Publ. RIMS, Kyoto Univ.
21 (1985), 297–354

A p_g-Formula and Elliptic Singularities

By

Masataka TOMARI*

Table of Contents

Introduction

In his paper [46], P. Wagreich introduced two numerical invariants, the geometric genus p_g and the arithmetic genus p_a, to the normal two-dimensional singularity (cf. Notation). After his definition, a normal two-dimensional singularity (V, p) is called elliptic if the condition $p_a(V, p)=1$ holds. In this decade, a great deal of work has been done on this singularity (e.g., [28], [34], [43], [48], [49], [50], [51],..., etc). Among others, H. B. Laufer has given the criterion for the absolute isolatedness (cf. Section 4 of this paper) and proved the theorem which identifies Artin's fundamental cycle with the maximal ideal cycle for the minimally elliptic singularity (Theorem 3.13, Theorem 3.15 of [28]) as analogies of the case of the rational singularities. The minimally elliptic singularity is a special elliptic singularity characterized as the Gorenstein elliptic

Communicated by S. Nakano, October 12, 1983. Revised September 21, 1984.
* Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan.

singularity with $p_g=1$. Following [28], S. S. -T. Yau developed his theory in the case of general elliptic singularities. Among his results, the criterion for the absolute isolatedness and the identification theorem are extended to more general classes, compared to the minimally elliptic singularities ([48], [50], [51]).

The goal of this paper is Theorems (7.4) and (7.8), which extend the results on the subjects mentioned above to some general forms in the case of normal two-dimensional Gorenstein singularities with $p_a=1$ (see also guidance for contents).

It seems that our method is quite different from those of H. B. Laufer and S. S. -T. Yau, even though we restricted our concern to the case of $p_g=1$. Ours is rather similar to the method of M. Reid [36]. One of the most important tools for our study is the p_g-formula, which is expressed in terms of the resolution process of the singularity by the composition of the blowing-ups with smooth centers. The geometric genus p_g can be computed by the infinitely near Hilbert-Samuel functions in the sense of (2.1). (This is rather conceptual. See also guidance for contents below.)

Let us explain the content of each section. We assume the singularity is defined over the infinite field after Section 3, the algebraically closed field after Section 4, and the complex number field after (7.8).

Section 1. For the later use, we shall collect some general results about blowing-up. Conceptually, the p_g-formula (2.7) belongs to the formula (1.3).

Section 2. A p_g-formula for the two-dimensional hypersurface isolated singularity is given in the terminologies in this section as follows ((2.1) and Theorem (2.7)):

$$-p_g(V, p)=\sum_{i=1}^{N}\{\chi(O_{\bar{V}_i})-\chi(O_{\bar{V}_{i-1}})\},$$

where

$$\chi(O_{\bar{V}_i})-\chi(O_{\bar{V}_{i-1}})=\frac{1}{6}(\rho_i-1)(2\rho_i-1)\sum_{j<i}\frac{\rho_j W_i\cdot W_j}{\rho_i}$$
$$+\frac{1}{6}((\rho_i)^2-1)(W_i)^2+\frac{1}{2}(\rho_i-1)\rho_i\chi(O_{\Gamma_i})$$

for all i.

Section 3. The virtual arithmetic genus $p_a(Y_\psi)$ of the maximal ideal cycle Y_ψ is discussed. Combining the equality (1.3) and the results by P. Wagreich and H. B. Laufer, we shall prove the formula (3.4) which relates the integer

$p_a(Y_\psi)$ with the ring theoretic information of the singularity.

Section 4. Based on the results of the previous sections, we shall prove the following decomposition theorem of Zariski's canonical resolution.

Theorem (4.6). *Let (V, p) be a normal two-dimensional Gorenstein singularity with $p_a(V, p)=1$. Then Zariski's canonical resolution is obtained by the composition of the blowing-ups with smooth centers as follows:*

$$(*)\qquad \begin{array}{ccccccccc} U=U_0 & \longleftarrow & U_1 & \longleftarrow & U_2 & \longleftarrow & \cdots & \longleftarrow & U_N \\ \cup\!\!\shortparallel \ \ \cup\!\!\shortparallel & \psi_1 & \cup\!\!\shortparallel & \psi_2 & \cup\!\!\shortparallel & \psi_3 & & \psi_N & \cup\!\!\shortparallel \\ V=V_0 & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow & \cdots & \longleftarrow & V_N, \end{array}$$

where $V\subset U$ is the minimal embedding, ψ_i the blowing-up of U_{i-1} with smooth center $\Gamma_i\subset V_{i-1}$, and V_i the strict transform of V_{i-1}, $1\leqq i\leqq N$. Moreover we have: There is an integer M $(\leqq N)$ such that (i) V_i is normal for $i\leqq M$, (ii) ψ_i is a blowing-up with point center p_i such that (V_{i-1}, p_i) is Gorenstein of maximal embedding dimension of multiplicity $\geqq 3$ (see Section 4 about the definition), for $i\leqq M$, (iii) at each stage, in which V_i is normal, there is at most one non-rational singularity, (iv) $\mathrm{mult}_q V_M\leqq 2$ for any point $q\in V_M$, and (v) in Zariski's canonical resolution for the singularities of V_M, each normalization is trivial or is obtained by one blowing-up along (reduced) $\boldsymbol{P}^1$.

Section 5. The adjunction formula (5.4) which induces the canonical bundle formula (5.2) in the resolution diagram $(*)$ of (4.6) (not necessarily with $p_a=1$) is established by following the method of J. Wahl and J. D. Sally.

Section 6. We shall collect general results on the dualgraph theoretic studies of the singularity with $p_a=1$. Among all the effective divisors we shall especially characterize the elliptic sequence of Yau by numerical condition, by which the elliptic sequence becomes more useful.

Section 7. By using the results in all sections, we shall reach the main results of this paper.

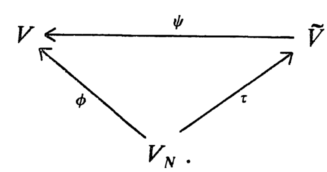
Theorem (7.4). *Let (V, p) be a normal two-dimensional Gorenstein singularity with $p_a(V, p)=1$, $\psi\colon (\tilde{V}, A)\to(V, p)$ the minimal resolution of (V, p) and $\boldsymbol{E}$ the minimal elliptic cycle on $(\tilde{V}, A)$.*

(i) (V, p) is absolutely isolated if and only if the inequality $(\boldsymbol{E})^2\leqq -3$ holds.

(ii) Zariski's canonical resolution gives the minimal resolution of (V, p) if and only if the inequality $(\boldsymbol{E})^2\leqq -2$ holds.

In addition we shall extend the identification theorem about the fundamental cycle to the following corresponding theorem between Zariski's canonical resolution and the minimal resolution without the condition about the integer $(\boldsymbol{E})^2$.

Theorem (7.8). *Let (V, p) be a normal two-dimensional Gorenstein singularity over $\boldsymbol{C}$ with $p_a(V, p)=1$. Let $\psi: (\tilde{V}, A)\to(V, p)$ be the minimal resolution of (V, p) and $\{\boldsymbol{Z}_{B_i};\ i=1,\dots, l\}$ the elliptic sequence on $(\tilde{V}, A)$ (cf. (6.3)). Let us consider Zariski's canonical resolution $\phi: V_N\to V$ and the following commutative diagram:*



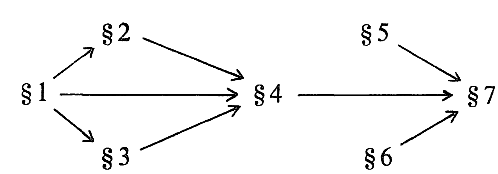
Let the sets of indexes $\{i_1,\dots, i_{p_g}\}$ and $\{j_1,\dots, j_{p_g}\}$ be the subsets of $\{1,\dots, N\}$ defined canonically in (4.7) and (7.7). (Note especially that $i_h\leqq j_h$ for all h.)

(i) There is a sequence of the integers $1\leqq k_1<\cdots<k_{p_g}=l$ such that the following equalities hold:

$$\sum_{i=1}^{k_1} \boldsymbol{Z}_{B_i}=\tau_*(W_{i_1})=\cdots=\tau_*(W_{j_1})\,,$$
$$\vdots$$
$$\sum_{i=k_{h-1}+1}^{k_h} \boldsymbol{Z}_{B_i}=\tau_*(W_{i_h})=\cdots=\tau_*(W_{j_h})\,,$$
$$\vdots$$
$$\sum_{i=k_{p_g-1}+1}^{k_{p_g}} \boldsymbol{Z}_{B_i}=\tau_*(W_{i_{p_g}})=\cdots=\tau_*(W_{j_{p_g}})\,.$$

(ii) The equality $k_{p_g-1}=l-1$ holds. Hence the divisor $\tau_*(W_{i_{p_g}})$ $(=\cdots=\tau_*(W_{j_{p_g}})$ by (i)) is the minimal elliptic cycle on $(\tilde{V}, A)$ (cf. (6.8)).

The following diagram gives a relationship of the sections in this paper.



For more details about the contents of the individual sections, we refer to their introductory remarks.

This article involves a certain amount of material of an expository nature for the convenience of the reader.

I am very grateful to Professor S. S. -T. Yau for the inspiring discussions during his stay in Kyoto. In particular, (i) of Theorem (7.4) was suggested to me by him as a Conjecture at that time. During the preparation for this paper, I have learned so many basic facts about blowing-up and others from Professor H. Hironaka. For that I am also very grateful to him. I would like to thank Professors K. Saito and I. Naruki and other members of the singularity seminar and the complex analysis seminar at the Research Institute for Mathematical Sciences of Kyoto University for their encouragement, interest, and useful suggestions.

After having submitted the early version of this paper, I received some papers about near topics of Chapter 1 of this paper from Professor M. Morales ([54], [55], [56], see (1.4) and (3.5) of this paper). For that I am very grateful to him.

Theorem (4.6) and (i) of Theorem (7.4) have already been announced in [44].

Notations. Let (V, p) be a normal two dimensional singularity and ψ: $(\tilde{V}, A)\to(V, p)$ a resolution with the decomposition into the irreducible components of the exceptional set $A=\bigcup_{j=1}^{m} A_j$. Then we use the following notations in this paper:

The *geometric genus* of the singularity (V, p) is the integer $p_g(V, p)$ defined by $p_g(V, p)=\dim R^1\psi_*O_{\tilde{V}}$ [46].

The *arithmetic genus* of the singularity (V, p) is the integer $p_a(V, p)$ defined by $p_a(V, p)=\sup\{p_a(D) \mid D$: the effective divisors on $\tilde{V}$ whose supports are contained in $A\}$. Here $p_a(D)$ is the virtual genus of D [46].

Artin's fundamental cycle for ψ is the divisor Z_0 which is the minimal element among the set $\{D$; the non-zero effective divisor on $\tilde{V}$ whose supports are contained in A and satisfies the conditions $D\cdot A_j\leqq 0$ for $j=1,\dots, m\}$ [4].

The *maximal ideal cycle* for ψ is the divisor Y_ψ defined by $Y_\psi=\sum_{j=1}^{m}(\min_{f\in m_{V,p}} \cdot v_j(f))\cdot A_j$, where v_j is the valuation on $O_{V,p}$ defined by $v_j(f)=$the vanishing order of $\psi^*(f)$ on A_j for $f\in O_{V,p}$, $1\leqq j\leqq m$ [48].

Let D be an effective divisor on $\tilde{V}$. Then we denote the supports of D by $|D|$.

Let E be an effective divisor on $\tilde{V}$ with $|E|\subseteqq A$. Then E is a *minimal elliptic cycle* if $p_a(E)=1$ and $p_a(D)<0$ for any non-zero effective divisor D such that $D<E$ [28].

For details about the materials above, we refer to the references cited after them. See [4], [8], [26], [28], [30], [31], [43], [46], [48], [52] for the basic facts on two-dimensional singularities and those numerical invariants.

