphi$.*

Proof. Let $\alpha, \beta \in R$ be roots s.t. $I(\alpha, \beta) \neq 0$ and $\mathbb{R}\alpha \neq \mathbb{R}\beta$. Put $F_2 := \mathbb{R}\alpha + \mathbb{R}\beta$ and $F_2' := \varphi(F_2)$. Then $R_2 := F_2 \cap R$ and $R_2' := F_2' \cap R'$ are root systems belonging to $I | F_2$ and $I' | F_2'$ respectively which are isomorphic by $\varphi | F_2$. Therefore due to the axiom 5) of irreducibility, one can reduce the proof of the lemma to the followings.

i) *Lemma is true for the case when rank* $F = 2$.
ii) Let *φ be an isomorphism of root systems R, R' which may not be irreducible in* rank 2 *vector spaces. If R is irreducible, then R' is irreducible.*

Proof of i). Let A_R and $A_{R'}$ be the accumulating sets in $\mathbb{P}(F)$ and $\mathbb{P}(F')$ of the image set of R and R' respectively. According as I is definite, semi-definite or indefinite, A_R is either void, one point or two points, which correspond to the isotropic vectors in F of I. (cf. (1.3) **Ex 1.** i), ii) and **Ex 3.** ([29] (1.6) *Note* 6)). Since φ induces a bijection among A_R and $A_{R'}$, φ maps the isotropic vectors of I to that of I'. This implies already the statement i) for the cases when I is semi-definite or indefinite.

In the case when I is definite, R is either of the types A_2, B_2, BC_2 or G_2. Then the coincidences of some geometric invariants (e.g. $\#R$, $\#(R/\mathbb{R}^\times)$, etc.) show that R

and R' are of the same type.

Proof of ii). Reducible root systems for rank $F=2$ is either $A_1\times A_1$, $A_1\times BC_1$ or $BC_1\times BC_1$. Thus they are characterized among all root systems by an equation $\#(R/\mathbf{R}^{\times})=2$, which is invariant by φ. q.e.d.

(1.5) **The dual root system** $R^{\vee}$.

For a given subset $R\subset F$,

put

$$R^{\vee}:=\{\alpha^{\vee}\in F:\ \alpha\in R,\ I(\alpha,\alpha)\neq 0\}.$$

Lemma. *If R is a root system belonging to I, then $R^{\vee}$ is a root system belonging to I.*

Proof. The axioms 2)-5) for $R^{\vee}$ are directly shown by definition. The axiom 1) for $R^{\vee}$ will be shown in (1.10) after a preparation of a finiteness **lemma** (1.9).

(1.6) **Irreducibility.**

Assertion. *For any $\alpha,\beta\in R$, there exists a sequence $\alpha_0:=\alpha, \alpha_1, \ldots, \alpha_k:=\beta$ of elements of R s.t.*

i) $I(\alpha_i,\alpha_j)\begin{cases}\neq 0 & \text{for } i=j, j\pm 1\\ =0 & \text{otherwise}\end{cases}$, ii) $k\leq \mu_+ + \mu_-$.

Proof. An existence of a sequence with a property i) is an immediate consequence of the axiom of irreducibility. The property i) implies that $\operatorname{rank}(I(\alpha_i,\alpha_j))_{i,j=0,\cdots,k}\geq k$. This implies ii). q.e.d.

Corollary 1. *Let R be a root system which may not be irreducible. Then R is irreducible iff it has the following property*:

*) *If a linear subspace G of F is invariant under W_R and $G\cap R\neq\phi$, then $G=F$.*

Proof. If R is irreducible and G is a W_R-invariant subspace with $\alpha\in G\cap R$. For any $\beta\in R$, take the sequence $\alpha_0, \ldots, \alpha_k$ of the assertion. Then $(w_{\alpha_k}-1)w_{\alpha_{k-1}}\cdots w_{\alpha_1}\alpha_0\in G$ is a non-zero multiple of β, and therefore $\beta\in G$. Then the axiom 1) implies $G=F$.

Conversely if R satisfies $*$) and $R=R_1 \amalg R_2$ with $R_1 \perp R_2$. Then $\mathbb{R}Q(R_1)$, $\mathbb{R}Q(R_2)$ are W_R-invariant subspaces which are orthogonal to each other. If $R_1 \neq \phi$, then by $*$), $\mathbb{R}Q(R_1)=F$ and hence $\mathbb{R}Q(R_2)$ is contained in the radical rad $I:=\{x \in F: I(x,y)=0$ for ${}^{\forall}y \in F\}$. Hence $I(\alpha,\alpha)=0$ for $\alpha \in R_2$. This implies $R_2=\phi$. q.e.d.

Corollary 2. *Let R be an irreducible root system belonging to I. For a constant $r \in \mathbb{R}$, let us put,*

$$R_r := \{\alpha \in R : I(\alpha,\alpha)=2r\}.$$

If $R_r \neq \phi$, then R_r is a root system belonging to I, which may not be irreducible.

Proof. All the axioms of (1.2) except the axiom 1) are trivially verified. For a proof of the axiom 1), we have only to show that $\mathbb{R}Q(R_r)=F$, since $Q(R_r) \subset Q(R)$. Since I is invariant by W_R, the axiom 3) implies $W_R R_r = R_r$. Hence $\mathbb{R}Q(R_r)$ is a W_R-invariant subspace of F containing $R_r \neq \phi$. Therefore $\mathbb{R}Q(R_r)=F$ due to (1.6) **Corollary 1**.

(1.7) **Rational structure of F.**

Assertion. *There exists a non-zero constant $c \in \mathbb{R}$ such that cI is an integral bilinear form on $Q(R) \times Q(R)$.*

Proof. Let us fix an element $\alpha \in R$ and put $c := I(\alpha,\alpha)^{-1}$. For any $\beta \in R$, take a sequence $\alpha_0, \ldots, \alpha_k$ of the **assertion** (1.6). Then

$$cI(\beta,\beta) = \Big(\prod_{j=1}^{k} I(\alpha_{j-1}^{\vee}, \alpha_j)\Big) / \Big(\prod_{j=1}^{k} I(\alpha_{j-1}, \alpha_j^{\vee})\Big) \in \mathbb{Q}.$$

Thus for any $\beta, \gamma \in R$,

$$cI(\beta,\gamma) = \frac{1}{2} cI(\beta,\beta) I(\beta^{\vee},\gamma) \in \mathbb{Q}.$$

Thus for a suitable integral $n \in \mathbb{N}$, ncI is an integral bilinear form on $Q(R)$.

q.e.d.

Definition. *A linear subspace G of F is said to be defined over $\mathbb{Q}$ if $G \cap Q(R)$ is a full lattice of G.*

Corollary. *Put*

$$\operatorname{rad} I = \{ x \in F : I(x, y) = 0 \text{ for any } y \in F \}.$$

Then $\operatorname{rad} I$ *is defined over* Q.

Proof. Since cI is rational valued on $Q(R)$ for a $c \neq 0$, the system of equations $cI(x, y) = 0$ $(y \in Q(R))$ are rational coefficients. q.e.d.

(1.8) **Quotient Root Systems.** Let G be a linear subspace of $\operatorname{rad} I$ defined over Q and let $p: F \to F/G$ be the linear projection map. The bilinear form on F/G induced from I on F is denoted by I_G so that $I_G(p(x), p(y)) = I(x, y)$ for $x, y \in F$.

Assertion. *Let us denote by* R/G *the image of* R *in* F/G. *Then* R/G *is a root system belonging to* I_G, *such that* $Q(R/G) \simeq Q(R)/(Q(R) \cap G)$.

Proof. The **axioms 2), ..., 5)** are trivially verified for R/G. The **axiom 1)** follows from $Q(R/G) = p(Q(R)) \simeq Q(R)/Q(R) \cap G$. q.e.d.

Definition. *We shall call* R/G *the quotient root system of* R *by* G.

Note. For the study of R, sometimes it is convenient to fix a flag $G_0 = 0 \subset G_1 \subset \cdots \subset G_r = \operatorname{rad} I$ defined over Q, which we shall call a marking of R. (cf. (2.1) **Def. 2.**)

This concept of a marking a root system comes from a study of primitive forms for the period mapping of simple elliptic singularities. (cf. [17] [20]).

(1.9) **Lemma of finiteness of root length.**

The following simple lemma plays an important role.

Lemma. *Let* R *be a root system belonging to* I. *Then there exists a non zero constant* c *such that the set*

$$\{ cI(a, a) : a \in R \} \tag{1.9.1}$$

is a finite set of integers.

Proof. Due to (1.7) **Cor.** and (1.8) **Assertion**, $R/\operatorname{rad} I$ is a root system belonging to $I_{\mathrm{rad}\,I}$ such that

$$\{I(\alpha, \alpha): \alpha \in R\} = \{I_{\mathrm{rad}\,I}(\beta, \beta): \beta \in R/\operatorname{rad} I\}.$$

Therefore without loss of generality, one assumes that I is non-degenerate.

Define, $P^\vee := \{f \in F: I(f, Q(R)) \subset \mathbb{Z}\}$. Since I is non-degenerate, $P^\vee$ is a full lattice of F. Due to (1.7) Assertion, there exists a non-zero constant c such that $cI(Q(R)), Q(R)) \subset \mathbb{Z}$. Therefore $cQ(R) \subset P^\vee$. Since both $cQ(R)$ and $P^\vee$ are full lattices of F, there exists an integer $N \neq 0$ such that $NP^\vee \subset cQ(R)$.

Hence $\quad I(P^\vee, P^\vee) \subset I(P^\vee, \frac{c}{N} Q(R)) = \frac{c}{N} I(P^\vee, Q(R)) \subset \frac{c}{N} \mathbb{Z}$.

Due to the axiom 4), $R^\vee \subset P^\vee$. In particular, $4/I(\alpha, \alpha) = I(\alpha^\vee, \alpha^\vee) \in \frac{c}{N}\mathbb{Z}$ for $\alpha \in R$. This implies $cI(\alpha, \alpha)$ is an integer which divides $4N$. This completes the proof of the finiteness. q.e.d.

Note. A modification of the proof shows that, if $Q(R)$ is an even lattice w.r.t. cI, then $cI(\alpha, \alpha) | 2N^2$.

(1.10) **The total tier number $t(R)$.**

Let R be a root system belonging to I, where cI is an integral bilinear form on $Q(R)$ for a positive constant c.

Due to (1.9) **Lemma**, we define now,

$$t(R) := \frac{l.c.m.\{cI(\alpha, \alpha): \alpha \in R\}}{g.c.d.\{cI(\alpha, \alpha): \alpha \in R\}}. \tag{1.10.1}$$

Let us call $t(R)$ *the total tier number* of R which takes the value in $\mathbb{N}$.

Proof of (1.5) **Lemma.**

Let us fix a root $\alpha \in R$. Then for any β, one computes,

$$\beta^\vee = \frac{2}{I(\beta, \beta)}\beta = \frac{2}{t(R)I(\alpha, \alpha)} \cdot t(R) \frac{cI(\alpha, \alpha)}{cI(\beta, \beta)}\beta \in \frac{2}{t(R)I(\alpha, \alpha)}\mathbb{Z}\beta.$$

This implies, $Q(R^\vee) \subset \frac{2}{t(R)I(\alpha, \alpha)} Q(R)$, so that $Q(R^\vee)$ is a lattice in F. q.e.d.

Note. The above proof shows also the following:

A linear subspace G of F is defined over $\mathbb{Q}$ with resp. to R, iff it is defined over $\mathbb{Q}$ w.r.t. $R^\vee$.

(1.11) **Even lattice structure.**

Let R be a root system belonging to I and c is a constant as in (1.9) Lemma. Put

$$I_R := \frac{2c}{g.c.d.\{cI(\alpha, \alpha):\ \alpha \in R\}} I, \tag{1.11.1}$$

$$I_{R^\vee} := \frac{l.c.m.\{cI(\alpha, \alpha):\ \alpha \in R\}}{2c} I. \tag{1.11.2}$$

Then we have following properties.

i) *$Q(R)$ (resp. $Q(R^\vee)$) is an even lattice w.r.t. I_R (resp. $I_{R^\vee}$).*

ii) *If $Q(R)$ (resp. $Q(R^\vee)$) is an even lattice w.r.t. $c'I$ for a suitable constant $c' \neq 0$, then $c'I$ is an integral multiple of I_R (resp. $I_{R^\vee}$).*

iii) *Formula*

$$I_R \otimes I_{R^\vee} = t(R) I \otimes I. \tag{1.11.3}$$

In particular,

$$\frac{I_R(\alpha, \alpha)}{2} \cdot \frac{I_{R^\vee}(\alpha^\vee, \alpha^\vee)}{2} = t(R) \quad \textit{for} \quad \alpha \in R. \tag{1.11.4}$$

(1.12) **Relative tier number.** Let the notations be as in (1.11). Let G, H are linear subspace of F defined over $\mathbb{Q}$ s.t. $G \supset H$.

We define tier numbers relative to G and H as follows.

$$t_R(G, H) := \# \frac{Q(R) \cap G / Q(R) \cap H}{Q(R^\vee) \cap G / Q(R^\vee) \cap H} \times (I_{R^\vee} : I)^{\operatorname{rank} G/H}, \tag{1.12.1}$$

$$t_{R^\vee}(G, H) := \# \frac{Q(R^\vee) \cap G / Q(R^\vee) \cap H}{Q(R) \cap G / Q(R) \cap H} \times (I_R : I)^{\operatorname{rank} G/H}. \tag{1.12.2}$$

Here in the above expression, $\# \frac{L}{M}$ for two lattices L and M means $|\det A|$ where A is a transformation matrix from a $\mathbb{Z}$-basis of L to a $\mathbb{Z}$-basis of M, and $A : B$ means a constant c s.t. $A = cB$.

Note that if one replace I by dI for a constant $d \neq 0$, then R and I_R are unchanged, but $R^\vee$ and $I_{R^\vee}$ are replaced by $d^{-1}R^\vee$ and $d^2 I_{R^\vee}$ so that the tier numbers in (1.12.1) and (1.12.2) are independent of d.

Assertion. i) *The relative tier numbers are positive integers.*
ii) $t_R(G, H)\cdot t_{R^\vee}(G, H) = t(R)^{\mathrm{rank}\, G/H}$.

Proof. i) Since the tier number does not depend on a constant factor of I, we take I to be $(l.c.m.\{I_R(\alpha, \alpha):\ \alpha\in R\})^{-1}2I_R$. Then we have $I_{R^\vee}=I$ and $\alpha^\vee=\frac{I(\alpha^\vee, \alpha^\vee)}{2}\alpha\in \mathbb{Z}\cdot\alpha\subset Q(R)$. Therefore $Q(R^\vee)\subset Q(R)$. These show that $t_R(G, H)$ is an integer. In a similar manner, one sees that $t_{R^\vee}(G, H)$ is an integer.
ii) Apply the formula (1.11.3). q.e.d.

(1.13) The following is a simple consequence of **the finiteness lemma.**

Assertion. *Let φ be an isomorphism of a root system R belonging to I. Then*

$$I\circ\varphi = \pm I.$$

Proof. Due to (1.4) **Lemma**, there exists a constant $c\neq 0$ $s.t.$ $I=cI\circ\varphi$. If $c\neq\pm 1$, then the set $\{I(\alpha, \alpha):\ \alpha\in R\}$ which is invariant by the multiplication by c is infinite. This contradicts to (1.9) **Lemma**. q.e.d.

(1.14) **Eichler Siegel presentation** E. For a study of the Weyl group W_R, we introduce a map $E:\ F\otimes(F/\mathrm{rad}\, I)\rightarrow \mathrm{End}(F)$ (**Def.** (1.14.1)) and its inverse E^{-1}: $W_R\rightarrow F\otimes(F/\mathrm{rad}\, I)$ (**Def.** (1.14.5)) in this paragraph.

Using E, we give a sufficient condition for R, G so that W_R splits into semi direct product in (1.15) and we study central extensions $\tilde{W}_{R,G}$ of W_R associated to hyperbolic extensions in (1.19).

Definition 1. *Eichler Siegel map E for (F, I) is the following,*

(1.14.1) $$E:\ F\otimes(F/\mathrm{rad}\, I)\longrightarrow \mathrm{End}(F),$$

where $$E(\sum_i f_i\otimes g_i)(u) := u-\sum_i f_i I(g_i, u).$$

2. *We define a semi group structure $\circ$ on $F\otimes(F/\mathrm{rad}\, I)$* by,

(1.14.2) $$\varphi\circ\psi := \varphi+\psi-I(\varphi, \psi),$$

where we use the following convention,

$$I:\ F\otimes(F/\operatorname{rad} I)\times\cdots\times F\otimes(F/\operatorname{rad} I)\to F\otimes(F/\operatorname{rad} I)$$
$$\varphi_1\times\cdots\times\varphi_k\mapsto I(\varphi_1,\cdots\varphi_k),$$
$$I(\varphi_1,\cdots,\varphi_k):=\sum_{i_1,\cdots,i_k} f^1_{i_1}\otimes I(g^1_{i_1}, f^2_{i_2})\cdots I(g^{k-1}_{i_{k-1}}, f^k_{i_k})g^k_{i_k},$$
$$\textit{for } \varphi_j:=\sum_{i_j} f^j_{i_j}\otimes g^j_{i_j}\in F\otimes(F/\operatorname{rad} I)\quad (j=1,\ldots,k).$$

From the definition directly we obtain following assertions.

i) *The map E is injective. It is bijective iff* $\operatorname{rad} I=0$.

ii) *E is a homomorphism of semi groups.*

$$E(\xi\circ\eta)=E(\xi)E(\eta). \tag{1.14.3}$$

iii) *For an non isotropic $\alpha\in F$, the reflection w_α* (1.1.2) *is given by*

$$w_\alpha=E(\alpha\otimes\alpha^\vee). \tag{1.14.4}$$

iv) *The inverse of the Eichler Siegel map on W_R is well defined.*

$$E^{-1}:\ W_R\longrightarrow F\otimes(F/\operatorname{rad} I). \tag{1.14.5}$$

The image of (1.14.5) *is contained in* $Q(R)\underset{\mathbf{Z}}{\otimes}(Q(R^\vee)/Q(R^\vee)\cap\operatorname{rad} I)$.
($\because$ The lattice $Q(R)\underset{\mathbf{Z}}{\otimes}(Q(R^\vee)/Q(R^\vee)\cap\operatorname{rad} I)$ of $F\otimes(F/\operatorname{rad} I)$ is closed under the product $\circ$ and contains $\alpha\otimes\alpha^\vee$ for all $\alpha\in R$. Taking on account of (1.14.4), W_R is contained in the image of the lattice by E.)

v) *The subspace* $\operatorname{rad} I\otimes(F/\operatorname{rad} I)$ *is closed under the product $\circ$, where $\circ$ coincides with the additive structure of the vector space on the subspace. We have*

$$(E(\psi)-id_F)^2=0\quad \textit{for}\quad \psi\in\operatorname{rad} I\otimes(F/\operatorname{rad} I). \tag{1.14.5}$$

For a later use, let us show an assertion.

Assertion. *Suppose $w:=E(\xi)\in O(F,I)$ for a $\xi\in F\otimes(F/\operatorname{rad} I)$. Then,*

$$\xi+{}^t\xi-I(\xi,{}^t\xi)\equiv 0\quad \operatorname{mod}\ \operatorname{rad} I\otimes F+F\otimes\operatorname{rad} I.$$

Proof. The orthogonality of w implies,

$$I(f,g)=I(wf,wg)=I(wf,g)-I(wf,\xi,g) \quad \text{for all } f,g\in F.$$

Hence

$$\begin{aligned}&f-wf+I(wf,\xi)\equiv 0 \quad \text{mod rad } I\\ &=I(\xi,f)+I(wf,\xi)=I(\xi w+{}^t\xi,wf)=I(\xi+{}^t\xi-I(\xi,{}^t\xi),wf)\end{aligned}$$

for all $f\in F$. This implies the assertion. q.e.d.

(1.15) **Splitting of W_R.** As in (1.8) let G be a linear subspace of rad I defined over $\mathbb{Q}$.

Define a lattice of $G\otimes(F/\text{rad } I)$ *by,*

(1.15.1) $$T_G := E^{-1}(W_R)\cap(G\otimes(F/\text{rad } I)).$$

(T_G is a lattice, due to (1.14) Assertion iv), v).)

Assertion. i) *Following is an exact sequence.*

(1.15.2) $$1\longrightarrow T_G \xrightarrow{E} W_R \xrightarrow{p_*} W_{R/G}\longrightarrow 1.$$

ii) *The adjoint action of* W_R *on* T_G *is given by*

(1.15.3) $$wE(\sum_i f_i\otimes g_i)w^{-1} = E(\sum_i f_i\otimes wg_i),$$
$$\textit{for} \quad w\in W_R,\ \sum_i f_i\otimes g_i\in G\otimes(F/\text{rad } I).$$

Proof. i) The following diagram is naturally commutative.

$$\begin{array}{ccccccc} 1\longrightarrow T_G & \xrightarrow{E} & W_R & \xrightarrow{p_*} & W_{R/G} & \longrightarrow 1\\ \cap & & \cap E^{-1} & & \cap E^{-1} & \\ 1\rightarrow G\otimes(F/\text{rad } I) & \longrightarrow & F\otimes(F/\text{rad } I) & \longrightarrow & (F/G)\otimes(F/\text{rad } I) & \rightarrow 1\end{array}$$

where the second line is exact.

ii) Let us show (1.15.3) for $w=E(\alpha\otimes\alpha^\vee)$, $\alpha\in R$.

$$\begin{aligned}E(\sum_i f_i\otimes wg_i)w&=E((\sum_i f_i\otimes(g_i-I(\alpha\otimes\alpha^\vee,g_i)))\circ\alpha\otimes\alpha^\vee)\\&=E(\sum_i f_i\otimes g_i+\alpha\otimes\alpha^\vee)\\&=E(\alpha\otimes\alpha^\vee\circ(\sum_i f_i\otimes g_i))=wE(\sum_i f_i\otimes g_i).\end{aligned}$$

q.e.d.

We state now a lemma giving a condition for (1.15.2) to split.

Lemma. *Let R be a root system belonging to I and let $G \subset \operatorname{rad} I$ be defined over $\mathbf{Q}$. Let L be a linear subspace of F such that*

$$F = L \oplus G \quad (\textit{i.e. } L \textit{ is complementary to } G.), \tag{1.15.4}$$

$$p_* | W_{R\cap L} : W_{R\cap L} \to W_{R/G} \quad \textit{is surjective.} \tag{1.15.5}$$

Then

i) *The homomorphism* (1.15.5) *is an isomorphism. Hence* (1.15.2) *splits into a semi direct product,*

$$W_R = W_{R\cap L} \ltimes T_G. \tag{1.15.6}$$

ii) *T_G is a full lattice of $G \underset{\mathbf{R}}{\otimes} (F/\operatorname{rad} I)$, which is generated by*

$$\alpha_G \otimes \alpha_L^\vee \quad \textit{for} \quad \alpha \in R. \tag{1.15.7}$$

Here $\alpha = \alpha_L + \alpha_G$ is the splitting (1.15.4) *for an $\alpha \in F$ and $\alpha_L^\vee$ means the element $(\alpha_L)^\vee = (\alpha^\vee)_L$ of L.*

Proof. i) The subgroup $E^{-1} W_{R\cap L}$ is contained in $L \otimes (F/\operatorname{rad} I)$, since it is generated (as a semi-group) by $\alpha \otimes \alpha^\vee \in L \otimes (F/\operatorname{rad} I)$ for $\alpha \in R \cap L$ and $L \otimes (F/\operatorname{rad} I)$ is closed under the product $\circ$.
Therefore,

$$E^{-1} W_{R\cap L} \cap T_G \subset (L \otimes (F/\operatorname{rad} I)) \cap (G \otimes (F/\operatorname{rad} I)) = 0.$$

Hence $W_{R\cap L} \cap E(T_G) = \{1\}$, which implies injectivity of (1.15.5) due to (1.15.2).
ii) Let $\alpha = \alpha_L + \alpha_G$ be the decomposition of (1.15.4) of $\alpha \in R$. Then in $F \otimes (F/\operatorname{rad} I)$, one computes as follows.

$$\alpha \otimes \alpha^\vee = (\alpha_L + \alpha_G) \otimes (\alpha_L)^\vee = \alpha_L \otimes \alpha_L^\vee + \alpha_G \otimes \alpha_L^\vee = (\alpha_L \otimes \alpha_L^\vee) \circ (\alpha_G \otimes \alpha_L^\vee).$$

Therefore we obtain a decomposition (1.15.6) of the reflection,

$$w_\alpha = E(\alpha_L \otimes \alpha_L^\vee) E(\alpha_G \otimes \alpha_L^\vee) \quad \text{for} \quad E(\alpha_L \otimes \alpha_L^\vee) \in W_{R\cap L}, \quad \alpha_G \otimes \alpha_L^\vee \in T_G. \tag{1.15.8}$$

Thus T_G is generated by $\alpha_G \otimes \alpha_L^\vee$ for $\alpha \in R$.

To show that T_G is a full lattice of $G \otimes (F/\operatorname{rad} I)$, it is enough to show that T_G

spans the whole vector space since T_G is already contained in a lattice $(Q(R)\cap G)\otimes(Q(R)/(Q(R)\cap \mathrm{rad}\, I))$.

Due to (1.15.3) the space $\mathbb{R}T_G$ is invariant under the action of $W_R/T_G \simeq W_{R/G}$. Therefore due to (1.6) Cor. 1 and (1.8) Assertion, $\mathbb{R}T_G$ contains $\alpha_G\otimes(F/\mathrm{rad}\, I)$ for any $\alpha\in R$. Since G is defined over $\mathbb{Q}$, G is spanned by α_G for $\alpha\in R$. This implies $\mathbb{R}T_G = G\otimes(F/\mathrm{rad}\, I)$. q.e.d.

Note. Under the same assumption of the lemma, *we have,*

i) *$R\cap L$ is a root system belonging to $I|L$.*

ii) *The linear isomorphism $p|_L: L\simeq F/G$ induces an injection map $R\cap L\to R/G$ among two root systems, which induces an isomorphism $W_{R\cap L}\simeq W_{R/G}$.*

Proof. i) We have only to show Axiom 1 and 5 for $R\cap L$. Since $R\cap L$ is invariant under the action of $W_{R\cap L}$, the image of $\mathbb{R}Q(R\cap L))$ in F/G by p is invariant under the action of $W_{R/G}\simeq p_* W_{R\cap L}$ which contains $p(R\cap L)\subset R/G$. Thus due to (1.6) Cor. 1 and (1.8) Assertion, $p|_L(\mathbb{R}Q(R\cap L))=F/G$. Hence $Q(R\cap L)$ spans L.

If $R\cap L$ were reducible, there exists a linear subspace $H\subsetneqq L$, which is invariant under the action of $W_{R\cap L}$ containing an element of $R\cap L$. Then $pH\subsetneqq F/G$ is invariant under $W_{R/G}$ containing an element of $p(R\cap L)\subset R/G$. This is a contradiction to the fact that R/G is an irreducible root system. q.e.d.

(1.16) **Counting** $K_G(\alpha)$. Let R be a root system belonging to I, and G be a subspace of rad I defined over $\mathbb{Q}$. To give a tentative description of the "extension" of the root system R/G to the root system R, we introduce K_G, which associates a subset $K_G(\alpha)$ of the lattice $Q(R)\cap G$ for each root $\alpha\in R$ as follows,

$$K_G(\alpha):=\{x\in G:\ \alpha+x\in R\}.$$

We shall call the $K_G(\alpha)$, *the counting set of* $\alpha\in R$.

Assertion 1. i) $0\in K_G(\alpha)$ and $K_G(\alpha)=-K_G(\alpha)$ *for* $\alpha\in R$.

ii) *$K_G(\varphi\alpha)=K_G(\alpha)$ for an automorphism φ of R and $\alpha\in R$.*

iii) *$K_G(\alpha)$ is closed under the reflexion centered at each point of $K_G(\alpha)$. (i.e. If $x,y\in K_G(\alpha)$, then $2x-y\in K_G(\alpha)$.)*

iv) *$K_G(\alpha)$ is closed under the translation by $I(\alpha,\beta^\vee)K_G(\beta)$ for $\alpha,\beta\in R$. (i.e. $K_G(\alpha)\supset K_G(\alpha)+I(\alpha,\beta^\vee)K_G(\beta)$.)*

v) $\frac{2}{I(\alpha, \alpha)}K_G(\alpha)=K_G^{\vee}(\alpha^{\vee})$, where $K_G^{\vee}(\alpha^{\vee}):=G\cap\{R^{\vee}-\alpha^{\vee}\}$.

Proof. i) is a consequence of (1.2) *Note* 1, i).
ii) and v) are trivial by definition.
iii) If α, $\alpha+x$, $\alpha+y\in R$, then $w_{\alpha+x}(\alpha+y)=-\alpha+y-2x\in R$.
iv) If α, β, $\alpha+x$, $\beta+y\in R$, then $w_{\beta+y}(\alpha+x)=w_{\beta}(\alpha)+x-I(\alpha, \beta^{\vee})y\in R$.

q.e.d.

Assertion 2. *Suppose that there exists a linear subspace L of F satisfying the conditions* (1.15.4), (1.15.5) *of* (1.15) **Lemma** *such that $R\cap L\rightarrow R/G$ is surjective. Then*

i) $K_G(\alpha)$ *contains a full lattice of* G *for* $\alpha\in R$.
ii) $R=\coprod_{\alpha\in R\cap L}\{\alpha+K_G(\alpha)\}$.
iii) $\bigcup_{\alpha\in R\cap L}K_G(\alpha)$ *generates the lattice* $Q(R)\cap G$.

Proof. By assumptions $R\subset(R\cap L)\oplus G$, so that we get the disjoint union presentation of R as ii). iii) is an immediate consequence of ii). i) is a consequence iii) taking in account the facts **Assertion 1.** iv) and the irreducibility of R. q.e.d.

(1.17) **Hyperbolic extension.** For the description of the central extension $\tilde{W}_{R,G}$ in (1.17.1), we prepare some linear algebraic assertions. Proofs are elementary and omitted.

Assertion 1. *Let F be a vector space over $\mathbb{R}$ of finite rank equipped with a symmetric bilinear form I. For a given linear subspace $G\subset\operatorname{rad}I$, there exists a triple $(\tilde{F}_G, \tilde{I}_G, \iota_G)$ of a vector space $\tilde{F}_G$ of rank*$=\operatorname{rank}F+\operatorname{rank}(\operatorname{rad}I/G)$, *a symmetric bilinear form $\tilde{I}_G$ on $\tilde{F}_G$ and an injective linear map $\iota_G: F\rightarrow\tilde{F}_G$, s.t.*

i) $I=\tilde{I}_G\circ\iota_G$,
ii) $\operatorname{rad}\tilde{I}_G=\iota_G(G)$.

Definition. *We call $(\tilde{F}_G, \tilde{I}_G, \iota_G)$ a hyperbolic extension of (F, I) w.r.t. G. So far there is no confusion, we identify F with the subspace $\iota_G(F)$ of $\tilde{F}_G$ and G with the radical of $\tilde{I}_G$.*

Assertion 2. *Hyperbolic extension is unique up to an isomorphism in the* *following sense. Let* G, H *be subspaces of* rad I *s.t.* $H \subset G$ *and let* $(\tilde{F}_G, \tilde{I}_G, \iota_G)$, $(\tilde{F}_H, \tilde{I}_H, \iota_H)$ *be hyperbolic extensions w.r.t. them. Then there exists an injective linear map* $\varphi: \tilde{F}_G \rightarrow \tilde{F}_H$ *such that*

i) $\tilde{I}_G = \tilde{I}_H \circ \varphi$,

ii) $\varphi \circ \iota_G = \iota_H$.

Assertion 3. *Let* $(\tilde{F}_G, \tilde{I}_G, \iota_G)$ *be a hyperbolic extension w.r.t.* G. *Then the automorphism group is given by,*

$$\mathrm{Aut}(\tilde{F}_G, \tilde{I}_G, \iota_G) = E_G(M_G), \tag{1.17.2}$$

where

$$E_G: \tilde{F}_G \otimes (\tilde{F}_G/G) \longrightarrow End(\tilde{F}_G),$$

is the Eichler Siegel map for $\tilde{F}_G$ *and* $\tilde{I}_G$ (1.14.1), *and*

$$\begin{array}{rcl} M_G := \ker(\mathrm{rad}\, I \otimes (\mathrm{rad}\, I/G) & \rightarrow & S^2(\mathrm{rad}\, I/G)) \\ \Downarrow & & \Downarrow \\ \xi & \longmapsto & \frac{1}{2}(\xi + {}^t\xi) \bmod G. \end{array} \tag{1.17.3}$$

(Here $S^2(V)$ is a symmetric tensor product of V.)
The rank of M_G *is given by,*

$$\mathrm{rank}(M_G) = \frac{1}{2}\mathrm{rank}(\mathrm{rad}\, I/G)(\mathrm{rank}(\mathrm{rad}\, I) + \mathrm{rank}\ G - 1). \tag{1.17.4}$$

The product $\circ$ *structure on* M_G *coincides with the addition structure on* M_G *as a linear space.*

(1.18) **The extension** $\tilde{W}_{R,G}$. Let R be a root system belonging to I and G be a subspace of rad I. We fix a hyperbolic extension $(\tilde{F}_G, \tilde{I}_G, \iota_G)$ and regard $\iota_G: F \rightarrow \tilde{F}_G$ as an inclusion map. Therefore as an element in $\tilde{F}_G$ each $\alpha \in R$ defines a reflexion of $\tilde{F}_G$ denoted by $\tilde{w}_\alpha = E_G(\alpha \otimes \alpha^\vee)$.
Put,

$$\tilde{W}_{R,G} := \text{the subgroup of } O(\tilde{F}_G, \tilde{I}_G) \text{ generated by } \tilde{w}_\alpha\ (\alpha \in R). \tag{1.18.1}$$

Lemma. *The group* $\tilde{W}_{R,G}$ *is a central extension of* W_R.

$$1 \longrightarrow \tilde{K}_G \xrightarrow{E_G} \tilde{W}_{R,G} \xrightarrow{p_*} W_R \longrightarrow 1, \tag{1.18.2}$$

where

i) p_* *is a surjective homomorphism induced from the restriction of the action of* $\tilde{W}_{R,G}$ *on* $\tilde{F}_G$ *to the subspace* F.

ii) $\tilde{K}_G$ *is a lattice of* M_G ((1.17.3)) *defined by*

$$\tilde{K}_G := E_G^{-1}(\tilde{W}_{R,G}) \cap M_G. \tag{1.18.3}$$

where E_G *is the Eichler Siegel map for* $\tilde{F}_G$ *and* $\tilde{I}_G$ *in* (1.17.2).

