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Weakly Hyperbolic Cauchy Problem for Second Order Operators

By

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§ 1. Introduction

We study the C^∞ well-posedness of the following Cauchy problem.

$$(1.1)\quad \begin{cases} P(t,x,D_t,D_x)u=D_t^2u-\sum_{j=0}^{2}Q_j(t,x,D_x)u+R(t,x,D_x)D_tu=f, \\ D_t^ju(0,x)=u_j(x), \quad j=0,1, \end{cases}$$

where $Q_j(t,x,D_x)$ and $R(t,x,D_x)$ are the differential operators of order j and 0 respectively with C^∞-coefficients defined in a neighborhood of the origin in $\mathbb{R}^{d+1}$, and

$$x=(x_1,\dots,x_d),\ \xi=(\xi_1,\dots,\xi_d)\in\mathbb{R}^d,\ D_x=\left(\frac{1}{i}\frac{\partial}{\partial x_1},\dots,\frac{1}{i}\frac{\partial}{\partial x_d}\right),\ D_t=\frac{1}{i}\frac{\partial}{\partial t}.$$

Denote by $P_2(t,x,\tau,\xi)$ and $P^s(t,x,\tau,\xi)$ the principal and subprincipal symbols of P. We formulate the assumptions and results in the global version. It is straightforward to give a microlocalized version of the assumptions and results. This is not carried out below.

Modifying $Q_2(t,x,\xi)$ near $\xi=0$ and extending it to the outside of a small ball $\{|x|>R\}$, we assume that

$$(1.2)\quad \begin{cases} Q_2(t,x,\xi)=\Phi(t,x,\xi)^{2p}E(t,x,\xi), \text{ with real } \Phi(t,x,\xi)\in S^0_{1,0}, \\ E(t,x,\xi)\geqq c_0\langle\xi\rangle^2, \text{ with } c_0>0,\ \langle\xi\rangle^2=1+\sum_{j=0}^{d}\xi_j^2,\ E(t,x,\xi)\in S^2_{1,0}. \end{cases}$$

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We also suppose one of the following conditions.

(1.3) $\partial_t \Phi(t, x, \xi)=B(t, x, \xi)\Phi(t, x, \xi)$, with $B(t, x, \xi)\in S^0_{1,0}$.

This is a global version of the involutive double characteristic case.

(1.4) $$\Phi(t, x, \xi)=(t-\phi(x, \xi))\psi(t, x, \xi),$$

where $\phi(x, \xi)$, $\psi(t, x, \xi)\in S^0_{1,0}$ are real valued and $|\psi(t, x, \xi)|\geqq c>0$. This case corresponds to the non-involutive double characteristic case.

Set $\Lambda_1=\tau-\Phi^p E^{1/2}$, $\Lambda_2=\tau+\Phi^p E^{1/2}$. Then we have

Theorem 1.1. *Assume* (1.2) (1.3) *or* (1.2) (1.4). *Then if*

(1.5) $P^s|_{\tau=0}=C_1\{\Lambda_1, \Lambda_2\}+C_2(\Lambda_1-\Lambda_2)$, *with* $C_i(t, x, \xi)\in S^0_{1,0}$,

the Cauchy problem (1.1) *is* C^∞ *well posed in a neighborhood of the origin. Here* $\{\Lambda_1, \Lambda_2\}$ *denotes the Poisson bracket of* Λ_1 *and* Λ_2.

In the case (1.3), the condition (1.5) is reduced to

(1.6) $P^s|_{\tau=0}=C(t, x, \xi)\Phi(t, x, \xi)^p$, with $C(t, x, \xi)\in S^1_{1,0}$.

Under this condition (with (1.2) (1.3)), C^∞ well-posedness of the Cauchy problem (1.1) was proved in [9], [15].

In the case (1.4), the condition (1.5) is reduced to

(1.7) $P^s|_{\tau=0}=C(t, x, \xi)\Phi(t, x, \xi)^{p-1}$, with $C(t, x, \xi)\in S^1_{1,0}$,

which is equivalent to

(1.7)′ $Q_1(t, x, \xi)=(t-\phi(x, \xi))^{p-1}q(t, x, \xi)$, with $q(t, x, \xi)\in S^1_{1,0}$.

If $p=1$, supposing (1.2) (1.4), the C^∞ well-posedness of (1.1) follows from [5] (note that (1.7)′ is satisfied for any Q_1). When $\phi(x, \xi)=\text{const.}$, $p\geqq 1$, the well-posedness of (1.1) was proved in [14], [8], [9], assuming (1.2) (1.4) and (1.7)′. However, we do not know how to reduce the general case to the case when $\phi=$ const., preserving support conditions of solutions.

Therefore, in this paper, we shall prove Theorem 1.1 under the conditions (1.2) (1.4) and (1.7)′ when $p\geqq 2$, establishing the energy inequalities, (Theorem 1.2 and

1.3).

We give some remarks on necessary conditions for the C^∞ well posedness of (1.1). Assume that (1.2) and (1.3) are satisfied locally. Then for the C^∞ well posedness of (1.1), it is necessary that $P^s|_{\tau=0}$ vanishes on $\{\Phi=0\}$ locally. Moreover, if $\mathrm{grad}_{t,x}\ \Phi=0$ on $\{\Phi=0\}$, then (1.6) is necessary for the well posedness. ([4]).

If (1.2) and (1.4) are satisfied locally, for the well posedness of (1.1), it is necessary that (1.7)′ holds locally ([4]).

Before stating the energy inequalities, we introduce some pseudodifferential operators which will be used throughout this paper. Set

$$(1.8)\qquad J(t, x, \xi)=\{(t-\phi(x, \xi))^2+\langle\xi\rangle^{-2\sigma}\}^{1/2},\ 0<\sigma=1/(p+1)\leqq 1/3.$$

We choose $\chi(y)\in C^\infty(\mathbb{R})$ so that $\chi(y)=1$ for $y\geqq -1/4$, $\chi(y)=0$ for $y\leqq -1/2$ and define

$$\begin{aligned} &a_n^\pm(t, x, \xi)=\chi(\pm n^{1/2}(t-\phi(x, \xi))\langle\xi\rangle^\sigma),\ a^\pm(t, x, \xi)=a_1^\pm(t, x, \xi),\\ &J_\pm(t, x, \xi)=(2a^\pm(t, x, \xi)-1)\{\pm(t-\phi(x, \xi))\}+\langle\xi\rangle^{-\sigma}\\ &\qquad=a^\pm(t, x, \xi)\{\pm(t-\phi(x, \xi))+\langle\xi\rangle^{-\sigma}\}\\ &\qquad\quad+(1-a^\pm(t, x, \xi))\{\mp(t-\phi(x, \xi))+\langle\xi\rangle^{-\sigma}\}. \end{aligned}$$

Using these, we introduce the following semi-norm.

$$|||u|||_{n,s,r}^2=\|\langle D\rangle^{2n\sigma}J_-(n-r)a_n^- u\|_s^2+\|J_+(-n-r)a_n^+ u\|_s^2,$$

where $J_\pm(k)$ and $a_n^\pm$ denote the pseudo-differential operator with symbols $J_\pm(t, x, \xi)^k$ and $a_n^\pm(t, x, \xi)$ respectively and $\|u\|_s$ denotes the usual Sobolev norm in $H^s(\mathbb{R}^d)$.

Theorem 1.2. *Suppose* (1.2), (1.4), (1.7)′ *and* $p\geqq 2$. *Then we have*

$$\begin{aligned} &c(n, N)\int e^{-2t\theta}\|Pu\|_{-2N}^2dt+c(n, N)\int e^{-2t\theta}|||Pu|||_{n,0,0}^2dt\\ &\geqq c_1n^{1/2}\int e^{-2t\theta}|||D_tu|||_{n,0,1}^2dt+c_2n\int e^{-2t\theta}|||u|||_{n,1,1-p}^2dt\\ &+c_3\theta^{1/2}\int e^{-2t\theta}|||D_tu|||_{n,0,1/2}^2dt+c_3\theta^{1/2}\int e^{-2t\theta}|||u|||_{n,1,1/2-p}^2dt\\ &+c_4\theta^{1+1/4}\int e^{-2t\theta}\|u\|_{-N}^2dt+c_4\theta^{2+3/4}\int e^{-2t\theta}\|D_tu\|_{-N}^2dt, \end{aligned}$$

for $n\geqq c(Q_2)C$, $\theta\geqq\theta_0(n, N)$, $N\geqq 1$, $u\in C_0^\infty((-T, T)\times\mathbb{R}^d)$, *where* $C=\sup|J(t, x, \xi)^{-p+1}\langle\xi\rangle^{-1}Q_1(t, x, \xi)|$, $c_i>0\ (1\leqq i\leqq 3)$, $c_4=c_4(n, N)>0$, *and*

$c(Q_2)$ depends only on $Q_2(t, x, \xi)$.

Theorem 1.3. *Assume the same hypothesis as above. Then the following inequality holds,*

$$c(n, s)\int e^{-2t\theta}\| Pu\|^2_{2n\sigma+s+1}dt \geqq \theta^{1+1/4}\int e^{-2t\theta}\| D_t u\|^2_s dt + \theta^{2+3/4}\int e^{-2t\theta}\| u\|^2_s dt,$$

for $\theta \geqq \theta_0(n, s)$, $n \geqq c(Q_2)C$, $s \in \mathbb{R}$, $u \in C_0^\infty((-T, T)\times \mathbb{R}^d)$.

Remark 1.1. We note that

$$\langle\xi\rangle^{2n\sigma}J_-(t, x, \xi)^n a_n^-(t, x, \xi) + J_+(t, x, \xi)^{-n} a_n^+(t, x, \xi)$$

is equivalent to 1 when $t-\phi(x, \xi) \geqq c$, to $\langle\xi\rangle^{n\sigma}$ when $|t-\phi(x, \xi)| \leqq c\langle\xi\rangle^{-\sigma}$ to $\langle\xi\rangle^{2n\sigma}$ when $t-\phi(x, \xi) \leqq -c$ with arbitrary positive c. Then, since this Cauchy problem (1.1) may cause a great loss of differentiability of solution, the use of this semi-norm $|||u|||_{n,s,r}$ seems to be natural.