Chapter I. p_g-Formula and Hilbert-Samuel Functions

§1. Generalities of Blowing-up

(1.1) In this section, we shall collect some general results concerning blowing-up which we will use in this paper. First we shall discuss the computation of the difference of Euler-Poincaré characteristics of the structure sheaves of the projective algebraic schemes by blowing-up. Second we shall discuss the description of the strict transforms of the varieties.

(1.2) Let $\bar{V}$ be a projective algebraic complete scheme and Γ a closed subspace of $\bar{V}$ which is defined by the $O_{\bar{V}}$ ideal sheaf $\mathscr{I}_\Gamma$. Let ψ: $\bar{V}_1 \to \bar{V}$ be the blowing-up of $\bar{V}$ with center $\mathscr{I}_\Gamma$ and $\mathscr{I}_\Theta$ the invertible $O_{\bar{V}_1}$ sheaf defined by $\mathscr{I}_\Theta = \psi^{-1}\mathscr{I}_\Gamma$. Then, for the function

$$k \longmapsto \chi(\mathscr{I}_\Gamma^k/\mathscr{I}_\Gamma^{k+1})\,;\ \boldsymbol{N} \cup \{0\} \longrightarrow \boldsymbol{Z}\,,$$

there exists a polynomial $P(t) \in \boldsymbol{Q}[t]$ such that

$$P(k) = \chi(\mathscr{I}_\Gamma^k/\mathscr{I}_\Gamma^{k+1}) \qquad k \gg 0\,.$$

In fact, this polynomial $P(t)$ is given in the following form. $P(t) = \chi(\Theta, \mathscr{I}_\Theta^t/\mathscr{I}_\Theta^{t+1})$ (see [7], [15]). We note the following equality.

Proposition (1.3). *Let the situation be as above. We have the following equality*:

$$\chi(O_{\bar{V}_1}) - \chi(O_{\bar{V}}) = \sum_{k \geqq 0} \{P(k) - \chi(\Gamma, \mathscr{I}_\Gamma^k/\mathscr{I}_\Gamma^{k+1})\}\,.$$

Here, if $\bar{V}_1$ is an empty set, regard $O_{\bar{V}_1}$ as the zero sheaf.

Proof. By the theorem of Grauert-Remmert (Theorem 2, 1 Chapter IV[6], or E. G. A. III (2.6.1) [15]), there is an integer k_1 ($\geqq 0$) such that $R^i\psi_*(\mathscr{I}_\Theta^k) = 0$ for any $k \geqq k_1$, and for any $i \geqq 1$. By the theorem (2.3.1) E. G. A. III [15], there is an integer k_2 ($\geqq 0$) such that the canonical morphism $\mathscr{I}_\Gamma^k \to \psi_*(\mathscr{I}_\Theta^k)$ is an isomorphism for any $k \geqq k_2$. We take an integer $k' \geqq \max(k_1, k_2)$. It follows that

$$\begin{aligned}\chi(O_{\bar{V}_1}) &= \sum_{k=0}^{k'-1} \{\chi(\bar{V}_1, \mathscr{I}_\Theta^k) - \chi(\bar{V}_1, \mathscr{I}_\Theta^{k+1})\} + \chi(\bar{V}_1, \mathscr{I}_\Theta^{k'}) \\ &= \sum_{k=0}^{k'-1} P(k) + \chi(\bar{V}_1, \mathscr{I}_\Theta^{k'})\,.\end{aligned}$$

By Leray's spectral sequence for ψ

$$\chi(\bar{V}_1, \mathscr{I}_\Theta^{k'}) = \sum_{i\geqq 0} (-1)^i \chi(\bar{V}, R^i\psi_*(\mathscr{I}_\Theta^{k'})) = \chi(\bar{V}, \mathscr{I}_\Gamma^{k'}) .$$

Here, the second equality follows from the assumption on the integer k'. On the other hand, we have

$$\begin{aligned}\chi(O_{\bar{V}}) &= \sum_{k=0}^{k'-1} \{\chi(\bar{V}, \mathscr{I}_\Gamma^k) - \chi(\bar{V}, \mathscr{I}_\Gamma^{k+1})\} + \chi(\bar{V}, \mathscr{I}_\Gamma^{k'}) \\ &= \sum_{k=0}^{k'-1} \chi(\mathscr{I}_\Gamma^k/\mathscr{I}_\Gamma^{k+1}) + \chi(\bar{V}, \mathscr{I}_\Gamma^{k'}) .\end{aligned}$$

The assertion follows from the above equalities.

Remark (1.4). Consider the function $k\mapsto\chi(\Gamma, O_{\bar{V}}/\mathscr{I}_\Gamma^{k+1})$; $\mathbb{N}\cup\{0\}\to\mathbb{Z}$, in the above situation. There is a polynomial $Q(t)\in\mathbb{Q}[t]$ such that

$$Q(k) = \chi(O_{\bar{V}}/\mathscr{I}_\Gamma^{k+1}) \qquad k\gg 0 .$$

Let the degree of $Q(t)$ be s and $\{a_i, i=0,\dots, s\}$ the integers defined by $Q(t) = \sum_{i=0}^{s} a_{s-i}\binom{i+t}{i}$. We have the equality $\sum_{k=0}^{k'} \{P(k) - \chi(\Gamma, \mathscr{I}_\Gamma^k/\mathscr{I}_\Gamma^{k+1})\} = -Q(-1) = -a_s$, for large integer k'.

Hence by (1.3), the equality

$$\chi(O_{\bar{V}_1}) - \chi(O_{\bar{V}}) = -a_s$$

holds. In this form, (1.3) is proved for the special cases by D. Kirby [24] and J. Lipman (23.2) [30] (see also Morales [54, 55, 56]).

(1.5) Let V be a scheme embedded in a scheme U and Γ a closed subspace of U defined by O_U ideal sheaf I_Γ. Let $\psi\colon U_1\to U$ be the blowing-up of U with center I_Γ and Θ the closed subspace of U_1 defined by $I_\Theta = \psi^{-1}I_\Gamma$. The strict transform of V by ψ is defined as the minimal closed subspace V_1 of U_1 such that $V_1|_{U_1-\Theta} = \psi^{-1}(V-\Gamma)$. Then the ideal sheaf I_{V_1} for V_1 is given by

$$\text{(1.5.1)} \qquad I_{V_1} = \bigcup_{k\geqq 0} (\psi^{-1}I_V : I_\Theta^k) \quad \text{in} \quad O_{U_1}$$

(see 0.42 of [11], [21]). The map $\psi|_{V_1}$, which is the restriction of ψ to V_1, is the blowing-up of V with center $I_\Gamma O_V$ ([21]).

We shall need the following lemma for later arguments.

Lemma (1.6). *Let the situation be as above. (i) The equality $I_\Theta^k \cap I_{V_1} = I_\Theta^k \cdot I_{V_1}$ holds for $k\geqq 0$. (ii) The equality $\psi_*(I_\Theta^k\cdot I_{V_1}) = \psi_*(I_\Theta^k)\cap\psi_*(I_{V_1})$ in $\psi_*(O_{U_1})$ holds for $k\geqq 0$.*

Proof. (i) The relation $I_\Theta^k \cap I_{V_1} \supseteqq I_\Theta^k \cdot I_{V_1}$ is clear. We shall show the converse inclusion relation. The assertion is local; hence we restrict ourselves to the relatively compact subdomain of U. Then there is an integer ν_0 such that the equality $I_{V_1} = (\psi^{-1}I_V : I_\Theta^k)$ holds for $k \geqq \nu_0$. Since I_Θ^k is an invertible O_{U_1}-ideal sheaf, we have the equality $I_\Theta^k \cdot I_{V_1} = I_\Theta^k \cap \psi^{-1}I_V$ for $k \geqq \nu_0$. We have $\psi^{-1}I_V \cap I_\Theta^{k+\nu_0} = I_\Theta^{k+\nu_0} \cdot I_{V_1} \subseteqq I_\Theta^{\nu_0} \cdot (I_\Theta^k \cap I_{V_1}) \subseteqq I_\Theta^{k+\nu_0} \cap I_\Theta^{\nu_0} I_{V_1} = I_\Theta^{k+\nu_0} \cap (I_\Theta^{\nu_0} \cap \psi^{-1}I_V) = I_\Theta^{k+\nu_0} \cap \psi^{-1}I_V$ for $k \geqq 0$. Hence we have the equality $I_\Theta^{k+\nu_0} \cdot I_{V_1} = I_\Theta^{\nu_0} \cdot (I_\Theta^k \cap I_{V_1})$ for $k \geqq 0$. Since $I_\Theta^{\nu_0}$ is O_{U_1}-invertible, the equality $I_\Theta^k \cdot I_{V_1} = I_\Theta^k \cap I_{V_1}$ follows for $k \geqq 0$.

(ii) The equality $\psi_*(I_\Theta^k \cap I_{V_1}) = \psi_*(I_\Theta^k) \cap \psi_*(I_{V_1})$ in $\psi_*(O_{U_1})$ can be easily checked. Thus the equality (i) induces the assertion. Q. E. D.

Lemma (1.7). *Let the situation be as in* (1.3). *Suppose that the center Γ of the blowing-up ψ is smooth and connected and that V is a locally hypersurface in a neighborhood of Γ.*

Then we have the equality $P(k) = \chi(\Gamma, \mathscr{I}_\Gamma^k / \mathscr{I}_\Gamma^{k+1})$ for $k \geqq \rho$. Here the integer ρ is the multiplicity of V at points in a Zariski open set of Γ.

Proof. We shall show that the integers k_1 and k_2 in the proof of (1.3) can be taken to be the integer ρ. Hence the problem is local. From now on, we restrict ourselves to the following situation. We employ the notations in (1.5) and assume that V is an open set of $\bar{V}$ such that V has an ambient manifold U and that the defining ideal I_V of V in O_U is O_U invertible. Since the center is smooth, we have the following equalities; $\psi_*(I_\Theta^k) = I_\Gamma^k$ for $k \geqq 0$, and $R^i\psi_*(I_\Theta^k) = 0$ for $k \geqq 0$, $i \geqq 1$. By (i) of (1.6), we have

$$0 \longrightarrow I_\Theta^k \cdot I_{V_1} \longrightarrow I_\Theta^k \longrightarrow I_\Theta^k \cdot O_{V_1} \longrightarrow 0, \qquad \text{for} \quad k \geqq 0.$$

Taking the direct image ψ, we have

$$\begin{aligned} 0 \longrightarrow \psi_*(I_\Theta^k \cdot I_{V_1}) \longrightarrow I_\Gamma^k \longrightarrow \psi_*(I_\Theta^k \cdot O_{V_1}) \\ \longrightarrow R^1\psi_*(I_\Theta^k \cdot I_{V_1}) \longrightarrow 0, \qquad \text{for} \quad k \geqq 0. \end{aligned}$$

Since I_V is O_U invertible, we have the equality $\psi^{-1}I_V = I_{V_1} \cdot I_\Theta^\rho$. Hence, for $k \geqq \rho$ and for $i \geqq 1$, we have $R^i\psi_*(I_\Theta^k \cdot I_{V_1}) = R^i\psi_*(I_\Theta^{k-\rho} \otimes \psi^{-1}I_V) = I_V \cdot R^i\psi_*(I_\Theta^{k-\rho}) = 0$ by the projection formula. Therefore we can take k_1 to be ρ.

