Publ. RIMS, Kyoto Univ.
21 (1985), 205-236

Partial *-Algebras of Closed Linear Operators in Hilbert Space

By

J.-P. ANTOINE* and W. KARWOWSKI**

Abstract

Given a dense domain $\mathscr{D}$ of a Hilbert space, we consider the class of all closed operators which, together with their adjoint, have $\mathscr{D}$ in their domain. A partial *-algebra of operators on $\mathscr{D}$ is a subset of that class, stable under suitable operations of involution, addition and multiplication, the latter when it is defined. We present two types of such objects and study their properties, both algebraic and topological.

§ 1. Introduction

Unbounded operators in a Hilbert space have rather awkward algebraic properties. The sum or the product of two operators is not always defined, and if one insists on defining these notions, then in general the resulting operator fails to possess the "nice" properties of the original operators (e.g. a dense domain or closedness). Yet these partially defined operations generate a considerable amount of structure on certain sets of unbounded operators, as we shall see.

Clearly, if one considers all unbounded operators, only trivial statements can be made. At the other extreme, the class of all bounded operators and subsets thereof are well under control, and we need not comment on the importance of the resulting theory of C*- and W*-algebras.

The next choice is the set of all closable operators which, together with their adjoints, are defined on a fixed dense domain and leave it invariant. In this case also there exists an elaborated theory, developed e.g. by Vasil'ev [1], Epifanio et al. [2], Powers [3], Lassner and his group [4], although many questions are still unanswered.

Communicated by H. Araki, June 25, 1984.
* Institut de Physique Théorique, Université Catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium.
** Institute of Theoretical Physics, University of Wrocław, PL-50 205 Wrocław, Poland.

Here, as in the case of bounded operators, one gets genuine *-algebras, the so-called Op*-algebras.

However, there are indications that more general objects could be important, too. Take for instance, in Quantum Mechanics, a single particle in a potential. In the Hilbert space $\mathscr{H}=L^2(\mathbb{R}^3)$, the position operator $\boldsymbol{x}$, the momentum operator $\boldsymbol{p}$, the angular momentum $\boldsymbol{x}\wedge\boldsymbol{p}$, the free Hamiltonian $H_0=\boldsymbol{p}^2/2m$ are all defined and essentially self-adjoint on Schwartz space $\mathscr{S}$, and they all leave it invariant. But in general the full Hamiltonian $H=H_0+V(x)$ will not, and one gets only $H: \mathscr{S}\to\mathscr{H}$. In such a case, algebras of unbounded operators leaving $\mathscr{S}$ invariant are of little use (the same difficulty arises frequently in the so-called Rigged Hilbert Space formulation of Quantum Mechanics [5,6]). As a second instance where operator algebras are not sufficient, one may think of Quantum Statistical systems for which the thermodynamical limit does not exist in a C*-topology on the algebra of local observables [7]. One may then use a locally convex algebra $\mathfrak{A}$. However, its completion $\tilde{\mathfrak{A}}$ is no longer an algebra, but a more general object called *quasi-algebra* by Lassner [8], since multiplication in $\tilde{\mathfrak{A}}$ is only defined for certain pairs of elements (see Example 2.5 below). As a third example, one may cite the generalized creation and annihilation operators introduced by Grossmann [9] in his study of unsmeared field operators; they too can no longer be multiplied freely. The same phenomenon appears systematically in the study of operators on partial inner product spaces [10].

In this paper we will restrict the analysis to unbounded linear operators in a given Hilbert space $\mathscr{H}$. More precisely, we will consider collections of closed operators with a fixed common dense domain $\mathscr{D}\subset\mathscr{H}$, which they do *not* necessarily leave invariant. This situation, although still restrictive, is sufficient to provide genuine generalizations of C*- and Op*-algebras.

The starting point is the set $\overline{C}(\mathscr{D},\mathscr{H})$ of all closed operators A in $\mathscr{H}$ such that $\mathscr{D}\subset D(A)\cap D(A^*)$ [2] (for an operator A, we denote its domain by $D(A)$, its range by $\mathrm{Ran}(A)$, its adjoint by A^* and its closure by $\bar{A}$). The central question is then, what kind of algebraic structure can one give to $\bar{C}(\mathscr{D},\mathscr{H})$?

The vector space structure is easy, but the product is not. What is needed is a definition such that, for some pairs $A, B\in\bar{C}(\mathscr{D},\mathscr{H})$ at least, the product is defined and belongs to $\bar{C}(\mathscr{D},\mathscr{H})$ also. So the restriction of the product to $\mathscr{D}$ must be closable and have an adjoint defined on $\mathscr{D}$. Then the obvious candidate for the product AB would be the closure of the restriction of the ordinary product to $\mathscr{D}$. But, first, B need not map $\mathscr{D}$ into $D(A)$, and, even if it does, its restriction to $\mathscr{D}$,

$A(B \mid \mathscr{D})$, need not be closable, nor have an adjoint defined on $\mathscr{D}$.

Lemma 1.1. *Let $A, B \in \bar{C}(\mathscr{D}, \mathscr{H})$ verify the following two conditions:*

(WM 1) $\mathrm{Ran}(B|\mathscr{D}) \subset D(A)$

(WM 2) $\mathrm{Ran}(A^*|\mathscr{D}) \subset D(B^*)$.

Then $A(B|\mathscr{D})$ is a closable operator and the domain of its adjoint contains $\mathscr{D}$.

Proof. By (WM1) and (WM2), one may write:

$$[A(B|\mathscr{D})]^* \supset (B|\mathscr{D})^* A^* \supset B^* A^*$$

which proves that $A(B|\mathscr{D})$ has an adjoint defined on $\mathscr{D}$ and is therefore closable.

□

Whenever conditions (WM1), (WM2) are satisfied, we say that A is a *weak left multiplier* of B, and B a *weak right multiplier* of A, and we write $A \in \mathrm{L}^W(B)$, or $B \in \mathrm{R}^W(A)$.

We emphasize that condition (WM2) is sufficient for the closability of $A(B|\mathscr{D})$, but not necessary. Take, for instance, $\mathscr{H} = L^2(\mathbb{R})$, $\mathscr{D} = C_0^\infty(\mathbb{R})$, A = multiplication by the characteristic function χ of some finite interval $[a, b]$ and $B = i\frac{d}{dx}$. Then $A^* = A$ and $B^* = B = \overline{(B|\mathscr{D})}$ both belong to $\bar{C}(\mathscr{D}, \mathscr{H})$, (WM1) is satisfied, but (WM2) is not. Yet the product $A(B|\mathscr{D}) = i\chi(x)\frac{d}{dx}|C_0^\infty$ is closable, since its adjoint is densely defined. Indeed, the domain of $[A(B|\mathscr{D})]^*$ contains all functions $f \in C_0^\infty$ such that $f(a) = f(b) = 0$, but only those! Hence $A(B|\mathscr{D})$ is a well-defined closable operator, but the domain of its adjoint (which equals B^*A^* since A is bounded) does not contain $\mathscr{D}$, so that $\overline{A(B|\mathscr{D})} \notin \bar{C}(\mathscr{D}, \mathscr{H})$.

Of course we want our products to be defined uniquely. But $\bar{C}(\mathscr{D}, \mathscr{H})$ does allow ambiguities: it happens quite often that A_1 and A_2 both belong to $\bar{C}(\mathscr{D}, \mathscr{H})$, with A_2 a proper closed extension of A_1, and then A_1 and A_2 coincide on $\mathscr{D}$. There are two canonical ways of defining a closed product in $\bar{C}(\mathscr{D}, \mathscr{H})$, always under the condition that $A \in \mathrm{L}^W(B)$:

(i) $A \cdot B \equiv \overline{A(B|\mathscr{D})}$ (1.1)

(ii) $A * B \equiv [B^*(A^*|\mathscr{D})]^*$. (1.2)

One sees easily that $(A \cdot B)\varphi = (A * B)\varphi = AB\varphi$ for every $\varphi \in \mathscr{D}$ so that both definitions are acceptable. In the first case, one gets a *$\mathscr{D}$-minimal operator* [11], that is $A \cdot B$ is the closure of its restriction to $\mathscr{D}$, whereas in the second case $A * B$ is a *$\mathscr{D}$-maximal operator* [11], i.e. the adjoint of a $\mathscr{D}$-minimal one. Not all operators in $\bar{C}(\mathscr{D}, \mathscr{H})$ are of these types, so it is useful to clarify this question before going further.

Let A be any operator in $\bar{C}(\mathscr{D}, \mathscr{H})$. Then its adjoint A^* is in $\bar{C}(\mathscr{D}, \mathscr{H})$ too. Furthermore, we define

$$A^{+} \equiv \overline{A^{*}|\mathscr{D}}, \quad A^{\dagger} \equiv (A|\mathscr{D})^{*} \tag{1.3}$$

so that

$$A^{+} \subset A^{*} \subset A^{\dagger}. \tag{1.4}$$

Then the following relations are easily verified:
$A^{\dagger\dagger} = (A^{+})^{*}$, $A^{++} = \overline{A|\mathscr{D}} = (A^{\dagger})^{*}$, $A^{\dagger\dagger\dagger} = A^{\dagger}$, $A^{+++} = A^{+}$. In brief we get the following picture:

$$\begin{array}{ccccccc} A|\mathscr{D} \subset & A^{++} & \subset & A = A^{**} & \subset & A^{\dagger\dagger} \\ & \updownarrow + & & \updownarrow * & & \updownarrow \dagger \\ A^{*}|\mathscr{D} \subset & A^{+} & \subset & A^{*} & \subset & A^{\dagger} \end{array} \tag{1.5}$$

The operators A^{+}, A^{++} are $\mathscr{D}$-minimal and interchanged by $+$, whereas $A^{\dagger}$, $A^{\dagger\dagger}$ are $\mathscr{D}$-maximal and interchanged by $\dagger$. In the case of a symmetric operator A, Eq. (1.5) becomes

$$A^{+} = A^{++} \subset A \subset A^{*} \subset A^{\dagger} = A^{\dagger\dagger} \tag{1.6}$$

All inclusions may be strict, as for instance in the well-known case of differential operators on a finite interval $[a, b] \subset \boldsymbol{R}$, with $\mathscr{H} = L^{2}(a, b)$ and $\mathscr{D} = C_{0}^{\infty}(a, b)$. Notice that in (1.6), A^{+} is symmetric with adjoint $A^{\dagger}$, whereas $A^{\dagger}$ is not symmetric unless $A^{+} = A^{\dagger}$, since its adjoint is A^{+}. We will have more to say about this in the sequel (see Sec. 4).

In conclusion, if we want to give to some subsets of $\bar{C}(\mathscr{D}, \mathscr{H})$ a structure of "partial *-algebra", two candidates arise naturally: the set $\mathfrak{C}(\mathscr{D})$ of all $\mathscr{D}$-minimal operators, with the partial product $A \cdot B$, and the set $\mathfrak{C}^{*}(D)$ of all $\mathscr{D}$-maximal operators, with the partial product $A * B$. These two cases will be studied in detail in Sections 3 and 4, respectively. However, the concept of partial *-algebra has general features which are independent of those particular realizations, so we will first study, in Section 2, the algebraic structure in full generality. Next, we will examine in Section 5 some topological aspects of our structure. In particular, we will address the problem of finding topologies on $\mathfrak{C}$ and $\mathfrak{C}^*$ such that the partial multiplication is (separately) continuous whenever it is defined. Finally, Section 6 will be devoted to some considerations about commutants in the present framework. Some of the results of this paper have been reported on in an earlier work [12].

Acknowledgments

We both want to thank gratefully Prof. L. Streit for the warm hospitality offered to us at the "Zentrum für interdisziplinäre Forschung" Universität Bielefeld, where this work was completed. One of us (W.K.) is indebted to GMBH Kernforschung Karlsruhe for financial support. We thank G. Lassner, W. Timmermann and C. Trapani for interesting discussions, and also the referee for pointing out some inaccuracies in a previous version of this paper.

§ 2. Abstract Partial *-Algebras

We begin with the study of partial *-algebras in full generality, following Borchers [13].

Definition 2.1. A *partial *-algebra* is a complex vector space $\mathfrak{A}$ with an involution $x \mapsto x^+$ (i.e. $(x+\lambda y)^+ = x^+ + \bar{\lambda} y^+$, $x^{++} = x$) and a subset $\Gamma \subset \mathfrak{A} \times \mathfrak{A}$ such that:

(i) $(x, y) \in \Gamma$ implies $(y^+, x^+) \in \Gamma$

(ii) (x, y_1) and $(x, y_2) \in \Gamma$ implies $(x, y_1 + \lambda y_2) \in \Gamma$

(iii) If $(x, y) \in \Gamma$, then there exists an element $x \circ y \in \mathfrak{A}$ with the usual properties of the product:

$$x \circ (y+z) = x \circ y + x \circ z \tag{2.1}$$

$$(x \circ y)^+ = y^+ \circ x^+. \tag{2.2}$$

Notice that we do *not* require the $\circ$-product to be associative.

Definition 2.2. The partial *-algebra $\mathfrak{A}$ is said to have a *unit* if there exists an element $\mathbb{1} \in \mathfrak{A}$ (necessarily unique) such that $\mathbb{1}^+ = \mathbb{1}$, $(\mathbb{1}, x) \in \Gamma$ and $\mathbb{1} \circ x = x \circ \mathbb{1} = x$ for every $x \in \mathfrak{A}$.

Whenever $(x, y)\in\Gamma$, we say that x is a *left multiplier* of y, and y a *right multiplier* of x. One may remark that those multipliers bear some resemblance to the centralizers introduced by Johnson [14] in the context of bounded operators. We denote by $\mathrm{L}(x)$, resp. $\mathrm{R}(x)$, the set of all left, resp. right, multipliers of x. Similarly we introduce for any subset $\mathfrak{N}\subset\mathfrak{A}$:

$$\mathrm{L}\mathfrak{N}=\bigcap_{x\in\mathfrak{N}}\mathrm{L}(x), \quad \mathrm{R}\mathfrak{N}=\bigcap_{x\in\mathfrak{N}}\mathrm{R}(x) \tag{2.3}$$

with the conventions $\mathrm{L}\{x\}\equiv\mathrm{L}(x)$, $\mathrm{R}\{x\}\equiv\mathrm{R}(x)$. These sets of multipliers will occupy us for the rest of this section. First we notice some immediate facts. We use the notation $\mathfrak{N}^{+}=\{x^{+}|x\in\mathfrak{N}\}$, for any subset $\mathfrak{N}\subset\mathfrak{A}$.