§ 2. Some Properties on Symbols in $J^r S^m$

We shall say that $a(t, x, \xi) \in C^\infty(I \times \mathbb{R}^{2d}), I=(-T, T)$ belongs to $S^m_{\rho,\delta}$ (Hörmander's class) if

$$|\partial_t^j a^{(\alpha)}_{(\beta)}(t, x, \xi)| \leqq C_{j,\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta(|\beta|+j)}, \tag{2.1}$$

for any multi-index $\alpha, \beta \in N^d$. We denote by $J^r S^m$ the set of all $a(t, x, \xi) \in C^\infty(I \times \mathbb{R}^{2d})$ such that

$$|\partial_t^j a^{(\alpha)}_{(\beta)}(t, x, \xi)| \leqq C_{j,\alpha,\beta} J(t, x, \xi)^{r-j-|\alpha+\beta|}\langle\xi\rangle^{m-|\alpha|}, \tag{2.2}$$

for all multi-indexes $\alpha, \beta \in N^d$. This class is a variant one treated in [2], also see [1]. But in this paper we always treat the variable t as a parameter.

The pseudodifferential operator with symbol $a(t, x, \xi)$ belonging to one of the above classes is realized by

$$a(t, x, D_x)u = \int e^{ix\xi}a(t, x, \xi)\hat{u}(\xi)d\xi, \quad \hat{u}(\xi) = \int e^{-ix\xi}u(x)dx.$$

For the pseudodifferential operator $a(t, x, D_x)$, we denote by $\sigma(a)(t, x, \xi)$ the symbol of $a(t, x, D_x)$. But sometimes we do not distinguish the operator and its symbol.

Proposition 2.1. $\langle\xi\rangle^{-\sigma}$, $(t-\phi(x,\xi))\in JS^0$, $a_n^{\pm}(t,x,\xi)\in J^0S^0$, $S^m_{1,0}\subset J^0S^m$, $J^rS^m\subset S^{m+\sigma r^-}_{1-\sigma,\sigma}$ *where* $r^-=\max(0,-r)$.

Proposition 2.2. $c_1J(t,x,\xi)\geqq J_\pm(t,x,\xi)\geqq c_2J(t,x,\xi)$ *with* $c_i>0$. *If* $a(t,x,\xi)\in J^rS^m$ then $\partial_t^j a^{(\alpha)}_{(\beta)}(t,x,\xi)\in J^{r-j-|\alpha+\beta|}S^{m-|\alpha|}$.

Proof. We shall show the first inequalities for $J_+(t,x,\xi)$. When $(t-\phi(x,\xi))\langle\xi\rangle^{\sigma}\geqq 0$ or $(t-\phi(x,\xi))\langle\xi\rangle^{\sigma}\leqq -1/2$, we have $J_+(t,x,\xi)=|t-\phi(x,\xi)|+\langle\xi\rangle^{-\sigma}$ from the definition, and the inequalities are immediate. When $-1/2\leqq(t-\phi(x,\xi))\langle\xi\rangle^{\sigma}\leqq 0$, it follows that

$$J_+(t,x,\xi)\geqq\langle\xi\rangle^{-\sigma}-|t-\phi(x,\xi)|\geqq 4^{-1}\{|t-\phi(x,\xi)|+\langle\xi\rangle^{-\sigma}\}.$$

Then we obtain the desired inequalities.

Proposition 2.3. *If* $a_i(t,x,\xi)\in J^{r_i}S^{m_i}(i=1,2)$, *then* $a_1(t,x,\xi)a_2(t,x,\xi)\in J^{r_1+r_2}S^{m_1+m_2}$. *If* $a(t,x,\xi)\in J^rS^m$ *and* $a(t,x,\xi)\geqq c\langle\xi\rangle^mJ(t,x,\xi)^r$ *with* $c>0$, *then* $a(t,x,\xi)^n\in J^{rn}S^{mn}$, *for* $n\in\mathbb{R}$.

Corollary 2.1. $J(t,x,\xi)^n$, $J_\pm(t,x,\xi)^n\in J^nS^0$, *for* $n\in\mathbb{R}$.

Proposition 2.4. *Let* $a_i(t,x,D_x)\in J^{r_i}S^{m_i}(i=1,2)$, *then* $a_1(t,x,D_x)a_2(t,x,D_x)\in J^{r_1+r_2}S^{m_1+m_2}$. *If* $a(t,x,D_x)\in J^rS^m$, *then* $a(t,x,D_x)^*$ *belongs to* J^rS^m, *where* $a(t,x,D_x)^*$ *denotes the adjoint of* $a(t,x,D_x)$ *with respect to the scalar product in* $L^2(\mathbb{R}^d)$.

Proof. (See [2], [1].) For the proof, we introduce the following weight function

$$d(y,\xi,\eta)=1+\langle\xi\rangle|y|^2+\langle\xi\rangle^{-1}|\eta|^2.$$

First we note the following inequalities.

(2.3) $$J(t,x+y,\xi)^s\leqq C_sJ(t,x,\xi)^sd(y,\xi,\eta)^{|s|/2},$$

(2.4) $$J(t,x,\xi+\theta\eta)^s\leqq C_sJ(t,x,\xi)^sd(y,\xi,\eta)^{|s|/2},\ 0\leqq\theta\leqq 1,\text{ for }|\xi|\geqq 2(1+|\eta|),$$

(2.5) $$J(t,x,\xi+\theta\eta)^s\leqq C_sJ(t,x,\xi)^s\langle\eta\rangle^{(1+\sigma)|s|},\ 0\leqq\theta\leqq 1,\text{ for }|\xi|\leqq 2(1+|\eta|),$$

(2.6) $$|a^{\gamma}_{\xi}(\langle\xi\rangle\Delta_\eta)^i(\langle\xi\rangle^{-1}\Delta_y)^jd(y,\xi,\eta)^{-K}|\leqq C_{K,i,j,\gamma}\langle\xi\rangle^{-|\gamma|}d(y,\xi,\eta)^{-K}.$$

Now we set $C_\theta(t,x,y,\xi,\eta)=a_1(t,x,\xi+\theta\eta)a_2(t,x+y,\xi)$ and consider the following oscillatory integral

$$I(C)=\iint e^{-iy\eta}C_\theta(t,x,y,\xi,\eta)dyd\eta=I_1(C_\theta)+I_2(C_\theta),$$

where $I_1(C_\theta)$ and $I_2(C_\theta)$ denote the integrals on $|\xi|\geqq 2(1+|\eta|)$ and $|\xi|\leqq 2(1+|\eta|)$ respectively. First we treat $I_1(C_\theta)$,

$$\partial_x^\alpha\partial_\xi^\gamma I_1(C_\theta)=\iint e^{-iy\eta}\partial_x^\alpha\partial_\xi^\gamma(1-\langle\xi\rangle\Delta_\eta-\langle\xi\rangle^{-1}\Delta_y)^K \times(d(y,\xi,\eta)^{-K}C_\theta(t,x,y,\xi,\eta))dyd\eta.$$

Using the estimates (2.3), (2.4) and (2.6), the integrand is estimated by

$$J(t,x,\xi)^{r_1+r_2-|\alpha+\gamma|}\langle\xi\rangle^{m_1+m_2-|\gamma|}d(y,\xi,\eta)^{-K+(|r_1|+|r_2|+|\alpha+\gamma|)/2},\ (|\xi|\geqq 2(1+|\eta|)).$$

Since

$$\iint(1+\langle\xi\rangle|y|^2+\langle\xi\rangle^{-1}|\eta|^2)^{-K+(|r_1|+|r_2|+|\alpha+\gamma|)/2}dyd\eta\leqq C,$$

with suitable $K=K(r_1,r_2,\alpha,\gamma)$, we have the desired estimates (that is, in $J^{r_1+r_2}S^{m_1+m_2}$) for $I_1(C_\theta)$.

We proceed to the next,

$$\partial_x^\alpha\partial_\xi^\gamma I_2(C_\theta)=\iint e^{-iy\eta}\partial_x^\alpha\partial_\xi^\gamma\langle\eta\rangle^{-2N}(1-\Delta_y)^N\langle y\rangle^{-2M}(1-\Delta_\eta)^M C_\theta(t,x,y,\xi,\eta)dyd\eta.$$

Remarking the inequalities (2.3) and (2.5), the above integrand is estimated by

$$J(t,x,\xi)^{r_1+r_2-|\alpha+\gamma|}\langle\xi\rangle^{m_1+m_2-|\gamma|}\langle y\rangle^{-d-1}\langle\eta\rangle^{-d-1}.$$

Here we have taken N, M so that $-2M+|r_2|+|\alpha+\gamma|\leqq -d-1$, $-N+(1+\sigma)\times(|r_1|+|r_2|+|\alpha+\gamma|)+|m_1|+|\gamma|\leqq -d-1$. Therefore it follows that

$$|\partial_x^\alpha\partial_\xi^\gamma I(C_\theta)|\leqq C_{\alpha,\gamma}J(t,x,\xi)^{r_1+r_2-|\alpha+\gamma|}\langle\xi\rangle^{m_1+m_2-|\gamma|},\ 0\leqq\theta\leqq 1.$$

Hence, taking $\theta=0$, we have the first assertion. The second assertion will be proved by the similar arguments.

This proof also shows that

Proposition 2.5. *Let $a_i(t,x,D_x)\in J^{r_i}S^{m_i}$ $(i=1,2)$, then $\sigma(a_1(t,x,$*

$D_x)a_2(t, x, D_x)) - \sum_{|\alpha| \leq N-1} (\alpha!)^{-1} a_1^{(\alpha)}(t, x, \xi) a_{2(\alpha)}(t, x, \xi) \in J^{r_1+r_2-2N} S^{m_1+m_2-N}$. *If* $a(t, x, D_x) \in J^r S^m$, *then* $\sigma(a(t, x, D_x)^*) - \sum_{|\alpha| \leq N-1} (\alpha!)^{-1} (-1)^{|\alpha|} \overline{a_{(\alpha)}^{(\alpha)}(t, x, \xi)} \in J^{r-2N} S^{m-N}$.

The standard method of constructing a symbol from its asymptotic expansion gives that

Proposition 2.6. *Let* $a_j(t, x, \xi) \in J^{r-j} S^{m-j/2}$, $j=0, 1, 2, \cdots$. *Then there is a symbol* $a(t, x, \xi) \in J^r S^m$ *such that*

$$a(t, x, \xi) - \sum_{j=0}^{N-1} a_j(t, x, \xi) \in J^{r-N} S^{m-N/2}, \text{ for } N=0, 1, 2, \cdots.$$

If $\operatorname{supp}[a_j(t, x, \xi)] \subset E$, $j=0, 1, 2, \cdots$, *we can take* $a(t, x, \xi)$ *so that* $\operatorname{supp}[a(t, x, \xi)] \subset E$.