On the other hand, for $k \geqq \rho$, we have the following equalities.

$$\begin{aligned} \psi_*(I_\Theta^k \cdot I_{V_1}) &= \psi_*(I_\Theta^k) \cap \psi_*(I_{V_1}) && \text{((ii) of (1.6))} \\ &= \psi_*(I_\Theta^k) \cap \psi_*(I_\Theta^\rho) \cap \psi_*(I_{V_1}) && \text{(since } k \geqq \rho) \\ &= I_\Gamma^k \cap \psi_*(I_\Theta^\rho \cdot I_{V_1}) && \text{((ii) of (1.6))} \\ &= I_\Gamma^k \cap \psi_*(\psi^{-1}I_V) = I_\Gamma^k \cap I_V && \text{(by the projection formula).} \end{aligned}$$

Hence we have

$$0 \longrightarrow I_\Gamma^k \cap I_V \longrightarrow I_\Gamma^k \longrightarrow \psi_*(I_\Theta^k \cdot O_{V_1}) \longrightarrow 0, \qquad \text{for} \quad k \geqq \rho.$$

By the isomorphism $I_\Gamma^k / I_\Gamma^k \cap I_V \cong (I_\Gamma^k + I_V)/I_V = I_\Gamma^k \cdot O_V$ and the above exact sequence, we can take k_2 to be ρ. Q.E.D.

(1.8) Let the situation be as in (1.5) and suppose that Γ is of dimension zero. We denote this point also by p. Let H be a closed subspace of U defined by the O_U ideal sheaf I_H. We say *V and H intersects tangentially at p* if the equality in the ring of the initial forms $gr_{m_p}(O_{U,p})$ (this is defined by $gr_{m_p}(O_{U,p}) = \bigoplus_{k \geqq 0} (m_p)^k/(m_p)^{k+1}$)

(1.8.1) $$gr_{m_p}(I_V) + gr_{m_p}(I_H) = gr_{m_p}(I_V + I_H)$$

holds. Here m_p $(=I_\Gamma)$ is the maximal ideal in $O_{U,p}$. Let H_1 (resp. $(H \cap V)_1$) be the strict transform of H (resp. $(H \cap V)$) by the blowing-up ψ.

Lemma (1.9). *If H and V intersect tangentially at p, then the equality $(H \cap V)_1 = H_1 \cap V_1$ holds.*

Proof. By taking Proj, the equality $(I_{V_1} + I_{H_1}) \cdot O_\Theta = I_{(V \cap H)_1} \cdot O_\Theta$ follows from (1.8.1) in $O_\Theta = \mathrm{Proj}(gr_{m_p}(O_{U,p}))$. Hence the equality $I_{H_1} + I_{V_1} + I_\Theta = I_{(H \cap V)_1} + I_\Theta$ holds in O_{U_1}. In general the inclusion relation $I_{H_1} + I_{V_1} \subseteqq I_{(V \cap H)_1}$ can be easily checked. By the equality above, we have

$$\begin{aligned} I_{(V \cap H)_1} &\subseteqq I_{H_1} + I_{V_1} + (I_\Theta \cap I_{(V \cap H)_1}) \\ &= I_{H_1} + I_{V_1} + (I_\Theta \cdot I_{(V \cap H)_1}) \qquad \text{(by (i) of (1.6))}. \end{aligned}$$

By Nakayama's lemma, the equality $I_{H_1} + I_{V_1} = I_{(V \cap H)_1}$ follows. Q.E.D.

Remark (1.10). In general, if the equality $(V \cap H)_1 = H_1 \cap V_1$ holds, graded $gr_{m_p}(O_{U,p})$ modules $gr_{m_p}(I_H) + gr_{m_p}(I_V)$ and $gr_{m_p}(I_H + I_V)$ are T. N. -isomorphic in the sense of E. G. A. II [15].

Assume that the equality $I_H = h \cdot O_{U,p}$ at p for some element h of $O_{U,p}$ holds. In this case, the following sufficient condition to satisfy (1.8.1) is given by H. Hironaka.

Proposition (1.11) (Proposition 6 [20]). *Let the situation be as above. The following conditions are equivalent to one another.*

(*i*) $H_{V \cap H}^{(t+1)} = H_V^{(t)}$ *for all* $t \geqq 0$.

(*ii*) *h is not a zero-divisor (and hence non-zero) in O_V and $(m_p)^{k+1} O_V \cap h \cdot O_V = h(m_p)^k O_V$ for all $k \geqq 0$.*

(*iii*) *the image* $\bar{h}$ *of* h *in* $gr^1_{m_p}(O_V)$ *is not a zero-divisor in* $gr_{m_p}(O_V)$.

(*iv*) $\bar{h}$ *is not a zero-divisor in* $gr_{m_p}(O_V)$ *and the natural homomorphism*

$$gr_{m_p}(O_V)/\bar{h}\cdot gr_{m_p}(O_V) \longrightarrow gr_{m_p}(O_{V\cap H}) \tag{1.11.1}$$

is bijective.

Note that the bijectivity of (1.11.1) is equivalent to the condition (1.8.1). Here the function $H^{(0)}_V$ is defined by

$$H^{(0)}_V(k)=\dim((m_p)^k\cdot O_V/(m_p)^{k+1}\cdot O_V).$$

Then the functions $\{H^{(t)}_V;\ t\geqq 0\}$ are defined inductively by the following rule:

$$H^{(t)}_V(k)=\sum_{i=0}^{k} H^{(t-1)}_V(i).$$

We will need the following fact for the later arguments (in the proof of Theorem (4.5)).

Proposition (1.12). *Let the situation be as in* (1.8). *Let* B *be the one-dimensional closed smooth subvariety of* V. *Let* $\psi\colon U_1\to U$ *be the blowing-up of* U *with center* B *and* V_1 (*resp.* H_1, *resp.* $(V\cap H)_1$) *the strict transform of* V (*resp.* H, *resp.* $V\cap H$). *Moreover we assume the following conditions*:

(*i*) H *and* V *intersect at* p *with the conditions in* (1.11).

(*ii*) V *is normally flat along* B.

(*iii*) *The equality* $\mathscr{I}_B+\mathscr{I}_H=m_p$ *holds in* O_V.

Then we have the equality $V_1\cap H_1=(V\cap H)_1$.

Proof. By (iii), we have the natural surjection

$$\alpha\colon (gr_{\mathscr{I}_B}(O_V)/m_p\cdot gr_{\mathscr{I}_B}(O_V)) \longrightarrow gr_{m_p}(O_{V\cap H}).$$

We shall show that α is bijective. By Corollary 2 of Chapter 2 [19], the normal flatness of V along B induces the equality $\sum_{j=0}^{k}\operatorname{rank}_{O_B}(gr^j_{\mathscr{I}_B}(O_V))=H^{(0)}_{(V,p)}(k)$ for $k\geqq 0$. On the other hand $H^{(0)}_{(V,p)}=H^{(1)}_{(V\cap H,p)}$ holds by (i). By the O_B-freeness of $gr^j_{\mathscr{I}_B}(O_V)$, we have

$$\dim(gr^j_{\mathscr{I}_B}(O_V)/m_p\cdot gr^j_{\mathscr{I}_B}(O_V))=\operatorname{rank}_{O_B}(gr^j_{\mathscr{I}_B}(O_V)) \qquad \text{for } j\geqq 0.$$

Combining the above equalities, we obtain

$$\dim(gr^j_{\mathscr{I}_B}(O_V)/m_p\cdot gr^j_{\mathscr{I}_B}(O_V))=H^{(0)}_{(V\cap H,p)}(j)=\dim(gr^j_{m_p}(O_{V\cap H})) \qquad \text{for } j\geqq 0.$$

Hence α is bijective. Taking the Proj, we obtain the equality $V_1\cap H_1\cap\Theta=(V\cap H)_1\cap\Theta$. By the same arguments in (1.9), we obtain the equality $V_1\cap H_1$

$=(V\cap H)_1$. Q. E. D.

§2. A p_g-Formula for Two-Dimensional Hypersurface Isolated Singularities

(2.1) Let (V, p) be a normal n-dimensional isolated singularity and $\psi\colon \tilde{V}\to V$ a resolution of the singularity (V, p). The geometric genus of the singularity (V, p) is the integer $p_g(V, p)$ defined by $p_g(V, p)=\dim_C(R^{n-1}\psi_*(O_{\tilde{V}}))$. This is independent of the choice of resolutions. By Theorem (3.8) of [5], we can take a representative V of (V, p) to be algebraic. Let $\bar{V}$ be a compactification of V such that V is an open subset of $\bar{V}$. We fix this compactification $\bar{V}$. The singularity (V, p) can be resolved by a succession of blowing-ups with smooth center as follows:

(2.1.1)
$$\begin{array}{ccccccccc} V=V_0 & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow & \cdots & \longleftarrow & V_N=\tilde{V} \\ \cap & & \cap\ \psi_1 & & \cap\ \psi_2 & & \cap\ \psi_3 & & \psi_N\ \cap \\ \bar{V}=\bar{V}_0 & \longleftarrow & \bar{V}_1 & \longleftarrow & \bar{V}_2 & \longleftarrow & \cdots & \longleftarrow & \bar{V}_N=\bar{\tilde{V}}\,. \end{array}$$

Here, $\psi_i\colon V_i\to V_{i-1}$ is the blowing-up with center $\mathscr{I}_{\Gamma_i}$ and Γ_i the closed submanifold of the singular locus of V_{i-1} defined by $O_{V_{i-1}}$ ideal sheaf $\mathscr{I}_{\Gamma_i}$. Then $\bar{\psi}_i\colon \bar{V}_i\to\bar{V}_{i-1}$ is the blowing-up of $\bar{V}_{i-1}$ which is canonically induced from ψ_i.

By Leray's spectral sequence for the map $\psi_1\circ\cdots\circ\psi_N$, we have the equality $\chi(O_{\bar{\tilde{V}}})-\chi(O_{\bar{V}})=\sum_{i\geq 1}(-1)^i\dim_C(R^i\psi_*(O_{\bar{\tilde{V}}}))$, since the equality $\psi_*(O_{\bar{\tilde{V}}})=O_{\bar{V}}$ holds. In particular, if $O_{V,p}$ is a Cohen-Macaulay local ring, we have the vanishings $R^i\psi_*(O_{\bar{\tilde{V}}})=0$ for $1\leq i\leq n-2$ (see [17], [52]). Hence we have the following equality.

$$(-1)^{n-1}p_g(V, p)=\chi(O_{\bar{\tilde{V}}})-\chi(O_{\bar{V}})=\sum_{i=1}^{N}\{\chi(O_{\bar{V}_i})-\chi(O_{\bar{V}_{i-1}})\}\,.$$

By (1.3), if we know all the functions

$$\boldsymbol{N}\cup\{0\}\longrightarrow \boldsymbol{Z};\ k\longmapsto \chi(\Gamma_i,\ \mathscr{I}^k_{\Gamma_i}\cdot O_{V_{i-1}}/\mathscr{I}^{k+1}_{\Gamma_i}\cdot O_{V_{i-1}})$$

in the resolution diagram (2.1.1), the geometric genus $p_g(V, p)$ can be computed. However, in general, it is difficult to describe the behavior of the above functions explicitly. We shall prove an equality, which we call "p_g-formula", in the case of the two-dimensional hypersurface isolated singularities (Theorem (2.7)).

Example (2.2). Before we proceed to study the hypersurface two-dimensional case, we shall consider the case where the dimension of the center Γ is zero. Then at this point, which we denote also by p, the function $k\mapsto\chi(\Gamma,\ \mathscr{I}^k_{\Gamma}/\mathscr{I}^{k+1}_{\Gamma})$

becomes the Hilbert function of $O_{V,p}$, and $P(t)$ the associated Hilbert polynomial. In some special cases, they are written explicitly as follows.

(i) Let (V, p) be of dimension n and embedded in $(\boldsymbol{C}^{n+2}, o)$ as a tangential complete intersection. Let the couple of integers (d_1, d_2) be the degrees of the standard basis, in the sense of H. Hironaka, of (V, p) with respect to $(\boldsymbol{C}^{n+2}, o)$. If $n=1$, we have $\chi(O_{\bar{V}_1})-\chi(O_{\bar{V}})=\frac{1}{2}d_1d_2(d_1+d_2-2)$ (Northcott [33]). If $n=2$, $\chi(O_{\bar{V}_1})-\chi(O_{\bar{V}})=-\frac{1}{12}d_1d_2\{2(d_1+d_2)^2-d_1d_2-9(d_1+d_2)+11\}$.