Proof. Since $\tilde{W}_{R,G}$ is generated by $E_G(\alpha\otimes\alpha^\vee)$ for $\alpha\otimes\alpha^\vee\in F\otimes(F/G)$ ($\alpha\in R$) and $F\otimes(F/G)$ is closed under the product $\circ$, the group $\tilde{W}_{R,G}$ is contained in $E_G(F\otimes(F/G))$. Hence the inverse Eichler Siegel map is well defined as an injective map

$$E_G^{-1}: \tilde{W}_{R,G} \longrightarrow F\otimes(F/G). \tag{1.18.4}$$

The action of $E_G(F\otimes(F/G))$ on $\tilde{F}_G$ leaves the subspace F invariant so that the restriction p_* is well defined to make the diagram commutative:

$$\begin{array}{ccc} \tilde{W}_{R,G} & \xrightarrow{p_*} & W_R \\ \cap E_G^{-1} & & \cap E^{-1} \\ F\otimes(F/G) & \rightarrow & F\otimes(F/\operatorname{rad} I). \end{array}$$

Therefore by putting $\tilde{K}_G := E_G^{-1}(\tilde{W}_{R,G})\cap(F\otimes(\operatorname{rad} I/G))$, one obtains the exact sequence (1.18.2).

Since $M_G\subset F\otimes(\operatorname{rad} I/G)$, it is enough to show the inclusion relation $\tilde{K}_G\subset M_G$ to show (1.17.2).

Let us apply (1.14) **Assertion** for $\tilde{F}_G$, $\tilde{I}_G$ and $\xi\in\tilde{K}_G$, since $\tilde{w} := E_G(\xi)\in\tilde{W}_{R,G}$ $\subset O(\tilde{F}_G, \tilde{I}_G)$. Hence

$$\xi + {}^t\xi \equiv I(\xi, {}^t\xi) \quad \operatorname{mod} \quad \tilde{F}_G\otimes G + G\otimes\tilde{F}_G.$$

Since $\xi\in F\otimes(\operatorname{rad} I/G)$ we have $I(\xi, {}^t\xi)=0$. Therefore

*) $\xi + {}^t\xi \equiv 0 \operatorname{mod} \tilde{F}_G\otimes G + G\otimes\tilde{F}_G$.

Again noting $\xi\in F\otimes(\operatorname{rad} I/G)$, *) implies

**) $\xi \in \operatorname{rad} I\otimes(\operatorname{rad} I/G)$.

Then *) and **) implies $\xi\in M_G$.

$\tilde{K}_G$ is discrete in M_G, since it is contained in a lattice $Q(R)\otimes_{\mathbf{Z}}(Q(R^\vee)/Q(R^\vee)\cap G)$. q.e.d.

(1.19) **Components of the Eichler-Siegel map E.** The sequence (1.18.2) does not split. For a more precise study of the sequence, we introduce in (1.20) a mapping, $r: \tilde{W}_{R,G} \to M_G$, which coincides with E_G^{-1} on the center $E_G(\tilde{K}_G)$. For a preparation to r, let us define some notations.

The definition of the r depends on a decomposition,

$$F = L \oplus H \oplus G \tag{1.19.1}$$

where L is a subspace of F complementary to rad$I$ and H is a subspace of rad I complementary to G.

Fixing one such decomposition, we introduce mappings,

$$\xi: \tilde{W}_{R,G} \longrightarrow L \otimes L, \tag{1.19.2}$$

$$p: \tilde{W}_{R,G} \longrightarrow H \otimes L, \tag{1.19.3}$$

$$q: \tilde{W}_{R,G} \longrightarrow G \otimes L, \tag{1.19.4}$$

as components of the inverse Eichler Siegel map (1.14.5),

$$E^{-1}(p_*(g)) = \xi(g) + p(g) + q(g) \quad \text{for} \quad g \in \tilde{W}_{R,G}, \tag{1.19.5}$$

where we use the following identification,

$$F \otimes (F/\operatorname{rad} I) \simeq (L \oplus H \oplus G) \otimes L = (L \otimes L) \oplus (H \otimes L) \oplus (G \otimes L).$$

The orthogonality of g implies the condition (cf. (1.18) **Assertion**),

$$\xi(g) + {}^t\xi(g) = I(\xi(g), {}^t\xi(g)) \quad \text{for} \quad g \in \tilde{W}_{R,G}. \tag{1.19.6}$$

Let us reformulate the relation (1.14.3) into some formulae for components of E.

$$\xi(g_1 g_2) = \xi(g_1) + \xi(g_2) - I(\xi(g_1), \xi(g_2)), \tag{1.19.7}$$

$$p(g_1 g_2) = p(g_1) E_0({}^t\xi(g_2)) + p(g_2), \tag{1.19.8}$$

$$q(g_1 g_2) = q(g_1) E_0({}^t\xi(g_2)) + q(g_2), \tag{1.19.9}$$

where

$$E_0: L \otimes L \longrightarrow \operatorname{End}(L),$$
$$E_0(\sum_i f_i \otimes g_i)(u) = u - \sum_i f_i I(g_i, u),$$

the Eichler Siegel map for L and $I|L$.

As a corollary to (1.19.7), we have,

$$(1.19.10)\qquad {}^t\xi(g) = \xi(g^{-1}) \quad \text{for} \quad g \in \tilde{W}_{R,G}.$$

($\because$ If (1.19.10) is true for g_1, g_2, (1.19.7) implies that it is also true for g_1g_2. On the other hand (1.19.10) is obviously true for reflexions $g := \tilde{w}_\alpha$.)

(1.20) **The map** r. We introduce the map r (1.20.1) and summarize its properties in the following **Assertions** 1—3.

Assertion 1. *There exists a map,*

$$(1.20.1)\qquad r:\ \tilde{W}_{R,G} \longrightarrow M_G$$

s.t. the inverse Eichler Siegel map (1.18.4) *is given by,*

$$(1.20.2)\qquad \begin{aligned} E_G^{-1}(g) = {} & \xi(g)+p(g)+q(g)-E_0(\xi(g))\,{}^tp(g) \\ & +\frac{1}{2}I(p(g), {}^tp(g))+r(g) \quad \text{for} \quad g \in \tilde{W}_{R,G}. \end{aligned}$$

Here the terms in the right hand corresponds to the following decomposition.

$$(1.20.3)\qquad \begin{aligned} & F\otimes(F/G) \simeq (L\otimes L)\oplus(H\otimes L)\oplus(G\otimes L)\oplus(L\otimes H)\oplus S^2(H)\oplus M_G, \\ & M_G \simeq A^2(H)\oplus G\otimes H, \end{aligned}$$

where $H\otimes H = S^2(H)\oplus A^2(H)$ is the direct sum decomposition of $H\otimes H$ into the symmetric and the anti-symmetric tensors.

Assertion 2. *The multiplicative law for* r *is given by,*

$$(1.20.4)\qquad \begin{aligned} r(g_1g_2) = {} & r(g_1)+r(g_2)-A(I(p(g_1), E_0(\xi(g_2))^tp(g_2))) \\ & -I(q(g_1), E_0(\xi(g_2))^tp(g_2)) \quad \textit{for} \quad g_1, g_2 \in \tilde{W}_{R,G}. \end{aligned}$$

Here $A(\eta) := \frac{1}{2}(\eta - {}^t\eta)$ *is the anti-symmetric part of* $\eta \in H\otimes H$.

Assertion 3. *The ranges of* ξ, p, q, r *are followings*

$$(1.20.5)\qquad \xi(\tilde{W}_{R,G}) \subset \sum_{\alpha,\beta\in R} \mathbf{Z}\,\mathrm{int}(\alpha, \beta)\alpha_L\otimes\beta_L,$$

$$(1.20.6)\qquad p(\tilde{W}_{R,G}) \subset \sum_{\alpha,\beta\in R} \mathbf{Z}\,\mathrm{int}(\alpha, \beta)\alpha_H\otimes\beta_L,$$

$$(1.20.7)\qquad q(\tilde{W}_{R,G}) \subset \sum_{\alpha,\beta\in R} \mathbf{Z}\,\mathrm{int}(\alpha, \beta)\alpha_G\otimes\beta_L,$$

$$(1.20.8)\qquad r(\tilde{W}_{R,G}) \subset \sum_{\alpha,\beta\in R} \mathbf{Z}\,\mathrm{int}(\alpha, \beta)A(\alpha_H\otimes\beta_H)$$

$$+\sum_{\alpha,\beta\in R}\mathbb{Z}\,\mathrm{int}(\alpha,\beta)\alpha_G\otimes\beta_H.$$

Here we denote by,

(1.20.9) $$\alpha=\alpha_L+\alpha_H+\alpha_G,$$

the decomposition (1.19.1) *for a root* $\alpha\in R$, *and by* $\mathrm{int}(\alpha,\beta)$ *for* $\alpha,\beta\in R$ *a positive real number defined by*

(1.20.10) $$\mathrm{int}(\alpha,\beta):=\begin{cases}2/I(\alpha,\alpha) & \text{for } \alpha=\beta\\ g.c.d.\left\{\dfrac{2^k\prod_{j=1}^{k-1}I(\alpha_j,\alpha_{j+1})}{\prod_{j=1}^{k}I(\alpha_j,\alpha_j)} : \begin{array}{l}\alpha_1,\ldots,\alpha_k\in R\\ s.t.\ \alpha_1=\alpha \ \text{ and } \ \alpha_k=\beta\end{array}\right\} & \end{cases}.$$

(Note that $\mathrm{int}(\alpha,\beta)=\mathrm{int}(\beta,\alpha)$ and $\dfrac{I(\alpha,\alpha)}{2}\mathrm{int}(\alpha,\beta)\in\mathbb{Z}$ for $\alpha,\beta\in R$.)

Proof for Assertions 1—3.

1. Let

$$E_G^{-1}(g)=\xi(g)+p(g)+q(g)+u(g)+v(g)+r(g),$$

be the decomposition of $E_G^{-1}(g)$ according to the splitting (1.20.3). Apply the criterium for the orthogonality (1.14) Assertion for this. Namely

$$\Xi+{}^t\Xi-I(\Xi,{}^t\Xi)\equiv0 \mod G\otimes F+F\otimes G \quad\text{for}\quad \Xi=E_G^{-1}(g).$$

An explicit calculation of this relation gives

$$u(g)=-{}^tp(g)+I(\xi(g),{}^tp(g)) \text{ and}$$
$$2v(g)=I(p(g),{}^tp(g)).$$

2. Apply (1.20.2) to the relation $E_G^{-1}(g_1g_2)=E_G^{-1}(g_1)\circ E_G^{-1}(g_2)$. An explicit calculation of this gives (1.20.4).

3. If $\alpha\in R$ is decomposed as (1.20.9), then

$$\alpha\otimes\alpha^\vee=\alpha_L\otimes\alpha_L^\vee+\alpha_H\otimes\alpha_L^\vee+\alpha_G\otimes\alpha_L^\vee+\alpha_L^\vee\otimes\alpha_H+\frac{2}{I(\alpha,\alpha)}\alpha_H\otimes\alpha_H+\frac{2}{I(\alpha,\alpha)}\alpha_G\otimes\alpha_H$$

in $F\otimes(F/G)$, so that we have

$$\xi(\tilde{w}_\alpha) = \alpha_L \otimes \alpha_L^\vee = \frac{2}{I(\alpha,\alpha)}\alpha_L \otimes \alpha_L,$$
$$p(\tilde{w}_\alpha) = \alpha_H \otimes \alpha_L^\vee = \frac{2}{I(\alpha,\alpha)}\alpha_H \otimes \alpha_L,$$
$$q(\tilde{w}_\alpha) = \alpha_G \otimes \alpha_L^\vee = \frac{2}{I(\alpha,\alpha)}\alpha_G \otimes \alpha_L,$$
$$r(\tilde{w}_\alpha) = \frac{2}{I(\alpha,\alpha)}\alpha_G \otimes \alpha_H.$$

Therefore for the reflexions $\tilde{w}_\alpha$ $(\alpha \in R)$, ξ, p, q, r takes value in the right hand of the formula of the assertion. Let us denote by M_ξ, M_p, M_q and by M_r the modules in the right hand of the formulae (1.20.5)$-$(1.20.8) respectively.

What is enough to show is that if for $g_1, g_2 \in \tilde{W}_{R,G}$ the mappings ξ, p, q, r take values in M_ξ, M_p, M_q, and in M_r respectively then so is also for $g_1 g_2$.

In view of formulae (1.19.7) (1.19.8) (1.19.9) and (1.20.4), it is sufficient to show,

$$I(M_\xi, M_\xi) \subset M_\xi, \quad M_p E_0(M_\xi) \subset M_p, \quad M_q E_0(M_\xi) \subset M_q, \quad \text{and}$$
$$AI(M_p, {}^tM_p) \subset M_r, \quad I(M_q, {}^tM_p) \subset M_r.$$

All these relations are reduced to a relation,

$$\text{(1.20.11)} \qquad \mathrm{int}(\alpha,\beta) I(\beta,\gamma) \mathrm{int}(\gamma,\delta) \in \mathbb{Z}\,\mathrm{int}(\alpha,\delta), \quad \text{for} \quad \alpha,\beta,\gamma,\delta \in R.$$

This is a direct consequence of the definition of $\mathrm{int}(\alpha,\delta)$ by noting the fact $I(\beta_L, \gamma_L) = I(\beta,\gamma)$ for $\beta, \gamma \in R$. q.e.d.

One can sharpen the **Assertion 3** by a slite modification of the proof as follows.

Assertion 3′. *The ranges of $p+q$, r are followings*

$$(p+q)(\tilde{W}_{R,G}) \subset \sum_{\alpha,\beta \in R} \mathbb{Z}\,\mathrm{int}(\alpha,\beta)(\alpha_H + \alpha_G) \otimes \beta_L,$$
$$r(\tilde{W}_{R,G}) \subset \sum_{\alpha,\beta \in R} \mathbb{Z}\,\mathrm{int}(\alpha,\beta)(A(\alpha_H \otimes \beta_H) + \alpha_G \otimes \beta_H).$$

§ 2. Marked Extended Affine Root System (R, G)

(2.1) Let us recall the definition of an extended affine root system ((1.3) **Ex. 1** iii)).

Definition 1. *A root system R belonging to a symmetric bilinear form I, whose* sign *is* $(l, 2, 0)$, *is called an extended affine root system of rank l.*

Namely ; I is a positive semi-definite bilinear form on a real vector space F

of rank $l+2$, *such that rank of* $\mathrm{rad} I := \{x \in F : I(x, y) = 0 \ {}^{\forall} y \in F\}$ *is equal two. R is a subset of F satisfying the* axioms 1), ..., 5) *for a root system belonging to I in* (1.2) Def.

For short we shall use an abbreviation e.a.r.s. for an extended affine root system.

2. *A marking G for an e.a.r.s. R is a linear subspace of* rad I *of* rank 1 defined over $\mathbb{Q}$ (*cf*. (1.8)). *The pair* (R, G) *is called a marked extended affine root system belonging to I or a m.e.a.r.s. for short.*

3. *Two e.a.r.s.'s* $R \subset F$ *and* $R' \subset F'$ *are said to be isomorphic, if there exists a linear isomorphism* $\varphi: F \simeq F'$ *s.t.* $\varphi R = R'$ ((1.2) Def. 2.). *Furthermore let G and G' be markings for R and R' respectively, and the map* φ *induces* $\varphi G = G'$. *Then we say that the two m.e.a.r.s.'s are isomorphic.*

$$\varphi: (R, G) \simeq (R', G').$$

Note 1. *If* φ *is an isomorphism between e.a.r.s.'s belonging to I and I', then there exists a positive constant* $c \in \mathbb{R}^+$ *s.t.* $I = cI' \circ \varphi$ (cf. (1.4) Lemma).

Note 2. The same e.a.r.s. can split into non isomorphic m.e.a.r.s.'s, by different choice of markings. In fact there are at most two isomorph classes of m.e.a.r.s.'s for the same e.a.r.s. (cf. (5.4) Appendix).

(2.2) **The dual** $(R^\vee, G)$. Let (R, G) be a m.e.a.r.s. belonging to I. Then $R^\vee := \{\alpha^\vee \in F : \alpha \in R\}$ is also an e.a.r.s. belonging to I (cf. (1.5) Lemma) and the same space G defines a marking for $R^\vee$ (cf. (1.10) *Note*). We call the pair $(R^\vee, G)$ *the dual m.e.a.r.s.* of (R, G).

(2.3) $\mathbb{Z}$**-basis** a, b **of** rad I. Let (R, G) be a m.e.a.r.s.. Recall that we denote by $Q(R)$ the lattice in F generated by R. ((1.2) Def. 1.1))

Then $Q(R) \cap \mathrm{rad}\, I$ is a full lattice of rad I, which has rank 2 (cf. (1.7) Cor.). We choose a $\mathbb{Z}$-basis a, b of the module in the following way,

(2.3.1) $$Q(R) \cap G^1 = \mathbb{Z}a,$$

(2.3.2) $$Q(R) \cap G^2 = \mathbb{Z}a + \mathbb{Z}b,$$

where

(2.3.3) $$G^2 := \operatorname{rad} I, \quad G^1 := G.$$

The ambiguity of such basis (a, b) is described by a group $\left\{\begin{bmatrix} \pm 1 & * \\ 0 & \pm 1 \end{bmatrix} : * \in \mathbb{Z}\right\}$.

§ 3. Quotient Root Systems of (R, G)

(3.1) Let (R, G) be a m.e.a.r.s. Put $G^1 := G$, $G^2 := \operatorname{rad} I$ as in (2.3). Denote by R/G^i the image set of R in F/G^i by the projection $p_i : F \to F/G^i$ and by I_{G^i} the metric on F/G^i induced from I, $i = 1, 2$.

Recall that we denote by W_B the group generated by reflexions w_a for $a \in B$ for a subset B in F (cf. (1.1.7)).

Lemma 1. i) *$R/\operatorname{rad} I := R/G^2$ is a finite root system belonging to $I_{\operatorname{rad} I}$.*
ii) *$R/G := R/G^1$ is an affine root system belonging to I_G.*

2. *The following is an exact sequence.*

(3.1.1) $$1 \longrightarrow T_{G^i} \xrightarrow{E} W_R \xrightarrow{p_{i*}} W_{R/G^i} \to 1, \quad \text{for} \quad i = 1, 2,$$

where

i) *p_{i_*} is a homomorphism induced naturally from the projection p_i.*

ii) $E : F \otimes (F/\operatorname{rad} I) \to \operatorname{End}(F)$, $E(\sum_i f_i \otimes g_i)(u) := u - \sum_i f_i I(g_i, u)$,
is the Eichler Siegel presentation for F and I, defined in (1.14).

iii) $T_{G_i} := E^{-1}(W_R) \cap (G^i \otimes F/\operatorname{rad} I)$,

3. i) *The sequence* (3.1.1) *splits into a semi-direct product.*

ii) *T_{G^i} is a full lattice of $G^i \otimes (F/\operatorname{rad} I)$ of rank il.*

Definition. *$R/\operatorname{rad} I$ and R/G are called the quotient finite root system and the quotient affine root system of (R, G) respectively.*

Proof 1. Due to (1.8) Assertion, R/G^i, $i = 1, 2$ are root systems. The examples (1.3) 1. i) and ii) show that they are either a finite root system or an affine root system respectively.

2. The exactness of (3.1.1) is shown in (1.15) Assertion.

3. Due to the **Lemma** of (1.15), it is enough to show the existences of linear subspaces L^l and L^{l+1} of F such that

i) *L^{l+2-i} is a complementary subspace of G^i in F.*

$$F = L^{l+2-i} \oplus G^i \quad \textit{for} \quad i=1, 2. \tag{3.1.2}$$

ii) *The following homomorphisms are surjective.*

$$p_{i*} | W_{R\cap L^{l+2-i}} : W_{R\cap L^{l+2-i}} \longrightarrow W_{R/G^i} \quad \textit{for} \quad i=1, 2. \tag{3.1.3}$$

Since the decomposition (3.1.2) and the splitting of (3.1.1) play an important role in the later study, we explain details of the construction of L^l and L^{l+1} in the next (3.2) and (3.3).

(3.2) **Subspace** L^l. *Let $\beta_1, \dots, \beta_l \in R/\mathrm{rad}\, I$ be a basis of the finite root system. For each $1 \leq i \leq l$, choose an element $\alpha_i \in R \cap p_2^{-1}(\beta_i)$ where $p_2 : F \to F/\mathrm{rad}\, I$ is the linear projection. Define,*

$$L^l := \bigoplus_{i=1}^{l} \mathbb{R}\alpha_i. \tag{3.2.1}$$

The fact that L^l satisfies (3.1.2), (3.1.3) for $i=2$ follows directly from the facts,

i) any root $\beta \in R/\mathrm{rad}\, I$ is a linear combination of $\beta_1, \dots, \beta_l$ with integral coefficients which are either all≥ 0 or all≤ 0,

ii) the reflexions w_{β_i}, $i=1, \dots, l$ generates $W_{R/\mathrm{rad}\, I}$.

Let us summarize direct consequences of this splitting. (cf. (1.15) **Lemma**)

Assertion. i) *$Q(R)$ splits over $\mathbb{Z}$ as follows,*

$$Q(R) = (Q(R) \cap L^l) \oplus (Q(R) \cap \mathrm{rad} I) \tag{3.2.2}$$

and

$$Q(R) \cap L^l = \bigoplus_{i=1}^{l} \mathbb{Z}\alpha_i. \tag{3.2.3}$$

ii) *The subgroup $W_{R\cap L^l}$ of W_R is generated by $w_{\alpha_1}, \dots, w_{\alpha_l}$ and is isomorphic to $W_{R/\mathrm{rad}\, I}$ so that*

$$W_R = W_{R\cap L^l} \ltimes T_{\mathrm{rad}\, I} \tag{3.2.4}$$

iii) *For any $\alpha \in R$, the reflexion w_α is decomposed as,*

$$w_\alpha = w_{\alpha_{L^l}} \cdot E(\alpha_{\mathrm{rad}\, I} \otimes \alpha_{L^l}^\vee), \quad \textit{for} \quad w_{\alpha_{L^l}} \in W_{R\cap L^l},\ \alpha_{\mathrm{rad}\, I} \otimes \alpha_{L^l}^\vee \in T_{\mathrm{rad}\, I},$$

where $\alpha=\alpha_{L^l}+\alpha_{\mathrm{rad}\,I}$ *is the splitting* (3.1.2) *and* $\alpha_{L^l}^{\vee}:=(\alpha_{L^l})^{\vee}=(\alpha^{\vee})_{L^l}=2I(\alpha,\alpha)^{-1}\alpha_{L^l}$.

iv) *The lattice* $T_{\mathrm{rad}\,I}$ *is generated by* $\alpha_{\mathrm{rad}\,I}\otimes\alpha_{L^l}^{\vee}$ *for* $\alpha\in R$.

Note 1. *If* $R/\mathrm{rad}\,I$ *is reduced, then* $p_2: F\to F/\mathrm{rad}\,I$ *induces an isomorphism,*

$$p_2:\ R\cap L^l\simeq R/\mathrm{rad}\,I.$$

$$(\because\quad R/\mathrm{rad}\,I=\bigcup_{i=1}^{l}W_{R/\mathrm{rad}\,I}\ \beta_i\overset{p_2}{\simeq}\bigcup_{i=1}^{l}W_{R\cap L^l}\alpha_i\subset R\cap L^l\subset R/\mathrm{rad}\,I.)$$

Note 2. *The group* W_R *acts nowhere properly on* F *and* F^*. ($\because$ The subgroup $E(T_{\mathrm{rad}\,I})$ is a free abelian group of rank$=2l>$rank $F=l+2$ for $l>2$, which cannot act properly anywhere on F and F^*. Remember that $\mathrm{rad}\,I$ is pointwisely fixed by W_R.)

(3.3) **Subspace** L^{l+1}. Let us recall the concept of basis $\beta_0,\dots,\beta_l$ for the affine root system R/G from [14] 4.

First, fix a sign of a generator $\overline{b}$ of $\mathrm{rad}\,I_G=\mathrm{rad}\,I/G$ and put,

$$E^l := \{x\in(F/G)^*:\ \bar{b}(x)=1\}. \tag{3.3.1}$$

The set E^l is an affine space of rank l, whose translation group is $(F/\mathrm{rad}\,I)^*$. F/G is identified with the vector space of affine linear functions on E^l. The contragradient action of $W_{R/G}$ on E^l is proper and the set of regular points of the action $E^l-\bigcup_{\beta\in R/G}H_\beta$. (Here $H_\beta:=\{x\in E^l:\ \beta(x)=0\}$) decomposes into open connected components, called chambers. $W_{R/G}$ acts faithfully and transitively on the set of chambers.

A basis $\{\beta_0,\dots,\beta_l\}$ of R/G is by definition a set of indivisible roots $\beta\in R/G$, such that $H_{\beta_0},\dots,H_{\beta_l}$ form the set of walls of a chamber C and $\beta_i(x)>0$ for all $x\in C$, $i=0,\dots,l$.

Let $\beta_0,\dots,\beta_l$ *be a basis for* R/G *and* $p_1: F\to F/G$ *be the projection. For each* $0\le i\le l$ *choose an element* $\alpha_i\in R\cap p_1^{-1}(\beta_i)$ *and we define,*

$$L^{l+1} := \bigoplus_{i=0}^{l}\mathbb{R}\alpha_i\,. \tag{3.3.2}$$

The fact that L^{l+1} satisfies (3.1.2), (3.1.3) for $i=1$, follows from the following properties i), ii) of the basis $\beta_0, \ldots, \beta_l$.

i) Any root $\beta \in R/G$ is a linear combination of $\beta_0, \ldots, \beta_l$ with integral coefficients which are either all≥ 0 or all≤ 0.

ii) The group $W_{R/G}$ is generated by w_{β_i}, $i=0, \ldots, l$.

iii) There exists positive integers $n_0, \ldots, n_l$ with gcd $(n_0, \ldots, n_l)=1$, such that

$$\sum_{i=0}^{l} n_i\beta_i \tag{3.3.3}$$

is a positive generator, say $\bar{b}$, of the integral constant functions $Q(R/G)\cap \mathrm{rad}\, I/G$ on E (cf. [14](6.7)).

Let us summarize direct consequences of this splitting (cf. (1.15)).

Assertion. i) *$Q(R)$ splits over $\mathbb{Z}$ as follows.*

$$\begin{aligned} Q(R) &= (Q(R)\cap L^{l+1})\oplus(Q(R)\cap G), \\ &= (\bigoplus_{i=0}^{l}\mathbb{Z}\alpha_i)\oplus\mathbb{Z}a. \end{aligned} \tag{3.3.4}$$

Put

$$b := \sum_{i=0}^{l} n_i\alpha_i. \tag{3.3.6}$$

Then

$$\begin{aligned} Q(R)\cap \mathrm{rad}\, I &= \mathbb{Z}b\oplus\mathbb{Z}a, \\ Q(R)\cap L^{l+1}\cap \mathrm{rad}\, I &= \mathbb{Z}b. \end{aligned}$$

ii) *The subgroup $W_{R\cap L^{l+1}}$ of W_R is generated by $w_{\alpha_0}, \ldots, w_{\alpha_l}$ and is isomorphic to $W_{R/G}$, so that*

$$W_R = W_{R\cap L^{l+1}} \ltimes T_G. \tag{3.3.7}$$

Note 1. In the above choice of $\alpha_0, \ldots, \alpha_l$, if necessary after a change of the ordering of them, one can always assume that $p_2(\alpha_1), \ldots, p_2(\alpha_l)$ is a basis for $R/\mathrm{rad}\, I$, and hence $L^l := \bigoplus_{i=1}^{l}\mathbb{R}\alpha_i$ is a splitting factor (3.2.1). Then n_0 of (3.3.3) is equal to 1 (cf. [14] (6.6)). Therefore,

$$\begin{aligned} \bigoplus_{i=0}^{l}\mathbb{Z}\alpha_i &= \left(\bigoplus_{i=1}^{l}\mathbb{Z}\alpha_i\right)\oplus\mathbb{Z}b, \\ Q(R) &= \left(\bigoplus_{i=1}^{l}\mathbb{Z}\alpha_i\right)\oplus\mathbb{Z}b\oplus\mathbb{Z}a. \end{aligned}$$

Note 2. If the affine root system R/G is reduced, then the projection $p_1 : F \to F/G$ induces an isomorphism,

$$p_1 : R \cap L^{l+1} \simeq R/G.$$

$$(\because \quad R/G = \bigcup_{i=0}^{l} W_{R/G}\, \beta_i \overset{p_1}{\simeq} \bigcup_{i=0}^{l} W_{R\cap L^{l+1}}\, \alpha_i \subset R \cap L^{l+1} \overset{p_1}{\subset} R/G.)$$

(3.4) **Definition.** *A set* $\{\alpha_0, \ldots, \alpha_l\}$ *of roots of* R *is called a basis for* (R, G), *if their image in* R/G *form basis for the affine root system* R/G.

We shall show the following assertion in (6.2).

Assertion. *Let* (R, G) *be a m.e.a.r.s. such that* R/G *is reduced. Let* $\{\alpha_0, \ldots, \alpha_l\}$ *and* $\{\beta_0, \ldots, \beta_l\}$ *be basis for* (R, G). *Then there exists an automorphism* φ *of* (R, G) *such that*

$$\{\varphi(\alpha_0), \ldots, \varphi(\alpha_l)\} = \{\beta_0, \ldots, \beta_l\}.$$

§ 4. Tier Numbers $t(R)$, $t_1(R, G)$, $t_2(R, G)$

For a m.e.a.r.s. (R, G), we introduce some numerical invariants $t(R)$, $t_1(R, G)$, $t_2(R, G)$ which we shall call tier numbers.

(4.1) **The total tier number.**

Definition. *Let* R *be an e.a.r.s. belonging to* I. *The total tier number of* R *is*,

$$t(R) := \max\{I(\alpha, \alpha)/I(\beta, \beta) : \alpha, \beta \in R\}.$$

Obviously this number depends only on the finite root system $R/\mathrm{rad}\, I$ as follows.

type of $R/\mathrm{rad}I$	A_l	B_l	C_l	BC_l	D_l	E_l	F_4	G_2
$t(R)$	1	2	2	4	1	1	2	3

(4.2) **Even lattice structure on** $Q(R)$. Since $R/\mathrm{rad}\, I$ is a finite root system, by suitable choice of positive constants c_1, c_2, the metrics $I_R := c_1 I$, $I_{R^\vee} := c_2 I$ have the following properties (cf. (1.11.1) (1.11.2) and (1.11.4)).

i) $Q(R)$ (*resp.* $Q(R^\vee)$) *is an even lattice w.r.t.* I_R (*resp.* $I_{R^\vee}$).

ii) $\inf\{I_R(\alpha, \alpha): \alpha\in R\} = \inf\{I_{R^\vee}(\alpha^\vee, \alpha^\vee): \alpha^\vee\in R^\vee\} = 2.$

iii) $t(R) = \frac{I_R(\alpha, \alpha)}{2} \cdot \frac{I_{R^\vee}(\alpha^\vee, \alpha^\vee)}{2}$ *for* $\alpha\in R.$

(4.3) **Relative tier numbers.** Let (R, G) be a m.e.a.r.s. Let a, b (resp. $a^\vee, b^\vee$) be a basis for $Q(R)\cap \operatorname{rad} I$ (resp. $Q(R^\vee)\cap \operatorname{rad} I$) as in (2.3).

Definition. *The first and the second tier numbers for* (R, G) *and* $(R^\vee, G)$ *are,*

$$\begin{aligned} t_1(R, G) &:= |(b^\vee \bmod a^\vee): (b \bmod a)|\times(I_{R^\vee}: I), \\ t_1(R^\vee, G) &:= |(b \bmod a): (b^\vee \bmod a^\vee)|\times(I_R: I), \\ t_2(R, G) &:= |(a^\vee: a)|\times(I_{R^\vee}: I), \\ t_2(R^\vee, G) &:= |(a: a^\vee)|\times(I_R: I). \end{aligned}$$

Here the notion $A: B$ *means the constant number* c *s.t.* $A=cB$.

The definition of the tier numbers above is invariant by a change of I by a constant multiple of I. A more general combinatorial definition of them is given in (1.12.1), (1.12.2).

(4.4) **Assertion.** i) *Tier numbers are positive integers.*

ii)
$$t(R) = t_1(R, G)\cdot t_1(R^\vee, G),$$
$$t(R) = t_2(R, G)\cdot t_2(R^\vee, G).$$

Proof (cf. **Assertion** (1.12)). i) Take I to be I_R. Since $\alpha=\frac{I_R(\alpha, \alpha)}{2}\alpha^\vee\in \mathbb{Z}\alpha^\vee\subset Q(R^\vee)$, we have $Q(R)\subset Q(R^\vee)$ and therefore $Q(R)\cap G^i\subset Q(R^\vee)\cap G^i$ for $i=1, 2$. Thus $a: a^\vee$, $b \bmod a: b^\vee \bmod a^\vee$ are integers. This implies $t_1(R^\vee, G)$, $t_2(R^\vee, G)$ are integers. Other cases are shown similarly by taking $I=I_{R^\vee}$.
ii) By definition, the right hand is equal to $(I_R: I)\cdot(I_{R^\vee}: I)$. Taking I to be equal to I_R and using (1.1.4) this number is equal to $(I_{R^\vee}(\alpha^\vee, \alpha^\vee)/2)\times(I_R(\alpha, \alpha)/2)$ for an $\alpha\in R$, which is equal to $t(R)$ by (4.2) iii). q.e.d.