Proposition 2.7. *Suppose that* $A_i(t, x, \xi) \in J^{r_i} S^{m_i}$ $(i=1, 2)$, $B \in J^r S^m$ *and* $A_i(t, x, \xi) \geq c_i \langle \xi \rangle^{m_i} J(t, x, \xi)^{r_i}$, *on* $\operatorname{supp}[B(t, x, \xi)]$, *with* $c_i > 0$ *for large* $|\xi|$. *Then there exists* $C(t, x, \xi) \in J^{r-(r_1+r_2)} S^{m-(m_1+m_2)}$ *such that* $A_1 C A_2 = B$, mod $S^{-\infty}$ *and* $\operatorname{supp}[C(t, x, \xi)] \subset \operatorname{supp}[B(t, x, \xi)]$.

Lemma 2.1. *Let* $J_1 \in J^k S^0$, $J_2 \in J^{k+1/2} S^0$ *and* $J_2(t, x, \xi) \geq c_2 J(t, x, \xi)^{k+1/2}$, *with* $c_2 > 0$ *for large* $|\xi|$. *Then if* $A(t, x, \xi) \in J^0 S^0$ *and* $|A(t, x, \xi)| \leq \hat{c}$, *we have*

$$\|A J_1 u\|^2 \leq \hat{c} \|J_1 u\|^2 + C(N) \|J_2 u\|^2 + C(N) \|u\|_{-N}^2 \text{ for any } N.$$

Proof. First it is clear from Proposition 2.5 that

$$A^* A = \operatorname{op}(|A(t, x, \xi)|^2) - B_1, \ B_1 \in J^{-2} S^{-1}.$$

The proof of the sharp Gårding inequality (see [7]) shows that

$$\begin{aligned} \sigma(\operatorname{op}(\hat{c}^2 - |A(t, x, \xi)|^2)) &\sim (\hat{c}^2 - |A(t, x, \xi)|^2)_F \\ &- \sum_{|\beta|=1} \psi_\beta(|A(t, x, \xi)|^2)_{(\beta)} - \sum_{|\alpha+\beta| \geq 2} \psi_{\alpha,\beta}(|A(t, x, \xi)|^2)_{(\beta)}^{(\alpha)}, \end{aligned}$$

where $\psi_\beta \in S_{1,0}^{-1}$, $\psi_{\alpha,\beta} \in S_{1,0}^{(|\alpha|-|\beta|)/2}$ and $A(t, x, \xi)_F$ denotes the Friedrichs symmetrization of $A(t, x, \xi)$. Remarking

$$\psi_\beta(|A(t, x, \xi)|^2)_{(\beta)} \in J^{-1} S^{-1}, \ \psi_{\alpha,\beta}(|A(t, x, \xi)|^2)_{(\beta)}^{(\alpha)} \in J^{-2-(|\alpha+\beta|-2)} S^{-(|\alpha+\beta|-2)/2-1},$$

and applying Proposition 2.6, we can write

$$\mathrm{op}(\hat{c}^2-|A(t,x,\xi)|^2)=(\hat{c}^2-|A(t,x,\xi)|^2)_F+B_2, \text{ mod } S^{-\infty}, \text{ with } B_2\in J^{-2}S^{-1}.$$

This gives that

$$\hat{c}^2-A^*A=(\hat{c}^2-|A(t,x,\xi)|^2)_F+B_3, \text{ mod } S^{-\infty}, \text{ with } B_3\in J^{-2}S^{-1},$$

and hence

$$\hat{c}^2\|v\|^2+|(B_3v,v)|+C(N)\|v\|^2_{-N-\sigma|k|}\geqq\|Av\|^2. \tag{2.7}$$

Whereas, from Proposition 2.7, one can write

$$J_1^*B_3J_1=J_2^*B_4J_2, \text{ mod } S^{-\infty}, \text{ with } B_4\in J^{-3}S^{-1}. \tag{2.8}$$

Here we note that

$$-1+3\sigma\leqq 0, \tag{2.9}$$

and this implies that $B_4\in J^0S^0\subset S^0_{1-\sigma,\sigma}$. Taking into account that (2.8) and $J_1\in J^kS^0\subset S^{\sigma|k|}_{1-\sigma,\sigma}$, we get this lemma by replacing v by J_1u in (2.7).

Next let us consider

$$Q_M(t,x,D_x)=Q_2(t,x,D_x)+M(J(p-1)^*\langle D\rangle J(p-1)+\langle D\rangle^{-2L}), \tag{2.10}$$

where $Q_2(t,x,\xi)$ is the symbol mentioned in Section 1.

Lemma 2.2. *Suppose that $c_0>0$ is given. Then there exists positive $c(Q_2,c_0)$ such that for any positive integer L, we have*

$$\mathrm{Re}(Q_Mu,u)+c_0\|J_\pm(-1)u\|^2\geqq c(Q_2,c_0)\|\langle D\rangle J_\pm(p)u\|^2,$$

with some positive M.

Proof. Let us set

$$L_\pm(t,x,\xi)=J_\pm(t,x,\xi)^{-2p}(Q_2(t,x,\xi)+c_0J_\pm^{-2}(t,x,\xi)),$$

and first show that

$$L_\pm(t,x,\xi)\in J^0S^2. \tag{2.11}$$

Propositions 2.1 and 2.3 imply that $Q_2(t, x, \xi)\in J^{2p}S^2$ and hence $J_\pm(t, x, \xi)^{-2p}Q_2(t, x, \xi)\in J^0S^2$. The other hand, the equality

$$2(p+1)\sigma=2, \tag{2.12}$$

shows $J_\pm(t, x, \xi)^{-2p-2}\in J^{-2p-2}S^0\subset J^0S^2$, then we get (2.11).

Next we observe $Q_2+c_0J_\pm(-1)^*J_\pm(-1)-J_\pm(p)^*L_\pm J_\pm(p)$. Taking (2.11) into account, Proposition 2.7 shows that

$$\begin{aligned} Q_2+c_0J_\pm(-1)^*J_\pm(-1)-J_\pm(p)^*L_\pm J_\pm(p) \\ =J(p-1)^*M_\pm J(p-1), \bmod S^{-\infty}, \end{aligned} \tag{2.13}$$

with $M_\pm\in J^0S^1$, here we have used the fact $J^{-4}S^{-1}\subset J^{2p-2}S^1$(follows from(2.12)). Therefore to prove this lemma, it suffices to consider $L_\pm$. Since

$$\begin{aligned} L_\pm(t, x, \xi)&=J_\pm(t, x, \xi)^{-2p-2}\langle\xi\rangle^2(Q_2(t, x, \xi)\langle\xi\rangle^{-2}J_\pm(t, x, \xi)^2+c_0\langle\xi\rangle^{-2}) \\ &\geqq c_1\langle\xi\rangle^2J_\pm(t, x, \xi)^{-2p-2}((t-\phi(x, \xi))^{2p+2}+\langle\xi\rangle^{-2}),\ c_1>0, \end{aligned}$$

in view of the identity $\langle\xi\rangle^{-2}=\langle\xi\rangle^{-2(p+1)\sigma}$, we have

$$L_\pm(t, x, \xi)\geqq c(Q_2, c_0)\langle\xi\rangle^2, \text{ with } c(Q_2, c_0)>0. \tag{2.14}$$

Let us put

$$Q_\pm(t, x, \xi)=L_\pm(t, x, \xi)-c(Q_2, c_0)\langle\xi\rangle^2\ (\in J^0S^2),$$

and ψ_β, $\psi_{\alpha,\beta}$ be the same as in the proof of Lemma 2.1. Then we see that

$$\psi_\beta Q_{\pm(\beta)}\in J^{-1}S^1,\ \psi_{\alpha,\beta}Q_{\pm(\beta)}^{(\alpha)}\in J^{-2-(|\alpha+\beta|-2)}S^{1-(|\alpha+\beta|-2)/2}.$$

From the same reasoning in Lemma 2.1, one has

$$Q_\pm(t, x, \xi)_F=Q_\pm(t, x, \xi)+K_1^\pm, \bmod S^{-\infty}, \text{ with } K_1^\pm\in J^{-2}S^1,$$

and then it follows that

$$\mathrm{Re}(L_\pm v, v)+C(L)\|v\|_{-L}^2+|(K_1^\pm v, v)|\geqq c(Q_2, c_0)\|\langle D\rangle v\|^2. \tag{2.15}$$

Take $v=J_\pm(p)u$ in (2.15) and remark that

$$J_\pm(p)^*K_1^\pm J_\pm(p)=J(p-1)^*K_2^\pm J(p-1), \bmod S^{-\infty}, \text{ with } K_2^\pm\in J^0S^1,$$

then it follows immediately that

$$(2.16)\quad C(L)\|u\|_{-L}^2+\mathrm{Re}(J_\pm(p)^*L_\pm J_\pm(p)u,\ u)+C(L)\|\langle D\rangle^{1/2}J(p-1)u\|^2 \geqq c(Q_2,\ c_0)\|\langle D\rangle J_\pm(p)u\|^2.$$

Thus (2.13) and (2.16) prove this Lemma.

§ 3. Energy Inequality

Set $I_n(m)=\langle D\rangle^{2n\sigma}J_-(m)$, $E_c^\pm=\{(t,\ x,\ \xi);\ \mp(t-\phi(x,\ \xi))\leqq c\langle\xi\rangle^{-\sigma}\}$, $E_c=E_c^+\cap E_c^-$. We start with the following identity,

$$(3.1)\quad \begin{aligned} -2\mathrm{Im}\int&(I_n(n-1/2)(D_t-i\theta)^2w,\ I_n(n-1/2)(D_t-i\theta)w)dt\\ &=2\theta\int\|I_n(n-1/2)(D_t-i\theta)w\|^2dt\\ &\quad-2\mathrm{Re}\int(\partial_tI_n(n-1/2)(D_t-i\theta)w,\ I_n(n-1/2)(D_t-i\theta)w)dt, \end{aligned}$$

where $w=a_n^-u$.