(ii) Let (V, p) be of dimension 2 and a Cohen-Macaulay of maximal embedding dimension; then the equality $\dim_{\boldsymbol{C}} m_p/m_p^2=\mathrm{mult}_p V+1$ holds (see Sally [40]). Then we have $\dim (m_p)^k/(m_p)^{k+1}=(\mathrm{mult}_p V)\cdot k+1$ for $k\geqq 0$. Hence $\chi(O_{\bar{V}_1})-\chi(O_{\bar{V}})=0$.

(iii) Let (V, p) be of dimension 2 and a Gorenstein of maximal embedding dimension of multiplicity greater than 2; $\dim m_p/(m_p)^2=\mathrm{mult}_p V>2$ (see Sally [40]). Then we have $\dim (m_p)^k/(m_p)^{k+1}=(\mathrm{mult}_p V)\cdot k$ for $k\geqq 1$. Hence $\chi(O_{\bar{V}_1})-\chi(O_{\bar{V}})=-1$.

(2.3) We need some information (2.3.1) from the ambient spaces in the resolution (2.1.1) for our p_g-formula.

$$
\begin{array}{ccccccccccc}
U= U_0 & \longleftarrow & U_1 & \longleftarrow & U_2 & \longleftarrow & \cdots & \longleftarrow & U_N & & \\
\cup\ \ \cup & \psi_1 & \cup & \psi_2 & \cup & \psi_3 & & \psi_N & \cup & & \\
V= V_0 & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow & \cdots & \longleftarrow & V_N & = & \tilde{V} \\
\cap\ \ \cap & & \cap & & \cap & & & & \cap & & \cap \\
\bar{V}= \bar{V}_0 & \longleftarrow & \bar{V}_1 & \longleftarrow & \bar{V}_2 & \longleftarrow & \cdots & \longleftarrow & \bar{V}_N & = & \bar{\tilde{V}}\,.
\end{array}
\tag{2.3.1}
$$

Here, V is embedded in U as in (1.5). ψ_i: $U_i \to U_{i-1}$ is the blowing-up of U_{i-1} with center I_{Γ_i}, which is defined as the kernel of the natural map $O_{U_{i-1}}\to O_{V_{i-1}}/\mathscr{I}_{\Gamma_i}\cdot O_{V_{i-1}}=O_{I_i}$. V_i is the strict transform of V_{i-1} by ψ_i. $\bar{V}_i$ is defined in the same way as in (2.1.1).

Now we shall go a step further by introducing the following notations:

On U_i, the divisor $\Theta_i^{(i)}$ is the exceptional set of ψ_i defined by $\psi_i^{-1}(I_{\Gamma_i})$. Then, on U_j $(i<j)$, the divisor $\Theta_i^{(j)}$ is the strict transform of $\Theta_i^{(i)}$ by $\psi_{i+1}\circ\cdots\circ\psi_j$. On $\Theta_i^{(i)}$, h_i is the line bundle on $\Theta_i^{(i)}$ defined by $I_{\Theta_i^{(i)}}/(I_{\Theta_i^{(i)}})^2$, where $I_{\Theta_i^{(i)}}$ is defined as $I_{\Theta_i^{(i)}}=\psi_i^{-1}(I_{\Gamma_i})$ above. On V_i, the divisor E_i is the exceptional set of $\psi_i|_{V_i}$: $V_i\to V_{i-1}$ defined by $\psi_i^{-1}(I_{\Gamma_i})\cdot O_{V_i}$. On $\tilde{V}$, the divisor W_i is defined by $(\psi_i\circ\cdots\circ\psi_N)^{-1}(I_{\Gamma_i})\cdot O_{\tilde{V}}$.

Lemma (2.4). *Let the situation be as above. Suppose that V is a hypersurface in U. We have the following equality*:

$$\chi(O_{\bar{V}_i})-\chi(O_{\bar{V}_{i-1}})=-\sum_{k=0}^{\rho_i-1}\chi(\Theta_i^{(i)},(I_{\Theta_i^{(i)}})^k/(I_{\Theta_i^{(i)}})^{k+1})\otimes I_{V_i}).$$

Here, the integer ρ_i *is the multiplicity of* V_{i-1} *at points in a Zariski open set of the center* Γ_i. I_{V_i} *is the defining ideal of* V_i *in* O_{U_i}.

Proof. By the equalities (1.3), (1.7) and (1.2), we have the following equality. $\chi(O_{\bar{V}_i})-\chi(O_{\bar{V}_{i-1}})=\chi(O_{\bar{V}_i}/(I_{\Theta_i^{(i)}})^{\rho_i}\cdot O_{\bar{V}_i})-\chi(O_{\bar{V}_{i-1}}/(I_{\Gamma_i})^{\rho_i}\cdot O_{\bar{V}_{i-1}})$. By (i) of Lemma (1.6), we have the following exact sequence.

$$0\longrightarrow I_{V_i}\otimes(O_{U_i}/(I_{\Theta_i^{(i)}})^{\rho_i})\longrightarrow O_{U_i}/(I_{\Theta_i^{(i)}})^{\rho_i}\longrightarrow O_{V_i}/(I_{\Theta_i^{(i)}})^{\rho_i}O_{\bar{V}_i}\longrightarrow 0.$$

Since $(I_{\Gamma_i})^{\rho_i}\supseteq I_{\bar{V}_{i-1}}$, we have $O_{\bar{V}_{i-1}}/(I_{\Gamma_i})^{\rho_i}O_{\bar{V}_{i-1}}\cong O_{U_{i-1}}/(I_{\Gamma_i})^{\rho_i}$. Furthermore, we have the equalities $\psi_*((I_{\Theta_i^{(i)}})^k)=(I_{\Gamma_i})^k$ for $k\geqq 0$, and $R^i\psi_*((I_{\Theta_i^{(i)}})^k)=0$ for $k\geqq 0$, $i\geqq 1$. Hence, by Leray's spectral sequence for ψ_i, we have

$$\chi(O_{U_i}/(I_{\Theta_i^{(i)}})^{\rho_i})=\chi(O_{U_{i-1}}/(I_{\Gamma_i})^{\rho_i})=\chi(O_{V_{i-1}}/(I_{\Gamma_i})^{\rho_i}\cdot O_{V_{i-1}}).$$

The assertion follows from the equalities above.

Proposition (2.5). *Let the situation be as above. Suppose* $\dim V=2$. *We have the following equalities.*

(*i*) *If* $\dim\Gamma_i=0$,

$$\chi(O_{\bar{V}_i})-\chi(O_{\bar{V}_{i-1}})=-\frac{1}{6}\rho_i(\rho_i-1)(\rho_i-2).$$

(*ii*) *If* $\dim\Gamma_i=1$,

$$\begin{aligned}\chi(O_{\bar{V}_i})-\chi(O_{\bar{V}_{i-1}})&=-\frac{1}{6}\rho_i\cdot(\rho_i+1)\cdot(\rho_i-1)(h_i)^2-\frac{1}{2}(\rho_i-1)\rho_i(g_i-1-r_i)\\&=\frac{1}{12}\rho_i\cdot(\rho_i-1)\cdot(\rho_i-2)\cdot(h_i)^2-\frac{1}{4}(\rho_i-1)\cdot(E_i)^2\\&\quad-\frac{1}{2}\rho_i\cdot(\rho_i-1)\cdot(g_i-1).\end{aligned}$$

Here the integer g_i *is the genus of the curve* Γ_i. *For the definition of* r_i, *note the following decompositions* $\mathrm{Pic}(\Theta_i^{(i)})=\boldsymbol{Z}\cdot h_i\oplus\psi_i^*\mathrm{Pic}(\Gamma_i)$ *and* $\mathrm{Num}(\Theta_i^{(i)})=\boldsymbol{Z}\cdot h_i\oplus\boldsymbol{Z}\cdot f_i$. *Here* f_i *is the fibre class of* $\Theta_i^{(i)}\to\Gamma_i$. *We have the decomposition by which we define the integer* r_i, $E_i\equiv\rho_i h_i-r_i f_i$ *in* $\mathrm{Num}(\Theta_i^{(i)})$. *The integers* $(h_i)^2$ *and* $(E_i)^2$ *are self-intersection numbers over* $\Theta_i^{(i)}$.

Proof. (i) We have the data $\Theta_i^{(i)}=\boldsymbol{P}^2$, $h_i=O_{\boldsymbol{P}^2}(1)$, the canonical line bundle of $\boldsymbol{P}^2=-3h_i$. From (2.4) and the Riemann-Roch formula (*p.* 433 [17]), we have the equality

$$\chi(O_{\bar{V}_i})-\chi(O_{\bar{V}_{i-1}})=-\sum_{k=0}^{\rho_i-1}\chi(\boldsymbol{P}^2, O_{\boldsymbol{P}^2}(k-\rho_i))$$
$$=-\frac{1}{6}\rho_i(\rho_i-1)(\rho_i-2).$$

(ii) We have the data $\Theta_i^{(i)}=\boldsymbol{P}((I_{\Gamma_i}/(I_{\Gamma_i})^2)^*)\xrightarrow{\psi_i}\Gamma_i$, the canonical line bundle of $\Theta_i^{(i)}\equiv -2\cdot h_i+(2g_i-2+(h_i)^2)\cdot f_i$ in $\mathrm{Num}(\Theta_i^{(i)})$ (cf. the arguments in §2 of Chapter V [17]). From (2.4) and the Riemann-Roch formula, we have the equality

$$\chi(O_{\bar{V}_i})-\chi(O_{\bar{V}_{i-1}})=-\sum_{k=0}^{\rho_i-1}\chi(\Theta_i^{(i)}, O_{\Theta_i^{(i)}}((-\rho_i+k)\cdot h_i+r_i\cdot f_i))$$
$$=-\frac{1}{6}\rho_i(\rho_i+1)(\rho_i-1)(h_i)^2-\frac{1}{2}\rho_i(\rho_i-1)(g_i-1-r_i).$$

Furthermore the equality $(E_i)^2=(\rho_i)^2(h_i)^2-2\rho_i r_i$ induces the later equality of (ii). Q. E. D.

The following lemma helps the computations of the integers $(h_i)^2$ and r_i.

Lemma (2.6). *Let the situation be as above. We have the following equalities.*

(*i*) *If* $\dim \Gamma_i=0$, $(W_i)^2=-\rho_i$.
If $\dim \Gamma_i=1$, $(W_i)^2=-\rho_i(h_i)^2+r_i$.

(*ii*) $r_i=\sum_{j<i}\frac{\rho_j W_i\cdot W_j}{\rho_i}$.

Here the integer $W_i\cdot W_j$ is the intersection number of W_i and W_j over $\bar{\bar{V}}$, for $j\leqq i$.