Proposition 2.3. *Let* $\mathfrak{N}$ *be any subset of* $\mathfrak{A}$. *Then*:

(i) $\mathrm{L}\mathfrak{N}$ *and* $\mathrm{R}\mathfrak{N}$ *are vector subspaces of* $\mathfrak{A}$, *and both contain the unit element* $\mathbb{1}$, *if any*;

(ii) $(\mathrm{L}\mathfrak{N})^{+}=\mathrm{R}\mathfrak{N}^{+}$, $(\mathrm{R}\mathfrak{N})^{+}=\mathrm{L}\mathfrak{N}^{+}$.

Next we consider the set of all spaces of multipliers. As we shall see, this set exhibits an interesting lattice structure, analogous to the one analyzed by Gustafson and one of us [15, 16] in the context of partial inner product spaces. The reason is in fact the same, namely in both cases one studies the structure generated on a certain set by a *binary relation*. Here it is the subset $\Gamma\subset\mathfrak{A}\times\mathfrak{A}$:

$$(x, y)\in\Gamma\Leftrightarrow x\in\mathrm{L}(y)\Leftrightarrow y\in\mathrm{R}(x). \tag{2.4}$$

This binary relation in turn defines a *Galois connection* on the complete lattice $\mathscr{V}(\mathfrak{A})$ of all vector subspaces of $\mathfrak{A}$. This means that the two maps $\mathrm{L}: \mathfrak{N}\mapsto\mathrm{L}\mathfrak{N}$ and $\mathrm{R}: \mathfrak{N}\mapsto\mathrm{R}\mathfrak{N}$ ($\mathfrak{N}$ may be an arbitrary subset or a subspace of $\mathfrak{A}$) both reverse order:

$$\mathfrak{N}_1\subset\mathfrak{N}_2\Rightarrow\mathrm{L}\mathfrak{N}_1\supset\mathrm{L}\mathfrak{N}_2 \text{ and } \mathrm{R}\mathfrak{N}_1\supset\mathrm{R}\mathfrak{N}_2$$

and have the closure property:

$$\mathfrak{N}\subset\mathrm{LR}\mathfrak{N} \quad\text{and}\quad \mathfrak{N}\subset\mathrm{RL}\mathfrak{N}.$$

This in turn implies the relations:

$$\mathrm{L}=\mathrm{LRL} \text{ and } \mathrm{R}=\mathrm{RLR}. \tag{2.5}$$

Then, from Proposition 2.3 and the general theory of Galois connections (see [15], [16] or any textbook on universal algebra), we obtain the following result. Let $\mathscr{F}_{\mathrm{L}}$

be the set of all subspaces $\mathfrak{N}$ of $\mathfrak{A}$ that verify the relation $\mathfrak{N}=\mathbb{L}\mathbb{R}\mathfrak{N}$, $\mathscr{F}_{\mathrm{R}}$ the set of those for which $\mathfrak{N}=\mathbb{R}\mathbb{L}\mathfrak{N}$. Both sets are partially ordered by inclusion.

Proposition 2.4. *Let $\mathscr{F}_{\mathrm{L}}$ and $\mathscr{F}_{\mathrm{R}}$ be as above. Then:*

(i) *$\mathscr{F}_{\mathrm{L}}$ is a complete lattice with respect to set intersection and $\mathbb{L}\mathbb{R}$-closure of vector sum. The elements of $\mathscr{F}_{\mathrm{L}}$ are exactly all spaces of left multipliers:*

$$\mathfrak{N}\in\mathscr{F}_{\mathrm{L}} \text{ iff } \mathfrak{N}=\mathbb{L}\mathfrak{B} \text{ for some } \mathfrak{B}\subset\mathfrak{A}.$$

The maximal element of $\mathscr{F}_{\mathrm{L}}$ is $\mathfrak{A}=\mathbb{L}\mathbb{R}\mathfrak{A}$ and the minimal one is $\mathbb{L}\mathfrak{A}$.

(ii) *$\mathscr{F}_{\mathrm{R}}$ is a complete lattice with respect to set intersection and $\mathbb{R}\mathbb{L}$-closure of vector sum. The elements of $\mathscr{F}_{\mathrm{R}}$ are exactly all spaces of right multipliers, and its extremal elements are $\mathfrak{A}=\mathbb{R}\mathbb{L}\mathfrak{A}$ and $\mathbb{R}\mathfrak{A}$.*

(iii) *$\mathbb{L}$ is a lattice anti-isomorphism from $\mathscr{F}_{\mathrm{R}}$ onto $\mathscr{F}_{\mathrm{L}}$, and similarly for $\mathbb{R}: \mathscr{F}_{\mathrm{L}}\rightarrow\mathscr{F}_{\mathrm{R}}$.*

(iv) *The involution $x\leftrightarrow x^{+}$ generates a lattice isomorphism $\mathfrak{N}\leftrightarrow\mathfrak{N}^{+}$ from* $\mathscr{F}_{\mathrm{L}}$ onto $\mathscr{F}_{\mathrm{R}}$ and vice-versa.

(v) *The set $\mathscr{F}_{\mathrm{L}}\cap\mathscr{F}_{\mathrm{R}}$ of those subspaces of $\mathfrak{A}$ that belong both to $\mathscr{F}_{\mathrm{L}}$ and to $\mathscr{F}_{\mathrm{R}}$ is invariant under involution; it contains in particular all $^{+}$-invariant elements of $\mathscr{F}_{\mathrm{L}}$ and of $\mathscr{F}_{\mathrm{R}}$.*

Remark. $\mathscr{F}_{\mathrm{L}}$ and $\mathscr{F}_{\mathrm{R}}$ are in general not sublattices of $\mathscr{V}(\mathfrak{A})$, since the supremums are different in all three cases.

A concise formulation of this result is obtained by considering the set $\mathscr{F}_{\mathrm{L}}\times\mathscr{F}_{\mathrm{R}}$ with the following partial order

$$(\mathfrak{N}_1, \mathfrak{M}_1)\leq(\mathfrak{N}_2, \mathfrak{M}_2) \text{ iff } \mathfrak{N}_1\subseteq\mathfrak{N}_2 \text{ and } \mathfrak{M}_1\supseteq\mathfrak{M}_2.$$

By Proposition 2.4 (i), (ii) $\mathscr{F}_{\mathrm{L}}\times\mathscr{F}_{\mathrm{R}}$ is a complete lattice with respect to the operations:

$$(\mathfrak{N}_1, \mathfrak{M}_1)\wedge(\mathfrak{N}_2, \mathfrak{M}_2)=(\mathfrak{N}_1\cap\mathfrak{N}_2, \mathbb{R}\mathbb{L}(\mathfrak{M}_1+\mathfrak{M}_2))$$
$$(\mathfrak{N}_1, \mathfrak{M}_1)\vee(\mathfrak{N}_2, \mathfrak{M}_2)=(\mathbb{L}\mathbb{R}(\mathfrak{N}_1+\mathfrak{N}_2), \mathfrak{M}_1\cap\mathfrak{M}_2).$$

The minimal element is $(\mathbb{L}\mathfrak{A}, \mathfrak{A})$, the maximal one $(\mathfrak{A}, \mathbb{R}\mathfrak{A})$. Consider now the following subset of $\mathscr{F}_{\mathrm{L}}\times\mathscr{F}_{\mathrm{R}}$:

$$\mathscr{F}_{\Gamma}=\{(\mathfrak{N}, \mathfrak{M})\mid\mathfrak{N}=\mathbb{L}\mathfrak{M}\}=\{(\mathfrak{N}, \mathfrak{M})\mid\mathfrak{M}=\mathbb{R}\mathfrak{N}\}.$$

Then statement (iii) of Proposition 2.4 means that $\mathscr{F}_\Gamma$ is a complete sublattice of $\mathscr{F}_L \times \mathscr{F}_R$. Finally, the involution on $\mathscr{F}_L \times \mathscr{F}_R$

$$(\mathfrak{N}, \mathfrak{M})^+ = (\mathfrak{M}^+, \mathfrak{N}^+)$$

is a lattice anti-isomorphism and $\mathscr{F}_\Gamma^+ = \mathscr{F}_\Gamma$. The elements of $\mathscr{F}_\Gamma$ are quite useful (see also Sec. 5) so we give them a special name, and call them *matching subspaces*. For instance, one sees readily that the following statements are equivalent:

(i) $(x, y) \in \Gamma$

(ii) there exists a pair $(\mathfrak{N}, \mathfrak{M}) \in \mathscr{F}_\Gamma$ such that $x \in \mathfrak{N}$, $y \in \mathfrak{M}$.

This formulation, which makes essential use of the Galois structure, is very convenient in practice. Comparing the present situation with that of partial inner product spaces [10, 16], we see that matching subspaces here correspond to dual pairs there.

For several purposes it is necessary to restrict further the concept of multiplier. We say that x is a *multiplier* of y if it is both a left and a right multiplier: $x \in \mathrm{M}(y)$ iff $x \in \mathrm{L}(y) \cap \mathrm{R}(y)$. Accordingly we define

$$\mathrm{M}\mathfrak{N} = \mathrm{L}\mathfrak{N} \cap \mathrm{R}\mathfrak{N} \tag{2.6}$$

and consider M as another map on $\mathscr{V}(\mathfrak{A})$. Again M reverses order and MM is a closure: $\mathfrak{N} \subset \mathrm{MM}\mathfrak{N}$. Thus $\mathrm{M} = \mathrm{MMM}$ and the set $\mathscr{F}_M$ of all subspaces $\mathfrak{N}$ such that $\mathfrak{N} = \mathrm{MM}\mathfrak{N}$ is a complete lattice, with an anti-isomorphism $\mathfrak{N} \leftrightarrow \mathrm{M}\mathfrak{N}$ and an isomorphism $\mathfrak{N} \leftrightarrow \mathfrak{N}^+$. This lattice $\mathscr{F}_M$ plays a rôle in the study of *commutants,* if one uses the natural definition, for any subset $\mathfrak{N} \subset \mathfrak{A}$ (see Sec. 6):

$$\mathfrak{N}' = \{x \in \mathfrak{A} \mid x \in \mathrm{M}\mathfrak{N},\ x \circ y = y \circ x,\ \forall y \in \mathfrak{N}\}. \tag{2.7}$$

Remark. The lattice $\mathscr{F}_M$ has nothing to do with the set $\mathscr{F}_L \cap \mathscr{F}_R$ considered in Proposition 2.4 (v).

We conclude this section with some examples of partial *-algebras.

Example 2.5. Let $\mathfrak{A}$ be a locally convex *-algebra, such that the multiplication $(x, y) \mapsto xy$ is separately continuous and the involution $x \mapsto x^+$ continuous. Then the completion $\tilde{\mathfrak{A}}$ is in general not an algebra, but only a partial *-algebra, called *quasi-*-algebra* by Lassner [8]. Given a pair $x, y \in \tilde{\mathfrak{A}}$, their product xy is defined (by continuity) only if one of them at least belongs to $\mathfrak{A}$. So, for every $x \in \mathfrak{A}$, $\mathrm{L}(x)$

$=\mathrm{R}(x)=\tilde{\mathfrak{A}}$; but if $y\in\tilde{\mathfrak{A}}\setminus\mathfrak{A}$, $\mathrm{L}(y)=\mathrm{R}(y)=\mathfrak{A}$. In other words the lattice structure is trivial, the two lattices $\mathscr{F}_{\mathrm{L}}$ and $\mathscr{F}_{\mathrm{R}}$ coincide and consist only of the two elements $\mathfrak{A}$, $\tilde{\mathfrak{A}}$ (the corresponding situation for a partial inner product space is the case of the so-called trivial compatibility [16]; this happens e.g. for spaces of distributions). A typical example [8] is $\mathfrak{A}=C(a, b)$, the algebra of all continuous functions on a compact interval $[a, b]$, with the topology given by an L^p-norm. Then $\tilde{\mathfrak{A}}=L^p(a, b)$, which is not an algebra, but a quasi-*-algebra.

Example 2.6. Take the space $\bar{C}(\mathscr{D}, \mathscr{H})$ itself. It has natural algebraic operations, namely the strong sum $A\mp B\equiv\overline{A+B}$, the involution $A\mapsto A^*$ and the $\circ$ product defined in (1.1). However it is not a partial *-algebra, because the strong sum is not associative. For instance, if $A\subset B$, $A\neq B$, then $(B\mp A)\mp(-B)=2A\mp(-B)=A$, whereas $B\mp(A\mp(-B))=B\mp 0=B$. Moreover, the identity operator is not a unit, since $A\circ I=I\circ A=A^+\subset A$, where the restriction may be proper.

The lesson of the last example is that strict extensions of operators are incompatible with the structure of partial *-algebra. Thus we are left with two solutions. Either we consider $\mathscr{D}$-minimal operators only, with the $\circ$ product (1.1), or only $\mathscr{D}$-maximal ones, with the $*$ product (1.2). This will be done in Sec. 3 and Sec. 4, respectively.

§ 3. Partial *-Algebras of $\mathscr{D}$-Minimal Operators

In this section, we will consider only $\mathscr{D}$-minimal operators, that is, closed linear operators for which $\mathscr{D}$ is a core. These are all elements of the set:

$$\mathfrak{C}(\mathscr{D})=\{A^{++}\equiv\overline{A|\mathscr{D}}\,|\,A\in\bar{C}(\mathscr{D}, \mathscr{H})\}. \tag{3.1}$$

Since the domain $\mathscr{D}$ is fixed throughout, we write simply $\mathfrak{C}$ for $\mathfrak{C}(\mathscr{D})$. We will show that $\mathfrak{C}$ is a partial *-algebra for appropriate operations.

(i) *Vector space structure*: for $A, B\in\bar{C}(\mathscr{D}, \mathscr{H})$, we define

$$A\hat{+}B=\overline{(A+B)|\mathscr{D}}\in\mathfrak{C}. \tag{3.2}$$

Then with the $\hat{+}$ addition and the corresponding multiplication by scalars, $\mathfrak{C}$ is a complex vector space (in particular, the $\hat{+}$ addition is associative and $A\hat{+}0=A$). Notice that, for $A, B\in\mathfrak{C}$, $A\hat{+}B=\overline{A+B}\supset A+B$, where the extension may be proper.

(ii) *Involution*: for $A\in\mathfrak{C}$, $A^+\equiv\overline{A^*|\mathscr{D}}$ belongs to $\mathfrak{C}$ again and $A\mapsto A^+$ is an involution. In particular, $A^{++}=A$.