(4.5) **The first tier number.** By definition, the first tier number $t_1(R, G)$ depends only on the affine root system R/G. If R/G is reduced, it is calculated as follows.

Let $\beta_0, \ldots, \beta_l$ (resp. $\beta_0^\vee, \ldots, \beta_l^\vee$) be a basis of the affine root system R/G (resp. $R^\vee/G$), so that $\bar{b}=\sum_{i=0}^{l} n_i\beta_i$ (resp. $\bar{b}^\vee=\sum_{i=0}^{l} n_i^\vee\beta_i^\vee$) is the generator (3.3.3) of $Q(R/G)\cap(\operatorname{rad} I/G)$. Then $\bar{b}:\bar{b}^\vee = 2n_i/n_i^\vee I(\alpha_i^\vee, \alpha_i^\vee)$ for $i=0, \ldots, l$ and therefore, (4.5.1) $$t_1(R, G) = \frac{n_i^\vee}{n_i} \frac{I_{R^\vee}(\alpha_i^\vee, \alpha_i^\vee)}{2}, \quad (i=0, \ldots, l).$$

The following is a table of the first tier number.

1. Reduced quotient affine root system case.

Type of R/G (according to [14])	A_l	B_l	$B_l^\vee$	C_l	$C_l^\vee$	BC_l	D_l	E_l	F_4	$F_4^\vee$	G_2	$G_2^\vee$
$t_1(R, G)$	1	1	2	1	2	2	1	1	1	2	1	3

2. Non reduced quotient affine root system case.

Type of R/G (according to [14])	BCC_l	$C^\vee BC_l$	$BB_l^\vee$	$C^\vee C_l$
$t_1(R, G)$	1	4	2	2

(4.6) *Note*. If R/G is reduced, then R/G is uniquely determined by the type of the quotient finite root system $R/\operatorname{rad} I$ and the tier number $t_1(R, G)$, as we see in the first table above.

§ 5. Classification of Marked Extended Affine Root Systems

In this paragraph (5.2), we present a complete list of isomorphism classes of marked extended affine root systems (R, G), which satisfy an assumption,

A) *the quotient affine root system R/G is reduced.*

For each isomorphism class of a m.e.a.r.s. with this assumption, we associate *the type of the isomorphism class in* (5.1). For each type of a m. e. a. r. s., we shall exhibit in the table of (5.2):

1) The first and the second tier numbers $t_1(R, G)$ and $t_2(R, G)$ (cf. (4.3)). The type of the isomorphism class of dual $(R^\vee, G)$ (cf. (2.2)).
2) The set R of roots in a vector space F with a metric I and a marking G (cf. *Notation* below) such that (R, G) is a m.e.a.r.s. of the type of the table.
3) A basis, $\alpha_0, \ldots, \alpha_l$ for (R, G) (cf. (3.4)).
4) The countings, $k(\alpha_0), \ldots, k(\alpha_l)$ (cf. (6.1), (6.4)). The exponents $m_{\alpha_0}, \ldots, m_{\alpha_l}$ (cf. (7.1.1)) w. r. t. the basis.
5) The Dynkin diagram $\Gamma_{R,G}$ (cf. (8.2)) of the type.

$\#|\Gamma|=l(R)+\mathrm{cod}(R, G)+1$ and $\mathrm{cod}(R, G)$: $=$codimension of (R, G) (cf. (8.1)).

Notation. Let $\varepsilon_0, \varepsilon_1, \varepsilon_2, \cdots$ be a sequence of the orthogonal vectors in a real Hilbert space H. The ambient space F and the marking G for a root system in the following table are defined as linear subspaces of $\mathbb{R}a \oplus \mathbb{R}b \oplus H$ by,

$$F := \mathbb{R}a \oplus \bigoplus_{i=0}^{l} \mathbb{R}\alpha_i = \mathbb{R}a \oplus \mathbb{R}b \oplus \bigoplus_{i=1}^{l} \mathbb{R}\alpha_i$$
$$G := \mathbb{R}a,$$

where $\alpha_0, \cdots, \alpha_l$ are the basis given in 3). The metric I on F is induced from that of H by regarding $\mathbb{R}a \oplus \mathbb{R}b$ as the radical.

(5.1) **Definition of the type for a marked extended affine root system.**

Using a classification of m.e.a.r.s.'s in § 6, we define the types for a m.e.a.r.s. in the following manner.

Let (P_l, t_1, t_2) be a triple of a type P_l of a finite root system of rank l and two positive integers t_1 and t_2. A m.e.a.r.s. (R, G) is said to belong to (P_l, t_1, t_2) if P_l is the type of $R/\mathrm{rad}\, I$ and $t_i = t_i(R, G)$, $i=1,2$.

i) Type $P^{(t_1,t_2)}$

Let t_1, t_2 be divisors of the total tier number $t(P_l)$ of a finite root system P_l (cf. (4.1)), and (P_l, t_1, t_2) corresponds to a reduced affine root system (cf. *Note* 1.).

Then there exists a unique isomorphism class of m.e.a.r.s.'s which belong to (P_l, t_1, t_2) except the following 4 cases; $(A_1, 1, 1)$, $(B_l, 2, 2)$ $l\geq 2$, $(C_l, 1, 1)$ $l\geq 2$, $(BC_l, 2, 2)$ $l\geq 1$. Except for these cases we shall call $P_l^{(t_1,t_2)}$ to be the type of the isomorphism class.

ii) Type $P^{(t_1,t_2)*}$

Let (P_l, t_1, t_2) be one of $(A_1, 1, 1)$, $(B_l, 2, 2)$ $l\geq 2$ or $(C_l, 1, 1)$ $l\geq 2$. Then there exist two isomorphism classes of m.e.a.r.s.'s which belong to (P_l, t_1, t_2). In this case, one isomorphism class is called to be of type $P_l^{(t_1,t_2)}$ and the other to be of type $P_l^{(t_1,t_2)*}$.

iii) Types $BC_l^{(2,2)}(1)$ and $BC_l^{(2,2)}(2)$

If (P_l, t_1, t_2) is $(BC_l, 2,2)$ for $l\geq 1$, then there are two isomorphism classes of m.e.a.r.s.'s which belong to $(BC_l, 2, 2)$. Then one isomorphism class is called to be of type $BC_l^{(2,2)}(1)$ and the other to be of type $BC_l^{(2,2)}(2)$.

iv) **Exceptional types.**

Let us call the above $A_1^{(1,1)*}$, $B_l^{(2,2)*}$ $(l\geq 2)$, $C_l^{(1,1)*}$ $(l\geq 2)$, $BC_l^{(2,2)}(1)(l\geq 2)$ and $BC_l^{(2,2)}(2)$ $(l\geq 1)$, exceptional types of marked extended affine root systems.

Note 1. The condition A discludes the cases (BC_l, t_1, t_2) such that $t_1=1$ or 4.

Note 2. The treatments and the studies of exceptional types in this and the following paragraphs are rather case by case study using the explicit description of the set of roots in (5.3). After introducing a concept of (mean) foldings of Dynkin diagrams in paragraph 12, we shall see that the exceptional types form naturally a group (cf. (12.5) **Hierarchy**).

Note 3. If P is a type for a m.e.a.r.s, let us denote also by $P^\vee$ the type for the dual m.e.a.r.s.. Then as a consequence of the classification, we see easily;

$$(P_l^{(t_1,t_2)})^\vee = P_l^{\vee(t/t_1,t/t_2)} \quad \text{for } t=t(P),$$
$$(P_l^{(t_1,t_2)*})^\vee = P_l^{\vee(t/t_1,t/t_2)*} \quad \text{for } t=t(P).$$

Hereafter we shall use a convention that if a statement on a m.e.a.r.s. (R, G) depends only on the isomorphism class of (R, G), then we use the type P instead of (R, G) in the statement.

(5.2) **Extended Affine Root Systems with Markings.**

Type $A_l^{(1,1)}$ $(l\geq 1)$

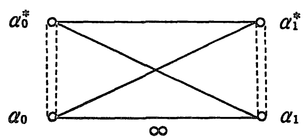
1) $t_1(A_l^{(1,1)})=1,\quad t_2(A_l^{(1,1)})=1,\quad (A_l^{(1,1)})^\vee=A_l^{(1,1)}$.

2) $R:\ \pm(\varepsilon_i-\varepsilon_j)+nb+ma\quad (0\leq i<j\leq l)\quad (n, m\in\mathbb{Z})$.

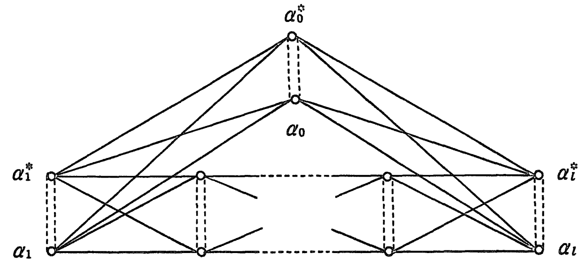
3) $\alpha_0=-\varepsilon_0+\varepsilon_l+b,\quad \alpha_i=\varepsilon_{i-1}-\varepsilon_i\ (1\leq i\leq l)$.

4) $k_i=1\ (0\leq i\leq l)$,
$m_i=1\ (0\leq i\leq l)$.

5) $\mathrm{cod}(A_l^{(1,1)})=l+1,\quad \#\Gamma(A_l^{(1,1)})=2l+2$.



($l=1$)



($l\geq 2$)

Type $A_1^{(1,1)*}$

1) $t_1(A_1^{(1,1)*})=1,\quad t_2(A_1^{(1,1)*})=1,\quad (A_1^{(1,1)*})^\vee=A_1^{(1,1)*}$

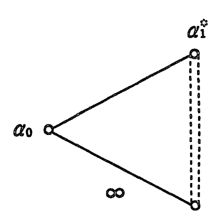
2) $R:\ \pm\varepsilon+nb+ma\quad (n, m\in\mathbb{Z}$ s.t. $n\cdot m\equiv 0 \bmod 2)$.

3) $\alpha_0=-\varepsilon+b,\quad \alpha_1=\varepsilon.$

4) $k_0=2,\quad k_1=1,$

$m_0=\dfrac{1}{2},\quad m_1=1.$

5) $\mathrm{cod}(A_1^{(1,1)*})=1,\ \#\Gamma(A_1^{(1,1)*})=3.$



Type $B_l^{(1,1)}\quad (l\geq 3)$

1) $t_1(B_l^{(1,1)})=1,\quad t_2(B_l^{(1,1)})=1,\quad (B_l^{(1,1)})^\vee=C_l^{(2,2)}.$

2) $R:\ \pm\varepsilon_i+nb+ma\quad (1\leq i\leq l)\ (n, m\in\mathbb{Z}),$

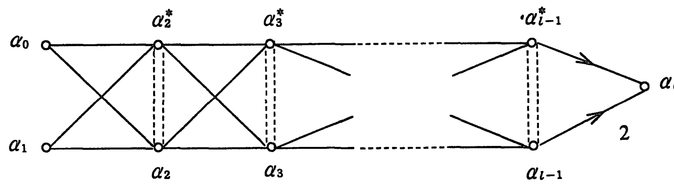
$\pm\varepsilon_i\pm\varepsilon_j+nb+ma\ (1\leq i<j\leq l)\ (n, m\in\mathbb{Z}).$

3) $\alpha_0=-\varepsilon_1-\varepsilon_2+b,\quad \alpha_i=\varepsilon_i-\varepsilon_{i+1}\quad (1\leq i\leq l-1),\quad \alpha_l=\varepsilon_l.$

4) $k_i=1\ (0\leq i\leq l),$

$m_0=2,\quad m_1=2,\quad m_i=4\ (2\leq i\leq l-1),\quad m_l=2.$

5) $\mathrm{cod}(B_l^{(1,1)})=l-2,\quad \#\Gamma(B_l^{(1,1)})=2l-1.$



Type $B_l^{(1,2)}$ $(l\geq 2)$

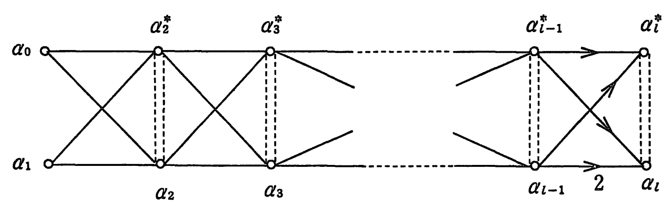
1) $t_1(B_l^{(1,2)})=1,\quad t_2(B_l^{(1,2)})=2,\quad (B_l^{(1,2)})^\vee=C_l^{(2,1)}.$

2) $R:\ \pm\varepsilon_i+nb+ma\ (1\leq i\leq l)\,(n,\ m\in\boldsymbol{Z}),$
$\quad\pm\varepsilon_i\pm\varepsilon_j+nb+2ma\ (1\leq i<j\leq l)\,(n,\ m\in\boldsymbol{Z}).$

3) $\alpha_0=-\varepsilon_1-\varepsilon_2+b,\quad \alpha_i=\varepsilon_i-\varepsilon_{i+1}\,(1\leq i\leq l-1),\quad \alpha_l=\varepsilon_l.$

4) $k_i=2\ (0\leq i\leq l-1),\quad k_l=1,$
$m_0=1,\quad m_1=1,\quad m_i=2\ (2\leq i\leq l).$

5) $\mathrm{cod}(B_l^{(1,2)})=l-1,\quad \#\Gamma(B_l^{(1,2)})=2l.$



Type $B_l^{(2,1)}$ $(l\geq 2)$

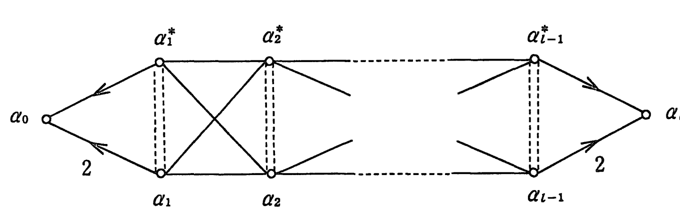
1) $t_1(B_l^{(2,1)})=2,\quad t_2(B_l^{(2,1)})=1,\quad (B_l^{(2,1)})^\vee=C_l^{(1,2)}.$

2) $R:\ \pm\varepsilon_i+nb+ma\quad (1\leq i\leq l)\ (n,\ m\in\boldsymbol{Z}),$
$\quad\pm\varepsilon_i\pm\varepsilon_j+2nb+ma\ (1\leq i<j\leq l)\,(n,\ m\in\boldsymbol{Z}).$

3) $\alpha_0=-\varepsilon_1+b,\quad \alpha_i=\varepsilon_i-\varepsilon_{i+1}\quad (1\leq i\leq l-1),\quad \alpha_l=\varepsilon_l.$

4) $k_i=1\ (0\leq i\leq l),$
$m_0=1,\quad m_i=2\ (1\leq i\leq l-1),\quad m_l=1.$

5) $\mathrm{cod}(B_l^{(2,1)})=l-1,\quad \#\,\Gamma(B_l^{(2,1)})=2l.$



Type $B_l^{(2,2)}$ $(l\geq 2)$

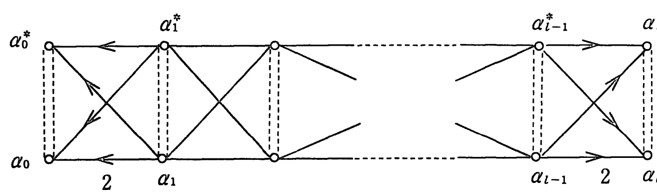
1) $t_1(B_l^{(2,2)})=2,\quad t_2(B_l^{(2,2)})=2,\quad (B_l^{(2,2)})^\vee=C_l^{(1,1)}$.

2) $R:\ \pm\varepsilon_i+nb+ma\ (1\leq i\leq l)\ (n,m\in\mathbb{Z})$,
$\pm\varepsilon_i\pm\varepsilon_j+2nb+2ma\ (1\leq i<j\leq l)\ (n,m\in\mathbb{Z})$.

3) $\alpha_0=-\varepsilon_1+b,\quad \alpha_i=\varepsilon_i-\varepsilon_{i+1}\ (1\leq i\leq l-1),\quad \alpha_l=\varepsilon_l$.

4) $k_0=1,\quad k_i=2\ (1\leq i\leq l-1),\quad k_l=1$,
$m_i=1\ (0\leq i\leq l)$.

5) $\mathrm{cod}(B_l^{(2,2)})=l+1,\quad \#\Gamma(B_l^{(2,2)})=2l+2$.



Type $C_l^{(1,1)}$ $(l\geq 2)$

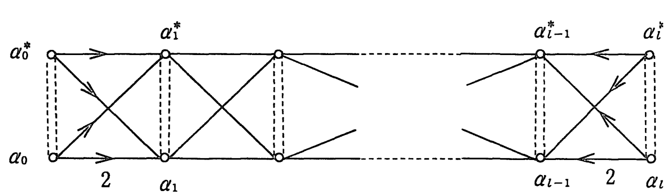
1) $t_1(C_l^{(1,1)})=1,\quad t_2(C_l^{(1,1)})=1,\quad (C_l^{(1,1)})^\vee=B_l^{(2,2)}$.

2) $R:\ \pm 2\varepsilon_i+nb+ma\ (1\leq i\leq l)\ (n,m\in\mathbb{Z})$.
$\pm\varepsilon_i\pm\varepsilon_j+nb+ma\ (1\leq i<j\leq l)\ (n,m\in\mathbb{Z})$.

3) $\alpha_0=-2\varepsilon_1+b,\quad \alpha_i=\varepsilon_i-\varepsilon_{i+1}\ (1\leq i\leq l-1),\quad \alpha_l=2\varepsilon_l$.

4) $k_i=1\ (0\leq i\leq l)$,
$m_i=2\ (0\leq i\leq l)$.

5) $\mathrm{cod}(C_l^{(1,1)})=l+1,\quad \#\Gamma(C_l^{(1,1)})=2l+2$.



Type $C_l^{(1,2)}$ $(l\geq 2)$

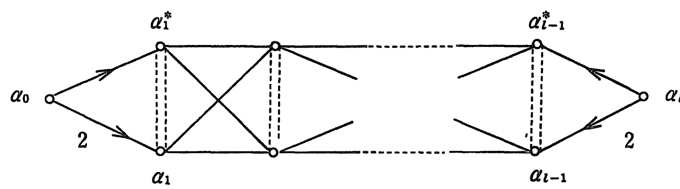
1) $t_1(C_l^{(1,2)})=1,\quad t_2(C_l^{(1,2)})=2,\quad (C_l^{(1,2)})^\vee=B_l^{(2,1)}$.

2) $R:\ \pm 2\varepsilon_i+nb+2ma\ (1\leq i\leq l)\ (n,m\in\mathbb{Z})$,
$\pm\varepsilon_i\pm\varepsilon_j+nb+ma\ (1\leq i<j\leq l)\ (n,m\in\mathbb{Z})$.

3) $\alpha_0=-2\varepsilon_1+b,\quad \alpha_i=\varepsilon_i-\varepsilon_{i+1}\ (1\leq i\leq l-1),\quad \alpha_l=2\varepsilon_l.$

4) $k_0=2,\quad k_i=1\ (1\leq i\leq l-1),\quad k_l=2,$
$m_0=1,\quad m_i=2\ (1\leq i\leq l-1),\quad m_l=1.$

5) $\mathrm{cod}(C_l^{(1,2)})=l-1,\quad \#\Gamma(C_l^{(1,2)})=2l.$



Type $C_l^{(2,1)}\quad (l\geq 2)$

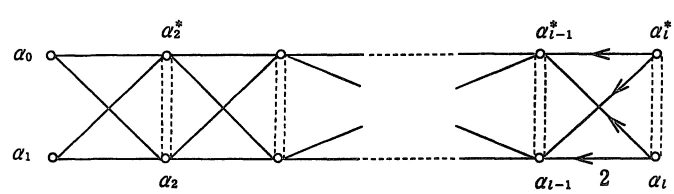
1) $t_1(C_l^{(2,1)})=2,\quad t_2(C_l^{(2,1)})=1,\quad (C_l^{(2,1)})^\vee=B_l^{(1,2)}.$

2) $R:\ \pm 2\varepsilon_i+2nb+ma\ (1\leq i\leq l)\ (n, m\in \mathbf{Z}),$
$\pm\varepsilon_i\pm\varepsilon_j+nb+ma\ (1\leq i<j\leq l)\ (n, m\in\mathbf{Z}).$

3) $\alpha_0=-\varepsilon_1-\varepsilon_2+b,\quad \alpha_i=\varepsilon_i-\varepsilon_{i+1}\ (1\leq i\leq l-1),\quad \alpha_l=2\varepsilon_l.$

4) $k_i=1\ (0\leq i\leq l),$
$m_0=1,\quad m_1=1,\quad m_i=2\ (2\leq i\leq l).$

5) $\mathrm{cod}(C_l^{(2,1)})=l-1,\quad \#\Gamma(C_l^{(2,1)})=2l.$



Type $C_l^{(2,2)}\quad (l\geq 3)$

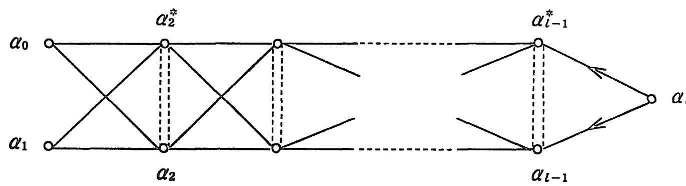
1) $t_1(C_l^{(2,2)})=2,\quad t_2(C_l^{(2,2)})=2,\quad (C_l^{(2,2)})^\vee=B_l^{(1,1)}.$

2) $R:\ \pm 2\varepsilon_i+2nb+2ma\ (1\leq i\leq l)\ (n, m\in\mathbf{Z}),$
$\pm\varepsilon_i\pm\varepsilon_j+nb+ma\ (1\leq i<j\leq l)\ (n, m\in\mathbf{Z}).$

3) $\alpha_0=-\varepsilon_1-\varepsilon_2+b,\quad \alpha_i=\varepsilon_i-\varepsilon_{i+1}\ (1\leq i\leq l-1),\quad \alpha_l=2\varepsilon_l.$

4) $k_i=1\ (0\leq i\leq l-1),\quad k_l=2,$
$m_0=1, m_1=1,\quad m_i=2\ (2\leq i\leq l-1),\quad m_l=1.$

5) $\mathrm{cod}(C_l^{(2,2)})=l-2,\quad \#\Gamma(C_l^{(2,2)})=2l-1.$



Type $B_l^{(2,2)*}$ $(l\geq 2)$

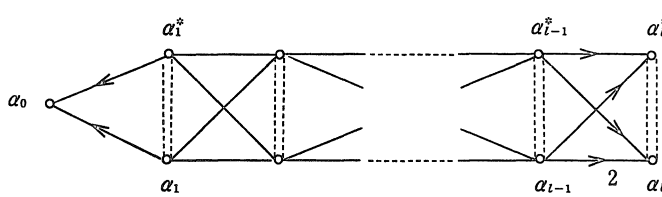
1) $t_1(B_l^{(2,2)*})=2,\quad t_2(B_l^{(2,2)*})=2,\quad (B_l^{(2,2)*})^\vee=C_l^{(1,1)*}.$

2) $R:\ \pm\varepsilon_i+nb+ma\ (1\leq i\leq l)\ (n, m\in\mathbb{Z}$ s.t. $nm\equiv 0 \bmod 2),$
$\pm\varepsilon_i\pm\varepsilon_j+2nb+2ma\ (1\leq i<j\leq l)\ (n, m\in\mathbb{Z}).$

3) $\alpha_0=-\varepsilon_1+b,\quad \alpha_i=\varepsilon_i-\varepsilon_{i+1}\quad (1\leq i\leq l-1),\quad \alpha_l=\varepsilon_l,$

4) $k_i=2\ (0\leq i\leq l-1),\quad k_l=1,$
$m_0=\frac{1}{2},\ m_i=1\ (1\leq i\leq l).$

5) $\mathrm{cod}(B_l^{(2,2)*})=l,\quad \#\Gamma(B_l^{(2,2)*})=2l+1,$



Type $C_l^{(1,1)*}$ $(l\geq 2)$

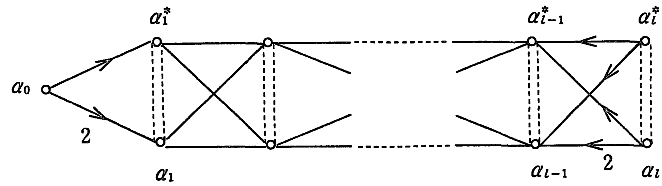
1) $t_1(C_l^{(1,1)*})=1,\quad t_2(C_l^{(1,1)*})=1,\quad (C_l^{(1,1)*})=B_l^{(2,2)*}.$

2) $R:\ \pm 2\varepsilon_i+nb+ma\ (1\leq i\leq l)\ (n, m\in\mathbb{Z}$ s.t. $nm\equiv 0 \bmod 2),$
$\pm\varepsilon_i\pm\varepsilon_j+nb+ma\ (1\leq i<j\leq l)\ (n, m\in\mathbb{Z}).$

3) $\alpha_0=-2\varepsilon_1+b,\quad \alpha_i=\varepsilon_i-\varepsilon_{i+1}\ (1\leq i\leq l-1),\quad \alpha_l=2\varepsilon_l.$

4) $k_0=2,\quad k_i=1\ (1\leq i\leq l),$
$m_0=1,\quad m_i=2\ (1\leq i\leq l).$

5) $\mathrm{cod}(C_l^{(1,1)*})=l,\quad \#\Gamma(C_l^{(1,1)*})=2l+1.$



Type $BC_l^{(2,1)}$ $(l \geq 1)$

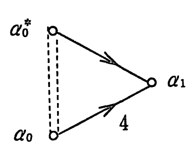
1) $t_1(BC_l^{(2,1)})=2, \quad t_2(BC_l^{(2,1)})=1, \quad (BC_l^{(2,1)})^\vee=BC_l^{(2,4)}.$

2) $R: \pm\varepsilon_i+nb+ma \ (1\leq i\leq l) \ (n,m\in\boldsymbol{Z}),$
$\pm 2\varepsilon_i+(2n+1)b+ma \ (1\leq i\leq l) \ (n,m\in\boldsymbol{Z}),$
$\pm\varepsilon_i\pm\varepsilon_j+nb+ma \ (1\leq i<j\leq l) \ (n,m\in\boldsymbol{Z}).$

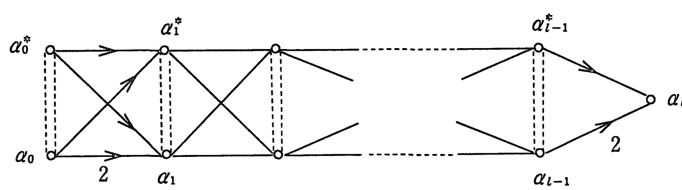
3) $\alpha_0=-2\varepsilon_1+b, \quad \alpha_i=\varepsilon_i-\varepsilon_{i+1} \ (1\leq i\leq l-1), \quad \alpha_l=\varepsilon_l.$

4) $k_i=1 \ (0\leq i\leq l),$
$m_i=4 \ (0\leq i\leq l-1), \quad m_l=2.$

5) $\mathrm{cod}(BC_l^{(2,1)})=l, \quad \#\Gamma(BC_l^{(2,1)})=2l+1.$



$(l=1)$



$(l\geq 2)$

Type $BC_l^{(2,4)}$ $(l \geq 1)$

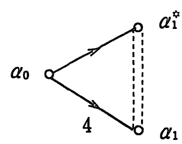
1) $t_1(BC_l^{(2,4)})=2, \quad t_2(BC_l^{(2,4)})=4, \quad (BC_l^{(2,4)})^\vee=BC_l^{(2,1)}.$

2) $R: \pm\varepsilon_i+nb+ma \ (1\leq i\leq l) \ (n,m\in\boldsymbol{Z}),$
$\pm 2\varepsilon_i+(2n+1)b+4ma \ (1\leq i\leq l) \ (n,m\in\boldsymbol{Z}),$
$\pm\varepsilon_i\pm\varepsilon_j+nb+2ma \ (1\leq i<j\leq l) \ (n,m\in\boldsymbol{Z}).$

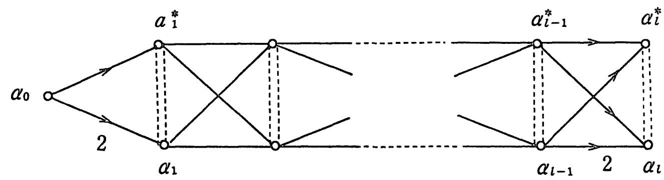
3) $\alpha_0=-2\varepsilon_1+b, \quad \alpha_i=\varepsilon_i-\varepsilon_{i+1} \ (1\leq i\leq l-1), \quad \alpha_l=\varepsilon_l.$

4) $k_0=4, \quad k_i=2 \ (1\leq i\leq l-1), \quad k_l=1,$
$m_0=1, \quad m_i=2 \ (1\leq i\leq l).$

5) $\mathrm{cod}(BC_l^{(2,4)})=l, \quad \#\Gamma(BC_l^{(2,4)})=2l+1.$



$(l=1)$



$(l\geq 2)$

Type $BC_l^{(2,2)}(1)$ $(l\geq 2)$

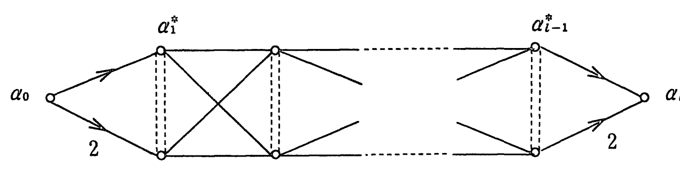
1) $t_1(BC_l^{(2,2)}(1))=2,\quad t_2(BC_l^{(2,2)}(1))=2,\quad (BC_l^{(2,2)}(1))^\vee=BC_l^{(2,2)}(1).$

2) $R:\ \pm\varepsilon_i+nb+ma\ (1\leq i\leq l)\ (n,m\in\mathbb{Z}),$
$\pm 2\varepsilon_i+(2n+1)b+2ma\ (1\leq i\leq l)\ (n,m\in\mathbb{Z}),$
$\pm\varepsilon_i\pm\varepsilon_j+nb+ma\ (1\leq i<j\leq l)\ (n,m\in\mathbb{Z}).$

3) $\alpha_0=-2\varepsilon_1+b,\quad \alpha_i=\varepsilon_i-\varepsilon_{i+1}\ (1\leq i\leq l-1),\quad \alpha_l=\varepsilon_l.$

4) $k_0=2,\quad k_i=1\ (1\leq i\leq l),$
$m_0=2,\quad m_i=4\ (1\leq i\leq l-1),\quad m_l=2.$

5) $\mathrm{cod}(BC_l^{(2,2)}(1))=l-1,\quad \#\Gamma(BC_l^{(2,2)}(1))=2l.$



Type $BC_l^{(2,2)}(2)$ $(l\geq 1)$

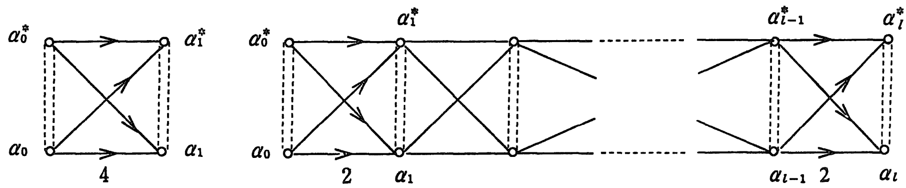
1) $t_1(BC_l^{(2,2)}(2))=2,\quad t_2(BC_l^{(2,2)}(2))=2,\quad (BC_l^{(2,2)}(2))^\vee=BC_l^{(2,2)}(2).$

2) $R:\ \pm\varepsilon_i+nb+ma\ (1\leq i\leq l)\ (n,m\in\mathbb{Z}),$
$\pm 2\varepsilon_i+(2n+1)b+2ma\ (1\leq i\leq l)\ (n,m\in\mathbb{Z}),$
$\pm\varepsilon_i\pm\varepsilon_j+nb+2ma\ (1\leq i<j\leq l)\ (n,m\in\mathbb{Z}).$

3) $\alpha_0=-2\varepsilon_1+b,\quad \alpha_i=\varepsilon_i-\varepsilon_{i+1}\ (1\leq i\leq l-1),\quad \alpha_l=\varepsilon_l.$

4) $k_i=2\ (0\leq i\leq l-1),\quad k_l=1,$
$m_i=2\ (0\leq i\leq l).$

5) $\mathrm{cod}(BC_l^{(2,2)}(2))=l+1,\quad \#\Gamma(BC_l^{(2,2)}(2))=2l+2.$



$(l=1)$ $(l\geq 2)$

Type $D_l^{(1,1)}$ $(l\geq 4)$

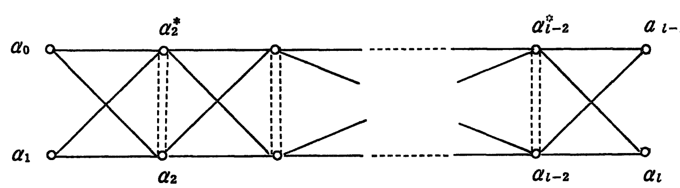
1) $t_1(D_l^{(1,1)})=1,\quad t_2(D_l^{(1,1)})=1,\quad (D_l^{(1,1)})^\vee=D_l^{(1,1)}.$

2) $R:\ \pm\varepsilon_i\pm\varepsilon_j+nb+ma\ (1\leq i<j\leq l)\ (n, m\in\boldsymbol{Z}).$

3) $\alpha_0=-\varepsilon_1-\varepsilon_2+b,\quad \alpha_i=\varepsilon_i-\varepsilon_{i+1}\ (1\leq i\leq l-1),\quad \alpha_l=\varepsilon_{l-1}+\varepsilon_l.$

4) $k_i=1\ (0\leq i\leq l),$
$m_0=1,\quad m_1=1,\quad m_i=2\ (2\leq i\leq l-2),\quad m_{l-1}=1,\quad m_l=1,$

5) $\mathrm{cod}(D_l^{(1,1)})=l-3,\quad \#\Gamma(D_l^{(1,1)})=2l-2.$



In the next three types ($E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$), let $\omega_i:=\varepsilon_i-\frac{1}{9}\sum_{j=0}^{8}\varepsilon_j\ (0\leq i\leq 8)$, so that $\sum_{i=0}^{8}\omega_i=0$ and $\mathrm{I}(\omega_i, \omega_j)=-\frac{1}{9}+\delta_{ij}$ for $0\leq i, j\leq 8$.