Proof. (Cf. the proof of Theorem 2.7 [46].) From the characteristics of the intersection number ([25]), we have the equality

$$W_i\cdot W_j=\chi(O_{\bar{V}_N})-\chi(\mathscr{I}_{W_i})-\chi(\mathscr{I}_{W_j})+\chi(\mathscr{I}_{W_i}\cdot\mathscr{I}_{W_j})\quad \text{on}\quad \bar{\bar{V}}=\bar{V}_N$$

for any couple of integers (i, j) with $j\leqq i$. This equals the number

$$\chi(O_{\bar{V}_i})-\chi(I_{\Theta_i^{(i)}}\cdot O_{\bar{V}_i})-\chi((\psi_j\circ\cdots\circ\psi_i)^{-1}(I_{\Gamma_j})\cdot O_{\bar{V}_i})$$
$$+\chi(I_{\Theta_i^{(i)}}\cdot(\psi_j\circ\cdots\circ\psi_i)^{-1}(I_{\Gamma_j})\cdot O_{\bar{V}_i})$$

by Proposition 6 in p. 299 of [25]. We have the following exact sequences:

$$0\longrightarrow I_{\Theta_i^{(i)}}\cdot O_{\bar{V}_i}\longrightarrow O_{\bar{V}_i}\longrightarrow O_{\Theta_i^{(i)}}\otimes O_{\bar{V}_i}\longrightarrow 0,$$

and

$$0\longrightarrow I_{\Theta_i^{(i)}}\cdot(\psi_j\circ\cdots\circ\psi_i)^{-1}(I_{\Gamma_j})\cdot O_{\bar{V}_i}\longrightarrow(\psi_j\circ\cdots\circ\psi_i)^{-1}(I_{\Gamma_j})\cdot O_{\bar{V}}$$
$$\longrightarrow O_{\Theta_i^{(i)}}\otimes((\psi_j\circ\cdots\circ\psi_i)^{-1}(I_{\Gamma_j})\cdot O_{\bar{V}_i})\longrightarrow 0.$$

These sequences induce the equality

$$W_i \cdot W_j = \chi(O_{\Theta_i^{(i)}} \otimes O_{V_i}) - \chi(O_{\Theta_i^{(i)}} \otimes ((\psi_j \circ \cdots \circ \psi_i)^{-1}(I_{\Gamma_j}) \cdot O_{V_i})).$$

In general we have the relation $I_{V_i} \otimes O_{\Theta_i^{(i)}} = I_{V_i} \cdot O_{\Theta_i^{(i)}}$ by (i) of Lemma (1.6). Hence we have the following exact sequence:

$$0 \longrightarrow I_{V_i} \otimes O_{\Theta_i^{(i)}} \longrightarrow O_{\Theta_i^{(i)}} \longrightarrow O_{V_i} \otimes O_{\Theta_i^{(i)}} \longrightarrow 0.$$

Tensoring the $O_{\Theta_i^{(i)}}$-invertible sheaf $(\psi_j \circ \cdots \circ \psi_i)^{-1}(I_{\Gamma_j}) \otimes O_{\Theta_i^{(i)}}$ to the above exact sequence over $O_{\Theta_i^{(i)}}$, we obtain the following exact sequence:

$$\begin{aligned} 0 &\longrightarrow ((\psi_j \circ \cdots \circ \psi_i)^{-1}(I_{\Gamma_j}) \otimes O_{\Theta_i^{(i)}}) \otimes (I_{V_i} \otimes O_{\Theta_i^{(i)}}) \\ &\longrightarrow (\psi_j \circ \cdots \circ \psi_i)^{-1}(I_{\Gamma_j}) \otimes O_{\Theta_i^{(i)}} \\ &\longrightarrow ((\psi_j \circ \cdots \circ \psi_i)^{-1}(I_{\Gamma_j}) \otimes O_{\Theta_i^{(i)}}) \otimes (O_{V_i} \otimes O_{\Theta_i^{(i)}}) \longrightarrow 0. \end{aligned}$$

Furthermore the relation $(\psi_j \circ \cdots \circ \psi_i)^{-1}(I_{\Gamma_j}) \cdot I_{V_i} = (\psi_j \circ \cdots \circ \psi_i)^{-1}(I_{\Gamma_j}) \cap I_{V_i}$ can be easily checked by using the fact that V_i is a hypersurface in U_i. Hence the equality $(\psi_j \circ \cdots \circ \psi_i)^{-1}(I_{\Gamma_j}) \cdot O_{V_i} = (\psi_j \circ \cdots \circ \psi_i)^{-1}(I_{\Gamma_j}) \otimes O_{V_i}$ holds. From this we obtain the equality $O_{\Theta_i^{(i)}} \otimes ((\psi_j \circ \cdots \circ \psi_i)^{-1}(I_{\Gamma_j}) \cdot O_{V_i}) = (O_{\Theta_i^{(i)}} \otimes O_{V_i}) \otimes ((\psi_j \circ \cdots \circ \psi_i)^{-1}(I_{\Gamma_j}) \otimes O_{\Theta_i^{(i)}})$.

Combining the equalities above, we have the following:

$$\begin{aligned} W_i \cdot W_j = \chi(O_{\Theta_i^{(i)}}) - \chi(I_{V_i} \otimes O_{\Theta_i^{(i)}}) - \chi((\psi_j \circ \cdots \circ \psi_i)^{-1}(I_{\Gamma_j}) \otimes O_{\Theta_i^{(i)}}) \\ + \chi(((\psi_j \circ \cdots \circ \psi_i)^{-1}(I_{\Gamma_j}) \otimes O_{\Theta_i^{(i)}}) \otimes (I_{V_i} \otimes O_{\Theta_i^{(i)}})). \end{aligned}$$

(i) In the case of $i = j$, the equality just above induces the equality $(W_i)^2 = -(E_i) \cdot h_i$ from the characteristics of the intersection number over $\Theta_i^{(i)}$ [25]. This is nothing but our assertion.

(ii) In the i-th stage, we have the decomposition of sheaves

$$(\psi_1 \circ \cdots \circ \psi_i)^{-1}(I_V) = I_{V_i} \cdot \prod_{j \leq i} \{(\psi_j \circ \cdots \circ \psi_i)^{-1}(I_{\Gamma_j})\}^{\rho_j}.$$

This induces the following equality in $\mathrm{Pic}(\Theta_i^{(i)})$:

$$0 = -E_i + \rho_i \cdot h_i + \sum_{j < i} \rho_j ((\psi_j \circ \cdots \circ \psi_i)^{-1}(I_{\Gamma_j}) \otimes O_{\Theta_i^{(i)}}).$$

We can write $(\psi_j \circ \cdots \circ \psi_i)^{-1}(I_{\Gamma_j}) \equiv s_j \cdot f_i$ in $\mathrm{Num}(\Theta_i^{(i)})$ by some integer s_j, for $j < i$. Then the equality $W_i \cdot W_j = -\rho_i s_j$ holds for the pair (i, j) with the condition $j < i$. Hence the assertion follows from the equality

$$0 = (-E_i) \cdot h_i + \rho_i \cdot (h_i)^2 + \sum_{j < i} \rho_j \cdot ((\psi_j \circ \cdots \circ \psi_i)^{-1}(I_{\Gamma_j}) \otimes O_{\Theta_i^{(i)}}) \cdot h_i.$$

Q. E. D.

Now we shall obtain the p_g-formula as a corollary.

Theorem (2.7). *Under the assumptions of Proposition* (2.5), *we have the following equality.*

$$\chi(O_{V_i})-\chi(O_{V_{i-1}})=\frac{1}{6}(\rho_i-1)\cdot(2\rho_i-1)\sum_{j<i}\frac{\rho_j W_i\cdot W_j}{\rho_i}$$
$$+\frac{1}{6}((\rho_i)^2-1)\cdot(W_i)^2+\frac{1}{2}(\rho_i-1)\rho_i\chi(O_{\Gamma_i}).$$

Remark (2.8). In the case of $\mathrm{mult}_p V=2$, the p_g-formula for the canonical resolution stated in Lemma 2 of [43] can be also induced from our formula (2.7) (cf. the argument in (4.8)).

Here, we shall give some examples computing the geometric genus from our formula.

Example (2.9). Let $(\bar{V}, p)$ be a complete two-dimensional scheme $\bar{V}$ over $\boldsymbol{C}$ with a reference point p in which the isomorphism $(\bar{V}, p)\cong(\{(x, y, z)\in \boldsymbol{C}^3 \mid z^3=x^5+y^5\}, o)$ exists. We shall construct the resolution of this singularity $(\bar{V}, p)$ by two different processes. The first one is a resolution by composition of blowing-ups with permissible centers (i.e., the blowing-ups with smooth centers, where the multiplicity is constant everywhere), we shall call this resolution (A). The other is a resolution by composition of blowing-ups with not necessarily permissible centers, we shall call this resolution (B). Consider $(\bar{V}, p)$ only as the germ $(\{z^3=x^5+y^5\})$ below.

(*A*), (*B*), *Step* 1. Blow up V at p, say ψ_1: $V_1\to V$. In V_1 the singularity appears along $\boldsymbol{P}^1$ as follows: The analytic space $\{z=0\}$ in U is denoted by H, and the strict transform of H in U_i is denoted by H_i (the other notations are as same as those in (2.3.1)). The singular locus $\mathrm{Sing}(V_1)$ of V_1 equals $|H_1 \cap V_1|$ $(=|\Theta_1^{(1)} \cap H_1|)$ as the analytic sets. There are five points, call them p_2, p_3, p_4, p_5, p_6, in which the singularity (V_1, p_i) is isomorphic to the singularity $(\{s^3=t^2u\}, o)$ for $i=2, 3, 4, 5, 6$. The multiplicity of V_1 at the points in $|H_1 \cap \Theta_1^{(1)}|-\{p_2, p_3, p_4, p_5, p_6\}$ is two.

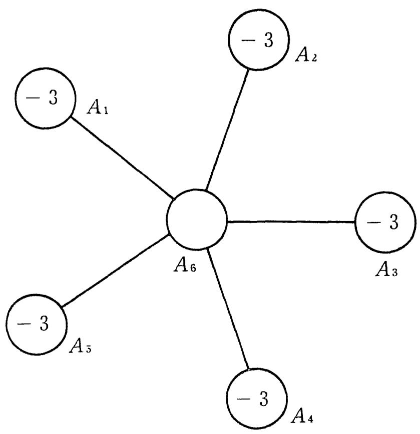
(*A*), *Step* 2. Blow up V_1 at the five points p_2, p_3, p_4, p_5, p_6, say the blowing-up ψ_i: $V_i\to V_{i-1}$ with center m_{p_i}, $2\leqq i\leqq 6$. In the 6-th stage, the singular locus $\mathrm{Sing}(V_6)$ of V_6 equals $|\Theta_1^{(6)} \cap V_6|$ which is isomorphic to $\boldsymbol{P}^1$. The multiplicity of V_6 at the points in $\mathrm{Sing}(V_6)$ is two.

(*A*), *Step* 3. Blow up V_6 along $\mathrm{Sing}(V_6)$ (with reduced center), say ψ_7: V_7

$\to V_6$. Then the composition of ψ_i $1 \leqq i \leqq 7$, gives a resolution ψ_A: $V_7 \to V$.

(B), *Step* 2. Blow up V_1 along $|H_1 \cap V_1|$ (with reduced center), say ϕ: $V_B \to V_1$. Then the composition of ψ_1 and ϕ gives a resolution ψ_B: $V_B \to V$.

These resolutions have the same dual graph of the exceptional sets as follows:



The fundamental cycle Z_0 is $A_1+A_2+A_3+A_4+A_5+3A_6$.
$p_a(Z_0)=2$ and $p_a(V, p)=2$.

W_i, and W_B (the divisor on V_B defined by $\phi^{-1}(I_{|\Theta_1^{(1)} \cap H_1|})$) are written as follows:

$$W_1 = A_1 + A_2 + A_3 + A_4 + A_5 + 3A_6,$$
$$W_i = A_{i-1}, \quad 2 \leqq i \leqq 6,$$
$$W_7 = 2A_6,$$
$$W_B = A_1 + A_2 + A_3 + A_4 + A_5 + 2A_6.$$

The numerical data ρ_i, ρ_B, r_7, and r_B are as follows: $\rho_i=3$ for $1 \leqq i \leqq 6$, $\rho_7=2$, $\rho_B=2$,

$$r_7 = \sum_{j \leqq 6} \frac{\rho_j W_7 \cdot W_j}{\rho_7} = 12, \quad \text{and} \quad r_B = \frac{\rho_1 W_B \cdot W_1}{\rho_B} = -3.$$

Let us consider the differences of the Euler-Poincaré characteristics of O_{V_i} and O_{V_B}.