Since $a^-(t,\ x,\ \xi)=1$ on $E_{1/4}^-$, we have

$$(3.2)\quad \partial_tJ_-(t,\ x,\ \xi)^{n-1/2}+(n-1/2)J_-(t,\ x,\ \xi)^{n-3/2}=0,\ \text{on } \mathrm{supp}[a_n^-]\subset E_{1/2\sqrt{n}}^-.$$

This means that $\partial_tI_n(n-1/2)a_n^-=-(n-1/2)I_n(n-3/2)a_n^-$, mod $S^{-\infty}$, and then it follows that

$$\begin{aligned} \int&(\partial_tI_n(n-1/2)(D_t-i\theta)w, I_n(n-1/2)(D_t-i\theta)w)dt\\ &=-(n-1/2)\int(I_n(n-3/2)(D_t-i\theta)w,\ I_n(n-1/2)(D_t-i\theta)w)dt,\\ &\quad\mathrm{mod}\ c(n)\left\{\int(\|(D_t-i\theta)u\|_{-N}^2+\|u\|_{-N}^2)dt\right\}=c(n)[u]_{-N}^2,\ (n\geqq4). \end{aligned}$$

Here we remark that the positive integer N can be taken arbitrarily large, and everywhere below we fix such one N.

Using this, the second term of the right-hand side of (3.1) is equivalent to

$$(3.3)\quad (2n-1)\mathrm{Re}\int(I_n(n-3/2)(D_t-i\theta)w,\quad I_n(n-1/2)(D_t-i\theta)w)dt,\qquad \mathrm{mod}\ c(n)[u]_{-N}^2.$$

On the other hand, from Proposition 2.7, one can find $B\in J^{-3}S^{-1}$ so that

$$I_n(n-1/2)^* I_n(n-3/2)=I_n(n-1)^* I_n(n-1)+I_n(n-1/2)^* B I_n(n-1/2), \quad \text{mod } S^{-\infty}.$$

In virtue of (2.9), B belongs to $S^0_{1-\sigma,\sigma}$ and then (3.3) is estimated from below by

$$(2n-1)\int\|I_n(n-1)(D_t-i\theta)w\|^2 dt$$
$$-c(n)\int\|I_n(n-1/2)(D_t-i\theta)w\|^2 dt, \text{ mod } c(n)[u]^2_{-N}.$$

Combining these inequalities, we get for $\theta\geqq\theta_0(n)$,

$$\text{(3.4)} \quad -2\mathrm{Im}\int(I_n(n-1/2)(D_t-i\theta)^2 w,\ I_n(n-1/2)(D_t-i\theta)w)dt$$
$$\geqq c_0\theta\int\|I_n(n-1/2)(D_t-i\theta)w\|^2 dt$$
$$+(2n-1)\int\|I_n(n-1)(D_t-i\theta)w\|^2 dt,$$
$$\text{mod } c(n)[u]^2_{-N}.$$

Next consider

$$\text{(3.5)} \quad -2\mathrm{Im}\int(J_-(-1)I_n(n-1/2)(D_t-i\theta)w,\ J_-(-1)I_n(n-1/2)w)dt$$
$$=2\theta\int\|J_-(-1)I_n(n-1/2)w\|^2 dt$$
$$-2\mathrm{Re}\int(\partial_t J_-(-1)I_n(n-1/2)w,\ J_-(-1)I_n(n-1/2)w)dt$$
$$-2\mathrm{Re}\int(J_-(-1)\partial_t I_n(n-1/2)w,\ J_-(-1)I_n(n-1/2)w)dt.$$

From the same reasoning as above, we can replace $\partial_t J_-(-1)$, $\partial_t J_-(n-1/2)$ by $-J_-(-2)$, $-(n-1/2)J_-(n-3/2)$ in (3.5) within modulo $c(n)[u]^2_{-N}$. Therefore, applying Proposition 2.7, the same arguments imply that the right-hand side of (3.5) is estimated from below by

$$\text{(3.6)} \quad c_0\theta\int\|J_-(-1)I_n(n-1/2)w\|^2 dt$$
$$+(2n-3)\int\|J_-(-1)I_n(n-1)w\|^2 dt, \text{ mod } [u]^2_{-N},$$

for $\theta\geqq\theta_0(n)$. Now we consider the left-hand side of (3.5). From Proposition

2.7, we get

$$I_n(n-1/2)^*J_-(-1)^*J_-(-1)I_n(n-1/2)=I_n(n-1)^*J_-(-1)^*I_n(n-1)$$
$$+I_n(n-1/2)^*J_-(-1)^*BI_n(n-1/2), \text{ with } B\in J^{-3}S^{-1}, \text{ mod } S^{-\infty},$$

and then the left side of (3.5) is estimated by

$$\int\|I_n(n-1)(D_t-i\theta)w\|^2dt+\int\|J_-(-1)I_n(n-1)w\|^2dt$$
$$+c(n)\theta^{-1/2}\int\|I_n(n-1/2)(D_t-i\theta)w\|^2dt$$
$$+c(n)\theta^{1/2}\int\|J_-(-1)I_n(n-1/2)w\|^2dt,$$
$$\text{mod } c(n)\{\theta^{1/2}\int\|u\|_{-N}^2dt+\theta^{-1/2}\int\|(D_t-i\theta)w\|_{-N}^2dt\}=c(n)[u]_{-N,\theta}^2.$$

Therefore it follows from (3.6) that

$$\int\|I_n(n-1)(D_t-i\theta)w\|^2dt\geqq c_0\theta\int\|J_-(-1)I_n(n-1/2)w\|^2dt \tag{3.7}$$
$$+(2n-4)\int\|J_-(-1)I_n(n-1)w\|^2dt$$
$$-c(n)\theta^{-1/2}\int\|I_n(n-1/2)(D_t-i\theta)w\|^2dt,$$

for $\theta\geqq\theta_0(n)$, mod $c(n)[u]_{-N,\theta}^2$.

Combining (3.4) with (3.7), we get,

Proposition 3.1.

$$-2\text{Im}\int(I_n(n-1/2)(D_t-i\theta)^2w, I_n(n-1/2)(D_t-i\theta)w)dt$$
$$\geqq c_1\theta\int\|I_n(n-1/2)(D_t-i\theta)w\|^2dt+n\int\|I_n(n-1)(D_t-i\theta)w\|^2dt$$
$$+c_1\theta^2\int\|J_-(-1)I_n(n)w\|^2dt+c_1n\theta\int\|J_-(-1)I_n(n-1/2)w\|^2dt$$
$$+2^{-1}n^2\int\|J_-(-1)I_n(n-1)w\|^2dt, \text{ mod } c(n)\theta[u]_{-N,\theta}^2, \text{ for } \theta\geqq\theta_0(n),\ n\geqq4.$$

§ 4. Energy Inequality (Continued)

We proceed to the next step of obtaining the energy inequality. Consider the identity,

$$\begin{aligned}(4.1)\quad & 2\mathrm{Im}\int(I_n(n-1/2)Q_M w, I_n(n-1/2)(D_t-i\theta)w)dt \\ &=2\theta\mathrm{Re}\int(Q_M I_n(n-1/2)w, I_n(n-1/2)w)dt \\ &-\int\{(Q_M\partial_t I_n(n-1/2)w, I_n(n-1/2)w)+(Q_M I_n(n-1/2)w, \partial_t I_n(n-1/2)w)\}dt \\ &-\int((\partial_t Q_M)I_n(n-1/2)w, I_n(n-1/2)w)dt \\ &+i\int(I_n(n-1/2)(D_t-i\theta)w, (Q_M-Q_M^*)I_n(n-1/2)w)dt \\ &+2\mathrm{Im}\int([I_n(n-1/2), Q_M]w, I_n(n-1/2)(D_t-i\theta)w)dt \\ &-\theta\int(I_n(n-1/2)w, (Q_M-Q_M^*)I_n(n-1/2)w)dt,\end{aligned}$$

where Q_M is defined by (2.10).

From the same reasoning as in Section 3, the second term of the right-hand of (4.1) is equivalent to

$$\begin{aligned}(4.2)\qquad & (n-1/2)\int\{(Q_M I_n(n-3/2)w, I_n(n-1/2)w) \\ &+(Q_M I_n(n-1/2)w, I_n(n-3/2)w)\}dt, \quad \mathrm{mod}\ c(n)[u]^2_{-N}.\end{aligned}$$

First we observe $I_n(n-1/2)^*Q_2 I_n(n-3/2)+I_n(n-3/2)^*Q_2 I_n(n-1/2)$. Since $Q_2(t, x, \xi)\in J^{2p}S^2$, from Proposition 2.7, it follows that

$$\begin{aligned}(4.3)\qquad & I_n(n-1/2)^*Q_2 I_n(n-3/2)+I_n(n-3/2)^*Q_2 I_n(n-1/2) \\ &=2I_n(n-1)^*Q_2 I_n(n-1) \\ &+I_n(n-1/2)^*J_-(p)^*AJ_-(-1)I_n(n-1/2), \\ &\text{with } A\in J^{p-2}S^1,\ \mathrm{mod}\ S^{-\infty}.\end{aligned}$$

Here the inequality $p\geqq 2$ implies that $A\in J^0S^1$. Remarking that $J_-(p-1)^*\langle D\rangle J_-(p-1)\in J^{2p-2}S^1$, from the similar arguments as above we have

$$\begin{aligned}(4.4)\qquad & I_n(n-1/2)^*J_-(p-1)^*\langle D\rangle J_-(p-1)I_n(n-3/2) \\ &+I_n(n-3/2)^*J_-(p-1)^*\langle D\rangle J_-(p-1)I_n(n-1/2) \\ &=2I_n(n-1)^*J_-(p-1)^*\langle D\rangle J_-(p-1)I_n(n-1) \\ &+I_n(n-1/2)^*J_-(p)^*AJ_-(-1)I_n(n),\end{aligned}$$

with $A\in J^{p-2-1/2}S^0$, mod $S^{-\infty}$. Using (4.3) and (4.4), one can estimate (4.2) from below by

$$(4.5)\qquad (2n-1)\int(Q_M I_n(n-1)w, I_n(n-1)w)dt$$
$$-c(n, M)\int\|\langle D\rangle J_-(p)I_n(n-1/2)w\|^2 dt$$
$$-c(n, M)\int\|J_-(-1)I_n(n-1/2)w\|^2 dt$$
$$-c(n, M)\int\|J_-(-1)I_n(n)w\|^2 dt,$$
$$\mod c(n, M)[u]_{-N}^2.$$