(iii) *Partial multiplication*: given $A, B\in\bar{C}(\mathscr{D},\mathscr{H})$, we say that A is a *left multiplier* of B and B a *right multiplier* of A if they verify the conditions:

(M1) $\mathrm{Ran}(B|\mathscr{D})\subset D(A^{++})$

(M2) $\mathrm{Ran}(A^+|\mathscr{D})\subset D(B^+)$.

We write $A\in\mathrm{L}(B)$, $B\in\mathrm{R}(A)$ and notice that $\mathrm{L}(B)\subset\mathrm{L}^W(B)$, for every $B\in\bar{C}(\mathscr{D},\mathscr{H})$. Hence we may use the $\circ$ product:

$$A\circ B=\overline{A(B|\mathscr{D})} \text{ for } A\in\mathrm{L}(B). \tag{3.3}$$

These three operations are compatible, for we have:

Proposition 3.1. *Given a dense domain $\mathscr{D}$, let $\mathfrak{C}\equiv\mathfrak{C}(\mathscr{D})$ be the set of all* $\mathscr{D}$-minimal operators. Equip $\mathfrak{C}$ with the $\hat{+}$ addition, *the involution $A\leftrightarrow A^+$ and the $\circ$ multiplication restricted to those pairs A, B where $A\in\mathrm{L}(B)$. Then $\mathfrak{C}$ is a partial *-algebra with the identity operator I as unit.*

Proof. We know already that $\mathfrak{C}$ is a vector space for $\hat{+}$ and that $A\leftrightarrow A^+$ is an involution. We have to verify the conditions of Def. 2.1:

(i) $A\in\mathrm{L}(B)\Leftrightarrow B^+\in\mathrm{L}(A^+)$ follows from conditions (M1), (M2).

(ii) Distributivity of multiplication, i.e. Eq. (2.1): $A\hat{+}B\in\mathrm{L}(C)$ if $A\in\mathrm{L}(C)$ and $B\in\mathrm{L}(C)$, and $(A\hat{+}B)\circ C=(A\circ C)\hat{+}(B\circ C)$, as can be checked readily.

(iii) Whenever $A\in\mathrm{L}(B)$, one has $A\circ B\in\mathfrak{C}$ and $(A\circ B)^+=B^+\circ A^+$. Indeed, for any ϕ, $\psi\in\mathscr{D}$, we may write:

$$\langle\psi,(A\circ B)\phi\rangle=\langle\psi, AB\phi\rangle=\langle A^+\psi, B\phi\rangle=\langle B^+A^+\psi,\phi\rangle=\langle(B^+\circ A^+)\psi,\phi\rangle$$

which shows that $\mathscr{D}$ is contained in the domain of $[A(B|\mathscr{D})]^*=(A\circ B)^*$ and then the assertion follows. Finally $I\in\mathfrak{C}$ and $I\circ A=A\circ I=A$, $\forall A\in\mathfrak{C}$. □

Although $\mathfrak{C}$ is a partial *-algebra with unit, multiplication in $\mathfrak{C}$ is not associative. Of course, in the context of a *partial* algebra, associativity means that, whenever $(A\circ B)\circ C$ is well-defined, then $A\circ(B\circ C)$ is also and the two are equal. This does not hold in $\mathfrak{C}$, but only a weaker statement (in the formulation of which we use a $\circ$ product of elements of $\bar{C}(\mathscr{D},\mathscr{H})$, not necessarily in $\mathfrak{C}$):

Proposition 3.2. *Let* $A, B, C \in \mathfrak{C}$, $A \in \mathrm{L}(B)$, $C \in \mathrm{R}(B)$ *and* $A \cdot B \in \mathrm{L}(C)$. *Then* $B \cdot C$ *is a weak right multiplier of* A^{+*} *and* $(A \cdot B) \cdot C = A^{+*} \cdot (B \cdot C)$.

Proof. Under the assumptions made, $(A \cdot B) \cdot C \in \mathfrak{C}$. Then we have, successively, for any $\phi, \psi \in \mathscr{D}$:

$$\begin{aligned}\langle \psi, ((A \cdot B) \cdot C)^{+} \phi \rangle &= \langle ((A \cdot B) \cdot C)\psi, \phi \rangle = \langle (A \cdot B) C \psi, \phi \rangle \\ &= \langle C\psi, (A \cdot B)^{+} \phi \rangle = \langle C\psi, (B^{+} \cdot A^{+})\phi \rangle = \langle C\psi, B^{+} A^{+} \phi \rangle \\ &= \langle BC\psi, A^{+}\phi \rangle = \langle (B \cdot C)\psi, A^{+}\phi \rangle = \langle A^{+*}(B \cdot C)\psi, \phi \rangle \\ &= \langle (A^{+*} \cdot (B \cdot C))\psi, \phi \rangle.\end{aligned}$$

The equality between the eighth term and the first two means that $B \cdot C \in \mathrm{R}^{W}(A^{+*})$, and then the last term gives the desired relation. □

A corresponding statement is obtained for $A \cdot (B \cdot C)$, using the involution $A \leftrightarrow A^{+}$. As shown by Proposition 3.2, the lack of associativity stems from the fact that A^{+} may be a proper restriction of A^{*}. Indeed, one has:

Corollary 3.3. (i) *Under the assumptions of Proposition* 3.2 *and, in addition, that* $A^{+} = A^{*}$, *one has* $B \cdot C \in \mathrm{R}^{W}(A)$ *and* $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

(ii) *The same conclusion holds, in particular, if* A *is self-adjoint*.

(iii) *If* $A \cdot B \in \mathrm{L}(C)$ *and* $B \cdot C \in \mathrm{R}(A)$, *then one has*

$$(A \cdot B) \cdot C = A \cdot (B \cdot C).$$

The discussion so far suggests a natural class of partial *-algebras of operators, namely partial *-subalgebras of $\mathfrak{C}$.

Definition 3.4. Let $\mathscr{D}$ be a dense domain. A *partial* **-algebra of* $\mathscr{D}$*-minimal operators* is a vector subspace $\mathfrak{M}$ of $\mathfrak{C}(\mathscr{D})$ such that:

(i) $\mathbb{1} \in \mathfrak{M}$

(ii) $\mathfrak{M}^{+} = \mathfrak{M}$

(iii) if $A, B \in \mathfrak{M}$ and $A \in \mathrm{L}(B)$, then $A \cdot B \in \mathfrak{M}$.

Let us give some examples:

(1) $\mathfrak{C} = \mathfrak{C}(\mathscr{D})$ itself.

(2) $B(\mathscr{H})$, the *-algebra of all bounded, everywhere defined operators on $\mathscr{H}$, and any *-subalgebra of $B(\mathscr{H})$ (this holds for *every* dense domain $\mathscr{D}$).

(3) The algebra $\overline{L^+(\mathscr{D})}=\{\bar{A}|A\in L^+(\mathscr{D})\}$, where $L^+(\mathscr{D})=\{A|\mathscr{D}|A\in \bar{C}(\mathscr{D},\mathscr{H}), A\mathscr{D}\subseteq\mathscr{D}, A^*\mathscr{D}\subseteq\mathscr{D}\}$ is the algebra of unbounded operators considered by Lassner [4] and also by Epifanio et al. [2] (who denote it $C_{\mathscr{D}}$). Indeed, statement (iii) of the definition follows from the relation $A\circ B=\overline{A(B|\mathscr{D})}=\overline{(AB)|\mathscr{D}}$.

(4) The algebra $\bar{\mathfrak{A}}=\{\bar{A}|A\in\mathfrak{A}\}$, for any Op*-algebra $\mathfrak{A}$ on $\mathscr{D}$, i.e. a *-subalgebra of $L^+(\mathscr{D})$ containing the identity.

(5) Any non-empty intersection of partial *-algebras of $\mathscr{D}$-minimal operators (all on the *same* domain $\mathscr{D}$).

(6) In particular, for any subset $\mathfrak{N}$ of $\mathfrak{C}(\mathscr{D})$, the set $\mathfrak{M}[\mathfrak{N}]\equiv\bigcap_\alpha\mathfrak{M}_\alpha$, where $\mathfrak{M}_\alpha$ runs over all partial *-algebras containing $\mathfrak{N}$. The partial *-algebra $\mathfrak{M}[\mathfrak{N}]$ is the smallest one of these, and will be called the partial *-algebra *generated* by $\mathfrak{N}$. Let us give a concrete example.

Example 3.5. Put $\mathscr{H}=L^2(\boldsymbol{R})$, $\mathscr{D}=C_0^\infty(\boldsymbol{R})$ and consider the following set of closed operators: $\mathfrak{N}=\left\{\mathbb{1}, e^x, \chi, x\frac{d}{dx}, S\right\}$. Here e^x and χ are the operators of multiplication by e^x and the characteristic function of a fixed interval, respectively; $x\frac{d}{dx}\equiv\overline{\left(x\frac{d}{dx}\right)|\mathscr{D}}$ and the operator S is defined as follows:

$$(Sf)(x)=\begin{cases}f(x+a), & x\geq 0\\ f(x-a), & x<0\end{cases}$$

where a is a fixed positive number. The operators I, e^x, and χ are selfadjoint; $\left(x\frac{d}{dx}\right)^+$ is the closure of $\left(\mathbb{1}+x\frac{d}{dx}\right)|\mathscr{D}$, that is, $\mathbb{1}\hat{+}\left(x\frac{d}{dx}\right)$, and S^+ is the tollowing operator:

$$(S^+f)(x)=\begin{cases}f(x-a), & x\geq a\\ 0 & , -a\leq x<a\\ f(x+a), & x<-a.\end{cases}$$

S^+ is an isometry, and S is a contraction: $SS^+=\mathbb{1}$, $S^+S<\mathbb{1}$. Accordingly, when $n\to\infty$, S^n tends to 0 and $(S^+)^n$ tends to 0 weakly. Next we compute products. Since S is bounded, $\operatorname{Ran}\left(x\frac{d}{dx}|\mathscr{D}\right)\subset D(S)$, but $\operatorname{Ran}(S^+|\mathscr{D})\not\subset D\left(\left(x\frac{d}{dx}\right)^+\right)$, so that $S\notin \mathrm{L}\left(x\frac{d}{dx}\right)$. On the other hand, $\operatorname{Ran}(S|\mathscr{D})\subset D\left(x\frac{d}{dx}\right)$ and of course

$\mathrm{Ran}\left(\left(x\frac{d}{dx}\right)^{+}|\mathscr{D}\right)\subset D(S^{+})$, so that $S\in \mathrm{R}\left(x\frac{d}{dx}\right)$. Furthermore, $\mathrm{Ran}(\chi|\mathscr{D})$ is contained neither in $D\left(x\frac{d}{dx}\right)$, nor in $D\left(\left(x\frac{d}{dx}\right)^{+}\right)$ so that χ and $x\frac{d}{dx}$ are not multipliers of each other. All other $\circ$ products are well-defined, including powers of each operator, which we define as $A^{(n)}\equiv A^{(n-1)}\circ A\,(n=2, 3, \ldots)$.

From these results, one may visualize the algebra $\mathfrak{M}[\mathfrak{N}]$ generated by $\mathfrak{N}$ as the set of all those polynomials in the operators of $\mathfrak{N}$ that contain only products allowed by the rules above.

Remark 3.6. A general operator $A\in\mathfrak{C}$ need not be a multiplier of itself! Of course, if it does, it is both a left and a right multiplier. Let indeed $A\in \mathrm{L}(A)$. Then $A^{(2)}\equiv A\circ A\in\mathfrak{C}$ and $(A^{(2)})^{+}=A^{+(2)}=(A^{2})^{+}$. Higher powers are trickier. For instance, if $A\circ A$ is *both* in $\mathrm{L}(A)$ and $\mathrm{R}(A)$, then, by Corollary 3.3. (iii), $A\circ(A\circ A)=(A\circ A)\circ A\equiv A^{(3)}$. Let in particular A be symmetric, $A\subset A^{*}$, which is equivalent to $A=A^{+}$. Then $A\circ A\in \mathrm{L}(A)$ iff $A\circ A\in \mathrm{R}(A)$, so that the usual rules of associativity apply.

Let $\mathfrak{M}$ be a partial *-algebra of $\mathscr{D}$-minimal operators. Its natural domain is the subspace $\mathscr{D}(\mathfrak{M})=\bigcap_{A\in\mathfrak{M}} D(A)$, which contains $\mathscr{D}$. On $\mathscr{D}(\mathfrak{M})$ we consider the $\mathfrak{M}$-topology $t_{\mathfrak{M}}$ defined by all seminorms $\phi\mapsto\|A\phi\|$, $A\in\mathfrak{M}$. Since every $A\in\mathfrak{M}$ is closed, its domain $D(A)$ is complete in the graph topology, thus $\mathscr{D}(\mathfrak{M})$ is $t_{\mathfrak{M}}$-complete, although $\mathscr{D}$ need not be. However, contrary to the case of Op*-algebras [2-4], $\mathscr{D}(\mathfrak{M})$ need not coincide with the completion $\tilde{\mathscr{D}}[t_{\mathfrak{M}}]$ of $\mathscr{D}$ (this fact was overlooked in [12]). A sufficient condition is that the seminorms $\phi\mapsto\|A\phi\|$, $A\in\mathfrak{M}$, be directed [17], i.e. that for every $A, B\in\mathfrak{M}$, there exists $C\in\mathfrak{M}$ such that $\|A\phi\|, \|B\phi\|\le\|C\phi\|$. This happens, for instance, if every element $A\in\mathfrak{M}$ has a "square" $A^{+}\circ A\in\mathfrak{M}$, i.e. $A\mathscr{D}\subset D(A^{+})$, $\forall A\in\mathfrak{M}$, in particular if $\mathfrak{M}$ is an Op*-algebra. For a general partial *-algebra $\mathfrak{M}$, the problem is open and we get:

$$\mathscr{D}\subset\tilde{\mathscr{D}}[t_{\mathfrak{M}}]\subset\mathscr{D}(\mathfrak{M}) \tag{3.4}$$

where both inclusions may be proper. Accordingly we extend the terminology familiar in the case of Op*-algebras [2-4] and make the following distinction:

Definition 3.7. A partial *-algebra $\mathfrak{M}$ of $\mathscr{D}$-minimal operators is said to be *closed* if $\mathscr{D}$ is complete in the $\mathfrak{M}$-topology, $\mathscr{D}=\tilde{\mathscr{D}}[t_{\mathfrak{M}}]$. It is called *fully closed* if, in addition, $\mathscr{D} = \mathscr{D}(\mathfrak{M})$.