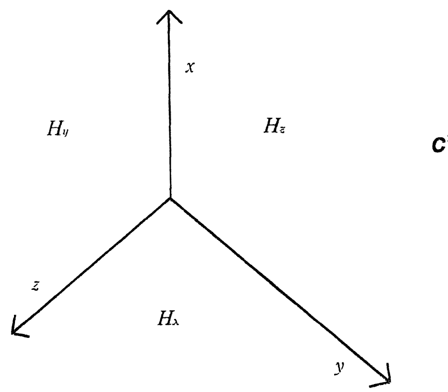
Step 1. $\chi(O_{V_1}) - \chi(O_V) = -1$. ($A$), *Step* 2. $\chi(O_{V_i}) - \chi(O_{V_{i-1}}) = -1$ for $2 \leqq i \leqq 6$. (A), *Step* 3. $\chi(O_{V_7}) - \chi(O_{V_6}) = 3$. ($B$), *Step* 2. $\chi(O_{V_B}) - \chi(O_{V_1}) = -2$.

Hence $p_g(V, p)=3$. The Euler-Poincaré characteristics for the resolution (A) oscillate.

Example (2.10). Let $(\bar{V}, p)$ be a complete two-dimensional scheme $\bar{V}$ over

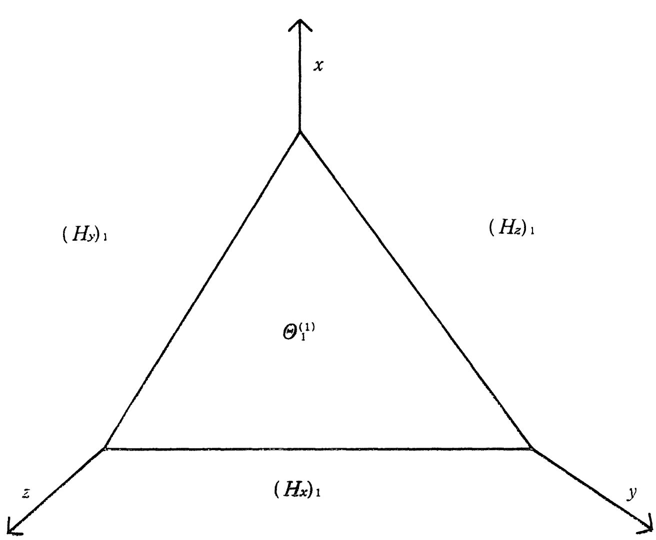
$\boldsymbol{C}$ with a reference point p in which the isomorphism $(\bar{V}, p) \cong (\{(x, y, z) \in \boldsymbol{C}^3 \mid x^8 + y^8 + z^8 + x^2y^2z^2 = 0\}, o)$ exists. Let us construct a resolution of (V, p) by a composition of blowing-ups with permissible centers and compute $p_g(V, p)$ from (2.5) without using Lemma (2.6).

We denote the coordinate system (x, y, z) by the following figure:



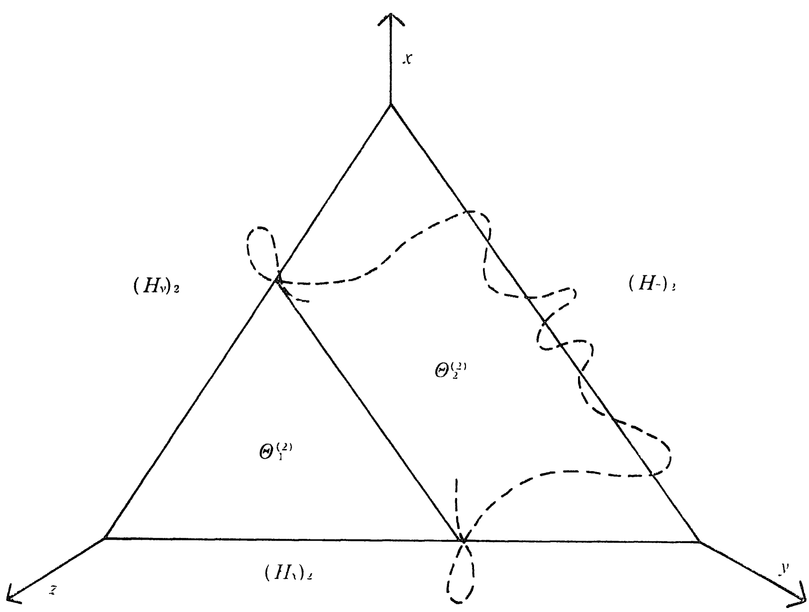
Here, H_x (resp. H_y, resp. H_z) is the hyperplane in $\boldsymbol{C}^3$ defined by x (resp. y, resp. z). Consider $(\bar{V}, p)$ only as the germ $(\{x^8 + y^8 + z^8 + x^2y^2z^2 = 0\}, o)$ below.

Step 1. Blow up V at p, say $\psi_1: V_1 \to V$. From (i) of (2.5), $\chi(\mathcal{O}_{V_1}) - \chi(\mathcal{O}_V) = -\frac{1}{6} \cdot 6 \cdot (6-1) \cdot (6-2) = -20$ follows. The singular locus $\mathrm{Sing}(V_1)$ of V_1 equals the union of three $\boldsymbol{P}^1$'s $|\Theta_1^{(1)} \cap (H_x)_1| \cup |\Theta_1^{(1)} \cap (H_y)_1| \cup |\Theta_1^{(1)} \cap (H_z)_1|$.



The multiplicity of V_1 at the points in $\mathrm{Sing}(V_1)$ is two.

Step 2. Set $\Gamma_2=|\Theta_1^{(1)}\cap(H_z)_1|$ as the center of the blowing-up of V_1, say $\psi_2\colon V_2\to V_1$.



The dotted curve in $\Theta_2^{(2)}$ is E_2.

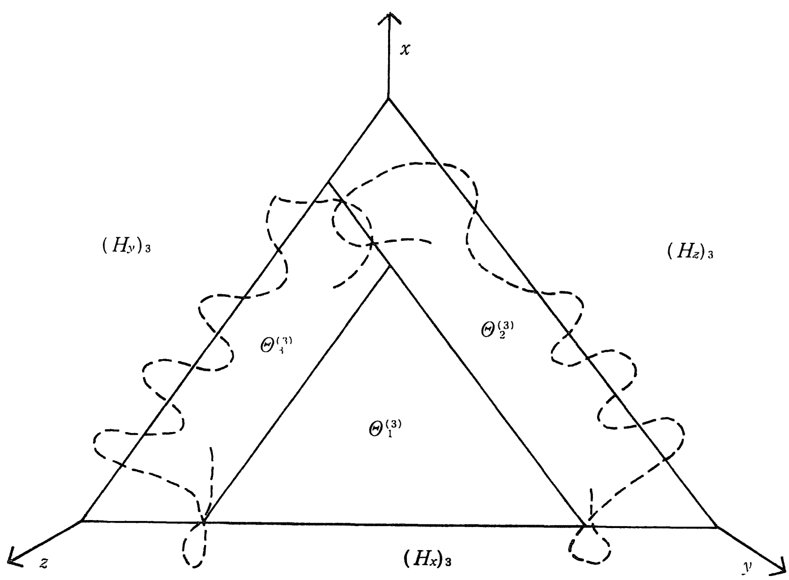
Let us employ the notations in (2.5). Since $\rho_2=2$ and $g_2=0$, $\chi(O_{\tilde V_2})-\chi(O_{\tilde V_1}) = -\frac{1}{4}(E_2)^2+1$ follows. Compute the integer $(E_2)^2$ as follows: Let D_2 be the divisor on $\Theta_2^{(2)}$ defined by $|\Theta_2^{(2)}\cap(H_z)_2|$. Let f_2 be the fiber class of $\psi_2\colon\Theta_2^{(2)}\to\Gamma_2$ in $\mathrm{Num}(\Theta_2^{(2)})$. We have the decomposition $\mathrm{Num}(\Theta_2^{(2)})=\boldsymbol{Z}\cdot D_2\oplus\boldsymbol{Z}\cdot f_2$ with the equality $D_2\cdot f_2=1$. Note the decomposition

$$I_{\Gamma_2}/(I_{\Gamma_2})^2\cong O_{\Gamma_2}((N_{\Gamma_2|\Theta_1^{(1)}})^*)\oplus O_{\Gamma_2}((N_{\Gamma_2|(H_z)_1})^*)$$

on Γ_2. D_2 is the divisor on $\Theta_2^{(2)}=\boldsymbol{P}((I_{\Gamma_2}/(I_{\Gamma_2})^2)^*)$ corresponding to $N_{\Gamma_2|(H_z)_1}$ in the terminologies of [13]. By (1.7) [13], the equalities $N_{D_2|\Theta_2^{(2)}}=N_{\Gamma_2|\Theta_1^{(1)}}\oplus((N_{\Gamma_2|(H_z)_1})^*)$ and $(D_2)^2=$ *degree of* $N_{D_2|\Theta_2^{(2)}}$ *over* $\boldsymbol{P}^1=2$ hold. Thus we can write $E_2\equiv 2\cdot D_2+b_2\cdot f_2$ in $\mathrm{Num}(\Theta_2^{(2)})$ for some integer b_2. In addition, by using the explicit defining equation, we can easily check the equality $E_2\cdot D_2=8$. Hence $b_2=4$, $(E_2)^2=24$, and $\chi(O_{\tilde V_2})-\chi(O_{\tilde V_1})=-5$.

The singular locus $\mathrm{Sing}(V_2)$ of V_2 equals the union of two $\boldsymbol{P}^1$'s, $|\Theta_1^{(2)}\cap(H_x)_2|\cup|\Theta_1^{(2)}\cap(H_y)_2|$. The multiplicity of V_2 at points in $\mathrm{Sing}(V_2)$ is two.

Step 3. Set $\Gamma_3 = |\Theta_1^{(2)} \cap (H_y)_2|$ as the center of the blowing-up of V_2, say $\psi_3\colon V_3 \to V_2$. As in Step 2, we decompose $\mathrm{Num}(\Theta_3^{(3)})$ by $\mathrm{Num}(\Theta_3^{(3)}) = \boldsymbol{Z}\cdot D_3 \oplus \boldsymbol{Z}\cdot f_3$. Here D_3 is the divisor defined by $|\Theta_3^{(3)} \cap (H_y)_3|$, and f_3 the fiber class of $\psi_3\colon \Theta_3^{(3)} \to \Gamma_3$.

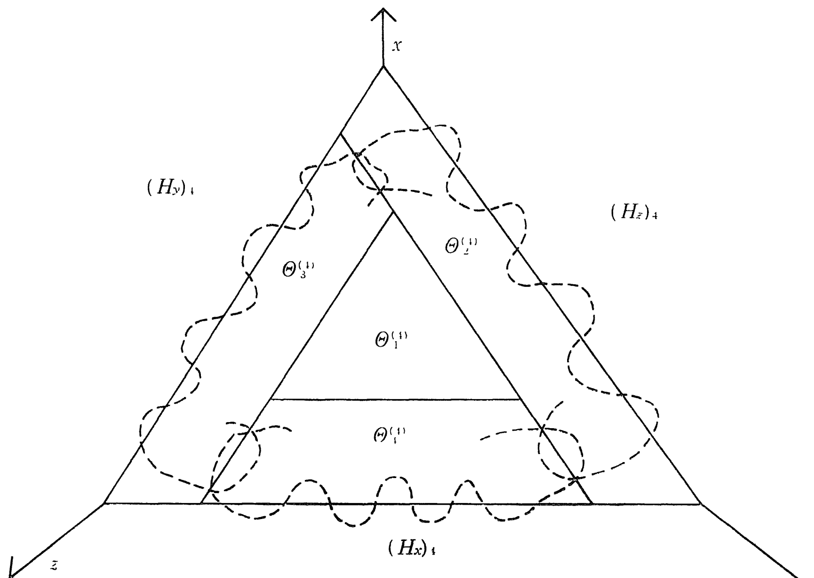


The dotted curve in $\Theta_3^{(3)}$ is E_3.