Next consider $I_n(n-1/2)^*(Q_M^*-Q_M)I_n(n-1/2)=I_n(n-1/2)^*(Q_2^*-Q_2)I_n(n-1/2)$. Writing $Q_2^*-Q_2=\tilde{Q}_1+\tilde{Q}_0$ with $\tilde{Q}_1\in J^{2p-2}S^{1+4n\sigma}$, $\tilde{Q}_0\in J^0 S^{4n\sigma}$, Proposition 2.7 gives that

$$\begin{aligned}I_n(n-1/2)^*&(Q_2^*-Q_2)I_n(n-1/2)\\&=I_n(n-1/2)^*J_-(p)^*AJ_-(-1)I_n(n)+I_n(n)^*J_-(-1)^*BJ_-(-1)I_n(n)\\&=I_n(n-1/2)^*J_-(p)^*\tilde{A}I_n(n-1/2)+I_n(n)^*J_-(-1)^*\tilde{B}I_n(n-1/2),\end{aligned}$$

with $A\in J^{p-2+1/2}S^1$, $\tilde{A}\in J^{p-2}S^1$, $B\in JS^0$, $\tilde{B}\in J^{1/2}S^0$, mod $S^{-\infty}$. From these, noting that A, $\tilde{A}\in S^1_{1-\sigma,\sigma}$, B, $\tilde{B}\in S^0_{1-\sigma,\sigma}$, it follows that

$$(4.6)\qquad \begin{aligned}\theta|(I_n(n-1/2)w, (Q_M-Q_M^*)I_n(n-1/2)w)|&\\ \leqq c(n)\theta^{1/2}\|\langle D\rangle J_-(p)I_n(n-1/2)w\|^2&\\ +c(n)\theta^{3/2}\|J_-(-1)I_n(n)w\|^2, \mod c(n)\theta^{1/2}|u|_{-N,\theta}^2,&\end{aligned}$$

$$(4.7)\qquad \begin{aligned}|(I_n(n-1/2)(D_t-i\theta)w, (Q_M-Q_M^*)I_n(n-1/2)w)|&\\ \leqq c(n)\|I_n(n-1/2)(D_t-i\theta)w\|^2&\\ +c(n)\|\langle D\rangle J_-(p)I_n(n-1/2)w\|^2&\\ +c(n)\|J_-(-1)I_n(n)w\|^2, \mod c(n)|u|_{-N}^2.&\end{aligned}$$

Where $|u|_{-N}^2=\|u\|_{-N}^2+\|(D_t-i\theta)u\|_{-N}^2$, $|u|_{-N,\theta}^2=\theta^{\frac{1}{2}}\|u\|_{-N}^2+\theta^{-1/2}\|(D_t-i\theta)u\|_{-N}^2$.

Now estimate
$[I_n(n-1/2), Q_M]=[I_n(n-1/2), Q_2]+M[I_n(n-1/2), J_-(p-1)^*\langle D\rangle J_-(p-1)]$.
From Propositions 2.5 and 2.7, taking into account that $p\geqq 2$, $Q_2\in J^{2p}S^2$ and $J_-(p-1)^*\langle D\rangle J_-(p-1)\in J^{2p-2}S^1$, it is easy to see that
$[I_n(n-1/2), Q_M]=AJ_-(p)I_n(n-1/2)+BJ_-(-1)I_n(n)$, with $A\in J^0S^1$, $B\in J^{1/2}S^0$, mod $S^{-\infty}$. Hence we have

(4.8) $\|[I_n(n-1/2), Q_M]w\|^2 \leqq c(n, M)\|\langle D\rangle J_-(p)I_n(n-1/2)w\|^2 + c(n, M)\|J_-(-1)I_n(n)w\|^2, \bmod c(n, M)|u|^2_{-N}.$

Finally we consider $I_n(n-1/2)^*(\partial_t Q_M)I_n(n-1/2)$. We set

$$A(t, x, \xi)=\partial_t Q_2(t, x, \xi)J_-(t, x, \xi)^{-2p+1}\in J^0S^2.$$

Remark that A does not depend on n. Since $\partial_t Q_2 \in J^{2p-1}S^2$, Propositions 2.5 and 2.7 imply that

(4.9) $I_n(n-1/2)^*(\partial_t Q_2)I_n(n-1/2)=I_n(n-1)^*J_-(p)^*AJ_-(p)I_n(n-1) + I_n(n-1/2)^*J_-(p)^*BJ_-(p)I_n(n-1/2),\ B\in J^{-3}S^1\subset J^0S^2, \bmod S^{-\infty}.$

Whereas taking into account that $\partial_t\{J_-(p-1)^*\langle D\rangle J_-(p-1)\}\in J^{2p-3}S^1$, we have

(4.10) $$I_n(n-1/2)^*\partial_t\{J_-(p-1)^*\langle D\rangle J_-(p-1)\}I_n(n-1/2) =I_n(n-1/2)^*J_-(p)^*\tilde{B}J_-(-1)I_n(n-1/2),$$

with $\tilde{B}\in J^{p-2}S^1\subset J^0S^1$, $\bmod S^{-\infty}$. Thus the following inequality follows immediately from (4.9), (4.10) and Lemma 2.1,

(4.11) $|((\partial_t Q_M)I_n(n-1/2)w, I_n(n-1/2)w)|\leqq C_1^2\|\langle D\rangle J_-(p)I_n(n-1)w\|^2 + c(n, M)\|\langle D\rangle J_-(p)I_n(n-1/2)w\|^2 + c(n, M)\|J_-(-1)I_n(n-1/2)w\|^2,$

$\bmod c(n, M)|u|^2_{-N}$, where $C_1=\sup|A(t, x, \xi)\langle\xi\rangle^{-2}|$.

Combining the inequalities (4.6), (4.7), (4.8) and (4.11), we get

Proposition 4.1.

$$2\mathrm{Im}\int(I_n(n-1/2)Q_M w, I_n(n-1/2)(D_t-i\theta)w)dt$$
$$\geqq 2\theta\mathrm{Re}\int(Q_M I_n(n-1/2)w, I_n(n-1/2)w)dt$$
$$+(2n-1)\mathrm{Re}\int(Q_M I_n(n-1)w, I_n(n-1)w)dt$$
$$-C_1^2\int\|\langle D\rangle J_-(p)I_n(n-1)w\|^2dt-c(n)\int\|I_n(n-1/2)(D_t-i\theta)w\|^2dt$$
$$-c(n, M)\int\|J_-(-1)I_n(n-1/2)w\|^2dt$$
$$-c(n, M)\theta^{1/2}\int\|\langle D\rangle J_-(p)I_n(n-1/2)w\|^2dt$$

$$-c(n,M)\theta^{3/2}\int\|J_-(-1)I_n(n)w\|^2dt, \bmod c(n,M)\theta^{1/2}[u]^2_{-N,\theta}.$$

Now take $L\geqq 2n\sigma+N$ and fix M so that Lemma 2.2 holds. Then Lemma 2.2 implies that

$$(2n-1)\mathrm{Re}(Q_MI_n(n-1)w, I_n(n-1)w)+2^{-1}n^2\|J_-(-1)I_n(n-1)w\|^2$$
$$\geqq c_1n\|\langle D\rangle J_-(p)I_n(n-1)w\|^2,\ c_1>0,$$

$$2\theta\mathrm{Re}(Q_MI_n(n-1/2)w, I_n(n-1/2)w)+c_2n\theta\|J_-(-1)I_n(n-1/2)w\|^2$$
$$\geqq \tilde{c}_1\theta\|\langle D\rangle J_-(p)I_n(n-1/2)w\|^2,\ \tilde{c}_1>0.$$

Hence using these inequalities, Propositions 3.1 and 4.1 show that

Lemma 4.1.

$$-2\mathrm{Im}\int(I_n(n-1/2)[(D_t-i\theta)^2-Q_M]a_n^-u, I_n(n-1/2)(D_t-i\theta)a_n^-u)dt$$
$$\geqq c_1n\int\|\langle D\rangle J_-(p)I_n(n-1)a_n^-u\|^2dt+c_1n\int\|I_n(n-1)(D_t-i\theta)a_n^-u\|^2dt$$
$$+c_1n^2\int\|J_-(-1)I_n(n-1)a_n^-u\|^2dt+c_2\theta\int\|I_n(n-1/2)(D_t-i\theta)a_n^-u\|^2dt$$
$$+c_2\theta\int\|\langle D\rangle J_-(p)I_n(n-1/2)a_n^-u\|^2dt+c_2n\theta\int\|J_-(-1)I_n(n-1/2)a_n^-u\|^2dt$$
$$+c_2\theta^2\int\|J_-(-1)I_n(n)a_n^-u\|^2dt, \bmod c(n,N)\theta^{1/2}[u]^2_{-N,\theta},$$

for $\theta\geqq\theta_0(n,N)$, $n\geqq n(Q_2)$.

From the analogous arguments for J_+, we have,

Lemma 4.2.

$$-2\mathrm{Im}\int(J_+(-n-1/2)[(D_t-i\theta)^2-Q_M]a_n^+u, J_+(-n-1/2)(D_t-i\theta)a_n^+u)dt$$
$$\geqq c_1n\int\|\langle D\rangle J_+(p)J_+(-n-1)a_n^+u\|^2dt+c_1n\int\|J_+(-n-1)(D_t-i\theta)a_n^+u\|^2dt$$
$$+c_1n^2\int\|J_+(-1)J_+(-n-1)a_n^+u\|^2dt+c_2\theta\int\|J_+(-n-1/2)(D_t-i\theta)a_n^+u\|^2dt$$
$$+c_2\theta\int\|\langle D\rangle J_+(p)J_+(-n-1/2)a_n^+u\|^2dt+c_2n\theta\int\|J_+(-1)J_+(-n-1/2)a_n^+u\|^2dt$$
$$+c_2\theta^2\int\|J_+(-1)J_+(-n)a_n^+u\|^2dt,$$

$$\bmod c(n,N)\theta^{1/2}[u]^2_{-N,\theta},\ \text{for } \theta\geqq\theta_0(n,N),\ n\geqq n(Q_2).$$

If we note that

$$\begin{aligned}&I_n(n-1/2)^*I_n(n-1/2)J(p-1)^*\langle D\rangle J(p-1)\\&\quad=I_n(n-1/2)^*J_-(p)^*A^-I_n(n-1/2),\\&J_+(-n-1/2)^*J_+(-n-1/2)J(p-1)^*\langle D\rangle J(p-1)\\&\quad=J_+(-n-1/2)^*J_+(p)^*A^+J_+(-n-1/2),\end{aligned}$$

with $A^{\pm}\in J^0S^1$, mod $S^{-\infty}$, we can replace Q_M by Q_2 in Lemmas 4.1 and 4.2 changing the constants c_2, $c(n, N)$ and $\theta_0(n, N)$.