Type $E_6^{(1,1)}$

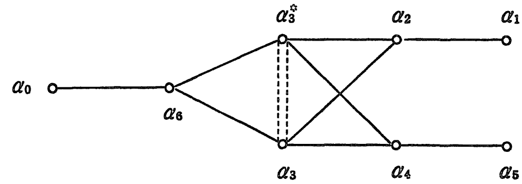
1) $t_1(E_6^{(1,1)})=1,\quad t_2(E_6^{(1,1)})=1,\quad (E_6^{(1,1)})^\vee=E_6^{(1,1)}.$

2) $R:\ \pm(\omega_i-\omega_j)+nb+ma\ (1\leq i<j\leq 6)\,(n, m\in\boldsymbol{Z}),$
$\pm(\omega_i+\omega_j+\omega_k)+nb+ma\ (1\leq i<j<k\leq 6)\ (n, m\in\boldsymbol{Z}),$
$\pm(\omega_1+\omega_2+\cdots+\omega_6)+nb+ma(n, m\in\boldsymbol{Z}).$

3) $\alpha_0=-(\omega_1+\cdots+\omega_6)+b,\quad \alpha_i=\omega_i-\omega_{i+1}\ (1\leq i\leq 5),\quad \alpha_6=\omega_4+\omega_5+\omega_6.$

4) $k_i=1\ (0\leq i\leq 6),$
$m_0=1,\quad m_1=1,\quad m_2=2,\quad m_3=3,\quad m_4=2,\quad m_5=1,\quad m_6=2.$

5) $\mathrm{cod}(E_6^{(1,1)})=1, \#\Gamma(E_6^{(1,1)})=8.$



Type $E_7^{(1,1)}$

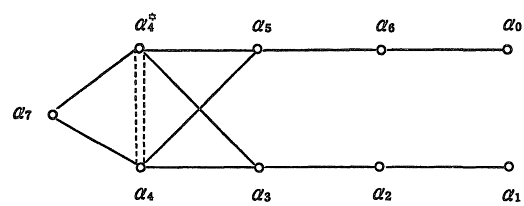
1) $t_1(E_7^{(1,1)})=1,\quad t_2(E_7^{(1,1)})=1,\quad (E_7^{(1,1)})^\vee=E_7^{(1,1)}$.

2) $R:\ \pm(\omega_i-\omega_j)+nb+ma\ (1\leq i<j\leq 7)\ (n,m\in\mathbb{Z}),$
$\pm(\omega_i+\omega_j+\omega_k)+nb+ma\ (1\leq i<j<k\leq 7)\ (n,m\in\mathbb{Z}),$
$\pm(\omega_1+\cdots+\hat{\omega}_i+\cdots+\omega_7)+nb+ma\ (1\leq i\leq 7)\ (n,m\in\mathbb{Z}).$

3) $\alpha_0=-(\omega_1+\cdots+\omega_6)+b,\quad \alpha_i=\omega_i-\omega_{i+1}\ (1\leq i\leq 6),\quad \alpha_7=\omega_5+\omega_6+\omega_7.$

4) $k_i=1\ (0\leq i\leq 7),$
$m_0=1,\quad m_1=1,\quad m_2=2,\quad m_3=3,\quad m_4=4,\quad m_5=3,\quad m_6=2,\quad m_7=2.$

5) $\mathrm{cod}(E_7^{(1,1)})=1,\quad \#\Gamma(E_7^{(1,1)})=9.$



Type $E_8^{(1,1)}$

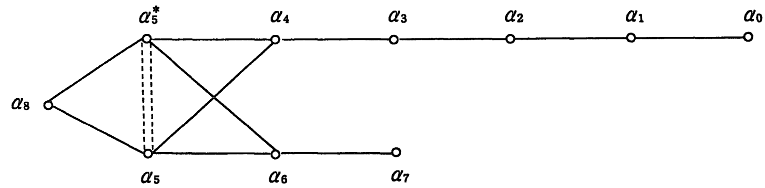
1) $t_1(E_8^{(1,1)})=1,\quad t_2(E_8^{(1,1)})=1,\quad (E_8^{(1,1)})^\vee=E_8^{(1,1)}$.

2) $R:\ \pm(\omega_i-\omega_j)+nb+ma\ (0\leq i<j\leq 8)\ (n,m\in\mathbb{Z}),$
$\pm(\omega_i+\omega_j+\omega_k)+nb+ma\ (0\leq i<j<k\leq 8)\ (n,m\in\mathbb{Z}).$

3) $\alpha_0=\omega_0-\omega_1+b,\quad \alpha_i=\omega_i-\omega_{i+1}\ (1\leq i\leq 7),\quad \alpha_8=\omega_6+\omega_7+\omega_8.$

4) $k_i=1\ (0\leq i\leq 8)$
$m_0=1,\quad m_1=2,\quad m_2=3,\quad m_3=4,\quad m_4=5,\quad m_5=6,\quad m_6=4,\quad m_7=2,\quad m_8=3.$

5) $\mathrm{cod}(E_8^{(1,1)})=1,\quad \#\Gamma(E_8^{(1,1)})=10.$



Type $F_4^{(1,1)}$

1) $t_1(F_4^{(1,1)})=1,\quad t_2(F_4^{(1,1)})=1,\quad (F_4^{(1,1)})^\vee=F_4^{(2,2)}$.

2) $R:\ \pm\varepsilon_i+nb+ma\ (1\le i\le 4)\ (n,m\in\boldsymbol{Z})$,

$\pm\varepsilon_i\pm\varepsilon_j+nb+ma\ (1\le i<j\le 4)\ (n,m\in\boldsymbol{Z})$,

$\frac{1}{2}(\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4)+nb+ma\ (n,m\in\boldsymbol{Z})$.

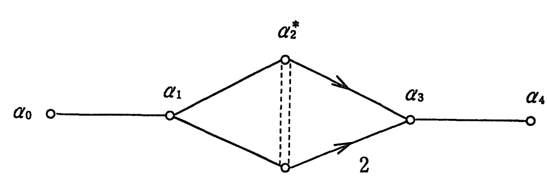
3) $\alpha_0=\varepsilon_1-\varepsilon_2+b,\quad \alpha_1=\varepsilon_2-\varepsilon_3,\quad \alpha_2=\varepsilon_3-\varepsilon_4,\quad \alpha_3=\varepsilon_4-\sigma,\quad \alpha_4=\sigma$,

where $\sigma:=\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)$.

4) $k_i=1\ (0\le i\le 4)$,

$m_0=2,\quad m_1=4,\quad m_2=6,\quad m_3=4,\quad m_4=2$.

5) $\mathrm{cod}(F_4^{(1,1)})=1,\quad \#\Gamma(F_4^{(1,1)})=6$.



Type $F_4^{(1,2)}$

1) $t_1(F_4^{(1,2)})=1,\quad t_2(F_4^{(1,2)})=2,\quad (F_4^{(1,2)})^\vee=F_4^{(2,1)}$.

2) $R:\ \pm\varepsilon_i+nb+ma\ (1\le i\le 4)\ (n,m\in\boldsymbol{Z})$,

$\pm\varepsilon_i\pm\varepsilon_j+nb+2ma\ (1\le i<j\le 4)\ (n,m\in\boldsymbol{Z})$,

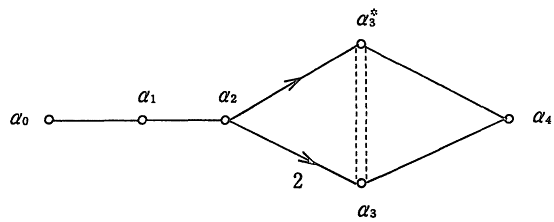
$\frac{1}{2}(\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4)+nb+ma\ (n,m\in\boldsymbol{Z})$.

3) $\alpha_0=\varepsilon_1-\varepsilon_2+b,\quad \alpha_1=\varepsilon_2-\varepsilon_3,\quad \alpha_2=\varepsilon_3-\varepsilon_4,\quad \alpha_3=\varepsilon_4-\sigma,\quad \alpha_4=\sigma$.

4) $k_0=2,\quad k_1=2,\quad k_2=2,\quad k_3=1,\quad k_4=1$,

$m_0=1,\quad m_1=2,\quad m_2=3,\quad m_3=4,\quad m_4=2$.

5) $\mathrm{cod}(F_4^{(1,2)})=1$, $\#\Gamma(F_4^{(1,2)})=6$.



Type $F_4^{(2,1)}$

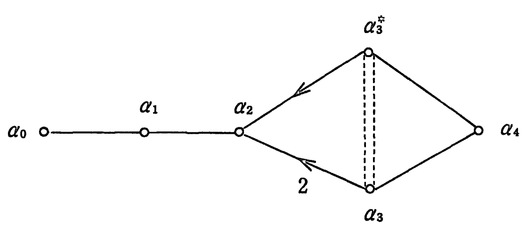
1) $t_1(F_4^{(2,1)})=2$, $t_2(F_4^{(2,1)})=1$, $(F_4^{(2,1)})^\vee=F_4^{(1,2)}$.

2) R: $\pm 2\varepsilon_i+2nb+ma$ $(1\leq i\leq 4)$ $(n, m\in\mathbb{Z})$,
$\pm\varepsilon_i\pm\varepsilon_j+nb+ma$ $(1\leq i<j\leq 4)$ $(n, m\in\mathbb{Z})$,
$\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4+2nb+ma$ $(n, m\in\mathbb{Z})$.

3) $\alpha_0=\varepsilon_1-\varepsilon_2+b$, $\alpha_1=\varepsilon_2-\varepsilon_3$, $\alpha_2=\varepsilon_3-\varepsilon_4$, $\alpha_3=2\varepsilon_4$, $\alpha_4=-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4$.

4) $k_i=1$ $(0\leq i\leq 4)$,
$m_0=1$, $m_1=2$, $m_2=3$, $m_3=4$, $m_4=2$.

5) $\mathrm{cod}(F_4^{(2,1)})=1$, $\#\Gamma(F_4^{(2,1)})=6$.



Type $F_4^{(2,2)}$

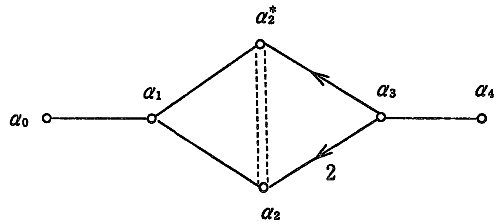
1) $t_1(F_4^{(2,2)})=2$, $t_2(F_4^{(2,2)})=2$, $(F_4^{(2,2)})^\vee=F_4^{(1,1)}$.

2) R: $\pm 2\varepsilon_i+2nb+2ma$ $(1\leq i\leq 4)$ $(n, m\in\mathbb{Z})$,
$\pm\varepsilon_i\pm\varepsilon_j+nb+ma$ $(1\leq i<j\leq 4)$ $(n, m\in\mathbb{Z})$,
$\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4+2nb+2ma$ $(n, m\in\mathbb{Z})$.

3) $\alpha_0=\varepsilon_1-\varepsilon_2+b$, $\alpha_1=\varepsilon_2-\varepsilon_3$, $\alpha_2=\varepsilon_3-\varepsilon_4$, $\alpha_3=2\varepsilon_4$, $\alpha_4=-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4$.

4) $k_0=1,\quad k_1=1,\quad k_2=1,\quad k_3=2,\quad k_4=2,$
$m_0=1,\quad m_1=2,\quad m_2=3,\quad m_3=2,\quad m_4=1.$

5) $\mathrm{cod}(F_4^{(2,2)})=1,\quad \#\Gamma(F_4^{(2,2)})=6.$



In the last four types $(G_2^{(1,1)}, G_2^{(1,3)}, G_2^{(3,1)}$ and $G_2^{(3,3)})$, let $\phi_i:=\varepsilon_i-\frac{1}{3}(\varepsilon_1+\varepsilon_2+\varepsilon_3)(i=1,2,3)$, so that $\sum_{i=1}^{3}\phi_i=0$ and $I(\phi_i,\phi_j)=-\frac{1}{3}+\delta_{ij}\ (1\le i,j\le 3)$.

Type $G_2^{(1,1)}$

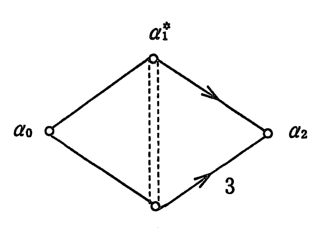
1) $t_1(G_2^{(1,1)})=1,\quad t_2(G_2^{(1,1)})=1,\quad (G_2^{(1,1)})^\vee=G_2^{(3,3)}.$

2) $R:\ \pm\phi_i+nb+ma\ (1\le i\le 3)\ (n,m\in\mathbb{Z}),$
$\pm(\phi_i-\phi_j)+nb+ma\ (1\le i<j\le 3)\ (n,m\in\mathbb{Z}).$

3) $\alpha_0=-\phi_1+\phi_3+b,\quad \alpha_1=\phi_1-\phi_2,\quad \alpha_2=\phi_2.$

4) $k_i=1\ (0\le i\le 2),$
$m_0=3,\quad m_1=6,\quad m_2=3.$

5) $\mathrm{cod}\ (G_2^{(1,1)})=1,\quad \#\Gamma(G_2^{(1,1)})=4.$



Type $G_2^{(1,3)}$

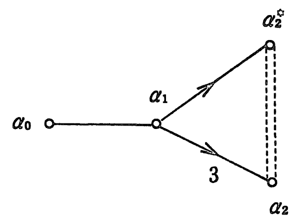
1) $t_1(G_2^{(1,3)})=1,\quad t_2(G_2^{(1,3)})=3,\quad (G_2^{(1,3)})^\vee=G_2^{(3,1)}.$

2) $R:\ \pm\phi_i+nb+ma\ (1\le i\le 3)\ (n,m\in\mathbb{Z}),$
$\pm(\phi_i-\phi_j)+nb+3ma\ (1\le i<j\le 3)\ (n,m\in\mathbb{Z}).$

3) $\alpha_0=-\phi_1+\phi_3+b, \quad \alpha_1=\phi_1-\phi_2, \quad \alpha_2=\phi_2.$

4) $k_0=3, \quad k_1=3, \quad k_2=1,$
$m_0=1, \quad m_1=2, \quad m_2=3.$

5) $\mathrm{cod}(G_2^{(1,3)})=1, \quad \#\Gamma(G_2^{(1,3)})=4.$



Type $G_2^{(3,1)}$

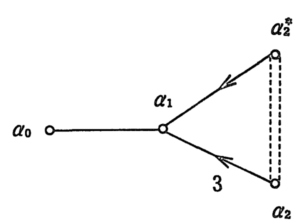
1) $t_1(G_2^{(3,1)})=3, \quad t_2(G_2^{(3,1)})=1, \quad (G_2^{(3,1)})^\vee=G_2^{(1,3)}.$

2) $R: \pm 3\phi_i+3nb+ma \ (1\leq i\leq 3) \ (n, m\in \mathbb{Z}),$
$\pm(\phi_i-\phi_j)+nb+ma \ (1\leq i<j\leq 3) \ (n, m\in \mathbb{Z}).$

3) $\alpha_0=-\phi_1+\phi_3+b, \quad \alpha_1=\phi_1-\phi_2, \quad \alpha_2=3\phi_2.$

4) $k_i=1 \ (1\leq i\leq 3)$
$m_0=1, \quad m_1=2, \quad m_2=3.$

5) $\mathrm{cod}\ (G_2^{(3,1)})=1, \quad \#\Gamma(G_2^{(3,1)})=4.$



Type $G_2^{(3,3)}$

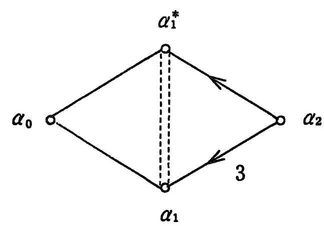
1) $t_1(G_2^{(3,3)})=3, \quad t_2(G_2^{(3,3)})=3, \quad (G_2^{(3,3)})^\vee=G_2^{(1,1)}.$

2) $R: \pm 3\phi_i+3nb+3ma \ (1\leq i\leq 3) \ (n, m\in \mathbb{Z}),$
$\pm(\phi_i-\phi_j)+nb+ma \ (1\leq i<j\leq 3) \ (n, m\in \mathbb{Z}).$

3) $\alpha_0=-\phi_1+\phi_3+b, \quad \alpha_1=\phi_1-\phi_2, \quad \alpha_2=3\phi_2.$

4) $k_0=1, \quad k_1=1, \quad k_2=3,$
$m_0=1, \quad m_1=2, \quad m_2=1.$

5) $\mathrm{cod}(G_2^{(3,3)})=1, \quad \# \Gamma(G_2^{(3,3)})=4.$



(5.3) **A description of the set of roots.**

Here we give a systematic description of the set R from the knowledge of the type of an. e.a.r.s. (R, G).

In the following we use the notation R_P as for the set of finite roots of type P. The Weyl group invariant metric is denoted by I_P, which is normalized s.t. $\inf \{I_P(\alpha, \alpha) : \alpha \in R_P\}=2$.

i) **Type** $P^{(1,1)}$

$$R=\bigcup_{\alpha \in R_P} \{\alpha+\boldsymbol{Z}b+\boldsymbol{Z}a\}.$$

ii) **Type** $P^{(1,t)}$ for $t=t(P)$

$$R=\bigcup_{\alpha \in R_P} \{\alpha+\boldsymbol{Z}b+\boldsymbol{Z}\frac{I_P(\alpha, \alpha)}{2}a\}.$$

iii) **Type** $P^{(t,1)}$ for $t=t(P)$

$$R=\bigcup_{\alpha \in R_P} \left\{\alpha+\boldsymbol{Z}\frac{I_P(\alpha, \alpha)}{2}b+\boldsymbol{Z}a\right\}.$$

iv) **Type** $P^{(t,t)}$ for $t=t(P)$

$$R=\bigcup_{\alpha \in R_P} \left\{\alpha+\boldsymbol{Z}\frac{I_P(\alpha, \alpha)}{2}b+\boldsymbol{Z}\frac{I_P(\alpha, \alpha)}{2}a\right\}.$$

v) **Type** $P^{(1,1)*}$ where $1=t(P)$ (*Case* $A_1^{(1,1)*}$)

$$R=\bigcup_{\alpha \in R_P} \left\{\alpha+nb+ma : \ m, n \in \boldsymbol{Z}, \ mn \equiv 0 \bmod 2\right\}.$$

vi) Type $P^{(1,1)*}$ where $1\neq t(P)$ (*Case* $C_l^{(1,1)*}, l\geq 2$)

$$R=\bigcup_{\alpha\in R_P}\{\alpha+nb+ma:\ m,\ n\in\mathbb{Z},\ mn\equiv 0 \bmod 2 \text{ if } \alpha \text{ is a long root}\}.$$

vii) Type $P^{(t,t)*}$ where $t=t(P)\neq 1$ (*Case* $B_l^{(2,2)*}, l\geq 2$)

$$R=\bigcup_{\alpha\in R_P}\left\{\begin{array}{l}\alpha+nb+ma:\ m,\ n\in\mathbb{Z},\ m\equiv n\equiv 0 \bmod \dfrac{I_P(\alpha,\alpha)}{2},\\ \qquad\qquad mn\equiv 0 \bmod 2 \text{ if } \alpha \text{ is a short root.}\end{array}\right\}.$$

viii) Type $BC_l^{(2,1)}$ $(l\geq 1)$

$$R=\bigcup_{\alpha\in R_{BC_l}}\{\alpha+nb+ma:\ m,\ n\in\mathbb{Z},\ n\equiv 1 \bmod 2 \text{ for } \alpha \text{ longest root}\}.$$

ix) Type $BC_l^{(2,4)}$ $(l\geq 1)$

$$R=\bigcup_{\alpha\in R_{BC_l}}\left\{\begin{array}{l}\alpha+nb+m\dfrac{I(\alpha,\alpha)}{2}a:\ m,\ n\in\mathbb{Z},\ n\equiv 1 \bmod 2\\ \qquad\qquad \text{for } \alpha \text{ longest root}\end{array}\right\}.$$

x) Type $BC_l^{(2,2)}(1)$ $(l\geq 2)$

$$R=\bigcup_{\alpha\in R_{BC_l}}\left\{\begin{array}{l}\alpha+nb+ma:\ m,\ n\in\mathbb{Z},\ n\equiv 1 \bmod 2 \text{ for } \alpha \text{ longest root,}\\ \qquad\qquad m\equiv 0 \bmod 2 \text{ for } \alpha \text{ longest root}\end{array}\right\}.$$

xi) Type $BC_l^{(2,2)}(2)$ $(l\geq 1)$

$$R=\bigcup_{\alpha\in R_{BC_l}}\left\{\begin{array}{l}\alpha+nb+ma:\ m,\ n\in\mathbb{Z},\ n\equiv 1 \bmod 2 \text{ for } \alpha \text{ longest root,}\\ \qquad\qquad m\equiv 0 \bmod 2 \text{ for } \alpha \text{ not shortest root}\end{array}\right\}.$$

(5.4) **Appendix. Isomorphism as Root Systems.**

In the following, we list up *all pair of non isomorphic marked extended affine root systems, which are isomorphic as root systems forgetting about their markings*. For each such pair, we give an isomorphism Φ explicitly by using the explicit description of the root system in (5.2).

1) $B_l^{(1,2)}\simeq B_l^{(2,1)}$ $(l\geq 3)$

$$\Phi:\ F_{B_l^{(1,2)}}\xrightarrow{\sim}F_{B_l^{(2,1)}}$$

$$\varepsilon_i\longmapsto\varepsilon_i \quad i=1,\dots,l$$

$$a \longmapsto b$$
$$b \longmapsto a$$

2) $C_l^{(1,2)} \simeq C_l^{(2,1)} \quad (l \geq 3)$

$$\Phi : F_{C_l^{(1,2)}} \tilde{\to} F_{C_l^{(2,1)}}$$

$$\varepsilon_i \longmapsto \varepsilon_i \quad i=1, \dots, l$$
$$a \longmapsto b$$
$$b \longmapsto a$$

3) $F_4^{(1,2)} \simeq F_4^{(2,1)}$

$$\Phi : F_{F_4^{(1,2)}} \tilde{\to} F_{F_4^{(2,1)}}$$

$$\varepsilon_1 \longmapsto \varepsilon_1 + \varepsilon_2$$
$$\varepsilon_2 \longmapsto \varepsilon_1 - \varepsilon_2$$
$$\varepsilon_3 \longmapsto \varepsilon_3 + \varepsilon_4$$
$$\varepsilon_4 \longmapsto \varepsilon_3 - \varepsilon_4$$
$$a \longmapsto b$$
$$b \longmapsto a$$

4) $G_2^{(1,3)} \simeq G_2^{(3,1)}$

$$\Phi : F_{G_2^{(1,3)}} \tilde{\to} F_{G_2^{(3,1)}}$$

$$\phi_1 \longmapsto \phi_2 - \phi_3$$
$$\phi_2 \longmapsto \phi_3 - \phi_1$$
$$\phi_3 \longmapsto \phi_1 - \phi_2$$
$$a \longmapsto b$$
$$b \longmapsto a$$

(To see that these 4 give a complete list of isomorphism among two m.e.a.r.s.'s (R, G) and (R', G'), we proceed the following. If $R \simeq R'$ then i) $R/\mathrm{rad}I \simeq R'/\mathrm{rad}I'$ and ii) the counting set $K_{\mathrm{rad}I}(\alpha)$ for $\alpha \in R$ and $K_{\mathrm{rad}I'}(\alpha')$ for $\alpha' \in R'$ (cf. (1.16)) should behave similarly. For a m.e.a.r.s. (R, G), such data are easily calculated from the description of (R, G) in (5.3). Thus in this way we can pick up all possible pair of marked e.a.r.s.'s from the table of (5.3), which might be isomorphic as root systems. Then as we see in the above list, one constructs explicitly the isomorphism Φ for each case.)

§ 6. The Second Tier Number $t_2(R,G)$ and the Counting $k(\alpha)$ $(\alpha\in R)$

In this paragraph we give a proof of the classification of m.e.a.r.s.'s stated in the last paragraph. The idea is to calculate $t_2(R,G)$ by introducing counting number $k(\alpha)$ for $\alpha\in R$ which is a positive integer s.t. $\{\alpha+\mathbb{Z}k(\alpha)a\}=R\cap\{\alpha+\mathbb{Z}a\}$ (cf. (6.1)). The counting number $k(\alpha)$ plays a role in the definition of exponents in § 7 and in the study of the Coxeter transformation in § 9.

(6.1) Let (R,G) be a m.e.a.r.s. such that R/G is reduced.

Assertion i) *For any root $\alpha\in R$, there exists a positive integer $k(\alpha)$, which we shall call the counting of α, such that*

$$\{\alpha+\mathbb{Z}k(\alpha)a\} = R\cap\{\alpha+\mathbb{Z}a\}. \tag{6.1.1}$$

ii) *For any two roots $\alpha,\beta\in R$,*

$$k(\beta)\,|\,I(\beta,\alpha^\vee)k(\alpha). \tag{6.1.2}$$

Particularly if $I(\beta,\alpha^\vee)=\pm 1$, then

$$1\,|\,k(\alpha)/k(\beta)\,|\,I(\alpha,\beta^\vee). \tag{6.1.3}$$

iii) *If φ is an automorphism of (R,G), then*

$$k(\alpha)=k(\varphi(\alpha)) \quad \textit{for} \quad \alpha\in R. \tag{6.1.4}$$

iv)

$$R=\coprod_{\alpha\in R\cap L^{l+1}}\{\alpha+\mathbb{Z}k(\alpha)a\}, \tag{6.1.5}$$

for a linear subspace L^{l+1} of F spanned by a basis (cf. (3.4)).

v)
$$\text{g.c.d.}\{k(\alpha):\ \alpha\in R\}=1,$$
$$\text{g.c.d.}\{k(\alpha_i):\ \alpha_0,\dots,\alpha_l \textit{ a basis for } (R,G)\}=1.$$

Proof. i) Put $K(\alpha):=\{x\in\mathbb{Z}:\ \alpha+xa\in R\}$ for a root $\alpha\in R$ (cf. (1.16)). Since $0\in K(\alpha)$ and $x,y\in K(\alpha)$ implies $2x-y\in K(\alpha)$ (cf. (1.16) Assertion 1. i) iii)), $K(\alpha)$ is an ideal of $\mathbb{Z}$, whose non negative generator is denoted by $k(\alpha)$. Then $k(\alpha)\neq 0$ due to (1.16) **Assertion 2** and (3.3) *Note* 2.

Then ii) iii) iv) and v) are direct consequences of (1.16) **Assertion 1**, **Assertion 2.**

(6.2) **Corollary.** *Let (R, G) and (R', G') be m.e.a.r.s.'s and let $\{\alpha_0, \ldots, \alpha_l\}$ and $\{\beta_0, \ldots, \beta_l\}$ be basis of (R, G) and (R', G') respectively* (*cf*. (3.4) **Def.**). *Suppose (R, G) and (R', G') are isomorphic. Then there exists an isomorphism $\varphi: (R, G) \simeq (R', G')$ such that $\{\varphi(\alpha_0), \ldots, \varphi(\alpha_l)\}=\{\beta_0, \ldots, \beta_l\}$.*

Proof. It is enough to show the case when $(R, G)=(R', G')$. Recall the projection $p_1: F\rightarrow F/G$. Since $p_1(\alpha_0), \ldots, p_1(\alpha_l)$ and $p_1(\beta_0), \ldots, p_1(\beta_l)$ are basis for the affine root system R/G, there exists an element $\bar{w}\in W_{R/G}$ and a sign $\varepsilon\in\{\pm 1\}$ such that $\{\varepsilon\bar{w}(p_1(\alpha_0)), \ldots, \varepsilon\bar{w}(p_1(\alpha_l))\}=\{p_1(\beta_0), \ldots, p_1(\beta_l)\}$. ($W_{R/G}$ acts transitively on the set of chambers.) Therefore if $w\in W_R$ is a lifting of $\bar{w}$, using the **Assertion** i) we see that there exists a permutation $\sigma\in\mathfrak{S}_{l+1}$ and integers $m_i\in\mathbb{Z}$ $(i=0, \ldots, l)$ such that

$$\varepsilon w(\alpha_{\sigma(i)}) = \beta_i+m_ik(\beta_i)a \quad i=0, \ldots, l.$$

Let us define two elements ψ and φ of $GL(F)$ by

$$\psi: F=\bigoplus_{i=0}^{l}\mathbb{R}\alpha_i\oplus\mathbb{R}a\ni\sum c_i\alpha_i+pa \mapsto \varepsilon(\sum c_i\alpha_i)+pa\in F,$$

$$\varphi: F=\bigoplus_{i=0}^{l}\mathbb{R}\beta_i\oplus\mathbb{R}a\ni\sum c_i\beta_i+pa \mapsto \sum c_i\beta_i+(p+\sum c_im_ik(\beta_i))a\in F.$$

The map ψ is shown to be an automorphism of (R, G) using the expression (6.1.5) for $L^{l+1}:=\bigoplus_{i=0}^{l}\mathbb{R}\alpha_i$ and the fact $k(\alpha)=k(-\alpha)$.

Let us show that φ is an automorphism of (R, G). First let us show that *for any $\beta\in R\cap\bar{L}^{l+1}$ with $\bar{L}^{l+1}:=\bigoplus_{i=0}^{l}\mathbb{R}\beta_i$, $\varphi(\beta)$ belongs to R such that $k(\beta)=k(\varphi(\beta))$.*

By definition $\varphi\beta_i=\beta_i+m_ik(\beta_i)a\in R$ and $\varphi w_{\beta_i}\varphi^{-1}=w_{\varphi\beta_i}\in W_R$ for $i=0, \ldots, l$. Therefore $\varphi(R\cap\bar{L}^{l+1})=\varphi(\bigcup_{i=0}^{l}W_{R\cap\bar{L}^{l+1}}\beta_i)=\varphi(\bigcup_{i=0}^{l}\langle w_{\beta_0}, \ldots, w_{\beta_l}\rangle\beta_i)=\bigcup_{i=0}^{l}\langle w_{\varphi\beta_0}, \ldots, w_{\varphi\beta_l}\rangle\varphi\beta_i\subset W_RR=R$. If $\beta=w\beta_i$ for $\beta\in R\cap\bar{L}^{l+1}$, $w\in W_{R\cap\bar{L}^{l+1}}$ and $0\leq i\leq l$, then using **Assertion** i) and iii) one computes $k(\varphi(\beta))=k(\varphi w\varphi^{-1}\beta_i)=k(\varphi\beta_i)=k(\beta_i)=k(w\beta_i)=k(\beta)$.

Using the expression (6.1.5) for $\bar{L}^{l+1}=\bigoplus_{i=0}^{l}\mathbb{R}\beta_i$ again, one computes $.\varphi R=\bigcup_{\beta\in R\cap\bar{L}^{l+1}}(\varphi(\beta)+\mathbb{Z}k(\beta)a)=\bigcup_{\beta\in R\cap\bar{L}^{l+1}}(\varphi(\beta)+\mathbb{Z}k(\varphi(\beta))a)\subset R$. Samely one computes that $\varphi^{-1}R\subset R$. This shows that φ is an automorphism of (R, G).

Therefore the composition $\varphi^{-1}w\psi$ is an automorphism of (R, G) such that $(\varphi^{-1}w\psi)\alpha_{\sigma(i)}=\beta_i$, $i=0, \dots, l$. This completes the proof of the corollary.

(6.3) Let us give a formula for the second tier number $t_2(R, G)$ using the counting $k(\alpha)$, $\alpha\in R$.

(6.3.1)
$$\begin{aligned} t_2(R, G) &= \text{g.c.d.}\{k(\alpha)\frac{I_{R^\vee}(\alpha^\vee, \alpha^\vee)}{2} : \alpha\in R\} \\ &= \text{g.c.d.}\{k(\alpha_i)\frac{I_{R^\vee}(\alpha_i^\vee, \alpha_i^\vee)}{2} : \alpha_0, \dots, \alpha_l \text{ a basis}\}. \end{aligned}$$

Proof. For the dual root system $(R^\vee, G)$, parallel to (6.1.1) let us define,

(6.3.2) $$k^\vee : R^\vee\to N \quad\text{by}\quad \{\alpha^\vee+\mathbb{Z}k^\vee(\alpha^\vee)a^\vee\}=R^\vee\cap\{\alpha^\vee+\mathbb{Z}a^\vee\}.$$

By definition, we have a relation,

(6.3.3) $$k^\vee(\alpha^\vee)a^\vee = \frac{2}{I(a, a)}k(\alpha)a = \frac{I(a^\vee, a^\vee)}{2}k(\alpha)a \quad\text{for}\quad \alpha\in R.$$

Taking the proportion of (6.3.3), for $\alpha, \beta\in R$, we have

$$\frac{k^\vee(\alpha^\vee)}{k^\vee(\beta^\vee)} = \frac{I(\alpha^\vee, \alpha^\vee)}{I(\beta^\vee, \beta^\vee)}\frac{k(\alpha)}{k(\beta)}.$$

Therefore

(6.3.4) $$k^\vee(\alpha^\vee) = \frac{1}{c}\frac{I(\alpha^\vee, \alpha^\vee)}{2}k(\alpha) \quad\text{for}\quad \alpha\in R,$$

where $c=\frac{I(\beta^\vee, \beta^\vee)}{2}\frac{k(\beta)}{k^\vee(\beta)}$.