By using similar methods to Step 2, we can obtain $(D_3)^2 = 3$ and $D_3 \cdot E_3 = 8$. Hence $E_3 \equiv 2\cdot D_3 + 2\cdot f_3$ in $\mathrm{Num}(\Theta_3^{(3)})$ and $\chi(O_{\overline{V}_3}) - \chi(O_{\overline{V}_2}) = -4$.

The singular locus $\mathrm{Sing}(V_3)$ of V_3 equals $|\Theta_1^{(3)} \cap (H_x)_3|$. The multiplicity of V_3 at points in $\mathrm{Sing}(V_3)$ is two.

Step 4. Set $\Gamma_4 = |\Theta_1^{(3)} \cap (H_x)_3|$ as the center of the blowing-up of V_3, say $\psi_4\colon V_4 \to V_3$. As in Step 2, we decompose $\mathrm{Num}(\Theta_4^{(4)})$ by $\mathrm{Num}(\Theta_4^{(4)}) = \boldsymbol{Z}\cdot D_4 \oplus \boldsymbol{Z}\cdot f_4$. Here D_4 is the divisor defined by $|\Theta_4^{(4)} \cap (H_x)_4|$, and f_4 the fiber class of $\psi_4\colon \Theta_4^{(4)} \to \Gamma_4$.



The dotted curve in $\Theta_4^{(4)}$ is E_4.

By repeating Step 2, we can obtain $(D_4)^2=4$ and $D_4 \cdot E_4=8$. Hence $E_4 \equiv 2D_4$ in $\mathrm{Num}(\Theta_4^{(4)})$ and $\chi(O_{\bar{V}_4})-\chi(O_{\bar{V}_3})=-3$.

V_4 is non-singular. Hence, the composition of $\psi_1,\ldots,\psi_4$ gives a resolution $V_4 \to V$. We have $p_g(V, p)=32$.

In (2.10), fortunatelly, we have a very explicit decomposition of $I_\Gamma/(I_\Gamma)^2$ which is useful in computing the integer $(E)^2$.

However, in general, we can not expect section D of Θ to be as good as it is above. Therefore we must try to compute p_g by using Lemma (2.6). Then the problem is deduced to the computation of the intersection numbers $(W_i \cdot W_j)$ and the genus of the centers of blowing-ups.

Remark (2.11). There is a different method to compute p_g of the hypersurface two-dimensional singularity in terms of numerical data appearing in embedded resolutions. This is the method to compute the Milnor number (e.g., by the method [10]) and to relate it to p_g by Laufer's formula [27].

§3. Maximal Ideal Cycles

(3.1) Let (V, p) be a two-dimensional isolated singularity and $\psi: (\tilde{V}, A) \to (V, p)$ a resolution of (V, p) with the decomposition into the irreducible components of the exceptional set $A=\bigcup_{j=1}^{m} A_j$. The *maximal ideal cycle* for the resolution ψ is the divisor Y_ψ on $\tilde{V}$ defined by

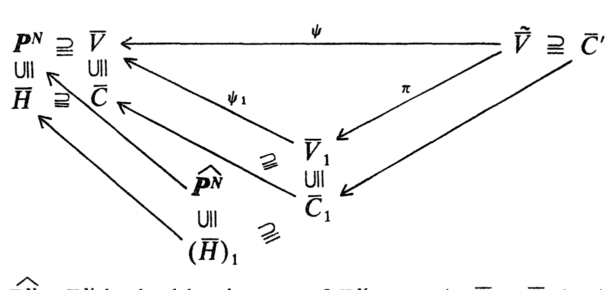
$$Y_\psi=\sum_{j=1}^{m}\left(\min_{f\in m_{V,p}} v_j(f)\right)A_j,$$

where v_j is the valuation on $O_{V,p}$ defined by $v_j(f)=$ the vanishing order of $\psi^*(f)$ on A_j for $f\in O_{V,p}$, $1\leqq j\leqq m$ (cf. (1.4) of [43]). In this section we shall study the virtual arithmetic genus $p_a(Y_\psi)$. Let us discuss the relation between the integer $p_a(Y_\psi)$ and the generic section of (V, p) by the hyperplane from the ambient space (3.4). This is a composition of (1.3) and the results by P. Wagreich and H. B. Laufer (see (3.5)). By noting the basic properties for one-dimensional Cohen-Macaulay local rings, we can obtain the non-negativity of $p_a(Y_\psi)$ for the normal two-dimensional singularity (V, p), which is similar to the non-negativity of $p_a(\boldsymbol{Z}_0)$ for the fundamental cycle $\boldsymbol{Z}_0$ ([3], [28]).

Throughout this section, let us assume that the base field k is infinite (see (3.3)) and that $O_{V,p}$ has at least one non-zero divisor in $m_{V,p}$. Hence (V, p) is a reduced isolated singularity (cf. [32]).

(3.2) Let us consider the following situation:

Let (V, p) be a two-dimensional isolated singularity. There is a compact projective algebraic surface $\bar{V}$ such that V is an open set of $\bar{V}$, $\bar{V}$ is embedded in $\boldsymbol{P}^N$ and that $\bar{V}-\{p\}$ is non-singular. Take a resolution $\psi\colon(\tilde{\bar{V}}, A)\to(\bar{V}, p)$ such that $\psi^{-1}(m_{V,p})$ is an invertible $O_{\tilde{\bar{V}}}$ ideal sheaf. Let $\bar{H}$ be a hyperplane in $\boldsymbol{P}^N$ containing p and set $\bar{C}=\bar{H}\cap\bar{V}$. Let $\bar{C}'$ be the divisor on $\tilde{\bar{V}}$ defined by the part of the divisor $\psi^{-1}(\bar{C})$ off A. By the universal mapping property of blowing-up, we have the following commutative diagram:



Here $\Phi\colon\widehat{\boldsymbol{P}^N}\to\boldsymbol{P}^N$ is the blowing-up of $\boldsymbol{P}^N$ at p, $\phi\colon\bar{V}_1\to\bar{V}$ the blowing-up of $\bar{V}$ at p, and $\bar{C}_1$ (resp. $\bar{H}_1$) the strict transform of $\bar{C}$ (resp. $\bar{H}$). The other arrows are induced morphisms.

Let us assume the following three conditions:

(3.2.1) *The equality* $\psi^{-1}(\mathscr{I}_{\bar{C}})=\mathscr{I}_{\bar{C}'}\psi^{-1}(m_{V,p})$ *holds.*

(3.2.2) $\bar{C}$ *has an isolated singularity at* p.

(3.2.3) *The equalities* $\operatorname{mult}_p \bar{C}=\operatorname{mult}_p \bar{V}$ *and* $\operatorname{embdim}_p \bar{C}=\operatorname{embdim}_p \bar{V}-1$ *hold.*

(3.3) From the arguments below, sufficiently many hyperplanes H may satisfy three conditions above.

First of all, choose a system of hyperplanes $\{H_1,\dots, H_M\}$ which satisfy the following two conditions: We denote the defining equation of H_i in $O_{P^N,p}$ by h_i.

(3.3.1) $(h_1,\dots, h_M)\cdot O_{\bar{V},p}=m_{V,p}$.

(3.3.2) $\min\limits_{1\leq i\leq M} v_{A_j}(h_i)=\min\limits_{f\in m_{V,p}} v_{A_j}(f)$ holds for any irreducible component A_j of A $(A=\bigcup\limits_{j=1}^{w} A_j)$. Here v_{A_j} is the discrete valuation defined by $O_{\bar{V},A_j}$, $1\leqq j\leqq w$ (see (1.4) of [43]).

(3.3.3) *There is a non-empty Zariski open set* T_1 *in* k^M *such that the equality* $v_{A_j}(\sum\limits_{i=1}^{M} a_ih_i)=\min\limits_{f\in m_{V,p}} v_{A_j}(f)$ *holds for any* A_j *and for any* $(a_i)\in T_1$.

We denote the hyperplane associated to $\sum\limits_{i=1}^{M} a_ih_i$ by $\bar{H}_{\boldsymbol{a}}$, where $\boldsymbol{a}=(a_1,\dots, a_M)\in k^M$. It is easily seen that the hyperplane $\bar{H}_{\boldsymbol{a}}$ satisfies (3.2.1) for $\boldsymbol{a}\in T_1$.

For (3.2.2), we note the following:

Theorem (3.3.4) (Flenner, Tessier, Bruns, Satz (4.1) [12]). *There is a non-empty Zariski open set* T_2 *in* k^M *such that the relation* $\operatorname{Sing}(H_{\boldsymbol{a}}\cap \bar{V})\subseteqq H_{\boldsymbol{a}}\cap \operatorname{Sing}(\bar{V})$ *at* p *for* $\boldsymbol{a}\in T_2$ *holds.*

For (3.2.3) it is sufficient to see the following:

(3.3.5) *There is a non-empty Zariski open set* T_3 *in* k^M *such that* $\sum\limits_{i=1}^{M} a_ih_i$ *defines a superficial element of* $O_{\bar{V},p}$ *for* $\boldsymbol{a}\in T_3$ (i.e., it defines non-zero element $\overline{\sum\limits_{i=1}^{M} a_ih_i}$ of $gr_m^1(O_{\bar{V},p})$ and there is an integer c such that $(0\colon (\overline{\sum a_ih_i})gr_m(O_{\bar{V},p}))\cap gr_m^k(O_{\bar{V},p})=0$ for $k\geqq c$).

For the properties of superficial element, see §3, Chapter 1 [38].

We may assume that the elements $\sum a_ih_i$ above are not contained in any associated prime of $O_{\bar{V},p}$ (i.e., it defines a non-zero divisor of $O_{\bar{V},p}$).

Note, by (3.2.2), that $\bar{C}_1$ is reduced and so the natural morphism $O_{\bar{C}_1}\to\pi_*(O_{\bar{C}'})$ is injective.

Proposition (3.4). *Let us assume the conditions* (3.2.1), (3.2.2) *and* (3.2.3) *for the hyperplane $\bar{H}$. Suppose h defines a non-zero divisor of $O_{\bar{V},p}$. Then the following equality holds.*

$$p_a(Y_\psi)=\chi(\pi_*(O_{\bar{C}'})/O_{\bar{C}_1})+\sum_{k\geqq 1}\{P_{(\bar{C},p)}(k)-H_{(\bar{C},p)}(k)\}.$$

Here $H_{(\bar{C},p)}(h)=\dim_k(m_{\bar{C},p})^h/(m_{\bar{C},p})^{h+1}$ is the Hilbert-Samuel function of $O_{\bar{C},p}$ and $P_{(\bar{C},p)}(h)$ the Hilbert-Samuel polynomial associated to $H_{(\bar{C},p)}$ (hence $P_{(\bar{C},p)}(h)$ $=\mathrm{mult}_p\,\bar{C}$ for $h\geqq 0$).

Proof. First we shall show the following equality: $\chi(O_{\bar{C}'})-\chi(O_{\bar{C}})=-\chi(\mathscr{I}_{\bar{C}'}\otimes O_{Y_\psi})$. By (3.2.1), we have

$$0\longrightarrow \psi^{-1}\mathscr{I}_{\bar{C}}\longrightarrow \mathscr{I}_{\bar{C}'}\longrightarrow \mathscr{I}_{\bar{C}'}\otimes O_{Y_\psi}\longrightarrow 0.$$

Combining with

$$0\longrightarrow \mathscr{I}_{\bar{C}'}\longrightarrow O_{\tilde{\bar{V}}}\longrightarrow O_{\bar{C}'}\longrightarrow 0,$$

we have the equalities $\chi(O_{\bar{C}'})=\chi(O_{\tilde{\bar{V}}})-\chi(\mathscr{I}_{\bar{C}'})=\chi(O_{\tilde{\bar{V}}})-\chi(\psi^{-1}(\mathscr{I}_{\bar{C}}))-\chi(\mathscr{I}_{\bar{C}'}\otimes O_{Y_\psi})$. By Leray's spectral sequence for ψ, we obtain $\chi(O_{\tilde{\bar{V}}})-\chi(O_{\bar{V}})=\sum_{q\geqq 0}\cdot(-1)^q\chi(R^q\psi_*O_{\tilde{\bar{V}}})-\chi(O_{\bar{V}})$.