Proposition 4.2. *After having replaced Q_M by Q_2, Lemmas* 4.1 *and* 4.2 *are also valid.*

§ 5. Estimates of the Commutators with $\alpha_n^{\pm}$

In this Section, we shall show that one can replace $[(D_t-i\theta)^2-Q_2]\alpha_n^{\pm}$ by $\alpha_n^{\pm}[(D_t-i\theta)^2-Q_2]$ in Lemmas 4.1 and 4.2. Since $\alpha_n^{\pm}(D_t-i\theta)^2=(D_t-i\theta)^2\alpha_n^{\pm}-2D_t\alpha_n^{\pm}(D_t-i\theta)-D_t^2\alpha_n^{\pm}$, to do so we investigate $D_t^2\alpha_n^{\pm}$ and $D_t\alpha_n^{\pm}$.

From Proposition 2.5, we see that

$$\sigma(I_n(n-1/2)^*I_n(n-1/2)D_t^2\alpha_n^-)\sim\langle\xi\rangle^{4n\sigma}J_-(t,x,\xi)^{2n-1}D_t^2\alpha_n^-+B,\quad \text{with } B\in J^{2n-5}S^{4n\sigma-1},$$

then we define $A(t, x, \xi)$ by

$$nJ_-^{n-1}J_+^{-n-2}\langle\xi\rangle^{2n\sigma}A(t,x,\xi)\alpha_n^+=\langle\xi\rangle^{4n\sigma}J_-^{2n-1}D_t^2\alpha_n^-,$$

that is

$$\begin{aligned}&A(t,x,\xi)=-(\langle\xi\rangle^{2\sigma}J_+J_-)^n\chi^{(2)}(-\sqrt{n}(t-\phi)\langle\xi\rangle^{\sigma})\langle\xi\rangle^{2\sigma}J_+^2,\\&(D_t^2\alpha_n^-=-n\chi^{(2)}(-\sqrt{n}(t-\phi)\langle\xi\rangle^{\sigma})\langle\xi\rangle^{2\sigma}).\end{aligned}$$

Since we have $J_+(t,x,\xi)J_-(t,x,\xi)=\langle\xi\rangle^{-2\sigma}(1-(t-\phi(x,\xi))^2\langle\xi\rangle^{2\sigma})$ on $E_{1/4}$, it follows that

$$\begin{aligned}(5.1)\qquad &c_1\leqq(1-1/4n)^{\pm n}\leqq(\langle\xi\rangle^{2\sigma}J_+(t,x,\xi)J_-(t,x,\xi))^{\pm n}\\&\leqq(1-1/16n)^{\pm n}\leqq c_2\quad\text{on } E_{1/4},\end{aligned}$$

with $c_i>0$ independent of n.

Proposition 5.1. *Let* $a(t, x, \xi)\in J^r S^m$ *and* $\mathrm{supp}[a(t, x, \xi)]\subset E_c$ *with some* $c>0$. *Then for any* $\mu\in\mathbb{R}$, *we have*

$$a(t, x, \xi)\in J^{r+\mu}S^{m+\sigma\mu}.$$

Thus, using this proposition and (5.1), we have

$$\begin{aligned}(5.2)\qquad & A(t, x, \xi)\in J^0 S^0,\\ & |A(t, x, \xi)|\leqq c \text{ (independent of } n), \ \mathrm{supp}[A]\subset\mathrm{supp}[D_t^2 \alpha_n^-].\end{aligned}$$

From Propositions 2.5 and 5.1, it follows that

$$\begin{aligned}&\sigma(I_n(n-1/2)^* I_n(n-1/2)D_t^2\alpha_n^- - nI_n(n-1)^* AJ_+(-1)J_+(-n-1)\alpha_n^+)\sim b,\\ &b\in J^\mu S^{-1+2n\sigma+5\sigma+\mu\sigma}, \quad \mu\in\mathbb{R},\ \mathrm{supp}[b]\subset\mathrm{supp}[D_t^2\alpha_n^-].\end{aligned}$$

Noting that $\alpha_n^+(t, x, \xi)=1$ on $\mathrm{supp}[D_t^2\alpha_n^-]$, Proposition 2.7 shows that

$$\begin{aligned}I_n(n-1/2)^* I_n(n-1/2)D_t^2\alpha_n^- &= nI_n(n-1)^* AJ_+(-1)J_+(-n-1)\alpha_n^+\\ &\quad + I_n(n-1/2)^* BJ_+(-1)J_+(-n-1/2)\alpha_n^+,\end{aligned}$$

with $B\in J^0 S^{-1+3\sigma}\subset S^0_{1-\sigma,\sigma}$, mod $S^{-\infty}$, and hence

$$\begin{aligned}&|(I_n(n-1/2)D_t^2\alpha_n^- u, I_n(n-1/2)(D_t-i\theta)w)|\\ &\quad\leqq n^{3/2}\|AJ_+(-1)J_+(-n-1)\alpha_n^+ u\|^2\\ &\quad + n^{1/2}\|I_n(n-1)(D_t-i\theta)w\|^2 + c(n)\|I_n(n-1/2)(D_t-i\theta)w\|^2\\ &\quad + c(n)\|J_+(-1)J_+(-n-1/2)\alpha_n^+ u\|^2,\end{aligned}$$

mod $c(n)|u|_{-N}^2$. Applying Lemma 2.1 to A, we get

$$\begin{aligned}(5.3)\qquad &|(I_n(n-1/2)D_t^2\alpha_n^- u, I_n(n-1/2)(D_t-i\theta)\alpha_n^- u)|\\ &\quad\leqq C^2 n^{3/2}\|J_+(-1)J_+(-n-1)\alpha_n^+ u\|^2\\ &\qquad + n^{1/2}\|I_n(n-1)(D_t-i\theta)\alpha_n^- u\|^2\\ &\qquad + c(n)\|I_n(n-1/2)(D_t-i\theta)\alpha_n^- u\|^2\\ &\qquad + c(n)\|J_+(-1)J_+(-n-1/2)\alpha_n^+ u\|^2, \ \mathrm{mod}\ c(n)|u|_{-N}^2,\end{aligned}$$

with C independent of n.

From the same procedure, we can write

$$\begin{aligned}&I_n(n-1/2)^* I_n(n-1/2)D_t\alpha_n^-\\ &\quad = nI_n(n-1)^* A_2 J_+(-n-1)\alpha_n^+ + I_n(n-1/2)^* B_2 J_+(-n-1/2)\alpha_n^+,\end{aligned}$$

with $B_2 \in J^0 S^{-1+3\sigma} \subset S^0_{1-\sigma,\sigma}$, mod $S^{-\infty}$, where

$$A_2(t, x, \xi) = i(\langle\xi\rangle^{2\sigma} J_+ J_-)^n \chi^{(1)}(-\sqrt{n}(t-\phi)\langle\xi\rangle^\sigma) J_+ \langle\xi\rangle^\sigma.$$

Since $\mathrm{supp}[A_2] \cap \mathrm{supp}[D_t a_n^+] = \phi$, it is clear that

$$\begin{aligned} &I_n(n-1)^* A_2 J_+(-n-1) a_n^+(D_t - i\theta) \\ &\quad = I_n(n-1)^* A_2 J_+(-n-1)(D_t - i\theta) a_n^+, \text{ mod } S^{-\infty}, \end{aligned}$$

and then we obtain

$$\begin{aligned} (5.4) \qquad &|(I_n(n-1/2) D_t a_n^-(D_t - i\theta) u, I_n(n-1/2)(D_t - i\theta) a_n^- u)| \\ &\leqq \sqrt{n} \| I_n(n-1)(D_t - i\theta) a_n^- u \|^2 \\ &\quad + C\sqrt{n} \| J_+(-n-1)(D_t - i\theta) a_n^+ u \|^2 \\ &\quad + c(n) \| I_n(n-1/2)(D_t - i\theta) a_n^- u \|^2 \\ &\quad + c(n) \| J_+(-n-1/2)(D_t - i\theta) a_n^+ u \|^2, \text{ mod } c(n) |u|^2_{-N}, \end{aligned}$$

where C does not depend on n.

Finally, we observe $[a_n^-, Q_2]$. It is easily seen from Proposition 2.5 that

$$\sigma([a_n^-, Q_2]) \sim a, \ a \in J^{2p-2} S^1, \ \mathrm{supp}[a] \subset E_0^+ \cap E^-_{1/2\sqrt{n}}.$$

Using Propositions 2.7 and 5.1, we have

$$\begin{aligned} &I_n(n-1/2)^* I_n(n-1/2)[a_n^-, Q_2] \\ &\quad = I_n(n-1/2)^* A J_+(p) J_+(-n-1/2) a_n^+ + I_n(n-1/2)^* B J_+(-1) J_+(-n) a_n^+, \end{aligned}$$

with $A \in J^{p-2} S^1$, $B \in J^{1/2} S^0$, mod $S^{-\infty}$. This shows that

$$\begin{aligned} (5.5) \qquad &|(I_n(n-1/2)[a_n^-, Q_2] u, I_n(n-1/2)(D_t - i\theta) a_n^- u)| \\ &\leqq c(n) \| I_n(n-1/2)(D_t - i\theta) a_n^- u \|^2 \\ &\quad + c(n) \| \langle D \rangle J_+(p) J_+(-n-1/2) a_n^+ u \|^2 \\ &\quad + c(n) \| J_+(-1) J_+(-n) a_n^+ u \|^2, \text{ mod } c(n) |u|^2_{-N}. \end{aligned}$$

After making the same procedure for J_+, we get

Lemma 5.1.

$$\begin{aligned} &-2\mathrm{Im} \int \{(I_n(n-1/2) a_n^- [(D_t - i\theta)^2 - Q_2] u, I_n(n-1/2)(D_t - i\theta) a_n^- u) \\ &\quad + (J_+(-n-1/2) a_n^+ [(D_t - i\theta)^2 - Q_2] u, J_+(-n-1/2)(D_t - i\theta) a_n^+ u)\} dt \\ &\geqq \hat{c} n \int \{ \| I_n(n-1)(D_t - i\theta) a_n^- u \|^2 + \| J_+(-n-1)(D_t - i\theta) a_n^+ u \|^2 \} dt \end{aligned}$$

$$+\hat{c}n\int\{\|\langle D\rangle J_-(p)I_n(n-1)a_n^- u\|^2+\|\langle D\rangle J_+(p)J_+(-n-1)a_n^+ u\|^2\}dt$$
$$+c_3n^2\int\{\|J_-(-1)I_n(n-1)a_n^- u\|^2+\|J_+(-1)J_+(-n-1)a_n^+ u\|^2\}dt$$
$$+c_4\theta\int\{\|\langle D\rangle J_-(p)I_n(n-1/2)a_n^- u\|^2+\|\langle D\rangle J_+(p)J_+(-n-1/2)a_n^+ u\|^2\}dt$$
$$+c_4\theta\int\{\|I_n(n-1/2)(D_t-i\theta)a_n^- u\|^2+\|J_+(-n-1/2)(D_t-i\theta)a_n^+ u\|^2\}dt$$
$$+c_4n\theta\int\{\|J_-(-1)I_n(n-1/2)a_n^- u\|^2+\|J_+(-1)J_+(-n-1/2)a_n^+ u\|^2\}dt$$
$$+c_4\theta^2\int\{\|J_-(-1)I_n(n)a_n^- u\|^2+\|J_+(-1)J_+(-n)a_n^+ u\|^2\}dt$$

for $\theta\geqq\theta_0(n, N)$, $n\geqq n(Q_2)$, $\mathrm{mod}\, c(n, N)\theta^{1/2}[u]^2_{-N,\theta}$.