Applying (6.3.4) to g.c.d. $\{k^\vee(\alpha^\vee): \alpha\in R^\vee\}=1$, one gets,

(6.3.5) $$c = \text{g.c.d.}\ \{\frac{I(\alpha^\vee, \alpha^\vee)}{2}k(\alpha): \alpha\in R\}.$$

By definition of $t_2(R, G)$ in (4.3), applying (6.3.3), (6.3.4), (6.3.5),

$$t_2(R, G) := |(a^\vee : a)|(I_{R^\vee} : I) = \frac{k(\alpha)}{k^\vee(\alpha^\vee)}\frac{I(\alpha^\vee, \alpha^\vee)}{2}(I_{R^\vee} : I)$$

$$= c(I_{R^\vee}: I) = \text{g.c.d.}\left\{\frac{I_{R^\vee}(\alpha^\vee, \alpha^\vee)}{2}k(\alpha): \alpha \in R\right\}$$
$$= \text{g.c.d.}\left\{\frac{I_{R^\vee}(\alpha_i^\vee, \alpha_i^\vee)}{2}k(\alpha_i): \alpha_0, \ldots, \alpha_l \text{ a basis}\right\}.$$

q.e.d.

Corollary.

(6.3.6) $$k^\vee(\alpha^\vee) = \frac{1}{t_2(R, G)} \frac{I_{R^\vee}(\alpha^\vee, \alpha^\vee)}{2}k(\alpha) \quad \text{for} \quad \alpha \in R.$$

(6.4) Let $\{\alpha_0, \ldots, \alpha_l\}$ be a basis for a m.e.a.r.s. (R, G), (3.4). Since it can be regarded as a basis for the affine root system $R \cap \bigoplus_{i=0}^{l} \mathbb{R}\alpha_i$, one can associate a Dynkin diagram Γ for this basis. (See for instance [14,5]). By $|\Gamma|$ let us denote the set of nodes of Γ, which is identified with the set $\{\alpha_0, \ldots, \alpha_l\}$ of the basis. At each node of the diagram corresponding to a base α_i, let us associate the counting integer $k(\alpha_i)$, so that we obtain weighted Dynkin diagram $(\Gamma, (k(\alpha))_{\alpha \in |\Gamma|})$.

Lemma i) *The weighted diagram* $(\Gamma, (k(\alpha)_{\alpha\in|\Gamma|})$ *is uniquely determined by the isomorphism class of* (R, G).

ii) *The weighted diagram* $(\Gamma, (k(\alpha))_{\alpha\in|\Gamma|})$ *for a* (R, G) *determines uniquely the isomorphism class of* (R, G).

Proof. i) (6.1) Assertion iii) and (6.2) Corollary.

ii) Let $\{\alpha_0, \ldots, \alpha_l\}$ be a basis for (R, G). Let us take the expression (6.1.5) for $L^{l+1} = \bigoplus_{i=0}^{l} \mathbb{R}\alpha_i$,

$$R = \bigcup_{\alpha \in R \cap L^{l+1}} \{\alpha + \mathbb{Z}k(\alpha)a\} = \bigcup_{i=0}^{l} \bigcup_{w \in \langle w_{\alpha_0}, \cdots w_{\alpha_l}\rangle} \{w\alpha_i + \mathbb{Z}k(\alpha_i)a\},$$

where the right hand is uniquely determined by the weighted diagram $(\Gamma, (k(\alpha))_{\alpha\in|\Gamma|})$. The marking G is given by $\mathbb{R}a$. q.e.d.

Note. Let $(\Gamma, (k(\alpha))_{\alpha\in|\Gamma|})$ be the weighted diagram for a m.e.a.r.s. (R, G). Then the weighted diagram for the dual $(R^\vee, G)$ is given by $(\Gamma^\vee, (k^\vee(\alpha^\vee))_{\alpha^\vee \in |\Gamma^\vee|})$, where $\Gamma^\vee$ is the dual of Γ (i.e. the set of nodes of $\Gamma^\vee$ is bijective to Γ and the arrows on the bonds are reversed) and $k^\vee(\alpha^\vee)$ is given by (6.3.6).

(6.5) Due to the **Lemma** (6.4), the classification of m.e.a.r.s.'s is reduced to *the* *classification of weighted diagram* $(\Gamma,(k(\alpha))_{\alpha\in|\Gamma|})$ *where*

i) Γ *is a Dynkin diagram for a reduced affine root system,*

ii) *the counting weights* $k(\alpha)$ $\alpha\in|\Gamma|$ *are positive integers s.t.*

$$k(\beta)|I(\beta,\alpha^\vee)k(\alpha) \quad \text{for} \quad \alpha,\beta\in|\Gamma| \quad \text{and}$$
$$\text{g.c.d.}\{k(\alpha):\ \alpha\in|\Gamma|\}=1.$$

In the following we list all such weighted diagrams. For each weighted diagram, we calculate the first and the second tier numbers, using (4.5.1) and (6.3.1). For each weighted diagram, let us define the type according to a similar rule as in (5.1).

(6.6) **Existence of m.e.a.r.s.'s.** To finish the classification of m.e.a.r.s.'s in § 5, we want to show the existence of a m.e.a.r.s. (R, G) for each weighted diagram $(\Gamma, (k(\alpha))_{\alpha\in|\Gamma|})$ in the list of (6.5) s.t. a basis for (R, G) gives the weighted diagram.

This can be achieved by showing the following steps.

i) *An explicit construction of* (R, G) *is given in* (5.2) (or (5.3)). (The author owes to the appendix of [14] for a description of affine root systems).

Therefore we need to show the following ii)$-$v).

ii) *Each* (R, G) *in* (5.2) *satisfies the axioms for an e.a.r.s. given in* (2.1).

iii) *The set* $\alpha_0,\ldots,\alpha_l$ *given in* 3) *of the table* (5.2) *is a basis for* (R, G) *in the sense of* (3.4).

iv) *The weighted Dynkin diagram* (*defined in* (6.4)) *associated to the basis belongs to the list in* (6.5).

v) *By this correspondence* $(R, G)\Rightarrow(\Gamma,(k(\alpha))_{\alpha\in|\Gamma|})$, *the type for* (R, G) *in* (5.2) *coincides with the type for* $(\Gamma,(k(\alpha))_{\alpha\in|\Gamma|})$ *in* (6.5).

To check all these steps for all types of m.e.a.r.s.'s is a rather long cumbersome routin work, which we do not proceed in this note.

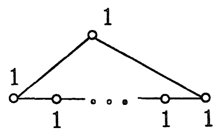
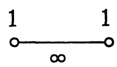
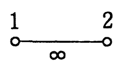
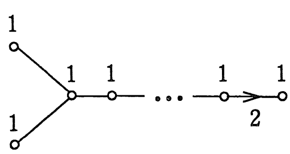
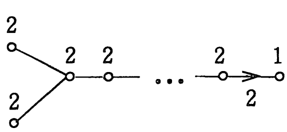
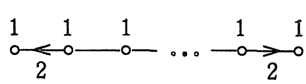
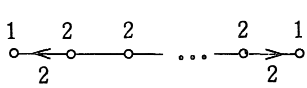
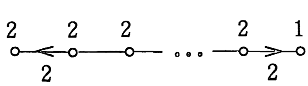
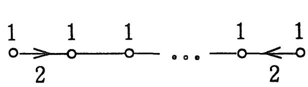
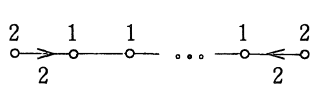
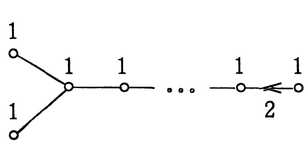
(6.7) As a result of the classification, we have the following.

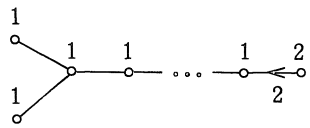
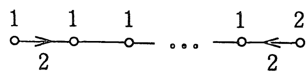
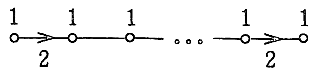
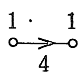
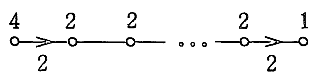
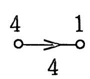
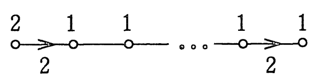
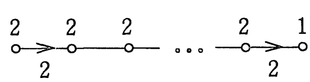
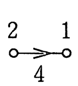
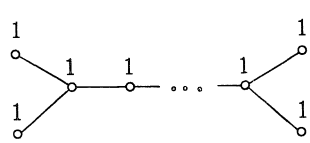
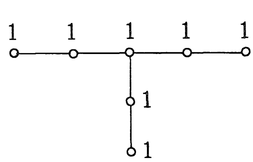
Assertion. *Let* (R, G) *be a m.e.a.r.s. of type* $P^{(t_1,t_2)}$.

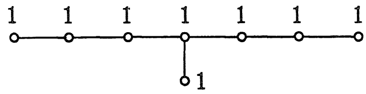
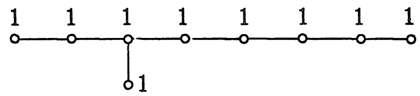
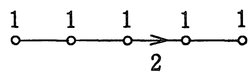
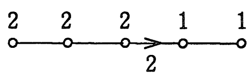
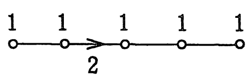
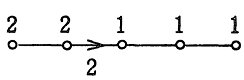
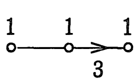
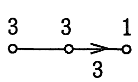
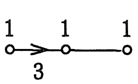
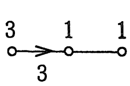
If $t_2=1$, *then* $k(\alpha)=1$ *for any* $\alpha\in R$.

If $t_2=t(P)$, *then* $k(\alpha)=\frac{1}{2}I_R(\alpha,\alpha)$ *for any* $\alpha\in R$.

(6.8) *Note.* It is curious to observe that the coefficients n_i $(i=0,\ldots,l)$ of b

weighted diagram	the first tier number	the second tier number	the type	the dual type
	1	1	$A_l^{(1,1)}$	$A_l^{(1,1)}$
	1	1	$A_1^{(1,1)}$	$A_1^{(1,1)}$
	1	1	$A_1^{(1,1)*}$	$A_1^{(1,1)*}$
	1	1	$B_l^{(1,1)}$	$C_l^{(2,2)}$
	1	2	$B_l^{(1,2)}$	$C_l^{(2,1)}$
	2	1	$B_l^{(2,1)}$	$C_l^{(1,2)}$
	2	2	$B_l^{(2,2)}$	$C_l^{(1,1)}$
	2	2	$B_l^{(2,2)*}$	$C_l^{(1,1)*}$
	1	1	$C_l^{(1,1)}$	$B_l^{(2,2)}$
	1	2	$C_l^{(1,2)}$	$B_l^{(2,1)}$
	2	1	$C_l^{(2,1)}$	$B_l^{(1,2)}$

	2	2	$C_l^{(2,2)}$	$B_l^{(1,1)}$
	1	1	$C_l^{(1,1)*}$	$B_l^{(2,2)*}$
	2	1	$BC_l^{(2,1)}$	$BC_l^{(2,4)}$
	2	1	$BC_1^{(2,1)}$	$BC_1^{(2,4)}$
	2	4	$BC_l^{(2,4)}$	$BC_l^{(2,1)}$
	2	4	$BC_1^{(2,4)}$	$BC_1^{(2,1)}$
	2	2	$BC_l^{(2,2)}(1)$	$BC_l^{(2,2)}(1)$
	2	2	$BC_l^{(2,2)}(2)$	$BC_l^{(2,2)}(2)$
	2	2	$BC_1^{(2,2)}$	$BC_1^{(2,2)}$
	1	1	$D_l^{(1,1)}$	$D_l^{(1,1)}$
	1	1	$E_6^{(1,1)}$	$E_6^{(1,1)}$

	1	1	$E_7^{(1,1)}$	$E_7^{(1,1)}$
	1	1	$E_8^{(1,1)}$	$E_8^{(1,1)}$
	1	1	$F_4^{(1,1)}$	$F_4^{(2,2)}$
	1	2	$F_4^{(1,2)}$	$F_4^{(2,1)}$
	2	1	$F_4^{(2,1)}$	$F_4^{(1,2)}$
	2	2	$F_4^{(2,2)}$	$F_4^{(1,1)}$
	1	1	$G_2^{(1,1)}$	$G_2^{(3,3)}$
	1	3	$G_2^{(1,3)}$	$G_2^{(3,1)}$
	3	1	$G_2^{(3,1)}$	$G_2^{(1,3)}$
	3	3	$G_2^{(3,3)}$	$G_2^{(1,1)}$

(3.3.6) and the counting coefficients $k(\alpha_i)$ ($i=0, \ldots, l$) behave somehow parallelly in connection with the first and the second tier numbers (cf. (4.5.1) and (6.3.6)).

This fact leads us to a definition and the studies of exponents in the following paragraphs. Particularly it becomes clear again in the study of foldings in § 12.

§ 7. Exponents m_i ($i=0, \ldots, l$) for a m.e.a.r.s.

We introduce in this paragraph exponents for a m.e.a.r.s. (R, G), which we shall use for a definition of the Dynkin diagram for (R, G). The exponents appear as eigen values for Coxeter transformation of (R, G) in §9 and as the degrees of $\tilde{W}_{R,G}$ invariant θ-functions in [20].

(7.1) Let (R, G) be a m.e.a.r.s. and $\{\alpha_0, \ldots, \alpha_l\}$ be a basis for (R, G) (cf. (3.4) Def.).

Definition. *The exponents for* (R, G) *are*

(7.1.1)
$$m_i := \frac{I_R(\alpha_i, \alpha_i)}{2k(\alpha_i)} n_i, \quad i=0, \ldots, l.$$

where n_i *are the coefficients of* (3.3.3) *and,* $k(\alpha)$ *is the counting in* (6.1) *and* I_R *is the normalized metric on* F *defining the even lattice structure on* $Q(R)$ (cf. (4.2)).

(7.2) For each type of isomorphism class of m.e.a.r.s.'s, the exponents are calculated and given in 4) of the table (5.2).

Example. *Let* (R, G) *be of type* $P^{(t_1,t_2)}$.

If $t_2=t(P)$, *then* $m_i=n_i \quad i=0, \ldots, l.$

If $t_2=1$, *then* $m_i=\frac{I_R(\alpha_i, \alpha_i)}{2} n_i \quad i=0, \ldots, l.$

($\because$ (6.7) Assertion.)

(7.3) Assertion. i) *The exponents are half integers.*

ii) *The set of exponents does not depend on the choice of the bases* $\{\alpha_0, \ldots, \alpha_l\}$.

($\because$ (6.2) Corollary and (6.1) Assertion iii).)

iii) *Let*

$$m_i^\vee := \frac{I_{R^\vee}(\alpha_i^\vee, \alpha_i^\vee)}{2k^\vee(\alpha_i^\vee)} n_i^\vee \qquad i=0, \dots, l,$$

be the exponents for the dual $(R^\vee, G)$. *Then* m_i*'s and* $m_i^\vee$*'s are proportional. Precisely,*

$$m_i^\vee = \frac{t_1(R, G)t_2(R, G)}{t(R)} m_i \qquad i=0, \dots, l.$$

Proof.

$$\begin{aligned} m_i^\vee &:= \frac{I_{R^\vee}(\alpha_i^\vee, \alpha_i^\vee)}{2k^\vee(\alpha_i^\vee)} n_i^\vee \\ &= t_1(R, G)\frac{n_i}{k^\vee(\alpha_i^\vee)} \qquad \text{(by (4.5.1))} \\ &= t_1(R, G)\frac{I(\alpha_i, \alpha_i)}{2k(\alpha_i)}(\alpha^\vee : \alpha) n_i \qquad \text{(by (6.3.3))} \\ &= \frac{t_1(R, G)}{t_2(R^\vee, G)} m_i = \frac{t_1(R, G)t_2(R, G)}{t(R)} m_i \qquad \text{(by (4.3), (4.4)).} \end{aligned}$$

q.e.d.

(7.4) We prepare an assertion which is rather of technical nature, but will be used in crucial steps of the proofs of (8.6) **Assertion**, (9.6) **Theorem** and (10.1) **Assertion 5** for the proof of **Lemma B.**'

Assertion. *Let* $\{\alpha_0, \dots, \alpha_l\}$ *be a basis for* (R, G) *and* $m_{\alpha_0}, \dots, m_{\alpha_l}$ *be the exponents. For* $\alpha, \beta \in \{\alpha_0, \dots, \alpha_l\}$ *suppose* $m_\beta < m_\alpha$ *and* $I(\alpha, \beta) \neq 0$. *Then*

$$k(\beta)/k(\alpha) = -I(\beta, \alpha^\vee).$$

Proof. From the formula (6.3.6), we have,

$$*) \qquad \frac{k(\alpha)}{k(\beta)} \frac{k^\vee(\beta^\vee)}{k^\vee(\alpha^\vee)} = \frac{I_R(\alpha, \alpha)}{I_R(\beta, \beta)} \left(= \frac{I_{R^\vee}(\beta^\vee, \beta^\vee)}{I_{R^\vee}(\alpha^\vee, \alpha^\vee)} \right) = 1 \text{ or } 2^{\pm 1}.$$

Except for the types $A_1^{(1,1)*}$ or $BC_1^{(2,2)}$, we have either $I(\alpha, \beta^\vee) = -1$ or $I(\beta, \alpha^\vee) = -1$. Thus taking in account of (6.1.3), $*$) implies,

$$\text{either} \quad \frac{k(\alpha)}{k(\beta)} = \frac{I_R(\alpha, \alpha)}{I_R(\beta, \beta)} \quad \text{or} \quad \frac{k^\vee(\beta^\vee)}{k^\vee(\alpha^\vee)} = \frac{I_{R^\vee}(\beta^\vee, \beta^\vee)}{I_{R^\vee}(\alpha^\vee, \alpha^\vee)}.$$

On the other hand due to the proportionality of exponents (7.3) iii) and (6.3.6), the statement of the assertion is equivalent for (R, G) and $(R^\vee, G)$ so that one may prove only one of them. Therefore, except the types $A_1^{(1,1)*}$ and $BC_1^{(2,2)}$, we may assume $I_R(\alpha, \alpha)/2k(\alpha)=I_R(\beta, \beta)/2k(\beta)$ by choosing one of (R, G), $(R^\vee, G)$.

Thus the problem is reduced to show,

**) *If* $n_\beta<n_\alpha$ *and* $I(\alpha, \beta)\neq 0$ *for* $\alpha, \beta\in|\Gamma|$, *then* $I(\alpha, \beta^\vee)=-1$, *where* Γ *is a Dynkin diagram for a reduced affine root system.*

If $I(\alpha, \beta^\vee)=-t\neq 0, -1$ for $\alpha, \beta\in|\Gamma|$, then $\overset{\alpha}{\circ}\xrightarrow[t]{}\overset{\beta}{\circ}$ and therefore $n_\beta\geq n_\alpha$ ($\because$ Folding of Dynkin diagrams). This proves **).

The cases $A_1^{(1,1)*}$ and $BC_1^{(2,2)}$ can be directly verified. q.e.d.

§ 8. Dynkin Diagram for a m.e.a.r.s.

In this paragraph, we introduce a Dynkin diagram $\Gamma_{R,G}$ for an isomorphism class of (R, G). The Dynkin diagram gives a most intrinsic way of describing the marked extended affine root system (R, G) as we see in § 9.

(8.1) **Codimension of** (R, G). Let $\{\alpha_0, \ldots, \alpha_l\}$ be a basis for a m.e.a.r.s. (R, G) (cf. (3.4)) and Γ be the Dynkin diagram for the affine root system $R\cap\bigoplus_{i=0}^{l}\mathbb{R}\alpha_i$. Recall the exponent $m_i:=\dfrac{I_R(\alpha_i, \alpha_i)}{2k(\alpha_i)}n_i$ at each node α_i of Γ.

Definition 1. *Let us denote by* Γ_m *the subdiagram of* Γ *consisting of nodes,*

$$|\Gamma_m|:=\{\alpha_i\in|\Gamma|:\ m_i=m_{\max}\}.$$

Here $$m_{\max}:=\max\{m_0, \ldots, m_l\}.$$

2. *Let us denote by* $\operatorname{cod}(R, G)$ *the number* $\#|\Gamma_m|$ *and call it the codimension of* (R, G).

Note 1. Due to the proportionality of exponents ((7.3) iii)), the set $|\Gamma_m|$ is naturally bijective to

$$|\Gamma_m^\vee|:=\{\alpha_i^\vee\in|\Gamma^\vee|:\ m_i^\vee=\max\{m_0^\vee, \ldots, m_l^\vee\}\}.$$

Therefore $\operatorname{cod}(R, G)=\operatorname{cod}(R^\vee, G)$.

Note 2. The name codimension for $\operatorname{cod}(R, G)$ is introduced here, since it is identified with the codimension of a Hamiltonian system introduced in [30, (1.12)] (i.e. $\mathrm{cod}(R, G)$ is the number of (energy) functions which is necessary to describe the system constructed from (R, G). For the precise identification, see [20].)

(8.2) Hereafter, we use the following notations

i) $a^*:=a+k(a)a$ for $a\in R$. ($k(a)$ is the counting defined in (6.1)).
ii) $|\Gamma_m^*|:=\{a^*:\ a\in|\Gamma_m|\}$.

Definition. *The Dynkin diagram $\Gamma_{R,G}$ for a m.e.a.r.s. (R, G) is defined as the intersection diagram for the set $|\Gamma|\cup|\Gamma_m^*|$. i.e.*
i) *The set of nodes $|\Gamma_{R,G}|$ of the diagram is identified with the set $|\Gamma|\cup|\Gamma_m^*|$ consisting of $l(R)+\mathrm{cod}(R, G)+1$ points.*
ii) *Bonds and arrows among nodes are inserted according to the same rules for a finite root system with an additional case. Namely as follows.*

$\overset{\alpha}{\circ}\qquad\overset{\beta}{\circ}$	*if* $I(\alpha,\beta^\vee)=0$ $(\Longleftrightarrow I(\beta,\alpha^\vee)=0)$,
$\circ\!\!-\!\!\!-\!\!\!-\!\!\circ$	*if* $I(\alpha,\beta^\vee)=I(\beta,\alpha^\vee)=-1$,
$\circ\!\underset{t}{\longleftarrow}\!\circ$	*if* $I(\alpha,\beta^\vee)=-1$, $I(\beta,\alpha^\vee)=-t$, *for* $t=2, 3, 4$,
$\circ\!\underset{\infty}{\text{———}}\!\circ$	*if* $I(\alpha,\beta^\vee)=I(\beta,\alpha^\vee)=-2$,
$\circ\!=\!=\!=\!=\!\circ$	*if* $I(\alpha,\beta^\vee)=I(\beta,\alpha^\vee)=2$.

(8.3) i) *The diagram $\Gamma_{R,G}$ depends only on the isomorphism class of (R, G).* ($\because$ (6.2) **Corollary**, (7.3) ii)).
ii) *The diagram $\Gamma_{R^\vee,G}$ for the dual $(R^\vee, G)$ is the dual diagram of $\Gamma_{R,G}$.* (*i.e. arrows are reversed.*). ($\because$ (8.1) *Note* 1.)
iii) *For each type of m.e.a.r.s., the Dynkin diagram is explicitly given in* 5) *of* **Table** (5.2).
iv) *The Dynkin diagrams distinguish the isomorphism classes of m.e.a.r.s.'s.* (*i.e. if two m.e.a.r.s.'s have the same diagram, they are isomorphic.*)
($\because$ cf. (9.6) **Theorem**)

Note 1. The diagrams for $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1.1)}$ appeared already in a study of the lattice for simple elliptic singularities. (See W. Ebeling [5], where he gave more generalized Dynkin diagrams for the lattice of singularities.) Also W. Ebeling has noticed to the author that the diagram appear also in a study of the presentation of algebras (cf. [2]).

Note 2. B. Verdier has told to the author that he got also some of the diagrams in his note [6].

(8.4) We collect here some elementary facts on the Dynkin diagram and exponents for a (R, G), which we shall use in later paragraphs freely.

i) *The diagram is isomorphic by a transposition of nodes* $\alpha \in \Gamma_m$ *and* $\alpha^* \in \Gamma_m^*$ *which are connected by the bond* $\Longleftrightarrow$.

ii) *The complement* $\Gamma_{R,G} - \Gamma_m \cup \Gamma_m^*$ *decomposes into a finite disjoint union of diagrams of type* A_{l_i}, $i = 1, \ldots, r$.

iii) *Each component* $\Gamma_{A_{l_i}}$ *is connected to* Γ_m *only at one node, say* $\beta_i \in |\Gamma_m|$, *at a terminal node of* $\Gamma_{A_{l_i}}$.

iv) *On the branch* $\Gamma_{A_{l_i}} \cup \{\beta_i\}$ *of* $\Gamma_{R,G}$, *the exponents* m_j*'s are in arithmetic progression. Namely*

$$m_j / m_{\max} = j / (l_i + 1) \quad \textit{for} \quad j = 1, \ldots, l_i + 1,$$

where $\alpha_1, \ldots, \alpha_{l_i+1}$ *are renumbered nodes of* $\Gamma_{A_{l_i}} \cup \{\beta_i\}$.

v) *Put*

$$l_{\max} := \max\{l_i : 1 \leq i \leq r\}.$$

Then

$$l_i + 1 \mid l_{\max} + 1 \quad \textit{for} \quad 1 \leq i \leq r.$$

This implies that $l_{\max} + 1$ *is the smallest common denominators of* $m_{\max}/m_i$, $i = 0, \ldots, l$.

(8.5) As an application of the above facts (8.4), we give a reduction of a calculation of some numerical invariants, which practically helps much.

Assertion. *Let the notations be as before. Then,*

(8.5.1) $$\text{g.c.d.}\{k(\alpha) : \alpha \in |\Gamma_m|\} = 1,$$

(8.5.2) $$t_2(R, G) = \text{g.c.d.}\{k(\alpha) \frac{I_{R^\vee}(\alpha^\vee, \alpha^\vee)}{2} : \alpha \in |\Gamma_m|\}.$$

Proof. The formulae were shown readily if we replace the running index set $\alpha \in |\Gamma_m|$ by $\alpha \in |\Gamma|$ ((6.1) **Assertion** v), (6.3.1)).

Therefore in account of (8.5) iv), it is enough to show the following.

*) *Let* $\alpha, \beta \in |\Gamma|$ *s.t.* $I(\alpha, \beta) \neq 0$ *and* $m_\alpha < m_\beta$.

Then $k(\alpha)/k(\beta)$ *and* $k(\alpha)\frac{I_{R^\vee}(\alpha^\vee, \alpha^\vee)}{2}/k(\beta)\frac{I_{R^\vee}(\beta^\vee, \beta^\vee)}{2}$ *are integers.*

(Proof of *) Due to the (7.4) **Assertion**, in the above situation, $k(\alpha)/k(\beta) = -I(\beta, \alpha^\vee)$ is an integer and hence $(k(\alpha)\frac{I_{R^\vee}(\alpha^\vee, \alpha^\vee)}{2})/(k(\beta)\frac{I_{R^\vee}(\beta^\vee, \beta^\vee)}{2}) = -I(\beta^\vee, \alpha)$ is also an integer. q.e.d.

Corollary. *Let* (R, G) *be a m.e.a.r.s. such that* $\mathrm{cod}(R, G)=1$. *Then*

$$t_2(R, G) = \frac{I_{R^\vee}(\alpha^\vee, \alpha^\vee)}{2} \qquad \text{where} \qquad \alpha \in |\Gamma_m|.$$

(8.6) **Discussions on the Dynkin diagrams.**

There does not exist an apriori definition of a Dynkin diagram for an extended affine root system or for a marked extended affine root system, since there does not exist a concept of a *Weyl chamber* (cf. (3.2) *Note* 2.) comparing to the case of a finite root system or an affine root system. There are several trials to understand the Dynkin diagrams for such a generalized situation by several authors. (See for instance W. Ebeling [5], F. Knörrer [8], P. Kluitmann [10], Van der Lek [26] for our restricted cases.)

The Dynkin diagrams, which we defined in this note, have several similarities and un-similarities with the classical one for finite or affine root systems. We shall list them in the following.

Unsimilarities

i) *The diagram* $\Gamma_{R,G}$ *depends not only on the roots* R *but also on the marking* $G \subset \mathrm{rad} I$. *(cf.* (5.4) **Appendix**)

ii) *The number* $l(R)+\mathrm{cod}(R, G)+1$ *of the nodes of the diagram is larger than the rank* $l(R)+2$ *of the ambient space* F *in general. The linear dependence relations among the nodes as elements of* F *are described by the unipotent part of the pre-Coxeter transformation. (cf.* (9.6))

iii) *The Cartan matrix* $(I(\alpha, \beta^\vee))_{\alpha,\beta \in |\Gamma_{R,G}|}$ *contains positive numbers* $2 = I(\alpha^*, \alpha^\vee)$ *for* $\alpha \in |\Gamma_m|$ *in its off diagonal part. (compare with* [33])

(Obviously all these three facts are related to the fact: *the high degeneration* $(\mu_0 = 2)$ of the metric I.)

Similarities

i) *The lattice* $Q(R)$ *is generated by* $\alpha\in|\Gamma_{R,G}|$. *The group* W_R *is generated by* w_α *for* $\alpha\in|\Gamma_{R,G}|$. *The set of roots* R *is equal to* $\bigcup_{\alpha\in|\Gamma|} W_R\,\alpha$. ((9.6) Theorem)

ii) *Coxeter transformation for* (R, G) *is naturally defined as a product of* w_α *for* $\alpha\in|\Gamma_{R,G}|$, *which is of finite order, whose eigenvalues describe the degrees of the invariants of* $\tilde{W}_R$. ((9.7))

iii) *Up to isomorphy, the root system* R *together with the marking* G *is reconstructed from the datum of* $\Gamma_{R,G}$. ((9.6))

iv) *The diagrams behave naturally with respect to the automorphism of* (R, G) *and foldings of* (R, G). (cf. § 12)

v) *The multiplicity of the discriminant* (= *the square of the fundamental anti-invariant of* $\tilde{W}_R$) *is equal to* $\#|\Gamma_{R,G}|$ $(=l(R)+\mathrm{cod}(R, G)+1)$.

From these phenomenon, one would naturally expect an existence of a theory of infinite dimensional Lie algebra associated to these diagrams. (cf. Slodowy [23])

In fact there exist *simple elliptic singularities* ([28]) of types called $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{D}_5$, $\tilde{A}_4$ such that the middle homology groups of their smoothing contain naturally root systems of types $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$, $D_5^{(1,1)}$ and $A_4^{(1,1)}$ as the set of vanishing cycles respectively. In these cases, the multiplicity of the discriminant of the unfolding of the singularities is equal to $l(R)+\mathrm{cod}(R, G)+1$, where $\mathrm{cod}(R, G)$ coincides with the codimension of the unfolding. Then by a suitable choice of pathes in the base spaces of the unfolding, one can find a "*strongly distinguished basis*" *for the middle homology group such that their intersection diagrams coincide with the Dynkin diagrams of the corresponding root systems defined in this note*. (cf. [31],[34])

In [20] we shall construct a quotient space by the natural action of $\tilde{W}_{R,G}$ (cf. § 11) on a complex half space. The space will be identified with the base space of the universal unfolding of the singularities $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{D}_5$, $\tilde{A}_4$ for the case $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$, $D_5^{(1,1)}$, $A_4^{(1,1)}$ (cf. also E. Looijenga [12], P. Slodowy [22]). This might give a strong justification for the definition of the Dynkin diagram in this note.

§ 9 Coxeter Transformation 1

(Construction of (R, G) from the diagram $\Gamma_{R,G}$)

In this paragraph, we construct the m.e.a.r.s. (R, G) up to an isomorphy from the data of its diagram $\Gamma_{R,G}$ ((9.6) **Theorem**). Here the concept of a (pre-) Coxeter transformation ((9.3) **Def.**) plays an essential role.

(9.1) **A decomposition of $\Gamma_{R,G}$.** Let $\Gamma_{R,G}$ be a Dynkin diagram for a m.e.a.r.s. (R, G). The diagram consists of $l(R)+\mathrm{cod}(R, G)+1$ number of nodes and contains $\mathrm{cod}(R, G)$ number of bonds $\Longleftrightarrow$.

Let us fix a decomposition of the set of $2\mathrm{cod}(R, G)$ nodes connected by such bonds into two subsets $|\Gamma_m|$ and $|\Gamma_m^*|$ so that two nodes at the terminal of any such bond are divided into the two sets. Such decomposition is unique up to isomorphism of the diagram (cf. (8.4) i)). We shall denote by α^* the node in $|\Gamma_m^*|$ which is connected to the node α of $|\Gamma_m|$ by the bond $\Longleftrightarrow$.

Recall that the complement $\Gamma_{R,G}-\Gamma_m\cup\Gamma_m^*$ decomposes into connected components of diagrams of types $A_{l_i}(i=1, \ldots, r)$, so that we obtain a decomposition of the diagram $\Gamma_{R,G}$.

$$\Gamma_{R,G} = \Gamma_m\cup\Gamma_m^*\cup\coprod_{i=1}^{r}\Gamma_{A_{l_i}}. \tag{9.1.1}$$

(9.2) **The vector space $\hat{F}$ with a metric $\hat{I}$.** Define,

1) $\hat{F}$:= *the vector space of rank* $l(R)+\mathrm{cod}(R, G)+1$ *spanned by* $\alpha\in|\Gamma_{R,G}|$.

To avoid a confusion, the base of $\hat{F}$ corresponding to a node $\alpha\in|\Gamma_{R,G}|$ will be denoted by $\hat{\alpha}$.

2) *There exists a symmetric bilinear form* $\hat{I}$ *on* $\hat{F}$ *s.t.*

i) $\hat{I}(\hat{\alpha}, \hat{\alpha})>0$ *for* $\alpha\in|\Gamma_{R,G}|$.

ii) $\hat{I}$ *satisfies the same rule for* I *stated in* (8.2) **Def.**

Such metric $\hat{I}$ is unique up to a positive constant factor.