We believe that there are closed partial *-algebras which are not fully closed, but we have no example so far. Anyway, such objects, if they exist at all, are likely to be quite pathological.

In the case of the partial *-algebra $\mathfrak{C}(\mathscr{D})$, it is interesting to compare it with its subalgebra $\overline{L^+(\mathscr{D})}$. Denoting $\underline{\mathscr{D}} \equiv \mathscr{D}(\overline{L^+(\mathscr{D})})$ and t_+ the $L^+(\mathscr{D})$-topology, we get the following inclusions:

$$\mathscr{D} \subset \tilde{\mathscr{D}}[t_{\mathfrak{C}}] \subset \mathscr{D}(\mathfrak{C}) \subset \tilde{\mathscr{D}}[t_+] = \underline{\mathscr{D}} \tag{3.5}$$

Thus, if the Op*-algebra $L^+(\mathscr{D})$ is closed in the usual sense that $\mathscr{D} = \underline{\mathscr{D}}$, the partial *-algebra $\mathfrak{C}(\mathscr{D})$ is fully closed. Of course the converse need not be true.

It is a standard result [2-4] that an Op*-algebra may always be embedded into a minimal (fully) closed algebra, called its closure. In our case the situation is more complicated. Let $\mathfrak{M}$ be a partial *-algebra on $\mathscr{D}$, non fully closed. Every $A \in \mathfrak{M}$ maps $\mathscr{D}(\mathfrak{M})$ into $\mathscr{H}$, continuously for the $\mathfrak{M}$-topology. Since $\mathscr{D}$ is a common core for all $A \in \mathfrak{M}$, so are $\tilde{\mathscr{D}}[t_{\mathfrak{M}}]$ and $\mathscr{D}(\mathfrak{M})$. Conversely, let A be an element of $\mathfrak{C}(\mathscr{D}(\mathfrak{M}))$, so that $A = \overline{A \mid \mathscr{D}(\mathfrak{M})}$, and construct the operator $\overline{A \mid \mathscr{D}}$. If the latter belongs to $\mathfrak{M}$, then its domain contains $\mathscr{D}(\mathfrak{M})$ and $\overline{A \mid \mathscr{D}} = \overline{A \mid \mathscr{D}(\mathfrak{M})} = A$. Therefore, $\mathfrak{M}$ may be identified with a vector subspace $\tilde{\mathfrak{M}}$ of $\mathfrak{C}(\tilde{\mathscr{D}}[t_{\mathfrak{M}}])$, and also with a subspace $\hat{\mathfrak{M}}$ of $\mathfrak{C}(\mathscr{D}(\mathfrak{M}))$, and all three involutions coincide. In particular, one has $\mathfrak{C}(\mathscr{D}) = \mathfrak{C}(\tilde{\mathscr{D}}[t_{\mathfrak{C}}]) = \mathfrak{C}(\mathscr{D}(\mathfrak{C}))$ as vector spaces with involution. Of course the argument does *not* apply to any domain larger than $\mathscr{D}(\mathfrak{C})$, such as $\underline{\mathscr{D}}$. If an operator $A \in \mathfrak{C}(\mathscr{D})$ does not belong to $L^+(\mathscr{D})$, its domain $D(A)$ need not contain $\underline{\mathscr{D}}$, and therefore, in general, $\mathfrak{C}(\underline{\mathscr{D}}) \subset \mathfrak{C}(\mathscr{D})$. For instance [18] if $\mathscr{D}$ is a set of second category in $\mathscr{H}$, $L^+(\mathscr{D})$ contains only bounded operators and thus $\underline{\mathscr{D}} = \mathscr{H}$, $\mathfrak{C}(\underline{\mathscr{D}}) = B(\mathscr{H})$.

So, if $\mathfrak{M}$ is any partial *-algebra on $\mathscr{D}$, its vector space structure extends to $\tilde{\mathscr{D}} \equiv \tilde{\mathscr{D}}[t_{\mathfrak{M}}]$ and $\mathscr{D}(\mathfrak{M})$. What about the structure of partial *-algebra ?

Let $A \in \mathrm{L}(B)$ in $\mathfrak{M}$; in particular B maps $\mathscr{D}$ into $D(A)$. The graph topology on $D(A)$ is a projective topology, namely the coarsest topology on $D(A)$ such that A and $\mathbb{1}$ map it continuously into $\mathscr{H}$. Therefore the map $B : \mathscr{D} \to D(A)$ is continuous iff the composed map $A \circ B : \mathscr{D} \to \mathscr{H}$ (remember that $(A \circ B)\phi = AB\phi$, $\forall \phi \in \mathscr{D}$) is continuous, and this is the case since $A \circ B \in \mathfrak{M}$ by assumption. Thus $B : \mathscr{D} \to D(A)$ may be extended by continuity to the respective completions, i.e. B maps $\tilde{\mathscr{D}}$ continuously into $D(A)$, and therefore B verifies the condition (M1) in

$\overline{\mathfrak{M}} \equiv \mathfrak{M}(\tilde{\mathscr{D}})$. The same is true for (M2) using the involution $A \leftrightarrow A^+$. Conversely, if $A \in \mathrm{L}(B)$ in $\overline{\mathfrak{M}}$, the same is true a fortiori in $\mathfrak{M}$. In conclusion $\overline{\mathfrak{M}}$ is a closed partial *-algebra on $\tilde{\mathscr{D}}$, and it has same structure of partial *-algebra as $\mathfrak{M}$.

The preceding argument does not extend to $\hat{\mathfrak{M}} \equiv \mathfrak{M}(\mathscr{D}(\mathfrak{M}))$, if $\mathscr{D}(\mathfrak{M}) \neq \tilde{\mathscr{D}}$. Indeed, $A \in \mathrm{L}(B)$ in $\hat{\mathfrak{M}}$ implies the same relation in $\mathfrak{M}$, but not conversely in general: $A\mathscr{D} \subseteq D(B)$ does not imply $A\mathscr{D}(\mathfrak{M}) \subseteq D(B)$ although A is well defined on $\mathscr{D}(\mathfrak{M})$. Of course, if $A \in \mathrm{L}(B)$ both in $\mathfrak{M}$ and in $\hat{\mathfrak{M}}$, then the product $A \circ B$ is the same in both cases. This situation is best described by introducing the following concept.

Definition 3.8. A *homomorphism* of a partial *-algebra $\mathfrak{M}$ into another one $\mathfrak{N}$ is a linear map $\sigma : \mathfrak{M} \to \mathfrak{N}$ such that

(i) $\sigma(x^+) = [\sigma(x)]^+$

(ii) if $x \in \mathrm{L}(y)$ in $\mathfrak{M}$, then $\sigma(x) \in \mathrm{L}(\sigma(y))$ in $\mathfrak{N}$ and $\sigma(x \circ y) = \sigma(x) \circ \sigma(y)$.

The map σ is an *isomorphism* if it is a bijection and the inverse map $\sigma^{-1} : \mathfrak{N} \to \mathfrak{M}$ is also a homomorphism.

Using this terminology, the discussion above may be summarized in the following statements.

Proposition 3.9. *Let $\mathfrak{M}$ be a partial *-algebra of $\mathscr{D}$-minimal operators, $\tilde{\mathscr{D}}$ the completion of $\mathscr{D}$ in the $\mathfrak{M}$-topology $t_{\mathfrak{M}}$, $\mathscr{D}(\mathfrak{M}) = \bigcap_{A \in \mathfrak{M}} D(A)$. Denote by $\overline{\mathfrak{M}}$, resp. $\hat{\mathfrak{M}}$, the same set of operators as $\mathfrak{M}$, but considered as closures of their restriction to $\tilde{\mathscr{D}}$, resp. $\mathscr{D}(\mathfrak{M})$. Then:*

(i) *$\overline{\mathfrak{M}}$ is a closed partial *-algebra over $\tilde{\mathscr{D}}$, isomorphic to $\mathfrak{M}$.*

(ii) *$\hat{\mathfrak{M}}$ is a fully closed partial *-algebra over $\mathscr{D}(\mathfrak{M})$, and the identity is a homomorphism of $\hat{\mathfrak{M}}$ onto $\mathfrak{M}$.*

Naturally the partial *-algebra $\overline{\mathfrak{M}}$ will be called the *closure* of $\mathfrak{M}$.

We conclude this section by discussing, for a given partial *-algebra $\mathfrak{M} \subset \mathfrak{C}(\mathscr{D})$, the spaces of multipliers described in Sec. 2. Let $\mathfrak{N}$ be any subset of $\mathfrak{M}$. Then we may consider left and right multipliers $\mathrm{L}\mathfrak{N}$, $\mathrm{R}\mathfrak{N}$, that is, multipliers in $\mathfrak{C}$, but also *internal* multipliers:

$$\mathrm{L}_{\mathfrak{M}}\mathfrak{N} = \mathrm{L}\mathfrak{N} \cap \mathfrak{M}, \quad \mathrm{R}_{\mathfrak{M}}\mathfrak{N} = \mathrm{R}\mathfrak{N} \cap \mathfrak{M}.$$

For these we may develop the whole lattice machinery of Sec. 2, and in particular obtain the minimal classes $\mathrm{L}_{\mathfrak{M}}\mathfrak{M}$, $\mathrm{R}_{\mathfrak{M}}\mathfrak{M}$. We give two examples.

(1) Take $\mathfrak{C}$ itself. Then $A\in\mathrm{L}\mathfrak{C}$ means $\mathrm{Ran}(B|\mathscr{D})\subset D(A)$ and $\mathrm{Ran}(A^{+}|\mathscr{D})\subset D(B^{+})$ for every $B\in\mathfrak{C}$. The first condition implies that A is bounded. Were it not, there would exist $f\notin D(A)$, and then the projection operator $|f><f|$ would map $\mathscr{D}$ on $f\notin D(A)$. Similarly the second condition implies that A^{+} maps $\mathscr{D}$ into $\mathscr{D}(\mathfrak{C})=\bigcap_{A\in\mathfrak{C}}D(A)$. In the same way, $B\in\mathrm{R}\mathfrak{C}$ means that B is bounded and maps $\mathscr{D}$ into $\mathscr{D}(\mathfrak{C})$. Thus we verify that $A\in\mathrm{L}\mathfrak{C}$ iff $A^{+}\in\mathrm{R}\mathfrak{C}$. Moreover, if $\mathfrak{C}$ is fully closed, $\mathscr{D}=\mathscr{D}(\mathfrak{C})$, then $\mathrm{M}\mathfrak{C}=\mathrm{L}\mathfrak{C}\cap\mathrm{R}\mathfrak{C}$ consists of the (closures of the) bounded elements of $L^{+}(\mathscr{D})$.

(2) Take for $\mathfrak{M}$ the partial *-algebra $\mathfrak{M}[\mathfrak{N}]$ of Example 3.5. Then $\mathrm{L}_{\mathfrak{M}}\mathfrak{M}$ consists of all polynomials in I, e^{x}, S^{+}, and $\mathrm{R}_{\mathfrak{M}}\mathfrak{M}$ of all polynomials in I, e^{x}, S.

Spaces of multipliers are convenient for discussing the associativity of the $\cdot$ multiplication, as in Proposition 3.2. Let $\mathfrak{N}$ be any subset of $\mathfrak{C}$. Then to say that $A\cdot B\in\mathrm{L}(C)$ for all $B\in\mathfrak{N}$, $A\in\mathrm{L}\mathfrak{N}$, $C\in\mathrm{R}\mathfrak{N}$ is equivalent to the relation $\mathrm{L}\mathfrak{N}\circ\mathfrak{N}\subset\mathrm{LR}\mathfrak{N}$. In addition, we have always $\mathfrak{N}\subset\mathrm{L}\mathfrak{N}\cdot\mathfrak{N}$, since $I\in\mathrm{L}\mathfrak{N}$. So if $\mathfrak{N}=\mathrm{LR}\mathfrak{N}$, i.e. $\mathfrak{N}\in\mathscr{F}_{\mathrm{L}}$, the statement is equivalent to $\mathfrak{N}=\mathrm{L}\mathfrak{N}\cdot\mathfrak{N}$, i.e. $\mathrm{L}\mathfrak{N}$ maps $\mathfrak{N}$ *onto* itself by multiplication. A similar discussion can be done of course for right multiplication and the two cases are interchanged by the involution $\mathfrak{N}\leftrightarrow\mathfrak{N}^{+}$. Thus we get sufficient conditions for associativity to hold.

Lemma 3.10. *Given any subset* $\mathfrak{N}\subset\mathfrak{C}$, *consider the following conditions*: $(A1)$ $\mathrm{L}\mathfrak{N}\cdot\mathfrak{N}\subset\mathrm{LR}\mathfrak{N}$; $(A2)$ $\mathfrak{N}\circ\mathrm{R}\mathfrak{N}\subset\mathrm{RL}\mathfrak{N}$; $(A3)$ $\mathfrak{N}=\mathfrak{N}^{+}$.

Then $(A1)+(A3)\Rightarrow(A2)$, $(A2)+(A3)\Rightarrow(A1)$, *and every pair of conditions implies associativity of multiplication on* $\mathrm{L}\mathfrak{N}\times\mathfrak{N}\times\mathrm{R}\mathfrak{N}$. *In particular, equality holds in* $(A1)$, *resp.* $(A2)$, *if* $\mathfrak{N}=\mathrm{LR}\mathfrak{N}$, *resp.* $\mathfrak{N}=\mathrm{RL}\mathfrak{N}$.

Similar statements can be made in any partial *-algebra $\mathfrak{M}$. In particular, conditions $(A1)$ and $(A2)$ always hold for $\mathfrak{M}$ itself, so that we have finally:

Proposition 3.11. *Let* $\mathfrak{M}$ *be a partial *-algebra on* $\mathscr{D}$, $\mathrm{L}_{\mathfrak{M}}\mathfrak{M}$ and $\mathrm{R}_{\mathfrak{M}}\mathfrak{M}$ *the corresponding spaces of internal multipliers. Then the* $\circ$ *multiplication is associative on* $\mathrm{L}_{\mathfrak{M}}\mathfrak{M}\times\mathfrak{M}\times\mathrm{R}_{\mathfrak{M}}\mathfrak{M}$.