§ 6. Estimates of Lower Order Terms

Let us consider $Q_1(t, x, \xi)=(t-\phi(x, \xi))^{p-1}q_1(t, x, \xi)$, $q_1(t, x, \xi)\in S^1_{1,0}$. Noting $Q_1\in J^{p-1}S^1$, we get

$$\begin{aligned}&I_n(n-1/2)^*I_n(n-1/2)Q_1\\&\quad=I_n(n-1)^*AJ_-(p)I_n(n-1)+I_n(n-1/2)^*BJ_-(p)I_n(n-1/2),\end{aligned}$$

with $A(t, x, \xi)=J_-(t, x, \xi)^{-p+1}Q_1(t, x, \xi)\in J^0S^1$, $B\in J^{-3}S^0\subset J^0S^{3\sigma}$, mod $S^{-\infty}$. Here, we note that $A(t, x, \xi)$ does not depend on n. From the same reasoning which we have used in Section 4, it follows that

$$\begin{aligned}(6.1)\qquad &|(I_n(n-1/2)Q_1w, I_n(n-1/2)(D_t-i\theta)w)|\\&\leqq 4^{-1}\hat{c}n\|I_n(n-1)(D_t-i\theta)w\|^2\\&\quad+4\hat{c}^{-1}n^{-1}C^2\|\langle D\rangle J_-(p)I_n(n-1)w\|^2\\&\quad+c(n)\|I_n(n-1/2)(D_t-i\theta)w\|^2\\&\quad+c(n)\|\langle D\rangle J_-(p)I_n(n-1/2)w\|^2,\ \mathrm{mod}\ c(n)|u|^2_{-N},\end{aligned}$$

with $C=\sup|J_-(t, x, \xi)^{-p+1}Q_1(t, x, \xi)\langle\xi\rangle^{-1}|$. On the other hand, Propositions 2.5, 2.7 and 5.1 show that

$$\begin{aligned}&I_n(n-1/2)^*I_n(n-1/2)[a_n^-, Q_1]\\&\quad=I_n(n-1/2)^*BJ_+(-1)J_+(-n-1/2)a_n^+,\ B\in J^{p-2}S^0,\ \mathrm{mod}\ S^{-\infty}.\end{aligned}$$

Combining these, we get

(6.2)
$$\begin{aligned}&|(I_n(n-1/2)a_n^- Q_1 u, I_n(n-1/2)(D_t-i\theta)a_n^- u)|\\&\leqq 4^{-1}\hat{c}n\|I_n(n-1)(D_t-i\theta)a_n^- u\|^2\\&\quad+4\hat{c}^{-1}n^{-1}C^2\|\langle D\rangle J_-(p)I_n(n-1)a_n^- u\|^2\\&\quad+c(n)\|I_n(n-1/2)(D_t-i\theta)a_n^- u\|^2\\&\quad+c(n)\|\langle D\rangle J_-(p)I_n(n-1/2)a_n^- u\|^2\\&\quad+c(n)\|J_+(-1)J_+(-n-1/2)a_n^+ u\|^2, \bmod c(n)|u|_{-N}^2.\end{aligned}$$

It is easy to see that for R, $Q_0\in S_{1,0}^0$,

(6.3)
$$\begin{aligned}&|(I_n(n-1/2)a_n^- Q_0 u, I_n(n-1/2)(D_t-i\theta)a_n^- u)|\\&\leqq c(n)\|I_n(n-1/2)(D_t-i\theta)a_n^- u\|^2\\&\quad+c(n)\|J_-(-1)I_n(n)a_n^- u\|^2\\&\quad+c(n)\|J_+(-1)J_+(-n)a_n^+ u\|^2, \bmod c(n)|u|_{-N}^2,\end{aligned}$$

(6.4)
$$\begin{aligned}&|(I_n(n-1/2)a_n^- R(D_t-i\theta)u, I_n(n-1/2)(D_t-i\theta)a_n^- u)|\\&\leqq c(n)\|I_n(n-1/2)(D_t-i\theta)a_n^- u\|^2\\&\quad+c(n)\|I_n(n-1/2)a_n^-(D_t-i\theta)u\|^2\\&\quad+c(n)\|J_+(-n-1/2)a_n^+(D_t-i\theta)u\|^2, \bmod c(n)|u|_{-N}^2.\end{aligned}$$

Here, we estimate $(D_t-i\theta)a_n^- u$ in terms of $a_n^-(D_t-i\theta)u$. Writing

$$I_n(n-1/2)D_t a_n^- = \sqrt{n}AJ_+(-1)J_+(-n-1/2)a_n^+ + BJ_+(-1)J_+(-n)a_n^+, \bmod S^{-\infty},$$

with $A(t,x,\xi)=(J_+J_-\langle\xi\rangle^{2\sigma})^n J_-^{-1/2}J_+^{3/2}\langle\xi\rangle^{\sigma}\chi^{(1)}(-\sqrt{n}(t-\phi)\langle\xi\rangle^{\sigma})$, $B\in J^{-2-1/2}S^{-1}$, the same arguments as in Section 5 give that

$$\begin{aligned}&\|I_n(n-1/2)D_t a_n^- u\|^2\\&\quad\leqq C^2 n\|J_+(-1)J_+(-n-1/2)a_n^+ u\|^2+c(n)\|J_+(-1)J_+(-n)a_n^+ u\|^2,\end{aligned}$$

with $C=\sup|A(t,x,\xi)|$. Hence

$$\begin{aligned}&\|I_n(n-1/2)(D_t-i\theta)a_n^- u\|^2\\&\quad\geqq 2^{-1}\|I_n(n-1/2)a_n^-(D_t-i\theta)u\|^2-Cn\|J_+(-1)J_+(-n-1/2)a_n^+ u\|^2\\&\qquad-c(n)\|J_+(-1)J_+(-n)a_n^+ u\|^2, \bmod c(n)|u|_{-N}^2,\end{aligned}$$

with C independent of n. Using this inequality, one obtains

(6.5)
$$\begin{aligned}n^{1/2}\|I_n(n-1)(D_t-i\theta)a_n^- u\|^2 &\geqq 2^{-1}n^{1/2}\|I_n(n-1)a_n^-(D_t-i\theta)u\|^2\\&\quad-Cn^{3/2}\|J_+(-1)J_+(-n-1)a_n^+ u\|^2\\&\quad-c(n)\|J_+(-1)J_+(-n-1/2)a_n^+ u\|^2,\end{aligned}$$

$$\begin{aligned}\theta^{1/2}\|I_n(n-1/2)(D_t-i\theta)a_n^- u\|^2 &\geqq 2^{-1}\theta^{1/2}\|I_n(n-1/2)a_n^-(D_t-i\theta)u\|^2\\ &-Cn\theta^{1/2}\|J_+(-1)J_+(-n-1/2)a_n^+ u\|^2\\ &-c(n)\theta^{1/2}\|J_+(-1)J_+(-n)a_n^+ u\|^2,\end{aligned}$$

mod $c(n)|u|^2_{-N,\theta}$. On the other hand, the following inequalities are easily verified.

(6.6)
$$\begin{aligned}n\|\langle D\rangle J_-(p)I_n(n-1)a_n^- u\|^2 &\geqq 2^{-1}n\|\langle D\rangle^{2n\sigma+1}J_-(n+p-1)a_n^- u\|^2\\ &-c(n)\|\langle D\rangle J_-(p)I_n(n-1/2)a_n^- u\|^2,\\ \theta\|\langle D\rangle J_-(p)I_n(n-1/2)a_n^- u\|^2 &\geqq c_0\theta^{1/2}\|\langle D\rangle^{2n\sigma+1}J_-(n+p-1/2)a_n^- u\|^2,\\ n^2\|J_-(-1)I_n(n-1)a_n^- u\|^2 &\geqq 2^{-1}n^2\|\langle D\rangle^{2n\sigma}J_-(n-2)a_n^- u\|^2\\ &-c(n)\|J_-(-1)I_n(n-1/2)a_n^- u\|^2,\\ n\theta\|J_-(-1)I_n(n-1/2)a_n^- u\|^2 &\geqq 2^{-1}n\theta\|\langle D\rangle^{2n\sigma}J_-(n-3/2)a_n^- u\|^2\\ &-c(n)\theta\|J_-(-1)I_n(n)a_n^- u\|^2,\\ \theta^2\|J_-(-1)I_n(n)a_n^- u\|^2 &\\ \geqq c_0\theta^{3/2}\|\langle D\rangle^{2n\sigma}J_-(n-1)a_n^- u\|^2, &\ \text{mod } c(n)\theta|u|^2_{-N,\theta},\end{aligned}$$

for $\theta\geqq\theta_0(n)$. The following is also immediate.