3) $\hat{I}$ *is positive semi-definite. The radical of* $\hat{I}$ *has rank* $\mathrm{cod}(R, G)+1$, *which is spanned by*

$$\hat{b} := \sum_{\alpha\in\Gamma_{R,G}-\Gamma_m^*} n_\alpha\hat{\alpha} \quad \text{and} \quad \hat{\alpha}-\hat{\alpha}^* \quad \textit{for} \quad \alpha\in|\Gamma_m|,$$

where n_α *are the coefficients in* (3.3.6).

Proof. By definition $\hat{\alpha}^*-\hat{\alpha}$ for $\alpha\in|\Gamma_m|$ belongs to rad $\hat{I}$. The quotient space $\hat{F}\,/\bigoplus_{\alpha\in|\Gamma_m|}\mathbb{R}(\hat{\alpha}^*-\hat{\alpha})$ may be regarded to be spanned by the diagram $\Gamma_{R,G}-\Gamma_m^*$ for a reduced affine root system, where the $\hat{I}$ induces a positive semi-definite metric, whose radical is spanned by $\hat{b}$. q.e.d.

(9.3) **Pre-Coxeter transformation.**

For a node $\alpha\in|\Gamma_{R,G}|$, let us define a reflexion $\hat{w}_\alpha\in GL(\hat{F})$,

$$\hat{w}_\alpha(u):=u-\hat{I}(u,\hat{\alpha}^\vee)\hat{\alpha}\quad\text{where}\quad \hat{\alpha}^\vee:=2\hat{\alpha}/\hat{I}(\hat{\alpha},\hat{\alpha}).$$

Definition. *An element $\hat{c}$ of $O(\hat{F},\hat{I})$ is called a pre-Coxeter transformation for (R,G), if it is a product of $\hat{w}_\alpha$ for $\alpha\in\Gamma_{R,G}-\Gamma_m\cup\Gamma_m^*$ and $\hat{w}_\alpha\hat{w}_{\alpha^*}$ for $\alpha\in\Gamma_m$ for any ordering of them.*

Due to **lemma 1** of [1] Ch. V, § 6, every pre-Coxeter transformations are conjugate in the group $\hat{W}:=\langle\hat{w}_\alpha:\ \alpha\in\Gamma_{R,G}\rangle$.

(9.4) **Jordan decomposition of $\hat{c}$.**

The following is a key Lemma in this paragraph.

Lemma. *Let $\hat{c}$ be a pre-Coxeter transformation for the diagram $\Gamma_{R,G}$. Let the Jordan decomposition of $\hat{c}$ be,*

(9.4.1) $$\hat{c}=SU.$$

Then, the set of eigenvalues of the semi-simple part S is given by

(9.4.2) $$\exp(2\pi\sqrt{-1}m_\alpha/m_{\max})\quad \textit{for}\quad \alpha\in|\Gamma_{R,G}|.$$

Hence

(9.4.3) $$S^{(l_{\max}+1)}=1.$$

(9.4.4) $\quad (U-1)^2=0,\quad$ (i.e. $\log U=U-1$).

(9.4.5) $$\mathrm{Image}(U-1)=\sum_{\alpha,\beta\in|\Gamma_m|}\mathbb{R}\left(\frac{\hat{\alpha}^*-\hat{\alpha}}{k(\alpha)}-\frac{\hat{\beta}^*-\hat{\beta}}{k(\beta)}\right)\subset \mathrm{rad}\,\hat{I}.$$

Proof. For any $\beta\in|\Gamma_m|$, define

(9.4.6) $$\hat{b}_\beta:=n_\beta\hat{\beta}+\textstyle\sum' n_\alpha\hat{\alpha},$$

(9.4.7) $$\hat{b}_\beta^*:=n_\beta\hat{\beta}^*+\textstyle\sum' n_\alpha\hat{\alpha}^*.$$

Here $\sum'$ means the summation over all nodes α belonging to components $\Gamma_{A_{l_i}}$ which is connected to $\beta\in|\Gamma_m|$. Due to (9.2) 3), rad $\hat{I}$ is spanned by $\hat{b}=\sum_{\beta\in|\Gamma_m|}\hat{b}_\beta$ and by $\hat{b}_\beta^*-\hat{b}_\beta$ for $\beta\in|\Gamma_m|$.

For each component $\Gamma_{A_{l_i}}$ of (9.1.1), define a space

$$\hat{F}_i := \bigoplus_{\alpha\in\Gamma_{A_{l_i}}} \mathbb{R}\hat{\alpha} \qquad (i=1,\ldots,r), \tag{9.4.8}$$

so that we obtain an orthogonal direct sum decomposition,

$$\hat{F} = \bigoplus_{i=1}^{r}\hat{F}_i \oplus \bigoplus_{\alpha\in|\Gamma_m|}\mathbb{R}\hat{b}_\alpha \oplus \bigoplus_{\alpha\in|\Gamma_m|}\mathbb{R}(\hat{b}^*_\alpha-\hat{b}_\alpha). \tag{9.4.9}$$

Since the statements of the lemma depends only on the conjugacy class of $\hat{c}$, we may choose one $\hat{c}$ as follows.

$$\hat{c} := \hat{c}_1\ldots\hat{c}_r\circ\hat{t}, \tag{9.4.10}$$

where

$$\hat{c}_i := \prod_{\alpha\in|\Gamma_{A_{l_i}}|}\hat{w}_\alpha \qquad (i=1,\ldots,r),$$

$$\hat{t} := \prod_{\alpha\in|\Gamma_m|}\hat{w}_\alpha\hat{w}_{\alpha^*}.$$

To analize the action of $\hat{c}$ on $\hat{F}$, we look at the actions of $\hat{c}_1,\ldots,\hat{c}_r$ and $\hat{t}$ on each factor spaces of (9.4.9).

Assertion 1.

i) $\hat{c}_i|\hat{F}_j = \begin{cases} \text{A Coxeter transformation for} \\ \quad A_{l_i} \quad \text{for} \quad i=j, \\ \quad id_{\hat{F}_j} \quad \text{for} \quad i\neq j. \end{cases}$

ii) $\hat{c}_i\hat{b}_\alpha = \hat{b}_\alpha$ *for* $\alpha\in|\Gamma_m|,\ 1\leqq i\leqq r,$

$\hat{c}_i\hat{b}^*_\alpha = \hat{b}^*_\alpha$ *for* $\alpha\in|\Gamma_m|,\ 1\leqq i\leqq r.$

iii) $\hat{t} = id_{\hat{F}} - B$

where

$$B:\hat{F}\longrightarrow\bigoplus_{\alpha\in|\Gamma_m|}\mathbb{R}(\alpha^*-\alpha)\subset\hat{F} \text{ is defined by } u\longmapsto\sum_{\beta\in|\Gamma_m|}\hat{I}(u,\beta^\vee)(\beta^*-\beta).$$

Proof. i) Trivial, since the spaces $\hat{F}_1,\ldots,\hat{F}_r$ are orthogonal to each other.

ii) We show only the first formula.

If Γ_{A_i} and $\alpha\in|\Gamma_m|$ are disconnected, α is orthogonal to $\hat{F}_i$ and hence $\hat{c}_i\hat{b}_\alpha = \hat{b}_\alpha$. If Γ_{A_i} and $\alpha\in|\Gamma_m|$ are connected, $\hat{c}_i\hat{b}_\alpha = \hat{c}_i(\hat{b}-\sum_{\substack{\beta\in|\Gamma_m|\\ \beta\neq\alpha}}\hat{b}_\beta) = \hat{b}-\sum_{\substack{\beta\in|\Gamma_m|\\ \beta\neq\alpha}}\hat{b}_\beta = \hat{b}_\alpha$.

iii) Using $\hat{I}(\hat{\alpha}^*,\hat{\alpha}^\vee)=2$ and $\hat{I}(u,\hat{\alpha}^*)=\hat{I}(u,\hat{\alpha})$, one computes,

$$\hat{w}_\alpha \hat{w}_{\alpha^*}(u) = u - \hat{I}\,(u, \hat{\alpha}^\vee)(\hat{\alpha}^* - \hat{\alpha}).$$

Since $\hat{\alpha}^* - \hat{\alpha} \in \mathrm{rad}\, \hat{I}$ for $\alpha \in |\Gamma_m|$,, we obtain,

$$\hat{t}\,(u) = u - \sum_{\alpha \in |\Gamma_m|} \hat{I}\,(u, \hat{\alpha}^\vee)(\hat{\alpha}^* - \hat{\alpha}). \qquad \text{q.e.d.}$$

Assertion 2. *Let the notations be as above. Define*

$$S := \hat{c}_1 \ldots \hat{c}_r (id_{\hat{F}} - B \circ p_1), \tag{9.4.11}$$

$$U := id_{\hat{F}} - B \circ p_2 \tag{9.4.12}$$

where p_1 and p_2 are linear projections from $\hat{F}$ to the direct factor spaces $\bigoplus_{i=1}^{r} \hat{F}_i$ and $\bigoplus_{\alpha \in |\Gamma_m|} \mathbb{R}\hat{b}_\alpha$ in (9.4.9) respectively.

Then $\hat{c} = SU$ gives the Jordan decomposition of $\hat{c}$.

Proof. Using the Assertion 1 one checks immediately $p_1 \circ B = p_2 \circ B = [B \circ p_2, \hat{c}_1 \ldots \hat{c}_r] = 0$ and $B(p_1 + p_2) = B$. Then by definition

$$\begin{aligned} SU &= \hat{c}_1 \ldots \hat{c}_r (id_{\hat{F}} - B \circ p_1)(id_{\hat{F}} - B \circ p_2) \\ &= \hat{c}_1 \ldots \hat{c}_r (id_{\hat{F}} - B \circ p_1 - B \circ p_2) = \hat{c}_1 \ldots \hat{c}_r (id_{\hat{F}} - B) = \hat{c}. \\ US &= (id_{\hat{F}} - B \circ p_2) \hat{c}_1 \ldots \hat{c}_r (id_{\hat{F}} - B \circ p_1) \\ &= (\hat{c}_1 \ldots \hat{c}_r (id_{\hat{F}} - B \circ p_2) - [B \circ p_2, \hat{c}_1 \ldots \hat{c}_r])(id_{\hat{F}} - B \circ p_1) \\ &= \hat{c}_1 \ldots \hat{c}_r (id_{\hat{F}} - B \circ p_2 - B \circ p_1) = \hat{c}_1 \ldots \hat{c}_r (id_{\hat{F}} - B) = \hat{c}. \end{aligned}$$

The Unipotency of U is a direct calculation as,

$$(U - id_{\hat{F}})^2 = B \circ p_2 \circ B \circ p_2 = 0. \tag{9.4.13}$$

To show the semi-simplicity of S, we show that S is conjugate to $\hat{c}_1 \ldots \hat{c}_r$, which is semi-simple since $\hat{c}_i$ is a Coxeter transformation on $\hat{F}_i$ for the type A_{l_i} and an identity on the other factors of the decomposition (9.4.9). q.e.d.

Assertion 3. *Put*

$$T := id_{\hat{F}} - B \circ d \circ p_1$$

where $d := (\hat{c}_1 \ldots \hat{c}_r | \bigoplus_{i=1}^{r} \hat{F}_i - id)^{-1} : \bigoplus_{i=1}^{r} \hat{F}_i \longrightarrow \bigoplus_{i=1}^{r} \hat{F}_i$. *Then T is invertible and*

$$T^{-1} S T = \hat{c}_1 \ldots \hat{c}_r. \tag{9.4.14}$$

Proof. T is well defined, since $\hat{c}_i$ does not have eigen value 1 on $\hat{F}_i$. T is invertible, since $(T-id_{\hat{F}})^2=0$.

$$
\begin{aligned}
ST&=\hat{c}_1\dots\hat{c}_r(id_{\hat{F}}-B\circ p_1)(id_{\hat{F}}-B\cdot d\cdot p_1)\\
&=\hat{c}_1\dots\hat{c}_r-B\circ p_1-B\cdot d\circ p_1\\
&=(id_{\hat{F}}-B\circ d\circ p_1)\hat{c}_1\dots\hat{c}_r+B(d\cdot p_1(\hat{c}_1\dots\hat{c}_r-1)-p_1)\\
&=T\hat{c}_1\dots\hat{c}_r.
\end{aligned}
$$

q.e.d.

Proof of the lemma (9.4).

(9.4.2) is a direct consequence of (9.4.14), **Assertion 1** i) ii) and (8.5) iv) v). The unipotency (9.4.4) is shown in (9.4.13).

Let us show (9.4.5). Due to (9.4.12), $Im(U-1)=Im(B\circ p_2)$.

$$
\begin{aligned}
B(\hat{b}_\alpha)&=\sum_{\beta\in|\Gamma_m|}\hat{I}(\hat{b}_\alpha,\hat{\beta}^\vee)(\hat{\beta}^*-\hat{\beta})\\
&=\hat{I}(\hat{b}_\alpha,\hat{\alpha}^\vee)(\hat{\alpha}^*-\hat{\alpha})+\sum_{\substack{\beta\in|\Gamma_m|\\ \beta\neq\alpha}}\hat{I}(\hat{b}_\alpha,\hat{\beta}^\vee)(\hat{\beta}^*-\hat{\beta})\\
&=\hat{I}(\hat{b}-\sum_{\substack{\beta\in|\Gamma_m|\\ \beta\neq\alpha}}\hat{b}_\beta,\hat{\alpha}^\vee)(\hat{\alpha}^*-\hat{\alpha})+\sum_{\substack{\beta\in|\Gamma_m|\\ \beta\neq\alpha}}\hat{I}(n_\alpha\hat{\alpha},\hat{\beta}^\vee)(\hat{\beta}^*-\hat{\beta})\\
&=\sum_{\substack{\beta\in|\Gamma_m|\\ \beta\neq\alpha}}\hat{I}(\hat{\beta},\hat{\alpha})n_\alpha n_\beta\left(\frac{2}{n_\beta\hat{I}(\hat{\beta},\hat{\beta})}(\hat{\beta}^*-\hat{\beta})-\frac{2}{n_\alpha\hat{I}(\hat{\alpha},\hat{\alpha})}(\hat{\alpha}^*-\hat{\alpha})\right)\\
&=\sum_{\beta\in|\Gamma_m|}\hat{I}(\hat{\beta},\hat{\alpha})n_\alpha n_\beta\frac{(I_R:\hat{I})}{m_{\max}}\left(\frac{\hat{\beta}^*-\hat{\beta}}{k(\beta)}-\frac{\hat{\alpha}^*-\hat{\alpha}}{k(\alpha)}\right).
\end{aligned}
$$

Here we use a relation $n_\alpha\hat{I}(\hat{\alpha},\hat{\alpha})=k(\alpha)m_\alpha(\hat{I}:I_R)$, and $m_\alpha=m_{\max}$ for $\alpha\in|\Gamma_m|$.

Using the last expression, it is a straightforward calculation in linear algebra that the fact that Γ_m is a connected linear diagram implies that $B(\hat{b}_\alpha)$ for $\alpha\in|\Gamma_m|$ spans the space $\sum_{\alpha,\beta\in|\Gamma_m|}\mathbb{R}\left(\frac{\hat{\beta}^*-\hat{\beta}}{k(\beta)}-\frac{\hat{\alpha}^*-\hat{\alpha}}{k(\alpha)}\right)$. q.e.d.

Note. The formula (9.4.5) implies that *the abstract data of the Dynkin diagram $\Gamma_{R,G}$ determine automatically the counting constant $k(\alpha):=(\alpha^*-\alpha):\alpha$ for $\alpha\in|\Gamma_m|$ in the following way.*

First, notice that $\mathrm{Im}(U-1)$ does not depend on the choice of the pre-Coxeter transformation. ($\because$ $\mathrm{Im}(U-1)\subset\mathrm{rad}\,\hat{I}$, so that it is pointwisely fixed by the group $\hat{W}$.) Through the formula (9.4.5), the relation $\mathrm{Im}(U-1)\equiv0$ determines the proportion among $k(\alpha)$ ($\alpha\in|\Gamma_m|$). Since g.c.d. $(k(\alpha):\alpha\in|\Gamma_m|)=1$ ((8.5.1)), these data determine $k(\alpha)$ ($\alpha\in|\Gamma_m|$) uniquely.

(9.5) Assertion. *Put*

$\hat{W}$: = *the sub-group of* $O(\hat{F}, \hat{I})$ *generated by* $\hat{w}_\alpha$ *for* $\alpha \in |\Gamma_{R,G}|$.

$\hat{R} := \bigcup_{\alpha \in |\Gamma_{R,G}|} \hat{W}\hat{\alpha}$.

Then $\hat{R}$ *is a root system belonging to* $\hat{I}$.

Proof. Recall the definition (1.2) for a root system belonging to $\hat{I}$. It is almost a routine work to check that $\hat{R}$ is a root system in the sense of (1.2). (cf. (1. 3) Ex. 4.). q.e.d.

(9.6) We arrived at a goal of this paragraph.

Let $\Gamma_{R,G}$ be the Dynkin diagram for a m.e.a.r.s. (R, G) w.r.t. a basis $\{\alpha_0, \ldots, \alpha_l\}$. Put

$F_\Gamma := \hat{F} / \mathrm{Im}(U-1)$,

I_Γ : = the metric on F_Γ induced from $\hat{I}$,

R_Γ : = the image of $\hat{R}$ in F_Γ,

W_Γ : = the subgroup of $O(F_\Gamma, I_\Gamma)$ induced from $\hat{W}$,

$G_\Gamma := \bigoplus_{\alpha \in |\Gamma_m|} \mathbb{R}(\hat{\alpha}^* - \hat{\alpha}) / \mathrm{Im}(U-1)$ (one dimensional subspace of F_Γ),

$Q_\Gamma := \bigoplus_{\alpha \in |\Gamma_{R,G}|} \mathbb{Z}\hat{\alpha} / ((\bigoplus_{\alpha \in |\Gamma_{R,G}|} \mathbb{Z}\hat{\alpha}) \cap \mathrm{Im}(U-1))$.

Here the notations $\hat{F}$, $\hat{I}$, U, $\hat{W}$, $\hat{R}$ are defined in (9.2) 1), 2), (9.4.1) and (9.5).

Theorem. *Let the notations be as above.*

1. *R_Γ is an extended affine root system belonging to I_Γ, such that $W_{R_\Gamma} = W_\Gamma$, $Q(R_\Gamma) = Q_\Gamma$. The subspace G_Γ defines a marking of R_Γ.*

2. *The marked extended affine root systems (R, G) and (R_Γ, G_Γ) are isomorphic.*

Proof. *1. Due to* (9.4.5), $\mathrm{Im}(U-1)$ *is a two codimensional subspace of* rad $\hat{I}$, which is defined over $\mathbb{Q}$. (i.e. $(\bigoplus_{\alpha \in |\Gamma_{R,G}|} \mathbb{Z}\hat{\alpha}) \cap \mathrm{Im}(U-1)$ is a full lattice in $\mathrm{Im}(U-1)$.) Then the image R_Γ of $\hat{R}$ in the quotient space F_Γ is a root system (cf. (1.8) Assertion). G_Γ is obviously a rank 1 module defined over $\mathbb{Q}$.

2. Since a change of the choice of the decomposition $\Gamma_m \cup \Gamma_m^*$ in (9.1) induces an

isomorphism of the diagram $\Gamma_{R,G}$ ((8.5) i)), it induces an isomorphism of the root system R_Γ. Therefore let us fix a choice of Γ_m and Γ_m^* as done in (8.1) (8.2).

Define a linear map $\hat{F} \to F$ by $\hat{\alpha} \mapsto \alpha$ for $\alpha \in |\Gamma_{R,G}|$. Since $\dfrac{\alpha^* - \alpha}{k(\alpha)} = a$ for any $\alpha \in |\Gamma_m|$ (cf. (8.2) i)), taking account of (9.4.5), one sees that the map is factorized by an isomorphism $F_\Gamma \xrightarrow{\sim} F$, inducing $G_\Gamma \simeq G$. Furthermore this identification induces $I_\Gamma = cI$ for some $c > 0$.

By this identification W_Γ becomes a subgroup of W_R and hence $R_\Gamma = \bigcup_{\alpha \in |\Gamma_{R,G}|} W_\Gamma \alpha$ is a subset of R. We want to show that $R_\Gamma \supset R$, which implies also $W_\Gamma = W_R$.

Let us put $\{\alpha_0, \ldots, \alpha_l\} := |\Gamma_{R,G}| - |\Gamma_m|^*$. Then $\alpha_0, \ldots, \alpha_l$ is a basis for the reduced affine root system $R \cap L^{l+1}$ where $L^{l+1} := \bigoplus_{i=0}^{l} \mathbb{R}\alpha_i$ (cf. (3.3)). Recalling (6.1.5) and (6.14), we have

$$\begin{aligned} R &= \bigcup_{\alpha \in R \cap L^{l+1}} (\alpha + \mathbb{Z}k(\alpha)a) \\ &= \bigcup_{i=0}^{l} W_{R \cap L^{l+1}} (\alpha_i + \mathbb{Z}k(\alpha_i)a). \end{aligned}$$

Since $W_{R \cap L^{l+1}} = \langle w_{\alpha_0}, \ldots, w_{\alpha_l} \rangle \subset W_\Gamma$, the last expression implies that if $\alpha_i + \mathbb{Z}k(\alpha_i)a \subset R_\Gamma$ for $i = 0, \ldots, l$, then $R \subset R_\Gamma$.

Let us show the inclusion $\alpha + k(\alpha)\mathbb{Z}a \subset R_\Gamma$ for $\alpha \in |\Gamma_{R,G}| - |\Gamma_m^*|$ by decent induction on the exponent m_α.

If $\alpha \in |\Gamma_m|$, then

$$\begin{aligned} (w_\alpha w_{\alpha^*})^m(\alpha) &= \alpha - 2m(\alpha - \alpha^*) = \alpha + 2mk(\alpha)a \quad \text{for} \quad m \in \mathbb{Z}. \\ (w_\alpha w_{\alpha^*})^m(\alpha^*) &= \alpha^* - 2m(\alpha - \alpha^*) = \alpha + (2m+1)k(\alpha)a \quad \text{for} \quad m \in \mathbb{Z}. \end{aligned}$$

If α, β are nodes of $\Gamma_{R,G}$ which are joint by a bond. Suppose $m_\beta < m_\alpha$. By induction we assume that $\alpha, \alpha^* \in R_\Gamma$. Then (7.4) **Assertion** implies,

$$\begin{aligned} (w_\alpha w_{\alpha^*})^m \beta &= \beta - mI(\beta, \alpha^\vee)(\alpha - \alpha^*) \\ &= \beta + mI(\beta, \alpha^\vee)k(\alpha)a \\ &= \beta - mk(\beta)a \quad \text{for} \quad m \in \mathbb{Z}. \end{aligned}$$

This completes the proof of the theorem.

(9.7) **The Coxeter transformation** c. Let $\alpha_0, \ldots, \alpha_l$ be a basis for a m.e.a.r.s. (R, G) and $\Gamma_{R,G}$ be the Dynkin diagram for (R, G) w.r.t. the basis (cf. (8.2)).

Definition. *A Coxeter transformation c for (R, G) w.r.t. the basis $\{\alpha_0, \ldots, \alpha_l\}$ is an element of W_R defined as a product of $w_\alpha w_{\alpha^*}$ for $\alpha \in |\Gamma_m|$ and w_α for $\alpha \in |\Gamma_{R,G}| - |\Gamma_m| \cup |\Gamma_m^*|$.*

As a conclusion of the study in this paragraph, we have;

Lemma A. i) *The conjugacy class of a Coxeter transformation in W_R depends only on the linear space $L^{l+1} := \bigoplus_{i=0}^{l} \mathbb{R}\alpha_i$ and the sign of the generator a of $Q(R) \cap G$ (cf. (2.3.1)), but neither on the order of the product for the expression of c nor on the choice of the Weyl chamber C in choosing a basis for R/G (cf. (3.3)).*

The change $a \mapsto -a$ of the sign of the generator a of G brings the conjugacy class of c to the conjugacy class of c^{-1}.

ii) *Let $\varphi : (F, G) \simeq (F, G)$ be an (outer) automorphism for (R, G) and c be a Coxeter transformation w.r.t. a basis $\{\alpha_0, \ldots, \alpha_l\}$. Then $ad_\varphi(c) := \varphi c \varphi^{-1} \in W_R$ is a Coxeter transformation w.r.t. the basis $\{\varphi\alpha_0, \ldots, \varphi\alpha_l\}$.*

iii) *A Coxeter transformation c for (R, G) is semi-simple of finite order $l_{\max}+1$. The set of eigenvalues of c is given by,*

$$\exp(2\pi\sqrt{-1}m_\alpha/m_{\max}) \quad \text{for } \alpha \in \Gamma_{R,G} - \Gamma_m^* \quad \text{and} \quad 1 = \exp(2\pi\sqrt{-1}0).$$

Proof. i) The conjugacy class of c does not depend on the order of the product to present c, due to **Lemma 1** [1] Ch. V, § 6.

The change of the sign a to $-a$, induces the change of $\alpha^* := \alpha + k(\alpha)(-a)$ (cf. (8.2) i)). Therefore $w_\alpha w_{\alpha^*}$ is changed to $w_{\alpha^*} w_\alpha = (w_\alpha w_{\alpha^*})^{-1}$. Therefore if c is a Coxeter transformation for a, then c^{-1} is a Coxeter transformation for the generator $-a$.

ii) Let $\varphi : F \xrightarrow{\sim} F$ be an automorphism of (R, G). Since $\varphi(\alpha^*) = \varphi(\alpha)^*$ (cf. (6.1.4)) and $\varphi w_\alpha \varphi^{-1} = w_{\varphi\alpha}$ for $\alpha \in R$, if c is a Coxeter transformation w.r.t. a basis $\alpha_0, \ldots, \alpha_l$ then $\varphi c \varphi^{-1}$ is a Coxeter transformation w.r.t. the basis $\varphi\alpha_0, \ldots, \varphi\alpha_l$.

iii) Any Coxeter transformation c is the image of the pre-Coxeter transformation $\hat{c}$ by the natural projection $\hat{W} \twoheadrightarrow W_R$. Since $\hat{c}$ act as identity on the kernel $\mathrm{Im}(U-1) \subset \mathrm{rad}\, \hat{I}$ of the projection $\hat{F} \twoheadrightarrow F$, the eigenvalues of c is equal to that of $\hat{c}$ minus $\{1, \ldots, 1 : k\text{-times}\}$. Then (9.4) **Lemma** i) implies the assertion. q.e.d.

§ 10. Coxeter Transformation 2

(The existence of regular eigenspaces of a Coxeter transformation)

(10.1) Let c be a Coxeter transformation of a m.e.a.r.s. (R, G) defined in (9.7). The following **Lemma B** is the main result of this paper, which plays a crucial role in the construction of flat ϑ-invariants (cf. [20]). The proof of the **Lemma B** is now a straight forward work.

Lemma B.

$$R\cap \mathrm{Im}(c-id_F) = \phi.$$

Proof. Recall that $c^{l_{\max}+1}=1$ ((9.7) **Lemma A** iii)). Hence $(c-1)(c^{l_{\max}}+c^{l_{\max}-1}+\cdots+1)=0$. Due to the semi-simplicity of c, $\mathrm{Im}(c-1)=\ker(c^{l_{\max}}+\cdots+1)$. Thus what we have to show is that *no root α of R satisfies the equation,*

$$P(c)\alpha=0, \quad \textit{where } P(c) := c^{l_{\max}}+c^{l_{\max}-1}+\cdots+1. \tag{10.1.1}$$

Due to (9.7) **Lemma A** i) and ii), the statement of the **Lemma B** does not depend on a choice of a Coxeter transformation. Therefore we take a Coxeter transformation c which is the image of the pre-Coxeter transformation $\hat{c}$ of (9.4.10) and we use freely the results and notations in (9.1)-(9.6).

Put,

(10.1.2)
$$c := c_1\cdots c_r t$$
where
$$c_i := \prod_{\alpha\in|\Gamma_{A_{l_i}}|} w_\alpha \qquad (i=1, \dots, r),$$
$$t := \prod_{\alpha\in|\Gamma_m|} w_\alpha w_{\alpha^*},$$

$$F_i := \bigoplus_{\alpha\in|\Gamma_{A_{l_i}}|} \mathbf{R}\alpha \qquad (i=1, \dots, r), \tag{10.1.3}$$

$$b_\alpha := n_\alpha\alpha+\textstyle\sum' n_\beta\beta \qquad (\alpha\in|\Gamma_m|). \tag{10.1.4}$$

Here $\prod$ in (10.1.2) means a product for a fixed linear ordering and $\sum'$ in (10.1.4) means the summation over all nodes of components $\Gamma_{A_{l_i}}$ which is connected to the $\alpha\in|\Gamma_m|$.

Assertion 1. $P(c)F_i \equiv 0 \mod G$ *for* $i=1, \dots, r$.

Proof. Due to **Assertion 1.** iii) in the proof of (9.4), $t\equiv id_F \mod G$. Also due to the same **Assertion 1.** i), $c|F_i \equiv c_i|F_i$ and hence $P(c)|F_i \equiv P(c_i)|F_i$ mod G. On the other hand $l_i+1 | l_{\max}+1$ ((8.5) v)) so that $P_i(c_i):=c_i^{l_i}+c_i^{l_i-1}+\cdots+1$ divides $P(c_i)$. Since $c_i|F_i$ is a Coxeter transformation for A_{l_i} so that $P_i(c_i)|F_i=0$ and hence $P(c_i)|F_i=0$. q.e.d.

Assertion 2. $P(c)b_\alpha \equiv (l_{\max}+1)b_\alpha$ mod G *for* $\alpha \in |\Gamma_m|$.

Proof. Using (9.4) **Assertion 1.** ii),

$$P(c)b_\alpha \equiv P(1)b_\alpha = (l_{\max}+1)b_\alpha \quad \text{for} \quad \alpha \in |\Gamma_m|.$$

Assertion 3. *If a root $\alpha \in R$ satisfies the equation* (10.1.1), *then $\alpha=\beta+ma$ where β is a root of A_{l_i} for some i and $m \in \mathbb{Z}$.*

Proof. One may choose $b_\beta\,(\beta \in |\Gamma_m|)$, $\alpha\,(\alpha \in |\Gamma_{A_{l_i}}|,\ i=1, \ldots, r)$ and a as for (rational) basis for the linear space F.
Put

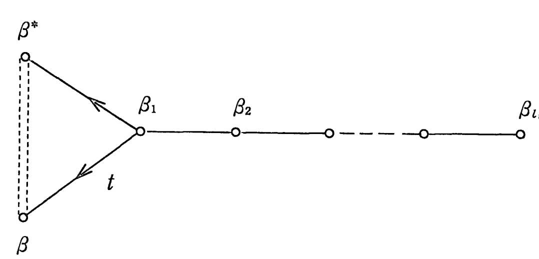
$$*)\quad \alpha = \sum_{\beta \in |\Gamma_m|} c_\beta b_\beta + \sum_{i=1}^{r} \sum_{\beta \in |\Gamma_{A_{l_i}}|} c_\beta \beta + da \quad \text{for } c_\beta,\ d \in \mathbb{Q}.$$

Apply $P(c)$ on $*$) and applying **Assertions 1** and **2**,

$$P(c)\alpha \equiv (l_{\max}+1) \sum_{\beta \in |\Gamma_m|} c_\beta b_\beta \quad \text{mod } G.$$

Since b_β ($\beta \in |\Gamma_m|$) are linearly independent mod G, $P(c)\alpha=0$ implies $c_\beta=0$ for $\beta \in |\Gamma_m|$. Therefore if α of $*$) is a root of R with $P(c)\alpha=0$, then $\sum_{i=1}^{r} \sum_{\beta \in |\Gamma_{A_{l_i}}|} c_\beta \beta$ is an affine root in $R \cap (\bigoplus_{i=0}^{l} \mathbb{R}\alpha_i)$. Since the set $\{\beta \in |\Gamma_{R,G}|-|\Gamma_m^*| : c_\beta \neq 0\}$ is connected, there exists some i, $1 \leq i \leq r$, so that $\{\beta : c_\beta \neq 0\} \subset |\Gamma_{A_{l_i}}|$. q.e.d.

4. **Notations.** Let the component $\Gamma_{A_{l_i}}$ be connected to $\beta \in |\Gamma_m|$, so that the diagram at that branch looks like,



where $t=1, 2^{\pm 1}, 3^{\pm 1}, 4^{\pm 1}$ or ∞. (For the convenience we use a convention $\circ\underset{1}{\longleftarrow}\circ =$ $\circ\!\!\longrightarrow\!\!\!\!-\circ$ and $\circ\underset{t}{\longleftarrow}\circ = \circ\underset{t^{-1}}{\longrightarrow}\circ$ for $t=2, 3, 4$.)

A positive root of A_{l_i} with respect to this basis $\beta_1, \ldots, \beta_{l_i}$ is given by,

$$\alpha_{u,v} := \beta_u + \beta_{u+1} + \ldots + \beta_v \text{ for some } \quad u, v \in \mathbf{Z} \text{ with } 1 \leq u \leq v \leq l_i.$$

Define a Coxeter transformation c_i for A_{l_i} by,

$$c_i := w_{\beta_{l_i}} \cdots w_{\beta_1}.$$

Direct from the definition, one calculates

$$\text{**)} \qquad c_i \alpha_{u,v} = \begin{cases} \alpha_{u-1,v-1} & \text{for } u > 1, \\ -\alpha_{v,l_i} & \text{for } u = 1. \end{cases}$$

Since all roots $\alpha_{u,v}$, $1 \leq u \leq v \leq l_i$ are conjugate each other the counting $k(\alpha_{u,v})$, $1 \leq u \leq v \leq l_i$ is a constant ((6.1) **Assertion** iii)), which we shall denote $k_i := k(\alpha_{u,v})$ $(1 \leq u \leq v \leq l_i)$.