§ 4. Partial *-Algebras of $\mathscr{D}$-Maximal Operators

A closed operator $A\in\bar{C}(\mathscr{D},\mathscr{H})$ is called $\mathscr{D}$-maximal iff its adjoint A^{*} is

$\mathscr{D}$ -minimal. Thus the set of all $\mathscr{D}$-maximal operators is:

$$\mathfrak{C}^*(\mathscr{D}) = \{A^* | A \in \mathfrak{C}(\mathscr{D})\} = \{A \in \bar{C}(\mathscr{D}, \mathscr{H}) | A = A^{\dagger\dagger}\}.$$

Again we often write simply $\mathfrak{C}^*$. As can be expected, the set $\mathfrak{C}^*$ is also a partial *-algebra with respect to appropriate operations, which are to some extent dual to those of $\mathfrak{C}$.

(i) *Vector space structure*: for $A, B \in \bar{C}(\mathscr{D}, \mathscr{H})$, define:

$$A \check{+} B = [(A^* + B^*) | \mathscr{D}]^* \in \mathfrak{C}^*. \tag{4.1}$$

Notice that we have always $A \check{+} B = (A+B)^{\dagger\dagger} = (A^\dagger + B^\dagger)^\dagger$, where $A^\dagger \equiv (A|\mathscr{D})^*$. Indeed:

$$\begin{aligned}(A+B)^{\dagger\dagger} &= [[(A+B)|\mathscr{D}]^* | \mathscr{D}]^* = [[(A|\mathscr{D})^* + (B|\mathscr{D})^*] | \mathscr{D}]^* \\ &= [(A^\dagger + B^\dagger)|\mathscr{D}]^* \equiv (A^\dagger + B^\dagger)^\dagger \\ &= [(A^* + B^*)|\mathscr{D}]^* \equiv A \check{+} B.\end{aligned}$$

It follows that $A \check{+} B \supset A \hat{+} B$ for all $A, B \in \bar{C}(\mathscr{D}, \mathscr{H})$.

(ii) *Involution*: *for* $A \in \mathfrak{C}^*$, *one has* $A^\dagger \in \mathfrak{C}^*$ *and* $A^{\dagger\dagger} = A$.

Lemma 4.1. $\mathfrak{C}^*$ *is a vector space with respect to* $\check{+}$ *and corresponding multiplication by complex numbers;* $A \leftrightarrow A^\dagger$ *is an involution on it.*

Proof. Let $A, B, C \in \mathfrak{C}^*$, $\lambda \in \mathbb{C}$. Then $\lambda A \equiv (\lambda A)^{\dagger\dagger} \in \mathfrak{C}^*$, $A \check{+} B \in \mathfrak{C}^*$, $A \check{+} 0 = 0 \check{+} A = A$ and addition is associative:

$$\begin{aligned}(A \check{+} B) \check{+} C &= [(A \check{+} B)^* | \mathscr{D} + C^* | \mathscr{D}]^* \\ &= [A^* | \mathscr{D} + B^* | \mathscr{D} + C^* | \mathscr{D}]^* = A \check{+} (B \check{+} C).\end{aligned}$$

Furthermore, $A^\dagger \in \mathfrak{C}^*$ and $(A \check{+} B)^\dagger = A^\dagger \check{+} B^\dagger$. □

(iii) *Partial multiplication*: for $A, B \in \bar{C}(\mathscr{D}, \mathscr{H})$, we say that A is a *left *-multiplier* of B and B a *right *-multiplier* of A if the following conditions are satisfied:

(* M1) $\mathrm{Ran}(B|\mathscr{D})\subset D(A^{\dagger\dagger})$
(* M2) $\mathrm{Ran}(A^{\dagger}|\mathscr{D})\subset D(B^{\dagger})$

We write $A\in \mathrm{L}^*(B)$, $B\in \mathrm{R}^*(A)$ and notice that $\mathrm{L}(B)\subset \mathrm{L}^W(B)\subset \mathrm{L}^*(B)$, for every $B \in \bar{C}(\mathscr{D},\mathscr{H})$. Then, we define the $*$-product:

$$A*B=[B^{\dagger}(A^{\dagger}|\mathscr{D})]^* \quad \text{for } A\in L^*(B). \tag{4.2}$$

We also notice that, in this case, $(A*B)\varphi=A^{\dagger\dagger}B\varphi$, for $\varphi\in\mathscr{D}$.

Proposition 4.2. *Let $\mathfrak{C}^*\equiv\mathfrak{C}^*(\mathscr{D})$ be the set of all $\mathscr{D}$-maximal operators. Then $\mathfrak{C}^*$ is a partial *-algebra with respect to the $\check{+}$ addition, the involution $A\leftrightarrow A^{\dagger}$ and the $*$-multiplication restricted to those pairs A, B for which $A\in L^*(B)$. Moreover, the identity operator is a unit for $\mathfrak{C}^*$.*

Proof. It remains only to verify the properties of the product.
(i) $A\in \mathrm{L}^*(B)$ iff $B^{\dagger}\in \mathrm{L}^*(A^{\dagger})$, by (* M1) and (* M2).
(ii) The involution property holds:

$$(A*B)^{\dagger}=[(A*B)|\mathscr{D}]^*=[A^{\dagger\dagger}(B|\mathscr{D})]^*=B^{\dagger}*A^{\dagger}.$$

(iii) The multiplication is distributive: $A\check{+}B\in L^*(C)$ if $A, B\in \mathrm{L}^*(C)$ and

$$\begin{aligned}(A\check{+}B)*C&=(A+B)^{\dagger\dagger}*C=[C^{\dagger}(A+B)^*|\mathscr{D}]^*\\&=[C^{\dagger}(A^*|\mathscr{D})+C^{\dagger}(B^*|\mathscr{D})]^*=[(C^{\dagger}*A^{\dagger})|\mathscr{D}+(C^{\dagger}*B^{\dagger})|\mathscr{D}]^*\\&=[C^{\dagger}*A^{\dagger}+C^{\dagger}*B^{\dagger}]^{\dagger}=[(A*C)^{\dagger}+(B*C)^{\dagger}]^{\dagger}\\&=(A*C)\check{+}(B*C).\end{aligned}$$

(iv) Finally, $I\in \mathrm{L}^*(A)$, $\forall A\in\mathfrak{C}^*$ and $I*A=A*I=A$. □

We have seen in Section 3 that the $\cdot$ multiplication is not associative on $\mathfrak{C}$. But the result of Proposition 3.2 suggests that the $*$ multiplication, which is less restrictive, might be associative on $\mathfrak{C}^*$.

Proposition 4.3. *Let $A, B, C\in\mathfrak{C}^*$, $A\in \mathrm{L}^*(B)$, $C\in \mathrm{R}^*(B)$ and $A*B\in \mathrm{L}^*(C)$. Then $B*C\in\mathrm{R}^*(A)$ and we have $(A*B)*C=A*(B*C)$.*

Proof. For any ϕ, $\psi\in\mathscr{D}$, we have:

$$\langle((A*B)*C)^*\psi, \phi\rangle=\langle\psi, ((A*B)*C)\phi\rangle=\langle\psi, (A*B)^{\dagger\dagger}C\phi\rangle$$
$$=\langle(A*B)^{\dagger}\psi, C\phi\rangle=\langle B^{\dagger}A^{\dagger}\psi, C\phi\rangle=\langle A^{\dagger}\psi, (B*C)\phi\rangle$$

which implies that $(B*C)\phi\in D(A^{\dagger\dagger})$, $A^{\dagger}\psi\in D((B*C)^{\dagger})$ and

$$((A*B)*C)\phi=A^{\dagger\dagger}(B*C)\phi=(A*(B*C))\phi.$$

This proves the two assertions (of course, we have $A^{\dagger\dagger}=A$ and $(A*B)^{\dagger\dagger}=A*B$ by definition). □

The natural definition of partial *-algebra is now obvious and entirely parallel to Definition 3.4.

Definition 4.4. Given a dense domain $\mathscr{D}$, a partial *-algebra of *$\mathscr{D}$-maximal operators* is a *-subalgebra of $\mathfrak{C}^*(\mathscr{D})$, that is, a vector subspace $\mathfrak{M}\subset\mathfrak{C}^*$, containing I, and stable under the involution $A\leftrightarrow A^{\dagger}$ and the $*$ multiplication.

Examples of such partial *-algebras are easy to give. If $\mathfrak{M}$ is any *-subalgebra of $\mathfrak{C}$, the set $\mathfrak{M}^*=\{A^*|A\in\mathfrak{M}\}$ is a *-subalgebra of $\mathfrak{C}^*$ called the *adjoint* of $\mathfrak{M}$. One gets in this way, for instance, $\mathfrak{C}^*$ itself, $B(\mathscr{H})$, $[L^+(\mathscr{D})]^*$. But in fact there are more, because the $*$ multiplication is less restrictive than the $\cdot$ multiplication, since in general $\mathrm{L}(B)\subsetneq\mathrm{L}^*(B)$. To understand the situation clearly, it is useful to make a comparison between $\mathfrak{C}$ and $\mathfrak{C}^*$.

Let $A, B\in\bar{C}(\mathscr{D}, \mathscr{H})$. Then $A^+, A^{++}\in\mathfrak{C}$, $A^{\dagger}$, $A^{\dagger\dagger}\in\mathfrak{C}^*$ and the two pairs are adjoint of each other by Eq. (1.5). The following relations are readily verified:

$$(A^{++}\hat{+}B^{++})^*=A^{\dagger}\check{+}B^{\dagger} \tag{4.3a}$$
$$(A^{\dagger\dagger}\check{+}B^{\dagger\dagger})^*=A^{+}\hat{+}B^{+}. \tag{4.3b}$$

This means that the involution $A\leftrightarrow A^*$ extends to an antilinear isomorphism of the *vector spaces* $(\mathfrak{C}, \hat{+})$ and $(\mathfrak{C}^*, \check{+})$. But for multiplication, the situation is not symmetric anymore. Let $A, B\in\mathfrak{C}$ with $A\in\mathrm{L}(B)$. Then $A^{\dagger\dagger}\in\mathrm{L}^*(B^{\dagger\dagger})$ and one has

$$(A\cdot B)^*=B^{\dagger}*A^{\dagger} \tag{4.4a}$$
$$(A^{\dagger\dagger}*B^{\dagger\dagger})^*=B^{+}\cdot A^{+} \tag{4.4b}$$

(notice that one has always $A*B=A*B^{\dagger\dagger}=A^{\dagger\dagger}*B=A^{\dagger\dagger}*B^{\dagger\dagger}$). But the converse does not hold. Let $A, B\in\mathfrak{C}^*$. Their adjoints are A^+, B^+, but $A\in\mathrm{L}^*(B)$

does not imply that $B^{+} \in \mathrm{L}(A^{+})$ or $A^{++} \in \mathrm{L}(B^{++})$. Therefore two corresponding partial *-algebras $\mathfrak{M} \subset \mathfrak{C}$ and $\mathfrak{M}^* \subset \mathfrak{C}^*$ are anti-isomorphic as vector spaces but *not* as partial algebras. More precisely, the linear map $A \mapsto A^{*\dagger} = A^{\dagger\dagger}$ is a one-to-one homomorphism (Def. 3.8) of $\mathfrak{M}$ onto $\mathfrak{M}^*$, but in general not an isomorphism.

These considerations clarify also the question of closedness of *-subalgebras of $\mathfrak{C}^*$. Let $\mathfrak{M}$ be a partial *-algebra of $\mathscr{D}$-minimal operators on some domain $\mathscr{D} \subset \mathscr{D}(\mathfrak{M})$. The adjoint partial *-algebra $\mathfrak{M}^*$, also defined on $\mathscr{D}$, has for natural domain $\mathscr{D}(\mathfrak{M}^*) = \bigcap_{A \in \mathfrak{M}} D(A^*)$, which is complete in its graph topology $t_{\mathfrak{M}^*}$. Obviously $\mathscr{D}(\mathfrak{M}^*) \supset \mathscr{D}(\mathfrak{M})$, and the topology $t_{\mathfrak{M}^*}$ coincides with $t_{\mathfrak{M}}$ on $\mathscr{D}(\mathfrak{M})$ and on $\mathscr{D}$. In general, $\mathscr{D}(\mathfrak{M})$ is smaller than $\mathscr{D}(\mathfrak{M}^*)$, but the condition that they are equal seems of little interest, in contrast with the case of Op*-algebras [3-4], where self-adjoint algebras have distincly better properties. In the case of $\mathfrak{C}^*$, however, the two notions are linked. Write, as usual [18], $\mathscr{D}_* \equiv \mathscr{D}([L^+(\mathscr{D})]^*) = \bigcap_{A \in L^+(\mathscr{D})} D(A^*)$. Then we have:

$$\mathscr{D} \subset \mathscr{D}(\mathfrak{C}) \subset \mathscr{D}(\mathfrak{C}^*) \subset \mathscr{D}_* \tag{4.5}$$

Thus, if $L^+(\mathscr{D})$ is self-adjoint, i. e. $\mathscr{D} = \mathscr{D}_*$, the partial *-algebras $\mathfrak{C}(\mathscr{D})$ and $\mathfrak{C}^*(\mathscr{D})$ are both fully closed.

Taking adjoints again, we obtain simply $\mathfrak{M}^{**} = \mathfrak{M}$. Of course the difference with Op*-algebras [3, 18] is that here every operator $A \in \mathfrak{M}$ is closed, and we consider the adjoints A^* themselves, not their restrictions to $\mathscr{D}(\mathfrak{M}^*)$. As a consequence, $\mathscr{D}(\mathfrak{M}^{**}) = \bigcap_{B \in \mathfrak{M}^*} D(B^*) = \mathscr{D}(\mathfrak{M})$, and the double adjoint $\mathfrak{M}^{**} = \mathfrak{M}$ offers nothing new.

As the reader will have noticed, our construction bears a strong resemblance to that used by Powers [3] for representations of algebras of unbounded operators. Although we won't discuss here the notion of representation of partial *-algebras, we want to stress the analogy. Let $\mathfrak{M}$ be a *-subalgebra of $\mathfrak{C}(\mathscr{D})$, and π the embedding (identity) of $\mathfrak{M}$ into $\mathfrak{C}$. Then π is a *-representation of $\mathfrak{M}$ in $\mathfrak{C}$, in the sense of Powers. The adjoint representation π^* is given by:

$$\pi^*(A) = \pi(A^+)^* = A^{\dagger\dagger}, \quad \pi^*(A^+) = \pi(A)^* = A^{\dagger}$$

that is, $\pi^*(\mathfrak{M}) = [\pi(\mathfrak{M})]^*$, a representation of $\mathfrak{M}$ in $\mathfrak{C}^*$. In this analogy, a representation by $\mathscr{D}$-minimal operators corresponds to a hermitian representation π.

In fact, this remark suggests what the genuine analog of self-adjoint representations might be in the case of partial *-algebras.