(6.7) $\quad I_n(n-1/2)^*I_n(n-1/2)=I_n(n-1)^*BI_n(n),\ B\in J^0S^0,\ \text{mod } S^{-\infty}.$

Using the inequalities (6.2) through (6.7) and the corresponding inequalities for J_+, we can rewrite Lemma 5.1 as follows,

Lemma 6.1. *Let* $P_\theta=(D_t-i\theta)^2-\sum_{j=0}^{2}Q_j+R(D_t-i\theta)$, *then*

$$\begin{aligned}&c(n,N)\int\|P_\theta u\|^2_{-2N}dt+c(n,N)\int|||P_\theta u|||^2_{n,0,0}dt\\ &\geqq\tilde c_1 n^{1/2}\int|||(D_t-i\theta)u|||^2_{n,0,1}dt+\tilde c_2 n\int|||u|||^2_{n,1,1-p}dt\\ &\quad+\tilde c_3\theta^{1/2}\int|||(D_t-i\theta)u|||^2_{n,0,1/2}dt+\tilde c_3\theta^{1/2}\int|||u|||^2_{n,1,1/2-p}dt\\ &\quad+\tilde c_3 n^2\int|||u|||^2_{n,0,2}dt+\tilde c_3 n\theta\int|||u|||^2_{n,0,3/2}dt+\tilde c_3\theta^{3/2}\int|||u|||^2_{n,0,1}dt,\end{aligned}$$

mod $c(n,N)\theta[u]^2_{-N,\theta}$, $n\geqq c(Q_2)C$, $\theta\geqq\theta_0(n,N)$, *where* $C=\sup|J(t,x,\xi)^{-p+1}\langle\xi\rangle^{-1}Q_1(t,x,\xi)|$.

§7. Estimates of Error Terms and Proofs of Theorems

Our aim in this section is to prove the following proposition, and complete the proof of theorems.

Proposition 7.1. *We have*

$$\int\|P_\theta u\|^2_{-2N}dt+\theta^{1/2}\int|||(D_t-i\theta)u|||^2_{n,0,1/2}dt+\theta^{3/2}\int|||u|||^2_{n,0,1}dt$$
$$\geqq c(n,N)\theta^{5/4}\int\|(D_t-i\theta)u\|^2_{-N}dt+c(n,N)\theta^{11/4}\int\|u\|^2_{-N}dt$$
$$\geqq c(n,N)\theta^{7/4}[u]^2_{-N,\theta},$$

for $\theta\geqq\theta_0(n,N)$, $N\geqq 1$, *with* $c(n,N)>0$.

Proof. First we note that

$$-2\mathrm{Im}\int((D_t-i\theta)^2u,(D_t-i\theta)u)dt\geqq\theta\int\|(D_t-i\theta)u\|^2dt+\theta^3\int\|u\|^2dt.$$

On the other hand, it is easily seen that

$$2\mathrm{Im}\int(Q_2u,(D_t-i\theta)u)dt=2\theta\mathrm{Re}\int(Q_2u,u)dt-\int((\partial_tQ_2)u,u)dt$$
$$-i\int((D_t-i\theta)u,(Q_2^*-Q_2)u)dt+\theta\int(u,(Q_2^*-Q_2)u)dt$$
$$\geqq-c\int\|(D_t-i\theta)u\|^2dt-c\int\|\langle D\rangle u\|^2dt-c\theta\int\|\langle D\rangle^{1/2}u\|^2dt.$$

Remarking that

$$\langle D\rangle^{-2N}P_\theta=P_\theta\langle D\rangle^{-2N}+q\langle D\rangle^{-2N}+r(D_t-i\theta)\langle D\rangle^{-2N},\ \text{with } q\in S^1_{1,0},\ r\in S^{-1}_{1,0},$$

we have

$$\int\|P_\theta u\|^2_{-2N}dt\geqq c_0\theta^2\int\|(D_t-i\theta)u\|^2_{-2N}dt+c_0\theta^4\int\|u\|^2_{-2N}dt \tag{7.1}$$
$$-c\theta\int\|u\|^2_{-2N+1}dt-c\theta^2\int\|u\|^2_{-2N+1/2}dt,\ \text{for } \theta\geqq\theta_0(N).$$

Next, we observe $|||u|||^2_{n,0,1}$ and $|||(D_t-i\theta)u|||^2_{n,0,1/2}$. From Proposition 2.6, one can find $A\in J^{n+1}S^0$ so that

$$\alpha^+_{1n}=AJ_+(-n-1)\alpha^+_n,\ \mathrm{mod}\, S^{-\infty}.$$

Since $a_n^+(t, x, \xi)=1$ on $\mathrm{supp}[a_{4n}^+(t, x, \xi)]$, taking into account that $\mathrm{supp}[1-a_{4n}^+]\subset \{t-\phi(x, \xi)\leqq 0\}$, Proposition 2.6 also gives that

$$1-a_{4n}^+=B\langle D\rangle^{2n\sigma}J_-(n-1)a_n^-$$
$$\text{with } B\in J^{-n+1}S^{-2n\sigma}\subset JS^{-n\sigma}, \bmod S^{-\infty}(n\geqq 0).$$

Preceding two equalities imply that

$$1=AJ_+(-n-1)a_n^+ + B\langle D\rangle^{2n\sigma}J_-(n-1)a_n^-, \bmod S^{-\infty}, \tag{7.2}$$

and hence

$$\|u\|^2\leqq c(n, N)|||u|||_{n,0,1}^2+c(n, N)\|u\|_{-N}^2,$$
$$\|(D_t-i\theta)u\|^2\leqq c(n, N)|||(D_t-i\theta)u|||_{n,0,1/2}^2+c(n, N)\|(D_t-i\theta)u\|_{-N}^2.$$

Whereas, the inequalities

$$2\theta^{11/4}\|u\|_{-N}^2\leqq \theta^{3/2}\|u\|^2+\theta^4\|u\|_{-2N}^2,\ 2\theta^{5/4}\|v\|_{-N}^2\leqq \theta^{1/2}\|v\|^2+\theta^2\|v\|_{-2N}^2,$$

show that

$$c(n, N)\theta^{11/4}\|u\|_{-N}^2\leqq \theta^{3/2}|||u|||_{n,0,1}^2+\theta^4\|u\|_{-2N}^2, \tag{7.3}$$
$$c(n, N)\theta^{5/4}\|(D_t-i\theta)u\|_{-N}^2\leqq \theta^{1/2}|||(D_t-i\theta)u|||_{n,0,1/2}^2+\theta^2\|(D_t-i\theta)u\|_{-2N}^2,$$

for $\theta\geqq\theta_0(n, N)$. In view of $-N\geqq -2N+1$ $(N\geqq 1)$, (7.1) and (7.3) prove this proposition.

Now, by virtue of Proposition 7.1, the following lemma follows immediately from Lemma 6.1.

Lemma 7.1. *For $n\geqq c(Q_2)C$, $N\geqq 1$, $\theta\geqq\theta_0(n, N)$, we have*

$$c(n,N)\int\|P_\theta u\|_{-2N}^2 dt+c(n,N)\int|||P_\theta u|||_{n,0,0}^2 dt$$
$$\geqq \hat{c}_1 n^{1/2}\int|||(D_t-i\theta)u|||_{n,0,1}^2 dt+\hat{c}_2 n\int|||u|||_{n,1,1-p}^2 dt$$
$$+\hat{c}_3\theta^{1/2}\int|||(D_t-i\theta)u|||_{n,0,1/2}^2 dt+\hat{c}_3\theta^{1/2}\int|||u|||_{n,1,1/2-p}^2 dt$$
$$+c_4\theta^{5/4}\int\|(D_t-i\theta)u\|_{-N}^2 dt+c_4\theta^{11/4}\int\|u\|_{-N}^2 dt,$$

where $C=\sup|J(t, x, \xi)^{-p+1}\langle\xi\rangle^{-1}Q_1(t, x, \xi)|$, $c_4=c_4(n, N)$ and $c(Q_2)$ depends *only on* Q_2.

We shall deduce Theorems 1.2 and 1.3 from Lemma 7.1. To show Theorem 1.2, it suffices to note that $(D_t - i\theta)e^{-\theta t}u = e^{-\theta t}D_t u$.

Next we shall prove Theorem 1.3. We operate $\langle D\rangle^{s+1}$ to P, then

$$\langle D\rangle^{s+1}P = \tilde{P}\langle D\rangle^{s+1},\ \tilde{P} = D_t^2 - Q_2 + (Q_1 + \tilde{Q}_1) + \tilde{Q}_0 + \tilde{R}D_t,$$
$$\tilde{Q}_0 \in S_{1,0}^0,\ \tilde{R} \in S_{1,0}^0,$$

where $\tilde{Q}_1 = [\langle D\rangle^{s+1}, Q_2]\langle D\rangle^{-s-1} \in J^{2p-2}S^1$. Thus the term $\tilde{Q}_1$ is the only one which must be considered. From Proposition 2.7, we get

$$I_n(n-1/2)^* I_n(n-1/2)\tilde{Q}_1 = I_n(n-1/2)^* BJ_-(p)I_n(n-1/2),$$
$$B \in J^{p-2}S^1,\ \mathrm{mod}\, S^{-\infty},$$

and then

$$|(I_n(n-1/2)\tilde{Q}_1 w, I_n(n-1/2)(D_t - i\theta)w)| \leq c(n)\| I_n(n-1/2)(D_t - i\theta)w\|^2$$
$$+ c(n)\|\langle D\rangle J_-(p)I_n(n-1/2)w\|^2,\ \mathrm{mod}\, c(n)|u|_{-N}^2.$$

Therefore in Lemma 7.1, we can replace P_θ by $\tilde{P}_\theta$ without any change. On the other hand, since $\langle D\rangle^{2n\sigma}J_-(n) \in J^n S^{2n\sigma} \subset S_{1-\sigma,\sigma}^{2n\sigma}$, $J_+(-n) \in J^{-n}S^0 \subset S_{1-\sigma,\sigma}^{n\sigma}$, it follows that $|||\langle D\rangle^{s+1}v|||_{n,0,0}^2 \leq c(n)\|v\|_{2n\sigma+s+1}^2$, and hence

$$|||\tilde{P}_\theta\langle D\rangle^{s+1}u|||_{n,0,0}^2 \leq c(n)\|P_\theta u\|_{2n\sigma+s+1}^2,$$
$$\|\tilde{P}_\theta\langle D\rangle^{s+1}u\|_{-2N}^2 = \|P_\theta u\|_{-2N+s+1}^2.$$

Then, replacing P_θ, u by $\tilde{P}_\theta$, $\langle D\rangle^{s+1}u$ and taking $N=1$ in Lemma 7.1, we have Theorem 1.3.

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