Assertion 5. *The Coxeter transformation c leaves the space $F_i \oplus G$ invariant for $i=1, \ldots, r$. We have a formula,*

$$P(c)\alpha_{u,v} = \frac{l_{\max}+1}{l_i+1}(v-u+1)k_i a \quad \text{for} \quad 1 \leq u \leq v \leq l_i.$$

Proof. Due to (9.4) **Assertion 1.** i) iii), $c|F_i \oplus G = c_i t_\beta | F_i \oplus G$ where $t_\beta = w_\beta w_{\beta^*}$ and $t_\beta(u) = u - I(u, \beta^\vee)k(\beta)a$.

By induction on m, we have,

$$(c_i t_\beta)^m u = c_i^m u - I\left(\sum_{n=0}^{m-1} c_i^n u, \beta^\vee\right)k(\beta)a \quad \text{for} \quad u \in F_i.$$

Hence for $u \in F_i$, we have

$$\begin{aligned} \text{***)} \quad P_i(c)u &:= \sum_{m=0}^{l_i} (c_i t_\beta)^m u = \sum_{m=0}^{l_i} c_i^m u - \sum_{m=0}^{l_i} I\left(\sum_{n=0}^{m-1} c_i^n u, \beta^\vee\right)k(\beta)a \\ &= -\sum_{n=0}^{l_i} (l_i - n) I(c_i^n u, \beta^\vee)k(\beta)a. \end{aligned}$$

On the other hand using **) in 4., $I(c_i^n \alpha_{u,v}, \beta^\vee)$ is either $I(\beta_1, \beta^\vee)$, $-I(\beta_1, \beta^\vee)$ or 0 according as $n = u-1$, $n = v$ or $n \neq u-1, v$. Therefore applying ***) for $u = \alpha_{u,v}$,

$$P_i(c)\alpha_{u,v} = -((l_i-u+1)-(l_i-v))I(\beta_1,\beta^\vee)k(\beta)a$$
$$= (v-u+1)k_i a.$$

Here in the last step of the calculation, we have applied (7.4) **Assertion** for $\alpha := \beta$, $\beta := \beta_1$ since $m_{\beta_1} < m_\beta = m_{\max}$.

Since $l_i+1 \mid l_{\max}+1$ and hence $P(c)=Q(c)P_i(c)$ for some polynomial $Q(c)$ such that $Q(1)=\frac{l_{\max}+1}{l_i+1}$, we have

$$P(c)\alpha_{u,v} = \frac{l_{\max}+1}{l_i+1}(v-u+1)k_i a. \qquad \text{q.e.d.}$$

6. Let $\alpha \in R$ be a root of the form,

$$\alpha = \alpha_{u,v} + mk_i a \quad \text{for} \quad 1 \le u \le v \le l_i, \quad m \in \mathbb{Z}.$$

If we have shown that $P(c)\alpha \neq 0$, we have completed the proof of the **Lemma** of (10.1).

Using the formula of **Assertion 5**,

$$P(c)\alpha = \frac{l_{\max}+1}{l_i+1}(v-u+1)k_i a + mk_i(l_{\max}+1)a$$
$$= \left(\frac{v-u+1}{l_i+1}+m\right)k_i(l_{\max}+1)a.$$

Since $1 \le u \le v \le l_i$, the value $\frac{v-u+1}{l_i+1}$ lies in the interval (0,1).

Hence $\frac{v-u+1}{l_i+1}+m \neq 0$ for any $m \in \mathbb{Z}$.

End of the proof of **Lemma B**.

§ 11. Coxeter Transformation 3
(The generator of the extension $\tilde{W}_{R,G}$ of W_R).

Let $\tilde{W}_{R,G}$ be the hyperbolic extension of W_R (cf. (1.18) or (11.1)) and $\tilde{c}$ be a hyperbolic Coxeter transformation ((11.2) **Def.**). We show that $\tilde{W}_{R,G}$ is a central extension of W_R by an infinite cyclic group which is generated by $\tilde{c}^{l_{\max}+1}$. ((11.3) **Lemma C**).

(11.1) Let (R, G) be a m.e.a.r.s.. Let us recall briefly the notion of a hyperbolic extension w.r.t. the marking G from (1.17), (1.18).

1. *There exists a triple* $(\tilde{F}, \tilde{I}, \iota)$ *where* $\tilde{F} :=$ *a vector space of rank* $l+3$, $\tilde{I} :=$ *a symmetric bilinear form on* $\tilde{F}$ *and* $\iota : F \to \tilde{F}$ *is an injective linear map s.t.* i) $I = \tilde{I} \cdot \iota$ ii) $\mathrm{rad}\, \tilde{I} = \iota G$.

Such triple is unique up to an isomorphism so that we fix one and regard ι as an inclusion map.

2. *Denote by* $\tilde{w}_\alpha \in O(\tilde{F}, \tilde{I})$ *the reflection of* $\alpha \in R$ *as an element in* $\tilde{F}$. *Denote by* $\tilde{W}_{R,G}$ *the group generated by* $\tilde{w}_\alpha$ $(\alpha \in R)$ *and call it the hyperbolic extension of* W_R.

3. *The restriction map* p_* *of the action of* $\tilde{W}_{R,G}$ *from* $\tilde{F}$ *to the subspace* F *induces a short exact sequence,*

$$1 \longrightarrow \tilde{K}_G \xrightarrow{E_G} \tilde{W}_{R,G} \xrightarrow{p_*} W_R \longrightarrow 1, \tag{11.1.1}$$

where

$$E_G : F \otimes F/G \to End(\tilde{F}),$$
$$E_G\Big(\sum_i f_i \otimes g_i\Big)(u) := u - \sum_i f_i I(g_i, u)$$

is the Eichler Siegel presentation (cf. (1.17.2)) *and*

$$\tilde{K}_G := E_G^{-1}(\tilde{W}_{R,G}) \cap M_G \tag{11.1.2}$$

is a lattice of $M_G := G \otimes (\mathrm{rad}\, I/G)$ (cf. (1.18) **Lemma**).

(11.2) Let $\{\alpha_0, \ldots, \alpha_l\}$ be a basis for (R, G) and $\Gamma_{R,G}$ be the Dynkin diagram for (R, G) w.r.t. the basis.

Definition. *A hyperbolic Coxeter transformation* $\tilde{c}$ *for* (R, G) *w.r.t. the basis* $\{\alpha_0, \ldots, \alpha_l\}$ *is an element of* $\tilde{W}_{R,G}$ *defined as a product of* $\tilde{w}_\alpha \tilde{w}_{\alpha^*}$ *for* $\alpha \in |\Gamma_m|$ *and* $\tilde{w}_\alpha$ *for* $\alpha \in |\Gamma_{R,G}| - |\Gamma_m| \cup |\Gamma_m^*|$.

Same argument as (9.7) **Lemma A** proves,

Assertion. i) *The conjugacy class of a hyperbolic Coxeter transformation* $\tilde{c}$ *in* $\tilde{W}_{R,G}$ *depends only on the linear space* $\bigoplus_{i=0}^{l} \mathbf{R}\alpha_i$ *and the choice of a generator* a *of* $Q(R) \cap G$, *but neither on the order of the product to present* $\tilde{c}$ *nor on the Weyl chamber* C *in choosing basis for* R/G (*cf*. (3.3)).

ii) *Let* $\varphi: F \simeq F$ *induce an automorphism of* (R, G), *whose orthogonal extension is denoted by* $\tilde{\varphi}: \tilde{F} \to \tilde{F}$. *If* $\tilde{c}$ *is a hyperbolic Coxeter transformation w.r.t. a basis* $\{\alpha_0, \ldots, \alpha_l\}$, *then* $ad_\varphi \tilde{c} := \tilde{\varphi}\tilde{c}\tilde{\varphi}^{-1} \in \tilde{W}_{R,G}$ *is a hyperbolic Coxeter transformation w.r.t. the basis* $\{\varphi\alpha_0, \ldots, \varphi\alpha_l\}$.

iii) $$\tilde{c}^{l_{\max}+1} \in E_G(\tilde{K}_G).$$

(11.3) The main purpose of this paragraph is to show the following $\mathbb{L}$emma $\mathbb{C}$.

$\mathbb{L}$emma $\mathbb{C}$. *Let the notations be as above.*

i) $\tilde{c}^{l_{\max}+1} = (I_R : I)\dfrac{l_{\max}+1}{m_{\max}} E_G(a \otimes b)$.

ii) $(I_R : I)\dfrac{l_{\max}+1}{m_{\max}} a \otimes b$ *generates* $\tilde{K}_G$ *as a cyclic group.*

i. e. The element $\tilde{c}^{l_{\max}+1}$ *generates the cyclic group* $\ker(p_*)$.
(*Here* a, b *are* $\mathbb{Z}$*-basis of* $Q(R) \cap \mathrm{rad} I$.)

Note. Let us choose the basis a, b of $Q(R) \cap \mathrm{rad} I$ as in (2.3.1) (2.3.2). Then as an element of M_G: $G \otimes (\mathrm{rad} I/G)$, $a \otimes b$ is unique up to a sign. The sign of the element is determined by the realization of the Dynkin diagram $\Gamma_{R,G}$ as a set of roots of R in the following way.

i) The choice of a basis $\{\alpha_0, \ldots, \alpha_l\}$ determines the sign of b by (3.3.6).

ii) The choice of Γ_m and Γ_m^* determines the sign of a by (8.2) i) ii).

(11.4) *Proof of* i) of $\mathbb{L}$emma $\mathbb{C}$.

Let us denote by $\tilde{I}_R$ the bilinear form $(I_R : I)\tilde{I}$. First let us show a formula,

(11.4.1) $$(\tilde{c}-1)\xi + \tilde{I}_R(\xi, b)\frac{1}{m_{\max}} a \in (c-1)F^{l+2} \qquad \text{for} \quad \xi \in \tilde{F}^{l+3}.$$

Note that the formula (11.4.1) does not depend on the choice of $\tilde{c}$ in the conjugacy class in $\tilde{W}_{R,G}$, since the change of $\tilde{c}$ by $w^{-1}\tilde{c}\,w$ for a $w \in \tilde{W}_{R,G}$ induces,

$$\begin{aligned} &((w^{-1}\tilde{c}\,w)-1)\xi + \tilde{I}_R(\xi, b)\frac{a}{m_{\max}} \\ &= w^{-1}\Big((\tilde{c}-1)\xi + \tilde{I}_R(\xi, b)a/m_{\max}\Big) + w^{-1}(\tilde{c}-1)(w-1)\xi \\ &\in w^{-1}(c-1)F^{l+2} = (w^{-1}cw-1)F^{l+2}. \end{aligned}$$

Case $\mathrm{cod}(R, G) = l+1$. This is the case when $\Gamma_{R,G} = \Gamma_m \cup \Gamma_m^*$. Since $\tilde{c} =$

$\prod_{a\in|\Gamma_m|} w_a w_{a*}$, similarly as in (9.4) **Assertion 1** iii),

$$\begin{aligned}\tilde{c}(u) &= u-\sum_{a\in|\Gamma_m|}\tilde{I}(u,a)\frac{2}{I(a,a)}(a^*-a)\\ &= u-\frac{1}{m_{\max}}\tilde{I}_R(u,\sum_{a\in|\Gamma_m|}n_a a)a.\end{aligned}$$

(Here we used the facts $m_a=\frac{I_R(a,a)n_a}{2k(a)}$ ((7.1.1)), $m_{\max}=m_a$ and $a^*-a=k(a)a$ ((8.2) i)) for $a\in|\Gamma_m|$.)

Case $\operatorname{cod}(R,G)<l+1$. Since $\Gamma_{R,G}-\Gamma_m\cup\Gamma_m^*\neq\phi$, let us take a node a_0 of the diagram $\Gamma_{A_{l_i}}$ for some i so that a_0 lies at terminal of the diagram. Take one $\tilde{c}$ by $\tilde{c}:=\tilde{w}_{a_0}\circ d$, where d is a product of $\tilde{w}_a\tilde{w}_{a^*}$ for $a\in|\Gamma_m|$ and w_a for $a\in|\Gamma_{R,G}|-|\Gamma_m|\cup|\Gamma_m^*|\cup\{a_0\}$.

Since (11.4.1) is obviously true for $\xi\in F^{l+2}$, we have only to show (11.4.1) for one $\xi\in\tilde{F}^{l+3}$ s.t. $\xi\notin F^{l+2}$. We may choose one such ξ satisfying $\tilde{I}(\xi,a_0)\neq0$, $\tilde{I}(\xi,a)=0$ for $a\in|\Gamma_{R,G}|$ $a\neq a_0$, so that $d(\xi)=\xi$. Then

$$\begin{aligned}*)\qquad \tilde{c}(\xi) &= \tilde{w}_{a_0}(\xi)=\xi-\tilde{I}(\xi,a_0^\vee)a_0\\ &= \xi-\tilde{I}(\xi,a_0^\vee)\left(a_0-\frac{k(a_0^\vee)}{l_i+1}a\right)-\frac{2I_R(\xi,a_0)}{I_R(a_0,a_0)}\frac{k(a_0)}{l_i+1}a.\end{aligned}$$

On one hand, applying **Assertion 5** in the proof of (10.1) **Lemma** for $u=v=1$, $P(c)\left(a_0-\frac{k(a_0)}{l_i+1}a\right)=0$. Therefore the semi-simplicity of c implies that the second term of $*$) $a_0-\frac{k(a_0^\vee)}{l_i+1}a$ belongs to $(c-1)F^{l+2}$.

On the other hand noting $\tilde{I}(\xi,b)=I(\xi,\sum_a n_a a)=n_{a_0}I(\xi,a_0)$, $(l_i+1)m_{a_0}=m_{\max}$ ((8.5) iv)) and $m_{a_0}=\frac{I_R(a_0,a_0)}{2k(a_0)}n_{a_0}$, the last term of $*$) becomes,

$$\frac{2I_R(\xi,a_0)}{I_R(a_0,a_0)}\frac{k(a_0)}{l_i+1}a=I_R(\xi,b)\frac{2k(a_0)}{n_{a_0}I_R(a_0,a_0)}\frac{1}{l_i+1}a=\frac{I_R(\xi,b)}{m_{\max}}a.$$

This completes the proof of (11.4.1).

Applying $P(\tilde{c})=\tilde{c}^{l_{\max}}+\cdots+1$ on (11.4.1), one obtains,

$$(\tilde{c}^{l_{\max}+1}-1)\xi=-I_R(\xi,b)\frac{l_{\max}+1}{m_{\max}}a\quad\text{for}\quad\xi\in\tilde{F}.$$

This is nothing but i) of **Lemma C**. q.e.d.

(11.5) For the proof of ii) of Lemma C, we need to recall a map $r: \tilde{W}_{R,G} \to M_G$ from (1.20).

Let $\{\alpha_0, \ldots, \alpha_l\}$ be a basis for (R, G) s.t. the images of $\alpha_1, \ldots, \alpha_l$ in $R/\mathrm{rad}\, I$ are basis for $R/\mathrm{rad}\, I$.
Put

$$L := \bigoplus_{i=1}^{l} \mathbb{R}\alpha_i,\ H := \mathbb{R}b,\ b = \sum_{i=0}^{l} n_i\alpha_i,\ G := \mathbb{R}a,$$

so that we fix a direct sum decomposition,

$$F = L \oplus H \oplus G. \tag{11.5.1}$$

Then, depending on the decomposition (11.5.1), we have introduced a map,

$$r: \tilde{W}_{R,G} \longrightarrow M_G,$$

(cf. (1.20.1)), such that

i) $r(g) = E_G^{-1}(g)$ for $g \in E_G(\tilde{K}_G)$,
ii) $r(\tilde{W}_{R,G}) \subset \sum_{\alpha,\beta\in R} \mathbb{Z}\,\mathrm{int}(\alpha,\beta)\alpha_G \otimes \beta_H$,

where
a) we denote by $\alpha = \alpha_L + \alpha_H + \alpha_G$ the decomposition (11.5.1) for $\alpha \in R$.
b) $\mathrm{int}(\alpha, \beta)$ for $\alpha, \beta \in R$ is a positive number defined by,

$$\mathrm{int}(\alpha,\beta) := \begin{cases} 2/I(\alpha,\alpha) & \text{if } \alpha = \beta, \\ \text{g.c.d.} \left\{ 2^k \dfrac{\prod_{j=1}^{k-1} I(\alpha_j, \alpha_{j+1})}{\prod_{j=1}^{k} I(\alpha_j, \alpha_j)} : \alpha_1, \ldots, \alpha_k \in R \text{ s.t. } \alpha_1 = \alpha, \alpha_k = \beta \right\}. \end{cases}$$

(See (1.20) Assertions 1-3 for more details and for the proofs about the map r.)

(11.6) *A proof of* ii) *of* Lemma C.

1. By definition of $\tilde{K}_G$ and i) of Lemma C we have the following commutative diagram.

$$\begin{array}{ccccccc} \{(\tilde{c}^{l_{\max}+1})^n : n \in \mathbb{Z}\} & \subset & E_G(\tilde{K}_G) & \subset & \tilde{W}_{R,G} & & \\ \wr\!\!\uparrow E_G & & \wr| & & \downarrow r & & \\ \mathbb{Z}(I_R : I)\dfrac{l_{\max}+1}{m_{\max}} a \otimes b & \subset & \tilde{K}_G & \subset & r(\tilde{W}_{R,G}) & \subset & M_G. \end{array}$$

Thus if we have shown an inclusion relation,

$$*)\qquad r(\tilde{W}_{R,G}) \subset \mathbb{Z}(I_R : I)\frac{l_{\max}+1}{m_{\max}} a\otimes b,$$

we get $\tilde{K}_G=\mathbb{Z}(I_R : I)\frac{l_{\max}+1}{m_{\max}} a\otimes b$ which proves ii) of the **Lemma C**.

For the purpose it is sufficient to show,

$$**)\qquad \mathrm{int}(\alpha,\beta)\alpha_G\otimes\beta_H \in \mathbb{Z}(I_R : I)\frac{l_{\max}+1}{m_{\max}} a\otimes b \quad \text{for} \quad \alpha,\beta\in R,$$

due to (11.5) ii).

2. First let us calculate int (α,β). As we see below, it is easy to see that the function int (α,β) depends only on the quotient root system $R/\mathrm{rad}\, I$.

Formula

$$(11.6.1)\qquad \mathrm{int}(\alpha,\beta) = t\frac{I_R : I}{\text{g.c.d.}\left(\frac{I_R(\alpha,\alpha)}{2}, \frac{I_R(\beta,\beta)}{2}\right)},$$

where

$t=1$ *for all cases except the following* 1)-4).

$t=2$ 1) $R/\mathrm{rad}\, I$ *is of type* A_1 *and* $\alpha\neq\beta$.

2) $R/\mathrm{rad}\, I$ *is of type* $B_l(l\geq 2)$ *and* $\alpha\neq\beta$ *are short roots of* R.

3) $R/\mathrm{rad}\, I$ *is of type* $C_l(l\geq 2)$ *and* $\alpha\neq\beta$ *are long roots of* R.

4) $R/\mathrm{rad}\, I$ *is of type* $BC_l(l\geq 2)$ *and either* $\alpha\neq\beta$ *are short roots of* R, *or* $\alpha\neq\beta$ *are long roots of* R, *or* $\alpha\neq\beta$ *are middle roots of* R *for* $l=2$.

A sketch of a proof for (11.6.1). Since $\frac{I(\alpha,\alpha)}{2}\mathrm{int}(\alpha,\beta)$, $\frac{I(\beta,\beta)}{2}\mathrm{int}(\alpha,\beta)\in\mathbb{Z}$, g.c.d. $\left(\frac{I(\alpha,\alpha)}{2}, \frac{I(\beta,\beta)}{2}\right)$ int $(\alpha,\beta)\in\mathbb{Z}$. This implies (11.6.1) for an integer t. If there exists a sequence $\alpha=\alpha_1,\dots,\alpha_k=\beta$ s.t. $I(\alpha_i,\alpha_{i+1})\neq 0$ $i=1,\dots,k-1$, $I(\alpha_1,\alpha_1)\leq I(\alpha_2,\alpha_2)\leq\cdots\leq I(\alpha_k,\alpha_k)$ and $\alpha_i\not\equiv\alpha_{i+1}$ mod rad I $i=1,\dots,k-1$, then $2^k\frac{\prod_{j=1}^{k-1}I(\alpha_j,\alpha_{j+1})}{\prod_{j=1}^{k}I(\alpha_j,\alpha_j)}=\frac{2}{I(\alpha_1,\alpha_1)}\prod_{j=1}^{k-1}I(\alpha_j,\alpha_{j+1}^\vee)=\pm\frac{2}{I(\alpha,\alpha)}$ and therefore $\mathrm{int}(\alpha,\beta)=$

$\frac{2}{I(\alpha, \alpha)}$ so that $t=1$ in this case. This will cover almost all cases except $R/\operatorname{rad} I$ is of type either one of A_1, B_l, C_l, BC_l. Some more careful study of the cases A_1, B_l, C_l, BC_l gives the results of the formula.

3. In the following we give a table of $\frac{l_{\max}+1}{m_{\max}}$ for all types of m.e.a.r.s.'s.

Type	$A_l^{(1,1)}$	$A_1^{(1,1)*}$	$B_l^{(1,1)}$	$B_l^{(1,2)}$	$B_l^{(2,1)}$	$B_l^{(2,2)}$	$C_l^{(1,1)}$	$C_l^{(1,2)}$	$C_l^{(2,1)}$	$C_l^{(2,2)}$
$l_{\max}+1$	1	2	2	2	2	1	1	2	2	2
$m_{\max}$	1	1	4	2	2	1	2	2	2	2

Type	$B_l^{(2,2)*}$	$C_l^{(1,1)*}$	$BC_l^{(2,1)}$	$BC_l^{(2,4)}$	$BC_l^{(2,2)}(1)$	$BC_l^{(2,2)}(2)$	$D_l^{(1,1)}$	$E_6^{(1,1)}$
$l_{\max}+1$	2	2	2	2	2	1	2	3
$m_{\max}$	1	2	4	2	4	2	2	3

Type	$E_7^{(1,1)}$	$E_8^{(1,1)}$	$F_4^{(1,1)}$	$F_4^{(1,2)}$	$F_4^{(2,1)}$	$F_4^{(2,2)}$	$G_2^{(1,1)}$	$G_2^{(1,3)}$	$G_2^{(3,1)}$	$G_2^{(3,3)}$
$l_{\max}+1$	4	6	3	4	4	3	2	3	3	2
$m_{\max}$	4	6	6	4	4	3	6	3	3	2

4. To show $**$) is a straight forward work now. Using the descriptions of R in (5.3) and the formula (11.6.1), one shows that $\operatorname{int}(\alpha, \beta)\ \alpha_G \otimes \beta_H \in \mathbb{Z}\ \frac{(I_R:\ I)}{t(R)}\ a \otimes b$ for $\alpha, \beta \in R$ of type $P^{(1,1)}$ and that $\operatorname{int}(\alpha, \beta) \alpha_G \otimes \beta_H \in \mathbb{Z}(I_R:\ I) a \otimes b$ for $\alpha, \beta \in R$ of type $P^{(t_1, t_2)}$ with $\max(t_1, t_2) > 1$.

The root systems of types $A_1^{(1,1)*}$, $B_l^{(2,2)*}$ and $C_l^{(1,1)*}$ are explained from 1), 2) and 3) of the formula (11.6.1).

This completes the proof of (11.3) Lemma $\mathbb{C}$.

(11.7) *Note. Let* $\alpha_1 = \alpha,\ \alpha_2 := \alpha + mb,\ \alpha_3 := \alpha + na,\ \alpha_4 := \alpha + na + mb$ belong to R.

Put $$g := \tilde{w}_{\alpha_1} \circ \tilde{w}_{\alpha_2} \circ \tilde{w}_{\alpha_3} \circ \tilde{w}_{\alpha_4} \in \widetilde{W}_{R,G}.$$
Then $$g := E_G((I_R:\ I) m \cdot n a \otimes b) \in \ker(p_*).$$

§ 12. Foldings of Dynkin Diagrams

In this paragraph, we study foldings of Dynkin diagrams for marked extended affine root systems. Precisely, we introduce two types ; folding and mean folding (cf. (12.2) Def.)

All Dynkin diagrams with multiple bonds are obtained by foldings of Dynkin diagrams with only simple bonds: $A_l^{(1,1)}$, $D_l^{(1,1)}$, $E_l^{(1,1)}$ ((12.4)). In case of finite or affine root systems, which are studied by P. Slodowy [21], T. Yano [27] and others, only one of the two types of foldings is enough to produce all diagrams, whereas in the case of m.e.a.r.s.'s, both types of foldings are necessary to obtain all diagrams.

In this way, we arrive *a hierarchy relation among m.e.a.r.s.'s,* where the exceptional types $A_1^{(1,1)*}$, $B_l^{(2,2)*}$, $C_l^{(1,1)*}$, $BC_l^{(2,2)}(1)$, $BC_l^{(2,2)}(2)$ form naturally one group (cf. (12.5)).

(12.1) Let (R, G) be a m.e.a.r.s. belonging to I. Let $\{\alpha_0, \ldots, \alpha_l\}$ be a basis of (R, G) and Γ be the Dynkin diagram for the affine root system $R \cap \bigoplus_{i=0}^{l} \mathbb{R}\alpha_i$ (cf. (6.4)).
Put

$$\mathrm{Aut}(R, \mathrm{rad}\, I) := \{\varphi \in \mathrm{Aut}(R) : \varphi | \mathrm{rad}\, I = \mathrm{id}_{\mathrm{rad}\, I}\}.$$

Lemma. *Assume that (R, G) is not of exceptional type.*
Then there exists a faithful representation,

$$r : \mathrm{Aut}(\Gamma) \longrightarrow \mathrm{Aut}(R, \mathrm{rad}\, I), \tag{12.1.1}$$

such that the action of $r(\mathrm{auto}(\Gamma))$ on F leaves the subspace $\bigoplus_{i=0}^{l} \mathbb{R}\alpha_i$ invariant.

Proof. We construct r explicitly as follows.

Let us regard $\mathrm{Aut}(\Gamma)$ as a subgroup of the permutation group of $\{0, \ldots, l\}$ of the indexes of $\alpha_0, \ldots, \alpha_l$.

Under the assumption of Lemma we discluded the types $A_1^{(1,1)*}$, $B_l^{(2,2)*}$, $C_l^{(1,1)*}$. Then we check easily,

$$k(\alpha_i) = k(\alpha_{\sigma(i)}) \quad (i=0, \ldots, l),$$
$$n_i = n_{\sigma(i)} \qquad (i=0, \ldots, l),$$

where k is the counting function ((6.1)) and n_i is the coefficients in (3.3). Hence, by the definition of exponents (7.1.1), we have,

$$m_i = m_{\sigma(i)} \quad (i=0, \ldots, l),$$

so that σ preserves the subset $|\Gamma_m|$ of $|\Gamma|$. Therefore the action of $\sigma \in \mathrm{Aut}(\Gamma)$ can be extended to $\Gamma_{R,G}$ (cf. (8.2)) by

$$\sigma(\alpha_i^*) = \alpha_{\sigma(i)}^* \quad \text{for} \quad \alpha_i \in |\Gamma_m|.$$

Then there exists a unique linear map $r_\sigma\colon F \simeq F$ such that $r_\sigma(\alpha_i)=\alpha_{\sigma(i)}$, $r_\sigma(\alpha_i^*)=\alpha_{\sigma(i)}^*$ $(0 \leq i \leq l)$ and $r_\sigma(a)=a$, $r_\sigma(b)=b$ for a, b the basis of rad I. ($\because$ The linear dependence relations among α_i's, α_i^*'s, a and b are generated by $\alpha_i - \alpha_i^* = k(\alpha_i)a$ $(0 \leq i \leq l)$ (cf. (8.2)) and $b=\sum_{i=0}^{l} n_i \alpha_i$ (cf. (3.3.6)). Since $k(\alpha_{\sigma(i)})=k(\alpha_i)$, $n_{\sigma(i)}=n_i$ $(0 \leq i \leq l)$, the same relations hold for $\alpha_{\sigma(i)}$'s, $\alpha_{\sigma(i)}^*$'s, a and b.)

Let us show that r_σ induces an automorphism of R.

$$\begin{aligned} r_\sigma(R) &= r_\sigma\Big(\bigcup_{i=0}^{l} \bigcup_{w \in \langle w_{\alpha_0}, \cdots w_{\alpha_l} \rangle} \{w\alpha_i + \mathbb{Z}k(\alpha_i)a\}\Big) \\ &= \bigcup_{i=0}^{l} \bigcup_{w \in \langle w_{\alpha_0}, \cdots w_{\alpha_l} \rangle} \{w\alpha_{\sigma(i)} + \mathbb{Z}k(\alpha_i)a\}) \\ &= R \text{ (cf. (6.4) Lemma ii).)} \end{aligned}$$

It is obvious that $\sigma \mapsto r_\sigma$ is a representation and is faithful. q.e.d.

Note. Under the assumption of the assertion, the (12.1.1) induces,

$$\mathrm{Aut}(\Gamma) \simeq \mathrm{Aut}(\Gamma_{R,G})/\, 2^{\mathrm{cod}(R,G)},$$

where $2^{\mathrm{cod}(R,G)}$ denotes the group generated by the transposition of α and α^* for $\alpha \in |\Gamma_m|$ (cf. (8.4) i)).

(12.2) Let the notations and the assumptions be as in (12.1). Let H be a subgroup of $\mathrm{Aut}(\Gamma)$, which acts on F through (12.1.1).

Define an invariant subspace of F,

$$F^H := \{x \in F : \ r_\sigma x = x \ \textit{for} \ \sigma \in H\}.$$

Clearly by definition, $\mathrm{rank}(F^H) = \#(|\Gamma|/H)+1$ and the sign of $I|F^H$ is $(\#(|\Gamma|/H)-1, 2, 0)$.

Define two mappings,

$$\mathrm{Tr}^H: F \longmapsto F^H, \quad \alpha \mapsto \sum_{\beta \in H\alpha} \beta,$$

$$\mathrm{Tr}_H: F \longmapsto F^H, \quad \alpha \mapsto (\# H\alpha)^{-1} \sum_{\beta \in H\alpha} \beta.$$

(12.3) *From now on we shall assume the following for the group H.*

$*$) *There exists at least a node of Γ so that H is contained in the isotropy subgroup of* $\mathrm{Aut}(\Gamma)$ *of the point.*

Under the assumption we have:

Lemma 1. *There exist extended affine root systems R_H and R^H in F^H belonging to $I|F^H$ such that the set $\mathrm{Tr}^H|\Gamma_{R,G}|$ and $\mathrm{Tr}_H|\Gamma_{R,G}|$ form Dynkin diagrams for (R^H, G) and (R_H, G) respectively.*

Together with the structure of the Dynkin diagram, we shall denote by $\mathrm{Tr}^H\Gamma_{R,G}$ and $\mathrm{Tr}_H\Gamma_{R,G}$ the sets, $\mathrm{Tr}^H|\Gamma_{R,G}|$ and $\mathrm{Tr}_H|\Gamma_{R,G}|$ respectively.

2. *The isomorphism classes of (R^H, G) and (R_H, G) depend only on $H \subset \mathrm{Aut}(\Gamma)$ but not on the choice of basis $\alpha_0, \dots, \alpha_l$.*

3. *(R^H, G) and $((R^\vee)_H, G)$ are dual of each other.*

Proof. The proof of **Lemma** will be done in the following 7 steps. The precise meaning of **Lemma 1** will be explained in the steps 4 and 5.

Since each of the steps are rather elementary, we do not give details of the calculations.

1. In the next (12.4), we shall list all groups H satisfying the **assumption** $*$).
As a result of the listing, we have the following.

i) $I(\alpha, \beta) = 0$ *for all* $\alpha, \beta \in H\gamma$ *s.t.* $\alpha \neq \beta$ *(for* $\gamma \in |\Gamma_{R,G}|$*).*

ii) *H acts transitively on the set of pairs $(\alpha, \beta) \in H\gamma \times H\delta$ with $I(\alpha, \beta) \neq 0$ (for $\gamma, \delta \in |\Gamma_{R,G}|$).*

2. *$Q(\mathrm{Tr}^H|\Gamma_{R,G}|)$ and $Q(\mathrm{Tr}_H|\Gamma_{R,G}|)$ are full lattices of F^H.*

($\because$ Since Tr_H is a linear map, we have

$$\mathrm{Tr}_H Q(R) = \mathrm{Tr}_H Q(|\Gamma_{R,G}|) = Q(\mathrm{Tr}_H|\Gamma_{R,G}|)$$

$$\subset \frac{1}{\# H} Q(\mathrm{Tr}^H|\Gamma_{R,G}|) \subset \frac{1}{\# H} Q(R) \cap F^H.)$$

3. Using 1 i) ii), one computes the lengths, intersection numbers and the duals of the elements of $\mathrm{Tr}^H|\Gamma_{R,G}|$ and $\mathrm{Tr}_H|\Gamma_{R,G}|$ as follows.

i) $I(\mathrm{Tr}^H\gamma, \mathrm{Tr}^H\gamma) = \# H\gamma I(\gamma, \gamma) \neq 0$ *for* $\gamma \in |\Gamma_{R,G}|$,

ii) $I(\mathrm{Tr}_H\gamma, \mathrm{Tr}_H\gamma) = (\# H\gamma)^{-1} I(\gamma, \gamma) \neq 0$ *for* $\gamma \in |\Gamma_{R,G}|$,

iii) $I(\mathrm{Tr}^H\gamma, (\mathrm{Tr}^H\delta)^\vee) = \dfrac{\# H(\gamma, \delta)}{\# H\delta} I(\gamma, \delta^\vee) \in \mathbb{Z}$ *for* $\gamma, \delta \in |\Gamma_{R,G}|$,

iv) $I(\mathrm{Tr}_H\gamma, (\mathrm{Tr}_H\delta)^\vee) = \dfrac{\# H(\gamma, \delta)}{\# H\gamma} I(\gamma, \delta^\vee) \in \mathbb{Z}$ *for* $\gamma, \delta \in |\Gamma_{R,G}|$.

v) $(\mathrm{Tr}^H\gamma)^\vee = \mathrm{Tr}_H(\gamma^\vee)$ *for* $\gamma \in |\Gamma_{R,G}|$,

vi) $(\mathrm{Tr}_H\gamma)^\vee = \mathrm{Tr}^H(\gamma^\vee)$ *for* $\gamma \in |\Gamma_{R,G}|$.