Definition 4.5. An operator $A \in \mathfrak{C}$ is called *standard* if $A^* = A^+$. A subset $\mathfrak{M} \subset \mathfrak{C}$ is *standard* if all its elements are standard operators.

Lemma 4.6. *An operator* $A \in \mathfrak{C}$ *is standard iff* $A \in \mathfrak{C}^*$. *A subset* $\mathfrak{M} \subset \mathfrak{C}$ *is standard iff* $\mathfrak{M} \subset \mathfrak{C}^*$.

So the set of all standard operators (on $\mathscr{D}$) is exactly $\mathfrak{C} \cap \mathfrak{C}^*$. In particular every bounded operator is standard, and every symmetric ($A = A^+$) standard operator is self-adjoint. However, even if A and B are standard, their sum $A \check{+} B$ need not be, neither their product $A \cdot B$ if it exists; in other words $A \check{+} B \neq A \hat{+} B$ and $A * B \neq A \cdot B$ are possible. Thus, for a partial *-subalgebra the condition of standardness must be imposed on each element separately. Then the following result is immediate.

Proposition 4.7. *Let* $\mathfrak{M}$ *be a standard* *-*subalgebra of* $\mathfrak{C}$, *i.e.* $\mathfrak{M} \subset \mathfrak{C} \cap \mathfrak{C}^*$. *Then the two additions* $\hat{+}$ *and* $\check{+}$ *coincide on* $\mathfrak{M}$ *and so do the two multiplications* $\cdot$ *and* $*$. *In particular the latter are both associative.*

Let us consider again Example 3.5. The operators I, χ, S, S^+ are bounded, e^x is self-adjoint, so they are all standard. Similarly $\left(x\frac{d}{dx}\right)^* = \left(x\frac{d}{dx}\right)^+ = \overline{\left(1+x\frac{d}{dx}\right)|\mathscr{D}}$ as can be checked easily, hence $\left(x\frac{d}{dx}\right)$ and $\left(x\frac{d}{dx}\right)^+$ are also standard. Finally the same is true of all powers of $x\frac{d}{dx}$, as well as all products $\left(x\frac{d}{dx}\right)^m \cdot S^n$, and all sums of such elements. In other words, the partial *-algebra $\mathfrak{M}[\mathfrak{N}]$ is standard.

Finally, we come back to spaces of multipliers in $\mathfrak{C}^*$, or in any *-subalgebra $\mathfrak{M}$ of it. As discussed in Sec. 2, the spaces $\mathrm{L}^*\mathfrak{N}$, for all subsets $\mathfrak{N} \subset \mathfrak{M}$, form a complete lattice, and so do the spaces $\mathrm{R}^*\mathfrak{N}$. The smallest elements are, respectively, $\mathrm{L}^*\mathfrak{M}$ and $\mathrm{R}^*\mathfrak{M}$. In the case of $\mathfrak{C}^*$ itself, these may be computed easily, exactly as we did in Sec. 3 for $\mathrm{L}\mathfrak{C}$ and $\mathrm{R}\mathfrak{C}$.

Lemma 4.8. (i) $\mathrm{R}^*\mathfrak{C}^*$ *consists of all bounded operators that map* $\mathscr{D}$ *into* $\mathscr{D}(\mathfrak{C}^*) = \bigcap_{A \in \mathfrak{C}^*} D(A) = \bigcap_{B \in \mathfrak{C}} D(B^*)$.

(ii) $\mathrm{L}^*\mathfrak{C}^*$ *consists of all bounded operators* B *such that their adjoint* B^* *maps* $\mathscr{D}$ *into* $\mathscr{D}(\mathfrak{C}^*)$.

§ 5. Topological Questions

In Section 3 we have discussed closedness of a partial *-algebra $\mathfrak{M}$ and the associated topology on the natural domain $\mathscr{D}(\mathfrak{M})$. We begin this section with some remarks concerning natural domains of subsets of $\bar{C}(\mathscr{D}, \mathscr{H})$ and the associated topologies. The first one is immediate.

Proposition 5.1. *If* $\mathfrak{A}\subset\bar{C}(\mathscr{D}, \mathscr{H})$ *contains an invertible operator, then* $\mathscr{D}$, *with the topology given by the seminorms*

$$\mathscr{D}\ni f\mapsto\|f\|_B=\|Bf\|,\ B\in\mathfrak{A},$$

is a Hausdorff space.

The same result applies to the natural domain $\mathscr{D}(\mathfrak{A})=\bigcap_{A\in\mathfrak{A}}D(A)$. Of course the condition is always satisfied if $\mathfrak{A}$ contains I, in particular for any *-subalgebra of $\mathfrak{C}$ or $\mathfrak{C}^*$.

Proposition 5.2. *Let* $\mathfrak{N}\subset\mathfrak{C}$ $(\mathfrak{N}\subset\mathfrak{C}^*)$ *and* $\tilde{\mathfrak{N}}$ *the vector space spanned by* $\mathfrak{N}$ *with the* $\hat{+}$ $(\check{+})$ *addition, then* $\mathscr{D}(\mathfrak{N})=\mathscr{D}(\tilde{\mathfrak{N}})$ *and the topologies defined on* $\mathscr{D}(\mathfrak{N})$ *by the two families of norms*

$$\|\varphi\|_A=\|A\varphi\|+\|\varphi\|,\ \varphi\in\mathscr{D}(\mathfrak{N})$$

with (a) A runs over $\mathfrak{N}$, *(b) A runs over* $\tilde{\mathfrak{N}}$, *are equivalent.*

Proof. Let $\mathfrak{N}\subset\mathfrak{C}$. Then, $\mathscr{D}(\tilde{\mathfrak{N}})=\bigcap_{B\in\tilde{\mathfrak{N}}}D(B)\subset\mathscr{D}(\mathfrak{N})=\bigcap_{B\in\mathfrak{N}}D(B)$ because $\mathfrak{N}\subset\tilde{\mathfrak{N}}$, but $D(A\hat{+}B)\supset D(A)\cap D(B)$, for $A, B\in\mathfrak{N}$. Hence $\mathscr{D}(\mathfrak{N})=\mathscr{D}(\tilde{\mathfrak{N}})$. The equivalence of topologies follows from the relation $\|\varphi\|_{A\hat{+}B}\leq\|\varphi\|_A+\|\varphi\|_B$, for any $\varphi\in\mathscr{D}(\mathfrak{N})$ and $A, B\in\mathfrak{N}$. The argument for $\mathfrak{N}\subset\mathfrak{C}^*$ is identical. □

We already know that in both $\mathfrak{C}$ and $\mathfrak{C}^*$ the corresponding distributive laws imply $\mathrm{L}^{(*)}\mathfrak{N}=\mathrm{L}^{(*)}\tilde{\mathfrak{N}}$ and $\mathrm{R}^{(*)}\mathfrak{N}=\mathrm{R}^{(*)}\tilde{\mathfrak{N}}$, and both are vector spaces.

Proposition 5.3. *If* $\mathfrak{N}\subset\mathfrak{C}$ $(\mathfrak{N}\subset\mathfrak{C}^*)$, *then* $\mathscr{D}(\mathfrak{N})=\mathscr{D}(\mathrm{L}^{(*)}\mathrm{R}^{(*)}\mathfrak{N})$.

Proof. Let $\mathfrak{N}\subset\mathfrak{C}$. We have $\mathscr{D}(\mathrm{LR}\mathfrak{N})\subset\mathscr{D}(\mathfrak{N})$ because $\mathfrak{N}\subset\mathrm{LR}\mathfrak{N}$. Let $\phi\in\mathscr{D}$,

$\psi \in \mathscr{D}(\mathfrak{N})$, both non zero. The rank one operator $U = |\psi\rangle\langle\phi|$ belongs to $\mathrm{R}\mathfrak{N}$ and hence $\psi \in \mathscr{D}(\mathrm{LR}\mathfrak{N})$, which means that $\mathscr{D}(\mathfrak{N}) \subset \mathscr{D}(\mathrm{LR}\mathfrak{N})$. The case $\mathfrak{N} \subset \mathfrak{C}^*$ is similar.
□

Although $\mathfrak{N}$ and $\mathrm{LR}\mathfrak{N}$ have the same natural domain $\mathscr{D}(\mathfrak{N}) = \mathscr{D}(\mathrm{LR}\mathfrak{N})$, the $\mathfrak{N}$-topology and the $\mathrm{LR}\mathfrak{N}$-topology, as defined in Proposition 5.2, need not coincide on it. However, since the domain is complete in both, the two topologies define the same bounded sets, exactly as for two closed Op*-algebras on the same domain [19].

In what follows we shall, as before, consider $\mathscr{D}$ as a topological vector space with the locally convex topology given by the seminorms $\|\phi\|_A = \|A\phi\|$, $\phi \in \mathscr{D}$ and A running over $\bar{C}(\mathscr{D}, \mathscr{H})$. Note that the set of seminorms remains the same if we let A run over $\mathfrak{C}$ or $\mathfrak{C}^*$ only. Let $\mathbb{B}(\mathscr{D})$ denote the set of all bounded subsets of $\mathscr{D}$. Unless stated explicitly (Props. 5.4 and 5.8), we do not assume $\mathscr{D}$ to be complete in the $\mathfrak{C}$-topology, although we could do so in view of Proposition 3.9 (the assumption is usually made when defining topologies on Op*-algebras).

We shall discuss first some topological properties of $\mathfrak{C}^*$ and its subsets. Let $\mathfrak{R}$, $\mathfrak{L}$, $\mathfrak{N} \subset \mathfrak{C}^*$ with $\mathfrak{R} \subset \mathrm{R}^*\mathfrak{N}$ and $\mathfrak{L} \subset \mathrm{L}^*\mathfrak{N}$. We define two quasi-uniform topologies [4, 20] on $\mathfrak{N}$, by the following seminorms:

$$\|B\|^{C,\mathscr{M}} = \sup_{\varphi \in \mathscr{M}} \|BC\varphi\| + \sup_{\varphi \in \mathscr{M}} \|C^\dagger B^\dagger \varphi\| \tag{5.1}$$

where C runs over $\mathfrak{R}$ and $\mathscr{M} \in \mathbb{B}(\mathscr{D})$, and

$$\|B\|^{A,\mathscr{M}} = \sup_{\varphi \in \mathscr{M}} \|AB\varphi\| + \sup_{\varphi \in \mathscr{M}} \|B^\dagger A^\dagger \varphi\| \tag{5.2}$$

where A runs over $\mathfrak{L}$ and $\mathscr{M} \in \mathbb{B}(\mathscr{D})$. We denote the first topology by $\tau_*^r(\mathfrak{R})$, and the second one by $\tau_*^l(\mathfrak{L})$.

We also equip $\mathfrak{C}^*$ with $\tau_* \equiv \tau_*^r(I) = \tau_*^l(I)$ (where I stands for the set $\{I\}$), given by the seminorms

$$\|A\|^{\mathscr{M}} = \sup_{\varphi \in \mathscr{M}} \|A\varphi\| + \sup_{\varphi \in \mathscr{M}} \|A^\dagger \varphi\|, \quad \mathscr{M} \in \mathbb{B}(\mathscr{D}). \tag{5.3}$$

This topology on $\mathfrak{C}^*$ is very natural, in view of the following result.

Proposition 5.4. *Let $\mathfrak{C}^*$ be fully closed. Then the three topologies $\tau_*^r(\mathrm{R}^*\mathfrak{C}^*)$, $\tau_*^l(\mathrm{L}^*\mathfrak{C}^*)$ and τ_* on $\mathfrak{C}^*$ are equivalent.*

Proof. Since $I \in \mathrm{R}^*\mathfrak{C}^*$, the topology $\tau_* = \tau_*^r(I)$ is not stronger than $\tau_*^r(\mathrm{R}^*\mathfrak{C}^*)$.

On the other hand, by Lemma 4.8, every $B\in \mathrm{R}^*\mathfrak{C}^*$ is bounded and maps $\mathscr{D}$ into itself, since $\mathscr{D}=\mathscr{D}(\mathfrak{C}^*)$ by assumption. Hence, given $B\in \mathrm{R}^*\mathfrak{C}^*$ and $\mathscr{M}\in \boldsymbol{B}(\mathscr{D})$, we have for every $A\in\mathfrak{C}^*$:

$$\|A\|^{B,\mathscr{M}}\leq \sup_{\varphi\in B\mathscr{M}}\|A\varphi\|+\|B^{\dagger}\|\sup_{\varphi\in\mathscr{M}}\|A^{\dagger}\varphi\|.$$

Clearly $B\mathscr{M}$ is bounded if $\mathscr{M}$ is bounded. Also there is no loss of generality to consider only those $B\in \mathrm{R}^*\mathfrak{C}^*$ with $\|B\|=1$. Thus we have

$$\|A\|^{B,\mathscr{M}}\leq \sup_{\varphi\in\mathscr{N}}\|A\varphi\|+\sup_{\varphi\in\mathscr{N}}\|A^{\dagger}\varphi\|=\|A\|^{\mathscr{N}}$$

where $\mathscr{N}=B\mathscr{M}\cup\mathscr{M}\in\boldsymbol{B}(\mathscr{D})$. Similarly for $\tau_*^{l}(\mathrm{L}^*\mathfrak{C}^*)$. □

Proposition 5.5. *$\mathfrak{C}^*$ is complete in τ_*.*

Proof. Let $\{A_\alpha\}\subset\mathfrak{C}^*(\mathscr{D})$ be a Cauchy net. Then for any $\varphi\in\mathscr{D}$ the vector nets $\{A_\alpha\varphi\}$ and $\{A_\alpha^{\dagger}\varphi\}$ are strongly convergent. Put ξ and ζ for the corresponding limits and define $A_0\varphi=\xi$, $B_0\varphi=\zeta$. If $\varphi,\psi\in\mathscr{D}$ then

$$\langle\psi, A_\alpha\varphi\rangle\rightarrow\langle\psi, A_0\varphi\rangle$$

and

$$\langle A_\alpha^{\dagger}\psi, \varphi\rangle\rightarrow\langle B_0\psi, \varphi\rangle.$$

Thus $\langle\psi, A_0\varphi\rangle=\langle B_0\psi, \varphi\rangle$, so that $B_0\subset A_0^*$. Hence A_0 has its adjoint densely defined and so it is closable. Let $A=A_0^{\dagger\dagger}$. Then $A^{\dagger}=B_0^{\dagger\dagger}$, A and $A^{\dagger}$ belong to $\mathfrak{C}^*$ and $A_\alpha\varphi\rightarrow A\varphi$, $A_\alpha^{\dagger}\varphi\rightarrow A^{\dagger}\varphi$, $\forall\varphi\in\mathscr{D}$. It remains to prove that $A_\alpha\rightarrow A$ in the topology τ_*, i.e. $\|A-A_\alpha\|^{\mathscr{M}}\rightarrow 0$ for all bounded sets $\mathscr{M}\in\boldsymbol{B}(\mathscr{D})$. Let $\psi\in\mathscr{M}$. Then:

$$\begin{aligned}\|(A-A_\alpha)\psi\|&\leq\|(A_\beta-A_\alpha)\psi\|+\|(A-A_\beta)\psi\|\\&\leq\|A_\beta-A_\alpha\|^{\mathscr{M}}+\|(A-A_\beta)\psi\|.\end{aligned}$$

For α, β sufficiently large, the first term on the r.h.s. is smaller than any given $\varepsilon/2$, and the second one tends to zero as $\beta\rightarrow\infty$. It follows that, for α large enough and all $\psi\in\mathscr{M}$, $\|(A-A_\alpha)\psi\|\leq\varepsilon/2$ and this implies $\|A-A_\alpha\|^{\mathscr{M}}\leq\varepsilon$. □

Proposition 5.6. *Let $\mathfrak{N}\subset\mathfrak{C}^*$. If we consider $\mathfrak{N}$, $\mathrm{R}^*\mathfrak{N}$ and $\mathfrak{N}*\mathrm{R}^*\mathfrak{N}$ as locally convex spaces with topologies $\tau_*^{r}(\mathrm{R}^*\mathfrak{N})$, $\tau_*^{l}(\mathfrak{N})$ and τ_* respectively, then the $*$ multiplication, considered as a map from $\mathfrak{N}\times\mathrm{R}^*\mathfrak{N}$ into $\mathfrak{C}^*$, is separately continuous.*

Proof. Let $A\in\mathfrak{N}$ be fixed. We have for $B\in\mathrm{R}^*\mathfrak{N}$

$$\|A*B\|^{\mathscr{M}}=\sup_{\varphi\in\mathscr{M}}\|AB\varphi\|+\sup_{\varphi\in\mathscr{M}}\|B^\dagger A^\dagger\varphi\|=\|B\|^{A,\mathscr{M}}.$$

Given $\varepsilon>0$, if $\|B\|^{A,\mathscr{M}}<\varepsilon$, so is $\|A*B\|^{\mathscr{M}}$. If $B\in\mathrm{R}^*\mathfrak{N}$ is fixed, then $\|A*B\|^{\mathscr{M}}=\|A\|^{B,\mathscr{M}}$ and $\|A\|^{B,\mathscr{M}}<\varepsilon$ implies the same for $\|A*B\|^{\mathscr{M}}$. □

Of course a corresponding statement holds, *mutatis mutandis*, for continuity of the multiplication $\mathrm{L}^*\mathfrak{N}\times\mathfrak{N}\to\mathfrak{C}^*$.

Proposition 5.7. *Let $\mathfrak{N}\subset\mathfrak{C}^*$ and assume that the topology $\tau_*^l(\mathfrak{N})$ on $\mathrm{R}^*\mathfrak{N}$ is stronger than τ_* restricted to $\mathrm{R}^*\mathfrak{N}$. Then $\mathrm{R}^*\mathfrak{N}$ is complete in $\tau_*^l(\mathfrak{N})$. Similarly $\mathrm{L}^*\mathfrak{N}$ is complete for $\tau_*^r(\mathfrak{N})$ whenever the latter is stronger than τ_* on $\mathrm{L}^*\mathfrak{N}$.*

Remark. The assumption that $\tau_*^l(\mathfrak{N})$ is stronger than τ_* on $\mathrm{R}^*\mathfrak{N}$ is satisfied, for example, when $I\in\mathfrak{N}$.

Proof of the proposition. Let $\{B_\alpha\}\subset\mathrm{R}^*\mathfrak{N}$ be a Cauchy net in $\tau_*^l(\mathfrak{N})$. Then it is also Cauchy in τ_* and by completeness of $\mathfrak{C}^*$ there is a τ_*-limit, say $B\in\mathfrak{C}^*$. On the other hand the fact that $\{B_\alpha\}$ is Cauchy and continuity of multiplication imply that for any $A\in\mathfrak{N}$ the net $\{A*B_\alpha\}$ is τ_*-Cauchy and again by completeness of $\mathfrak{C}^*$ there is a τ_*-limit

$$\lim_\alpha A*B_\alpha=Q\in\mathfrak{C}^*.$$

Let $\varphi\in\mathscr{D}$. We have

$$\lim_\alpha\|A*B_\alpha\varphi-Q\varphi\|=0. \tag{5.4}$$

But $A*B_\alpha\varphi=AB_\alpha\varphi$. Put $B_\alpha\varphi=\varphi_\alpha$. Clearly $\varphi_\alpha\xrightarrow{s}\psi=B\varphi$. Since A is closed we conclude that

$$A\varphi_\alpha\xrightarrow{s}Q\varphi=A\psi=AB\varphi. \tag{5.5}$$

Thus $Q|\mathscr{D}=A(B|\mathscr{D})$ which means that

$$\mathrm{Ran}(B|\mathscr{D})\subset D(A). \tag{5.6}$$

But $Q\in\mathfrak{C}^*$ implies $\mathscr{D}\subset D(Q^*)$. If $\varphi,\eta\in\mathscr{D}$ we get

$$\langle Q^*\varphi,\eta\rangle=\langle\varphi,Q\eta\rangle=\langle\varphi,AB\eta\rangle=\langle A^*\varphi,B\eta\rangle.$$

This shows that

$$A^*\varphi\in D((B|\mathscr{D})^*)=D(B^\dagger). \tag{5.7}$$

The formulas (5.6) and (5.7) imply $B\in \mathrm{R}^*\mathfrak{N}$. On the other hand, it follows from (5.4) and (5.5) that the τ_*-limit of $A*B_\alpha$ is $A*B$ or that the $\tau_*^l(\mathfrak{N})$ limit of B_α is B.

The proof of the second statement is identical. □

Finally we consider spaces of multipliers, $\mathrm{M}^*\mathfrak{N}\equiv \mathrm{L}^*\mathfrak{N}\cap \mathrm{R}^*\mathfrak{N}$. If $\mathfrak{N}^\dagger=\mathfrak{N}$, the two topologies $\tau_*^r(\mathfrak{N})$ and $\tau_*^l(\mathfrak{N})$ coincide on $\mathrm{M}^*\mathfrak{N}$ as can be seen from the expressions (5.1) and (5.2) of the seminorms. Then, by Proposition 5.7, $\mathrm{M}^*\mathfrak{N}$ is complete for that topology. We will come back to these multipliers in Section 6.

In the definition of the topologies τ_*^r, τ_*^l, τ_* as given by Eqs. (5.1) - (5.3), we let $\mathscr{M}$ run over *all* bounded sets of $\mathscr{D}$. We may therefore weaken each topology by restricting the family of bounded sets to some subset $\mathbb{B}_0(\mathscr{D})\subset\mathbb{B}(\mathscr{D})$ (the *same* for all topologies). Then Propositions 5.5, 5.6 and 5.7 remain valid: $\mathfrak{C}^*$ and its spaces of multipliers remain complete, and the $*$ multiplication remains separately continuous. As for Proposition 5.4, it remains true, provided $B\in\mathrm{R}^*\mathfrak{C}^*, \mathscr{M}\in\mathbb{B}_0(\mathscr{D})$ implies $B\mathscr{M}\in\mathbb{B}_0(\mathscr{D})$.

All four propositions are valid, in particular, if one takes for $\mathbb{B}_0(\mathscr{D})$ the set of all finite subsets of $\mathscr{D}$. In that case, the topology τ_* on $\mathfrak{C}^*$ reduces to the familiar strong *-topology [21], for which completeness of $\bar{C}(\mathscr{D},\mathscr{H})$ or $\mathfrak{C}^*$ has been proven before [22].

We shall now discuss similar problems concerning $\mathfrak{C}$. Let $\mathfrak{R}, \mathfrak{L}, \mathfrak{N}\subset\mathfrak{C}$ be such that $\mathfrak{R}\subset \mathrm{R}\mathfrak{N}$, $\mathfrak{L}\subset \mathrm{L}\mathfrak{N}$. As before we define two quasi-uniform topologies on $\mathfrak{N}$, by the seminorms:

$$\|B\|^{C,\mathscr{M}}=\sup_{\varphi\in\mathscr{M}}\|BC\varphi\|+\sup_{\varphi\in\mathscr{M}}\|C^+B^+\varphi\| \tag{5.8}$$

where $C\in\mathfrak{R}$ and $\mathscr{M}\in\mathbb{B}(\mathscr{D})$, and

$$\|B\|^{A,\mathscr{M}}=\sup_{\varphi\in\mathscr{M}}\|AB\varphi\|+\sup_{\varphi\in\mathscr{M}}\|B^+A^+\varphi\| \tag{5.9}$$

where $A\in\mathfrak{L}$ and $\mathscr{M}\in\mathbb{B}(\mathscr{D})$, and denote them by $\tau_*^r(\mathfrak{R})$ and $\tau_*^l(\mathfrak{L})$ respectively. In analogy to the previous case we equip $\mathfrak{C}$ with the topology $\tau_*=\tau_*^r(I)=\tau_*^l(I)$, given by the seminorms

$$\|A\|^{\mathscr{M}}=\sup_{\varphi\in\mathscr{M}}\|A\varphi\|+\sup_{\varphi\in\mathscr{M}}\|A^+\varphi\|,\quad \mathscr{M}\in\mathbb{B}(\mathscr{D}). \tag{5.10}$$

Here also we may replace $\mathbb{B}(\mathscr{D})$ by some subset $\mathbb{B}_0(\mathscr{D})$ and obtain weaker topologies, in particular τ_* reduces to the strong *-topology on $\mathfrak{C}$ if we take only finite subsets of $\mathscr{D}$.

Then, with proofs entirely similar to those of the corresponding propositions in the $\mathfrak{C}^*$ case, we have:

Proposition 5.8. *Let $\mathfrak{C}$ be fully closed. Then, the topologies $\tau_*^r(\mathrm{R}\mathfrak{C})$, $\tau_*^l(\mathrm{L}\mathfrak{C})$ and τ_* on $\mathfrak{C}$ are equivalent.*

Proposition 5.9. *$\mathfrak{C}$ is complete in τ_*.*

Proposition 5.10. *Let $\mathfrak{N}\subset\mathfrak{C}$. If $\mathfrak{N}$, $\mathrm{R}\mathfrak{N}$ and $\mathfrak{N}\circ\mathrm{R}\mathfrak{N}$ are equipped with topologies $\tau_*^r(\mathrm{R}\mathfrak{N})$, $\tau_*^l(\mathfrak{N})$ and τ_* respectively, then the $\circ$ multiplication, considered as a map from $\mathfrak{N}\times\mathrm{R}\mathfrak{N}$ into $\mathfrak{C}$, is separately continuous.*

Let $\mathfrak{N}\subset\mathfrak{C}$ and $\mathbb{1}\in\mathfrak{N}$. Assume that $\{B_\alpha\}\subset\mathrm{R}\mathfrak{N}$ is a Cauchy net in the $\tau_*^l(\mathfrak{N})$ topology. Exactly as before we can show that there is a τ_*-limit $B\in\mathfrak{C}$ of B_α and that $A\circ B\in\mathfrak{C}$ is the τ_*-limit of $A\circ B_\alpha$. However, in general we can conclude only that $(A\circ B)^+=B^\dagger\circ A^+$. Hence B is not necessarily a $\tau_*^l(\mathfrak{N})$-limit of B_α, so that $\mathrm{R}\mathfrak{N}$ does not have to be complete.

Here again a similar statement holds, with the appropriate topologies, for the multiplication $\mathrm{L}\mathfrak{N}\times\mathfrak{N}\to\mathfrak{C}$.

This brings us to a further remark, namely comparing sets $\mathfrak{N}$ and $\mathrm{RL}\mathfrak{N}$ or $\mathrm{LR}\mathfrak{N}$, and similarly in $\mathfrak{C}^*$. Since $\mathfrak{N}$ and $\mathrm{LR}\mathfrak{N}$ have the same set $\mathrm{R}\mathfrak{N}$ of right multipliers, the topology $\tau_*^r(\mathrm{R}\mathfrak{N})$ on $\mathfrak{N}$ is simply the one induced by $\mathrm{LR}\mathfrak{N}$. On the other hand, $\mathrm{R}\mathfrak{N}$ has now two natural topologies, namely $\tau_*^l(\mathfrak{N})$ and $\tau_*^l(\mathrm{LR}\mathfrak{N})$. *A priori* the latter is stronger, but they might be equivalent in certain cases. This is true, for instance, for $\mathfrak{N}=\{\mathbb{1}\}$. Then $\mathrm{R}\mathfrak{N}=\mathfrak{C}^*$, and Proposition 5.8 applies (and also 5.4). The general case is open.

Actually this would be a reason for considering only RL- and LR-closed subspaces, namely *matching subspaces* $(\mathfrak{N},\mathfrak{M})$, $\mathfrak{N}=\mathrm{L}\mathfrak{M}$, $\mathfrak{M}=\mathrm{R}\mathfrak{N}$, as defined in Sec. 2. Then the topological situation becomes completely symmetrical between left and right, and moreover, in the $\mathfrak{C}^*$ case at least, these subspaces are automatically complete in their mutual topologies.

Our last observation follows from comparison of the τ_* topologies on $\mathfrak{C}$ and $\mathfrak{C}^*$.

Corollary 5.11. *The linear map $A\mapsto A^{*+}$ from $\mathfrak{C}^*[\tau_*]$ onto $\mathfrak{C}[\tau_*]$ is* *continuous and one-to-one. The same is true for the inverse map* $B \mapsto B^{*\dagger}$.

Finally we may remark that other topologies can be introduced on $\mathfrak{C}$, $\mathfrak{C}^*$ and their subsets. For instance one can consider $\mathfrak{C}$ as a set of continuous linear maps from $\mathscr{D}$ or $\mathscr{D}(\mathfrak{C})$ into $\mathscr{H}$, with the (strong) topology inherited from $L(\mathscr{D}, \mathscr{H})$. For topologies of this kind, we proved in [12] that the $\circ$ multiplication is jointly sequentially continuous and separately continuous.