By definition of Tr^H, Tr_H directly,

vii) $\mathrm{Tr}^H\gamma^* - \mathrm{Tr}^H\gamma = (\# H\gamma) k(\gamma) a$ *for* $\gamma \in |\Gamma|$,
viii) $\mathrm{Tr}_H\gamma^* - \mathrm{Tr}_H\gamma = k(\gamma) a$ *for* $\gamma \in |\Gamma|$.

4. *The sets* $\mathrm{Tr}^H|\Gamma_{R,G}|$ *and* $\mathrm{Tr}_H|\Gamma_{R,G}|$ *have naturally the structure of m.e.a.r. s.'s.*

By this statement we mean the followings.
i) *On the sets* $\mathrm{Tr}^H|\Gamma|$ *and* $\mathrm{Tr}_H|\Gamma|$ in F^H we define structures of Dynkin diagrams according to the rule (8.2) ii) *using the intersection numbers* 3. iii), iv). *Then one verifies that they become diagrams for affine root systems, denoted by* $\mathrm{Tr}^H\Gamma$ *and* $\mathrm{Tr}_H\Gamma$ *respectively.*
ii) *On the diagrams* $\mathrm{Tr}^H\Gamma$ *and* $\mathrm{Tr}_H\Gamma$ we define the counting weights using 3 vii), viii) *as follows.*

$$\begin{cases} k(\mathrm{Tr}^H\alpha) := (H \circ k)^{-1} \# H\alpha \circ k(\alpha) & \text{for } \alpha \in |\Gamma|, \\ k(\mathrm{Tr}_H\alpha) := k(\alpha) & \text{for } \alpha \in |\Gamma|. \end{cases}$$

Here

$$(H \circ k) := \text{g.c.d.}\{\# H\alpha \circ k(\alpha) : \alpha \in |\Gamma|\}.$$

Then one verifies that the weighted diagrams $(\mathrm{Tr}^H\Gamma, (k(\mathrm{Tr}^H\alpha))_\alpha)$ *and* $(\mathrm{Tr}_H\Gamma,$

$(k(\mathrm{Tr}_H\alpha))_\alpha)$ *belong to the table of* (6.5) *for weighted diagrams of m.e.a.r.s.'s.*

iii) *The coefficients* n (3.3.6) *for the diagrams* $\mathrm{Tr}^H\Gamma$ *and* $\mathrm{Tr}_H\Gamma$ *are calculated as follows.*

$$\begin{cases} n_{\mathrm{Tr}^H\alpha} := n_\alpha & \text{for } \alpha\in|\Gamma|, \\ n_{\mathrm{Tr}_H\alpha} := (H\cdot n)^{-1}\#H\alpha\cdot n_\alpha & \text{for } \alpha\in|\Gamma|. \end{cases}$$

Here $(H\circ n):=\text{g.c.d.}\{\#H\alpha\circ n_\alpha:\ \alpha\in|\Gamma|\}$.

By putting $b^H:=\sum n_{\mathrm{Tr}^H\alpha}\cdot\mathrm{Tr}^H\alpha$ *and* $b_H:=\sum n_{\mathrm{Tr}_H\alpha}\circ\mathrm{Tr}_H\alpha$, *we have*

$$b^H = b, \quad b_H = (H\circ n)^{-1}b$$

iv) *We normalize the metric* I *in the following so that they define a minimal even lattice structure on* $Q(\mathrm{Tr}^H\Gamma)$ *and* $Q(\mathrm{Tr}_H\Gamma)$. (*cf*. 3. i), ii)).

$$\begin{aligned} I^H &:= (H\cdot I_R)^{-1}I_R, \\ I_H &:= (H^{-1}\cdot I_R)^{-1}I_R. \end{aligned}$$

Here

$$\begin{aligned} (H\circ I_R) &:= \text{g.c.d.}\{\#H\alpha\cdot I_R(\alpha,\alpha)/2:\ \alpha\in|\Gamma|\}, \\ (H^{-1}\circ I_R) &:= \text{g.c.d.}\{(\#H\alpha)^{-1}\circ I_R(\alpha,\alpha)/2:\ \alpha\in|\Gamma|\}. \end{aligned}$$

v) *Using the preceeding* ii), iii) *and* iv), *the exponents for* $(\mathrm{Tr}^H\Gamma, (k(\mathrm{Tr}^H\alpha))_\alpha)$ *and* $(\mathrm{Tr}_H\Gamma, (k(\mathrm{Tr}_H\alpha))_\alpha)$ *are calculated as follows.*

$$\begin{cases} m_{\mathrm{Tr}^H\alpha} := m_\alpha\dfrac{(H\cdot k)}{(H\cdot I_R)} & \alpha\in|\Gamma|, \\ m_{\mathrm{Tr}_H\alpha} := m_\alpha\dfrac{1}{(H\circ n)(H^{-1}\circ I_R)} & \alpha\in|\Gamma|. \end{cases}$$

vi) *Particularly* v) *implies*,

$$\begin{cases} (\mathrm{Tr}^H\Gamma)_m = \mathrm{Tr}^H\Gamma_m, \\ (\mathrm{Tr}_H\Gamma)_m = \mathrm{Tr}_H\Gamma_m. \end{cases} \qquad \text{(Recall \textbf{Definition} (8.1))}.$$

vii) *Together with the fact* ii), *we get*,

$$\begin{cases} (\mathrm{Tr}^H\Gamma)^*_m = \mathrm{Tr}^H\Gamma^*_m, \\ (\mathrm{Tr}_H\Gamma)^*_m = \mathrm{Tr}_H\Gamma^*_m. \end{cases}$$

Here we put $a^H:=(H\cdot k)a$ *and* $a_H:=a$.

viii) *The preceeding* i)~vii) *altogether imply that the set* $\mathrm{Tr}^H|\Gamma_{R,G}|$ *and* $\mathrm{Tr}_H|\Gamma_{R,G}|$ *form Dynkin diagrams for some m.e.a.r.s.'s*. (cf. (8.2) *for the definition*), which we shall denote by $\mathrm{Tr}^H\Gamma_{R,G}$ *and* $\mathrm{Tr}_H\Gamma_{R,G}$ *respectively.*

5. Let us show that *there exist m.e.a.r.s.'s, say* R^H *and* R_H *in* F^H *belonging to* $I|F^H$, *so that* $\mathrm{Tr}^H\Gamma_{R,G}$ *and* $\mathrm{Tr}_H\Gamma_{R,G}$ *are the Dynkin diagrams for them respectively.*

Put

$$R^H := \bigcup_{\gamma\in|\Gamma_{R,G}|} \langle w_\alpha:\ \alpha\in\mathrm{Tr}^H|\Gamma_{R,G}|\rangle \mathrm{Tr}^H\gamma,$$

$$R_H := \dot{\bigcup_{\gamma\in|\Gamma_{R,G}|}} \langle w_\alpha:\ \alpha\in\mathrm{Tr}_H|\Gamma_{R,G}|\rangle \mathrm{Tr}_H\gamma.$$

Then 1 and 2 iii) iv) v) vi) imply that R^H and R_H are root systems belonging to $I|F^H$ (cf. (1.3) Ex. 4). Since $I|F^H$ is positive semi-definite s.t. $\mathrm{rad}(I|F^H)=\mathrm{rad}\,I$, R^H and R_H are extended affine root systems by definition. The space $G:=\mathbb{R}a$ defines markings for R^H and R_H.

Let $(R_{\Gamma^H}, G_{\Gamma^H})$ and $(R_{\Gamma_H}, G_{\Gamma_H})$ be m.e.a.r.s.'s associated to the diagrams $\mathrm{Tr}^H\Gamma_{R,G}$ and $\mathrm{Tr}_H\Gamma_{R,G}$ respectively. (The constructions of them is given in (9.6)). Then, *there exists natural isomorphisms*

$$(R_{\Gamma^H}, G_{\Gamma^H}) \simeq (R^H, G),$$

and

$$(R_{\Gamma_H}, G_{\Gamma_H}) \simeq (R_H, G).$$

Proof. Define a map $\hat{F}_{\Gamma^H} := \bigoplus_{\alpha\in\mathrm{Tr}^H|\Gamma_{R,G}|} \mathbb{R}\hat{\alpha} \to F^H$ by $\hat{\alpha}\mapsto\alpha$. Then due to (9.4.5) and 3 vii), 4 ii), it is factorized by an isomorphism $(\hat{F}_{\Gamma^H})/\mathrm{Image}(U-1)\simeq F^H$, inducing $G_{\Gamma^H}:=\mathbb{R}a^H\simeq G:=\mathbb{R}a$.

Then the map induces a bijection,

$$R_{\Gamma^H} \simeq R^H$$

by definitions of them.

6. Since the Dynkin diagrams $\mathrm{Tr}^H\Gamma_{R,G}$ and $\mathrm{Tr}_H\Gamma_{R,G}$ are determined only by $H\subset\mathrm{Aut}(\Gamma)$ (cf. 3. iii) iv)), the isomorphism classes of (R^H, G) and (R_H, G) are determined by H.

7. The formula 3 v), vi) imply that, $(R^H)^\vee=(R^\vee)_H$ and $(R_H)^\vee=(R^\vee)^H$.

These complete the proof of (12.3) **Lemma**.

Definition. *Let the notations and the assumptions be the same as in the above* **Lemma**.

We call the diagram $\mathrm{Tr}^H \Gamma_{R,G}$ *the folding of* $\Gamma_{R,G}$ *and* $\mathrm{Tr}_H \Gamma_{R,G}$ *the mean folding of* $\Gamma_{R,G}$ *by the group* H.

We shall not make no distinction between two (mean) foldings by H and H', if the decomposition of the set $|\Gamma_{R,G}|$ into the orbits of H and that of H' coincides each other.

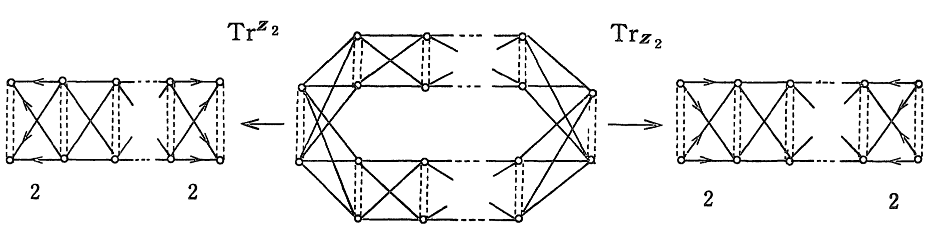
(12.4) The following are the complete list of foldings and mean foldings of Dynkin diagrams for m.e.a.r.s.'s.

The orbits of the nodes of $\Gamma_{R,G}$ by the action of H are drawn as nodes lying in the vertical position in the diagram. Thus if the action of H is obvious from the drawn picture, we have not explicitly mentioned about the group H.

If two subgroups $H, H' \subset \mathrm{Aut}(\Gamma)$ satisfying the assumption $*$) of **Lemma**, commute each other and $H \cap H' = \{1\}$, then Tr^H and $\mathrm{Tr}_{H'}$ commute. Let us denote by $\mathrm{Tr}^H_{H'}$ the product $\mathrm{Tr}^H \mathrm{Tr}_{H'} = \mathrm{Tr}_{H'} \mathrm{Tr}^H$ and by $\Rightarrow$ the correspondence of diagrams.

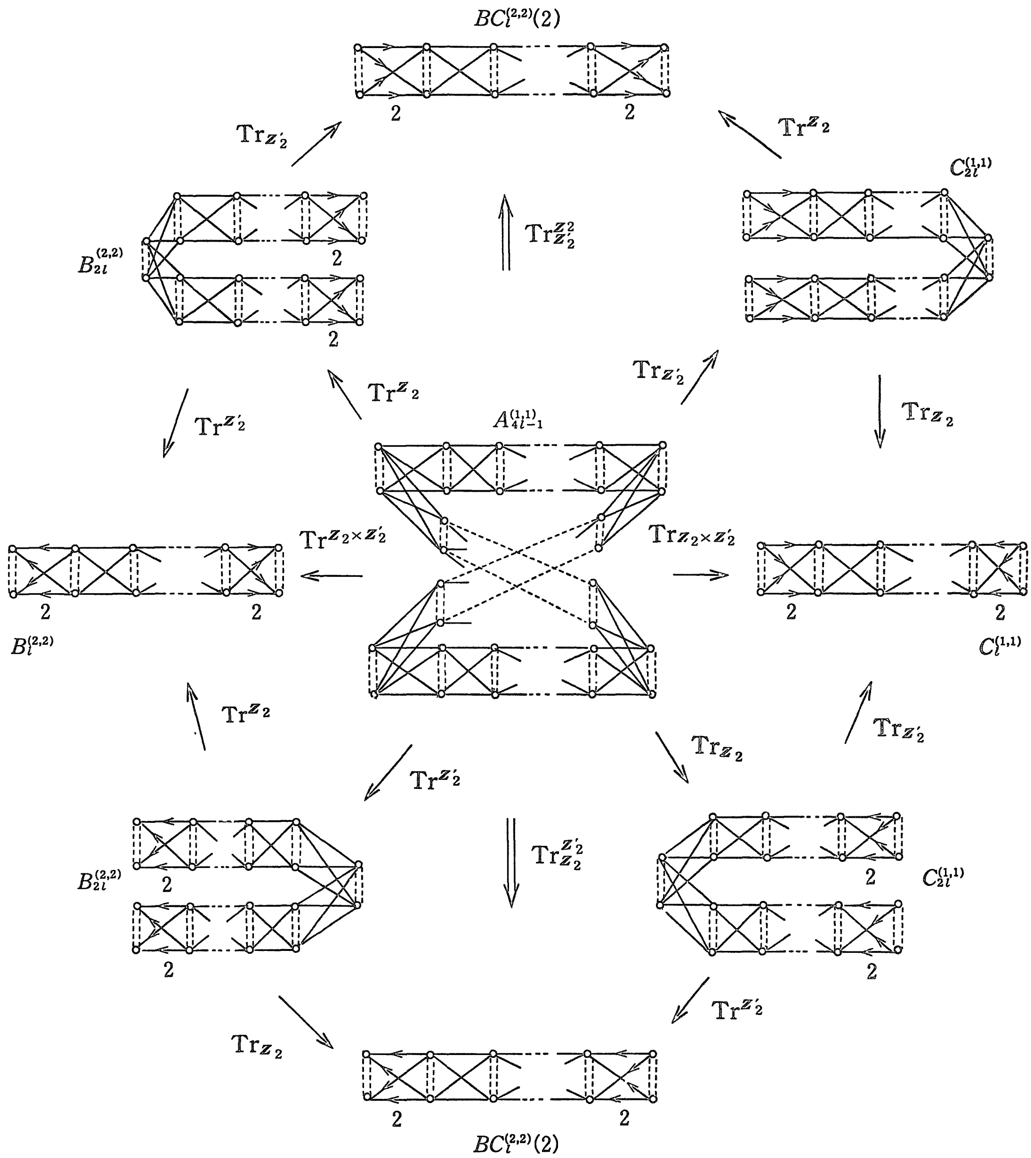
In each figure, the diagrams in antipodal positions are dual of each other.

Table 1. ($l \geq 2$)



$B_l^{(2,2)}$ $A_{2l-1}^{(1,1)}$ $C_l^{(1,1)}$

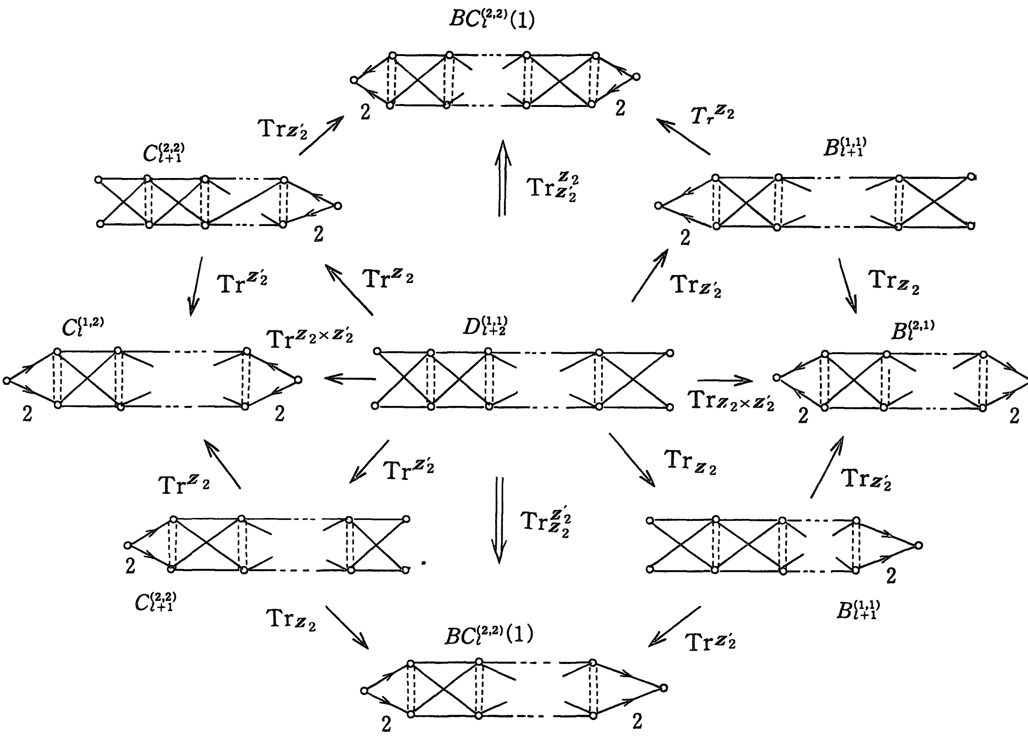
Table 2. ($l \geq 1$)



In the table 2, $\mathbb{Z}_2$ means the group generated by a involution which fixes the nodes on the right terminals of the diagrams and $\mathbb{Z}'_2$ means the group generated by the involution which fixes the nodes on the left terminals of the diagrams.

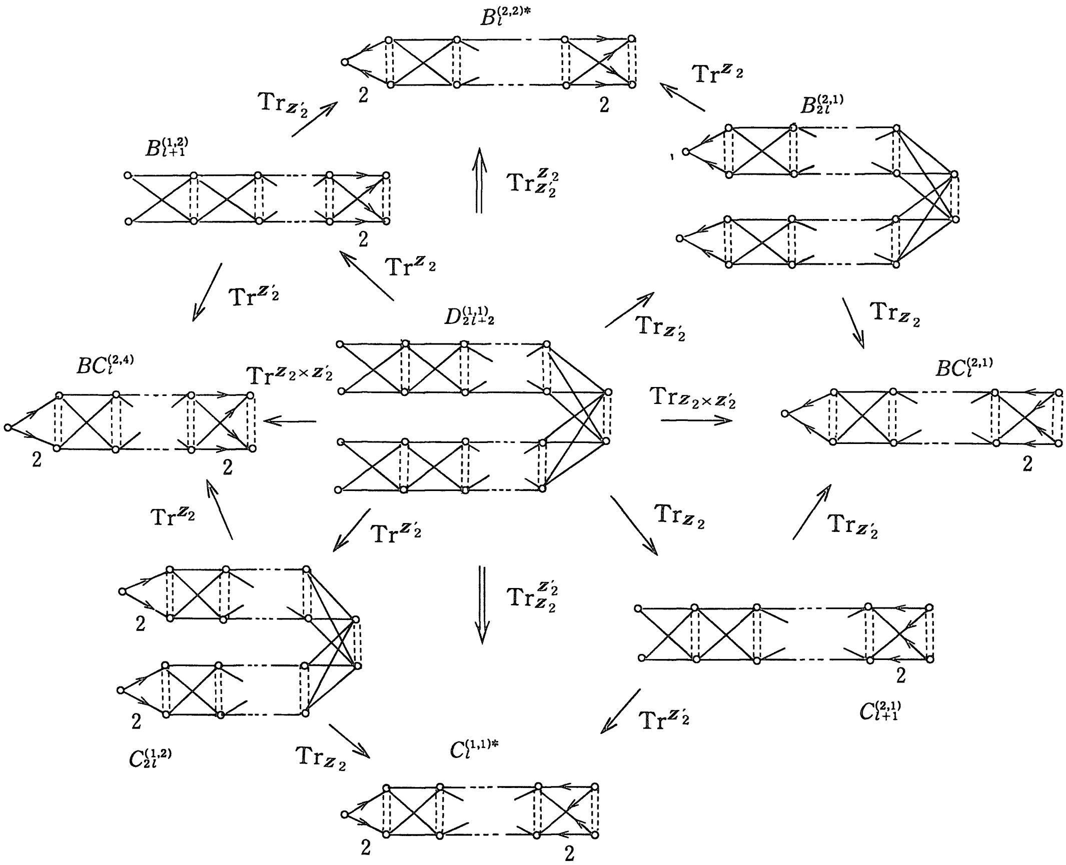
In case $l=1$, we formally define $B_1^{(2,2)} := A_1^{(1,1)}$, $C_1^{(1,1)} := A_1^{(1,1)}$.

Table 3. $(l \geq 2)$



In the **table 3**, $\mathbb{Z}_2$ means the group generated by the transposition of the nodes on the right terminals of the diagrams and $\mathbb{Z}_2'$ means the group generated by the transposition of the nodes on the left terminals of the diagrams.

Table 4. ($l \geq 2$)



In the table 4, $\mathbb{Z}_2$ means a group generated by an involution which fixes the nodes in the right terminals of the diagrams and $\mathbb{Z}_2'$ means a group generated by a (product of two) transposition (s) which exchanges nodes on the left terminals of the diagrams.

Table 5.

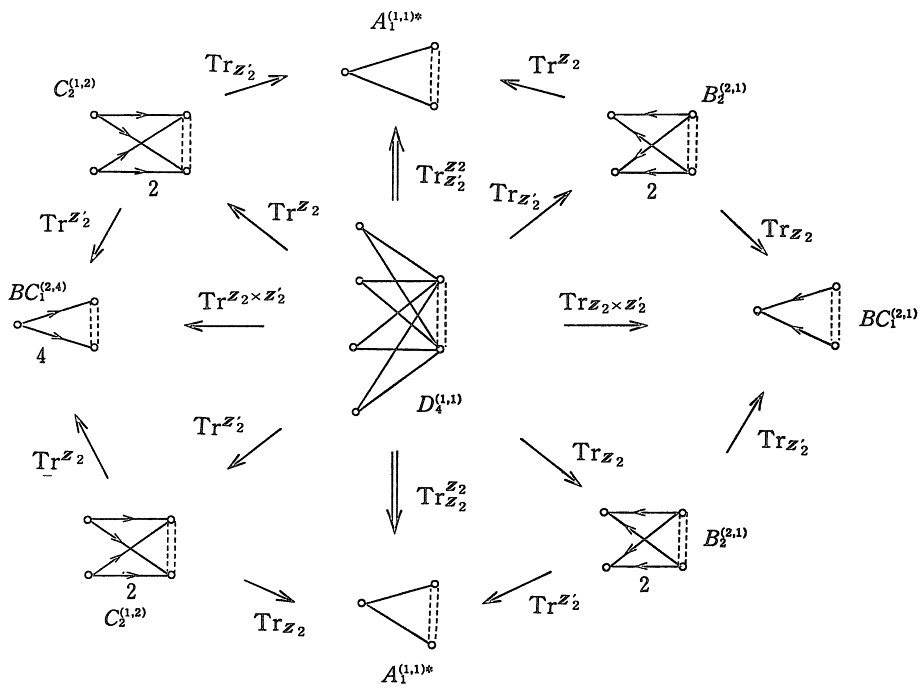


Table 6.

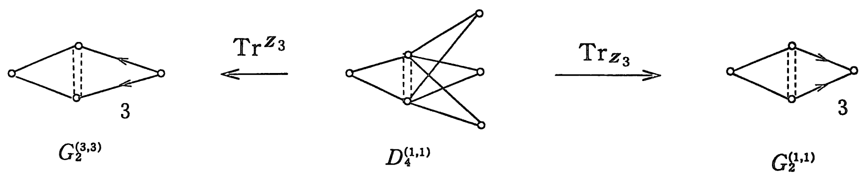


Table 7.

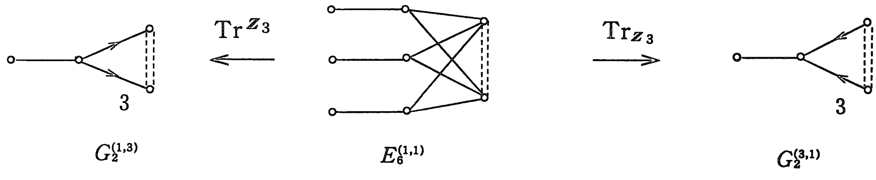
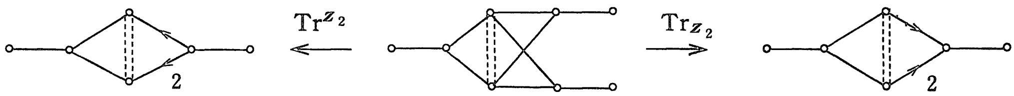
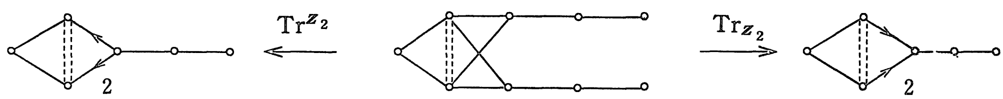


Table 8.



$F_4^{(2,2)}$ $E_6^{(1,1)}$ $F_4^{(1,1)}$

Table 9.



$F_4^{(1,2)}$ $E_7^{(1,1)}$ $F_4^{(2,1)}$

(12.5) **Hierarchy among Dynkin diagrams.**

The tables in (12.4) classify the diagrams for marked extended affine root systems into the following four groups.

I. *The Dynkin diagrams, which have no multiple bonds.*

$A_l^{(1,1)}$ $(l\geq 1)$, $D_l^{(1,1)}$ $(l\geq 4)$, $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$.

These diagrams are characterized as diagrams which can not be expressed neither as a folding nor as a mean folding of some other diagrams.

II. *The Dynkin diagrams of type* $P^{(t_1,t(P))}$ *(for* $t(P)>1$, $t_1|t(P)$*).*

$B_l^{(1,2)}$ $(l\geq 3)$, $B_l^{(2,2)}$ $(l\geq 2)$, $C_l^{(1,2)}$ $(l\geq 2)$, $C_l^{(2,2)}$ $(l\geq 3)$,
$BC_l^{(2,4)}$ $(l\geq 1)$, $F_4^{(1,2)}$, $F_4^{(2,2)}$, $G_2^{(1,3)}$, $G_2^{(3,3)}$.

These diagrams are characterized as diagrams which can be obtained by a folding of the diagrams of the group I.

III. *The Dynkin diagrams of type* $P^{(t_1,1)}$ *(for* $t(P)>1$, $t_1|t(P)$*).*

$B_l^{(1,1)}$ $(l\geq 3)$, $B_l^{(2,1)}$ $(l\geq 2)$, $C_l^{(1,1)}$ $(l\geq 2)$, $C_l^{(2,1)}$ $(l\geq 3)$,

$BC_l^{(2,1)}$ $(l \geq 1)$, $F_4^{(1,1)}$, $F_4^{(2,1)}$, $G_2^{(1,1)}$, $G_2^{(3,1)}$.

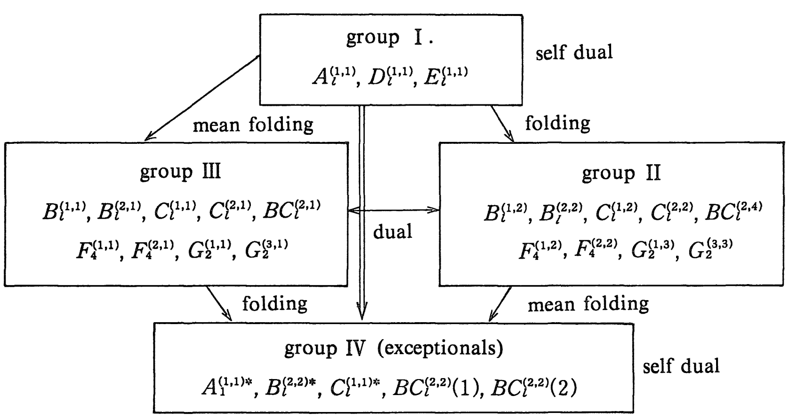
These diagrams are characterized as diagrams, which can be obtained by mean foldings of the diagrams of the group I.

IV. *The Dynkin diagrams of exceptional types.*

$A_1^{(1,1)*}$, $B_l^{(2,2)*}$ $(l \geq 2)$, $C_l^{(1,1)*}$ $(l \geq 2)$, $BC_l^{(2,2)}(1)$ $(l \geq 2)$,
$BC_l^{(2,2)}(2)$ $(l \geq 1)$.

These diagrams are characterized as diagrams, which can be obtained by foldings of the diagrams of the group III *and also by mean foldings of the diagrams of the group* II. *In the other words, diagrams of this group can be obtained from the diagrams of the group* I *by a succession of a folding and a mean folding which are commutative.*

As a summary, let us give a table of hierarchy relations among the Dynkin diagrams.



(12.6) **Assertion.** *Let us give formulae for calculating the tier numbers and the exponents, for the foldings and mean foldings.*

i) *The total tier number.*

$$t(Tr^H \Gamma_{R,G}) = t(\Gamma_{R,G}) \frac{1}{(H^{-1} \cdot I_{R^\vee})(H \cdot I_R)},$$

$$t(\mathrm{Tr}_H\Gamma_{R,G}) = t(\Gamma_{R,G})\frac{1}{(H^{-1}\circ I_R)(H\circ I_{R^\vee})}.$$

ii) *The first tier number.*

$$t_1(\mathrm{Tr}^H\Gamma_{R,G}) = t_1(\Gamma_{R,G})\frac{1}{(H\circ n^\vee)(H^{-1}\circ I_{R^\vee})},$$
$$t_1(\mathrm{Tr}_H\Gamma_{R,G}) = t_1(\Gamma_{R,G})\frac{(H\circ n)}{(H\circ I_{R^\vee})}.$$

iii) *The second tier number.*

$$t_2(\mathrm{Tr}^H\Gamma_{R,G}) = t_2(\Gamma_{R,G})\frac{1}{(H\circ k)(H^{-1}\circ I_{R^\vee})},$$
$$t_2(\mathrm{Tr}_H\Gamma_{R,G}) = t_2(\Gamma_{R,G})\frac{(H\circ k^\vee)}{(H\circ I_{R^\vee})}.$$

iv) *The exponents*

$$m_{\mathrm{Tr}^H\alpha} = m_\alpha\frac{(H\circ k)}{(H\circ I_R)},$$
$$m_{\mathrm{Tr}_H\alpha} = m_\alpha\frac{1}{(H\circ n)(H^{-1}\circ I_R)}.$$

Proof. Samely as in the proof of (12.3) Lemma we calculate as follows.

$$\begin{cases} k^\vee((\mathrm{Tr}^H\alpha)^\vee) = k^\vee(\alpha^\vee), \\ k^\vee((\mathrm{Tr}_H\alpha)^\vee) = (H\circ k^\vee)^{-1}\#H\alpha\cdot k^\vee(\alpha^\vee). \end{cases}$$

$$\begin{cases} n^\vee{}_{(\mathrm{Tr}^H\alpha)^\vee} = (H\circ n^\vee)^{-1}\#H\alpha\circ n^\vee_{\alpha^\vee}, \\ n^\vee{}_{(\mathrm{Tr}_H\alpha)^\vee} = n^\vee_{\alpha^\vee}. \end{cases}$$

$$\begin{cases} I_{(R^H)^\vee} = (H^{-1}\circ I_{R^\vee})^{-1}I_{R^\vee}, \\ I_{(R_H)^\vee} = (H\circ I_{R^\vee})^{-1}I_{R^\vee}. \end{cases}$$

Then applying these to the formulae for tier numbers (4.2) iii), (4.5.1) and (6.3.1) and for the exponents (7.1.1), we obtain the formulae in Assertion. q.e.d.

(12.7) **Corollary.** *Let the notations be as before. Then we have the following proportionalities.*

(12.7.1) $$\frac{m_{\mathrm{Tr}^H\alpha}}{m_\alpha} = \frac{t_2(\mathrm{Tr}_H\Gamma_{R^\vee,G})}{t_2(\Gamma_{R^\vee,G})} \quad \textit{for} \quad \alpha\in|\Gamma|,$$

$$\frac{m_{\mathrm{Tr}H\alpha}}{m_\alpha}=\frac{t_1(\mathrm{Tr}^H\Gamma_{R^\vee,G})}{t_1(\Gamma_{R^\vee,G})}\quad for\quad \alpha\in|\Gamma|. \tag{12.7.2}$$

Proof. Use ii), iii) and iv) of the Assertion. One may also check the proportionality, directly from the tables of (12.4).

These proportionalities are the last statements of this paper.

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Added in Proof.

1. The complete intersection of quadric cones defines a sequence of singularities, called $\tilde{D}_l$ (see F. Knörrer [8]). Recently in a private letter [34], W. Ebeling has informed to the author that he found a strongly distinguished basis of the middle homology group of the Milnor fiber of the singularities, which are intersecting in the form of the diagram as defined in (8.2) Definition of the present paper and that the Milnor's monodromies of the singularities are identified with the Coxeter transformations as defined in (9.7). (cf. also [31])

2. It might be worthwhile to notice that the Lemmas A, B of the present paper is an analog of a result of Coleman [36], who has calculated systematically the Betti numbers (exponents) for simple Lie groups, by showing the existence of regular eigen vectors for a Coxeter transformation. Another analog of the result for the case of indefinite root systems will be shown in a forthcoming paper.