§ 6. Commutants

Let $A \in \mathfrak{C}^*$. According to the general definition, Eq. (2.7), we define its *commutant* in $\mathfrak{C}^*$ as the set:

$$\{A\}'_* = \{X \in \mathfrak{C}^* \mid X \in \mathrm{M}^*(A) \text{ and } X * A = A * X\} \tag{6.1}$$

where, as in Sec. 2, $\mathrm{M}^*(A) \equiv \mathrm{L}^*(A) \cap \mathrm{R}^*(A)$. Similarly, given any subset $\mathfrak{N} \subset \mathfrak{C}^*$, its commutant in $\mathfrak{C}^*$ is:

$$\mathfrak{N}'_* = \bigcap_{A \in \mathfrak{N}} \{A\}'_*. \tag{6.2}$$

From Eqs. (6.1) and (6.2), it follows that $\mathfrak{N} \subset \mathfrak{M}$ implies $\mathfrak{N}'_* \supset \mathfrak{M}'_*$.

The *bicommutant* is defined in the obvious way:

$$\mathfrak{N}''_{**} = (\mathfrak{N}'_*)'_*. \tag{6.3}$$

Then one has, as usual, that $\mathfrak{N} \subset \mathfrak{N}''_{**}$ and $\mathfrak{N}'_* = \mathfrak{N}'''_{***}$.

What is the structure of the commutant $\mathfrak{N}'_*$ of a subset $\mathfrak{N} \subset \mathfrak{C}^*$? First the distributive law (see Proposition 4.2) implies that $\mathfrak{N}'_*$ is a vector space. Next, given $X, Y \in \{A\}'_*$, if $X * Y$ exists *and* belongs to $\mathrm{L}^*(A)$ *or* $\mathrm{R}^*(A)$, then $X * Y \in \{A\}'_*$ ($X \in \mathrm{L}^*(Y)$ alone does not imply that $X * Y \in \mathrm{L}^*(A)$!). Indeed we have, by associativity (Proposition 4.3):

$$\begin{aligned}(X * Y) * A = X * (Y * A) = X * (A * Y) = (X * A) * Y \\ = (A * X) * Y = A * (X * Y).\end{aligned}$$

Similarly, when $X, Y \in \mathfrak{N}'_*$, if $X * Y$ exists and belongs either to $\mathrm{L}^*\mathfrak{N}$ or to $\mathrm{R}^*\mathfrak{N}$, then $X * Y \in \mathfrak{N}'_*$. If (X, Y_1) and (X, Y_2) are two such pairs, then $X * (Y_1 \check{+} \lambda Y_2) \in \mathfrak{N}'_*$.

If, in addition, $\mathfrak{N} = \mathfrak{N}^\dagger$, then also $\mathfrak{N}'_* = (\mathfrak{N}'_*)^\dagger$. This means in particular that, if

X, Y and $X*Y$ belong to $\mathfrak{N}'_*$, then also $(X*Y)^\dagger = Y^\dagger * X^\dagger \in \mathfrak{N}'_*$. Indeed, if $X\in\mathfrak{N}'_*$, then for any $A\in\mathfrak{N}$ we have $X*A=A*X$. But $A^\dagger$ is also in $\mathfrak{N}$, so that $X*A^\dagger = A^\dagger * X$, which implies $A*X^\dagger = X^\dagger * A$.

We collect those facts in the following

Proposition 6.1. *If $\mathfrak{N}\subset\mathfrak{C}^*$, then $\mathfrak{N}'_*$ is a vector space. If moreover $\mathfrak{N}=\mathfrak{N}^\dagger$, then $\mathfrak{N}'_*$ is a partial *-algebra with addition $\check{+}$, involution $\dagger$, and multiplication $*$, with the restriction that a pair of elements $X, Y\in\mathfrak{N}'_*$ is multiplicable only if $X*Y$ exists and belongs to* $\mathrm{L}^*\mathfrak{N}$ *or to* $\mathrm{R}^*\mathfrak{N}$.

In other words, $\mathfrak{N}'_*$ is a partial *-algebra, but *not* a subalgebra of $\mathfrak{C}^*$ in general, since it is not stable under $*$ multiplication without the additional restriction: $\mathfrak{N}'_*$ is a subalgebra of $\mathfrak{C}^*$ iff every element of $\mathfrak{N}'_* * \mathfrak{N}'_*$ belongs either to $\mathrm{L}^*\mathfrak{N}$ or to $\mathrm{R}^*\mathfrak{N}$.

Since $\mathfrak{N}'_*\subset\mathrm{L}^*\mathfrak{N}\cap\mathrm{R}^*\mathfrak{N}$ it is equally natural to equip $\mathfrak{N}'_*$ with the $\tau^r_*(\mathfrak{N})$ or the $\tau^l_*(\mathfrak{N})$ topology. However, since the elements of $\mathfrak{N}$ and $\mathfrak{N}'_*$ commute, the corresponding norms are equal, so that the two topologies coincide; we will write simply $\tau_*(\mathfrak{N})$. Let $I\in\mathfrak{N}$ and $\{X_\alpha\}\in\mathfrak{N}'_*$ be a Cauchy net for $\tau_*(\mathfrak{N})$. Then, since $\mathfrak{C}^*$ is complete, the following τ_*-limits exist:

$$\begin{array}{ll} X_\alpha\to X \quad , & X_\alpha^\dagger\to X^\dagger \\ A*X_\alpha\to A*X, & A^\dagger * X_\alpha^\dagger \to A^\dagger * X^\dagger \\ X_\alpha * A\to X*A, & X_\alpha^\dagger * A^\dagger \to X^\dagger * A^\dagger, \end{array}$$

which immediately implies that $X_\alpha\to X$ in $\tau_*(\mathfrak{N})$. Moreover $A*X=X*A$ means $X\in\mathfrak{N}'_*$. Thus we have proven:

Proposition 6.2. *If $I\in\mathfrak{N}\subset\mathfrak{C}^*$, then $\mathfrak{N}'_*$ is complete in the topology $\tau_*(\mathfrak{N})$.*

Let again $\mathfrak{N}=\mathfrak{N}^\dagger$, and consider its bicommutant $\mathfrak{N}''_{**}$. Again the topologies $\tau^l_*(\mathfrak{N}'_*)=\tau^r_*(\mathfrak{N}'_*)\equiv\tau_*(\mathfrak{N}'_*)$ coincide on $\mathfrak{N}''_{**}$, and $\mathfrak{N}''_{**}$ is complete, since $\mathfrak{N}'_*=(\mathfrak{N}'_*)^\dagger$. Thus, as before:

Proposition 6.3. *Let $\mathfrak{N}=\mathfrak{N}^\dagger\subset\mathfrak{C}^*$. Then the bicommutant $\mathfrak{N}''_{**}$ is a partial *-algebra, complete in the topology $\tau_*(\mathfrak{N}'_*)$. It is a partial *-subalgebra of $\mathfrak{C}^*$ iff every element of $\mathfrak{N}''_{**} * \mathfrak{N}''_{**}$ belongs either to* $\mathrm{L}^*\mathfrak{N}'_*$ *or to* $\mathrm{R}^*\mathfrak{N}'_*$.

When the last condition is satisfied, $\mathfrak{N}''_{**}$ is a partial *-subalgebra of $\mathfrak{C}^*$

containing $\mathfrak{N}$, hence it contains also the partial *-algebra $\mathfrak{M}[\mathfrak{N}]$ generated by $\mathfrak{N}$:

$$\mathfrak{N}\subset\mathfrak{M}[\mathfrak{N}]\subset\mathfrak{N}''_{**}. \tag{6.4}$$

It would be interesting to find under which conditions they coincide, i.e. $\mathfrak{M}[\mathfrak{N}]=\mathfrak{N}''_{**}$. We leave this problem open, as well as the question as to whether $\mathfrak{N}''_{**}$ is the closure of $\mathfrak{N}$ in some topology. Results in this direction have been obtained recently by F. Mathot for Op*-algebras [22]. Clearly a similar analysis is needed in the present case.

We turn now to the study of commutants in $\mathfrak{C}$, which is entirely similar. Given $A\in\mathfrak{C}$ and $\mathfrak{N}\subset\mathfrak{C}$, we define naturally:

$$\{A\}'=\{X\in\mathfrak{C}\mid X\in \mathrm{M}(A) \text{ and } X\circ A=A\cdot X\} \tag{6.5}$$

$$\mathfrak{N}'=\bigcap_{A\in\mathfrak{N}}\{A\}'. \tag{6.6}$$

Exactly as in the case of $\mathfrak{C}^*$, we can prove:

Proposition 6.4. *If $\mathfrak{N}\subset\mathfrak{C}$, then $\mathfrak{N}'$ is a vector space. If moreover $\mathfrak{N}=\mathfrak{N}^+$, then $\mathfrak{N}'$ is a partial *-algebra with addition $\hat{+}$, involution $\mp$ and multiplication $\circ$, with the restriction that a pair of elements $X, Y\in\mathfrak{N}'$ is multiplicable only if $X\circ Y$ exists and belongs either to $\mathrm{L}\mathfrak{N}$ or to $\mathrm{R}\mathfrak{N}$.*

As before the topologies $\tau_*^r(\mathfrak{N})$ and $\tau_*^l(\mathfrak{N})$ coincide on $\mathfrak{N}'$, but the result about completeness is different. Namely we have now:

Proposition 6.5. *If $\mathfrak{N}\subset\mathfrak{C}$ and $\mathfrak{N}=\mathfrak{N}^+$, then $\mathfrak{N}'=(\mathfrak{N}')^+$ and $\mathfrak{N}'$ is complete in $\tau_*(\mathfrak{N})$.*

The necessity to assume that $\mathfrak{N}=\mathfrak{N}^+$ comes from the fact that completeness of $\mathfrak{C}$ implies the existence of the τ_*-limits

$$\begin{gathered} X_\alpha\to X\\ A\cdot X_\alpha\to A\circ X\\ X_\alpha\cdot A\to X\circ A\\ X_\alpha^+\circ A^+\to X^+\circ A^+. \end{gathered}$$

If however $\mathfrak{N}=\mathfrak{N}^+$, which implies that $\mathfrak{N}'=(\mathfrak{N}')^+$, we have

$$(A\cdot X_\alpha)^+=X_\alpha^+\cdot A^+\to X^+\circ A^+$$

which is sufficient to conclude that X is a $\tau_*(\mathfrak{N})$-limit of the Cauchy net $\{X_\alpha\}$ and belongs to $\mathfrak{N}'$.

Let us give a (trivial) example. Let $\mathfrak{N}=\overline{L^+(\mathscr{D})}\subset\mathfrak{C}$. Then, $\mathrm{M}\mathfrak{N}$ consists of all operators $A\in\mathfrak{C}$ such that A and A^+ map $\mathscr{D}$ into $\mathscr{D}(\mathfrak{N})$, i.e. $\mathrm{M}\mathfrak{N}$ is the closure of $\mathfrak{N}$, since here $\tilde{\mathscr{D}}[t_{\mathfrak{N}}]=\mathscr{D}(\mathfrak{N})$. From this one sees easily that $\mathfrak{N}'=\{\lambda I\,|\,\lambda\in\mathbb{C}\}$ and $\mathfrak{N}''=\mathfrak{C}$. Similarly for $\mathfrak{M}=[L^+(\mathscr{D})]^*\subset\mathfrak{C}^*$, $\mathfrak{M}'_*=\{\lambda I\}$ and $\mathfrak{M}'_{**}{}'=\mathfrak{C}^*$. This gives a natural definition of *irreducible* partial *-algebra.

References

[1] A.N. Vasil'ev, *Theor. Math. Phys.* **2** (1970), 113.

[2] R. Ascoli, G. Epifanio, A. Restivo, *Commun. Math. Phys.* **18** (1970), 291; *Rivista Mat. Univ. Parma* **3** (1974), 21.

[3] R.T. Powers, *Commun. Math. Phys.* **21** (1971), 85; *Trans. Amer. Math. Soc.* **187** (1974), 261.

[4] G. Lassner, *Rep. Math. Phys.* **3** (1972), 279; *Wiss. Z. Karl-Marx-Universität Leipzig, Math.-Naturwiss. R.* **24** (1975), 465; and subsequent papers.

[5] J.E. Roberts, *J. Math. Phys.* **7** (1966), 1097; *Commun. Math. Phys.* **3** (1966), 98.

[6] A. Böhm, *The Rigged Hilbert Space and Quantum Mechanics*, Lecture Notes in Physics, **78**, Springer, Berlin 1978.

[7] See, for instance, G.G. Emch, H.J.F. Knops, *J. Math. Phys.* **11** (1970), 3008 or H. Narnhofer, *Acta Phys. Austr.* **49** (1978), 207.

[8] G. Lassner, *Physica A* **124** (1984), 471.

[9] A. Grossmann, *Commun. Math. Phys.* **4** (1967), 203.

[10] J -P. Antoine, A. Grossmann, *J. Funct. Anal.* **23** (1976), 369, 379.

[11] G. Epifanio, C. Trapani, *J. Math. Phys.* **20** (1979), 148.

[12] J.-P. Antoine, W. Karwowski, in *Quantum Theory of Particles and Fields*, J. Lukierski and B. Jancewicz (eds.), World Scientific Publ. Co., Singapore (1983) 13-30.

[13] H.J. Borchers, in *RCP* 25 (*Strasbourg*) **22** (1975), 26, and also in "*Quantum Dynamics: Models and Mathematics*", L. Streit (ed.), *Acta Phys. Austr. Suppl.* **16**, (1976), 15.

[14] B. E. Johnson, *Proc. London Math. Soc.* (3) **14** (1964), 299, and subsequent papers.

[15] J.-P. Antoine and K. Gustafson, *Adv. in Math.* **41** (1981), 281.

[16] J.-P. Antoine, *J. Math. Phys.* **21** (1980), 268.

[17] S. Gudder, W. Scruggs, *Pacific J. Math.* **70** (1977), 369.

[18] G. Lassner, *Beiträge zur Analysis* **8** (1976), 85.

[19] K. Schmüdgen, *Math. Nachr.* **85** (1978), 161.

[20] G. Lassner, *Wiss. Z. Karl-Marx-Univ. Leipzig*, *Math.-Naturwiss.* R. **30** (1981), 572; also in "*Algèbres d'opérateurs et leurs applications en physique mathématique*" (Proceedings

Marseille 1977), A. Connes, D. Kastler and D.W. Robinson (eds.), Editions du CNRS, Paris (1979), 249–260.

[21] See, for instance, O. Bratteli, D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics* I, Springer, Berlin 1979.

[22] F. Mathot, *J. Math. Phys.* **26** (1985), to appear.