The projection formula induces the equality at p;

$$R^q\psi_*(O_{\tilde{\bar{V}}})\cong\mathscr{I}_{\bar{C}}\otimes R^q\psi_*(O_{\tilde{\bar{V}}})\cong R^q\psi_*(\psi^{-1}(\mathscr{I}_{\bar{C}}))\quad\text{for}\quad q\geqq 0.$$

Tensoring $\mathscr{I}_{\bar{C}}$ to the usual Ker-Coker exact sequence

$$0\longrightarrow \mathscr{K}\longrightarrow O_{\bar{V}}\longrightarrow \psi_*(O_{\tilde{\bar{V}}})\longrightarrow \mathscr{H}\longrightarrow 0,$$

we obtain the following equality at p:

$$\begin{array}{ccccccccc}0\longrightarrow & \mathscr{I}_{\bar{C}}\otimes\mathscr{K} & \longrightarrow \mathscr{I}_{\bar{C}}\longrightarrow & \psi_*(\psi^{-1}(\mathscr{I}_{\bar{C}})) & \longrightarrow & \mathscr{I}_{\bar{C}}\otimes\mathscr{H} & \longrightarrow 0.\\ & \wr\wr & & & & \wr\wr & \\ & \mathscr{K} & & & & \mathscr{H} & \end{array}$$

Since the supports of sheaves $R^q\psi_*(O_{\tilde{\bar{V}}})$ $(q\geqq 1)$, $\mathscr{K}$ and $\mathscr{H}$ are contained in $\{p\}$, we have the following equalities: $\chi(R^q\psi_*(O_{\tilde{\bar{V}}}))=\chi(R^q\psi_*(\psi^{-1}(\mathscr{I}_{\bar{C}})))$ $(q\geqq 1)$. $\chi(\psi_*(O_{\tilde{\bar{V}}}))-\chi(O_{\bar{V}})=\chi(\psi_*(\psi^{-1}(\mathscr{I}_{\bar{C}})))-\chi(\mathscr{I}_{\bar{C}})$. Furthermore Leray's spectral sequence induces the following equality:

$$\chi(\psi^{-1}(\mathscr{I}_{\bar{C}}))-\chi(\mathscr{I}_{\bar{C}})=\sum_{q\geqq 0}(-1)^q\chi(R^q\psi_*(\psi^{-1}(\mathscr{I}_{\bar{C}})))-\chi(\mathscr{I}_{\bar{C}}).$$

Combining the equalities above, we obtain $\chi(O_{\tilde{\bar{V}}})-\chi(O_{\bar{V}})=\chi(\psi^{-1}(\mathscr{I}_{\bar{C}}))-\chi(\mathscr{I}_{\bar{C}})$ and $\chi(O_{\bar{C}'})=\chi(O_{\bar{C}})-\chi(\mathscr{I}_{\bar{C}'}\otimes O_{Y_\psi})$.

By a characteristic of the intersection number of Cartier divisors (Kleiman

[25]), we have the following equalities: $-(Y_\psi)^2 = Y_\psi \circ \bar{C}' = \chi(O_{\tilde{V}}) - \chi(\mathscr{I}_{\bar{C}'}) - \chi(\mathscr{I}_{Y_\psi}) + \chi(\mathscr{I}_{\bar{C}'} \cdot \mathscr{I}_{Y_\psi}) = \chi(O_{Y_\psi}) - \chi(\mathscr{I}_{\bar{C}'} \otimes O_{Y_\psi})$. Hence we obtain the following equality:

$$\chi(O_{\bar{C}'}) - \chi(O_{\bar{C}}) = -\chi(O_{Y_\psi}) - (Y_\psi)^2. \tag{3.4.1}$$

By (1.3), (3.4.1) can be written as follows:

$$\begin{aligned} p_a(Y_\psi) - (Y_\psi)^2 - 1 &= \chi(O_{\bar{C}'}) - \chi(O_{\bar{C}_1}) + \chi(O_{\bar{C}_1}) - \chi(O_{\bar{C}}) \\ &= \chi(O_{\bar{C}'}) - \chi(O_{\bar{C}_1}) + \sum_{k \geq 0} \{P_{(\bar{C},p)}(k) - H_{(\bar{C},p)}(k)\}. \end{aligned}$$

By Leray's spectral sequence for π, $\chi(O_{\bar{C}'}) - \chi(O_{\bar{C}_1}) = \chi(\pi_*(O_{\bar{C}'})/O_{\bar{C}_1})$. By (3.2.3), $P_{(\bar{C},p)}(k) = \mathrm{mult}_p \bar{C} = \mathrm{mult}_p \bar{V}$. By Wagreich's theorem (Theorem (2.7) [46], see also our proof of (2.6)), $\mathrm{mult}_p \bar{V} = -(Y_\psi)^2$. Note the equality $H_{(\bar{C},p)}(0) = 1$. Therefore we obtain the desired equality from the equalities above.

Remark (3.5). When the characteristic of the base field k is zero, we may choose $\bar{H}$ such that $\bar{C}'$ is regular by the Bertini second theorem on $\bar{\bar{V}}$. Then the left side of (3.4.1) is the conductor number of $(\bar{C}, p)$; we shall write it $\delta(\bar{C}, p)$. From the Riemann-Roch theorem on $\bar{\bar{V}}$, the equality (3.4.1) can be written as follows:

$$\delta(\bar{C}, p) = \frac{K_{\bar{\bar{V}}} \cdot Y_\psi - (Y_\psi)^2}{2}.$$

Here $K_{\bar{\bar{V}}}$ is the canonical divisor of $\bar{\bar{V}}$. In this form, the equality (3.4.1) had already been announced by H. B. Laufer [29], and is proved by J. Giraud [53] and M. Morales [54].

Remark (3.6). In the equality (3.4), the isolatedness of the singularity $(\bar{V}, p)$ is essential, as can be seen in the following: Let $(\bar{V}, p)$ be a reduced (not necessarily isolated) singularity. Then we can prove the following equality easily as in (3.4):

$$\begin{aligned} p_a(Y_\psi) = \chi(\pi_*(O_{\bar{C}'})/O_{\bar{C}_1}) &+ \sum_{k \geq 1} \{P_{(\bar{C},p)}(k) - H_{(\bar{C},p)}(k)\} \\ &- \chi^{O_{P^N}}(O_{\bar{H}}, \psi_*(O_{\bar{\bar{V}}})/O_{\bar{V}}), \end{aligned}$$

where the integer $\chi^A(B, C)$ is the intersection number in the sense of Serre [41]. Here we have the inequality $\chi^{O_{P^N}}(O_{\bar{H}}, \psi_*(O_{\bar{\bar{V}}})/O_{\bar{V}}) \geq 0$. The equality holds if and only if (V, p) is an isolated singularity (see [41]).

(3.7) Next let us discuss the relation between $p_a(Y_\psi)$ and $p_a(Y_\phi)$ for two different resolutions ψ and ϕ. Let us consider the following commutative diagram:

$$\begin{array}{ccc} (V,p) & \xleftarrow{\quad\psi\quad} & (\tilde{V},A) \\ & {\scriptstyle\phi}\nwarrow \quad \nearrow{\scriptstyle\tau} & \\ & (V',A') & \end{array} .$$

Here ψ: $(\tilde{V}, A)\to(V, p)$ (resp. ϕ: $(V', A')\to(V, p)$) is a resolution of (V, p) with the exceptional set A (resp. A'), and τ is the holomorphic birational map such that $\phi=\psi\circ\tau$. Let $A=\bigcup_{j=1}^{m} A_j$ (resp. $A'=\bigcup_{j=1}^{m'} A'_j$) be the decomposition of the exceptional set A (resp. A') into the irreducible components. Assume A'_j is the strict transform of A_j by τ, for $j=1,\dots, m$. Let D be the divisor on V' of the form $D=\sum_{j=1}^{m'} d'_j\cdot A'_j$, where $d'_j\in \boldsymbol{Z}$ for $j=1,\dots, m'$. Then we shall denote the divisor $\sum_{j=1}^{m} d'_j\cdot A_j$ on $\tilde{V}$ by $\tau_*(D)$.

Lemma (3.8). *Let D be an effective divisor on V', whose support is contained in A'. Then the following equality holds*:

$$p_a(D)=p_a(\tau_*(D))-\dim R^1\tau_*(\mathscr{I}_D)-\dim(\mathscr{I}_{\tau_*(D)}/\tau_*(\mathscr{I}_D)).$$

Proof. Taking the direct image τ_* of

$$0\longrightarrow\mathscr{I}_D\longrightarrow O_{V'}\longrightarrow O_D\longrightarrow 0,$$

we obtain

$$\begin{aligned}0\longrightarrow \tau_*(\mathscr{I}_D)\longrightarrow O_{\tilde{V}}\longrightarrow \tau_*(O_D)\\ \longrightarrow R^1\tau_*(\mathscr{I}_D)\longrightarrow R^1\tau_*(O_{V'})\longrightarrow R^1\tau_*(O_D)\longrightarrow 0.\end{aligned}$$

Here note $R^1\tau_*(O_{V'})=0$. Hence we obtain

$$\begin{aligned}p_a(D)&=1-\chi(O_D)=1-\chi(\tau_*(O_D))-\chi(R^1\tau_*(O_D))\\ &=1-\chi(O_{\tilde{V}}/\tau_*(\mathscr{I}_D))-\dim R^1\tau_*(\mathscr{I}_D).\end{aligned}$$

By noting the relations $\tau_*(\mathscr{I}_D)\subseteqq\mathscr{I}_{\tau_*(D)}$ and $p_a(\tau_*(D))=1-\chi(O_{\tilde{V}}/\mathscr{I}_{\tau_*(D)})$, we obtain the desired equality. Q.E.D.

Proposition (3.9). *Let the situation be as above. (i) The inequality $p_a(Y_\phi)\leqq p_a(Y_\psi)$ holds. (ii) Assume that $\phi^{-1}(m_{V,p})$ is $O_{V'}$-invertible. Then the equality $p_a(Y_\phi)=p_a(Y_\psi)$ holds if and only if $\psi^{-1}(m_{V,p})$ is $O_{\tilde{V}}$-invertible.*

Proof. (i) We have the relation $\tau_*(Y_\phi)=Y_\psi$. Hence the assertion is clear by (3.8).

(ii) We have the relation $Y_\phi \cdot A'_j \leqq 0$ for $j=1,\ldots, m'$ (see Section 1 of [43], (3.2.1)). By Theorem 12 of [30], we have the vanishing $R^1\tau_*(\mathscr{I}_{Y_\phi})=0$. Hence by (3.8), $p_a(Y_\phi)=p_a(Y_\psi)-\dim(\mathscr{I}_{Y_\psi}/\tau_*(\phi^{-1}(m_{V,p})))$. If $p_a(Y_\phi)=p_a(Y_\psi)$, then $\mathscr{I}_{Y_\psi} =\tau_*(\phi^{-1}(m_{V,p}))=\tau_*(\tau^{-1}(\psi^{-1}(m_{V,p})))$ holds. Hence $\tau^{-1}(\mathscr{I}_{Y_\psi})=\tau^{-1}(\tau_*(\tau^{-1}(\psi^{-1}(m_{V,p}))))=\tau^{-1}(\psi^{-1}(m_{V,p}))$. By Lemma (5.3) of [46], the equality $\mathscr{I}_{Y_\psi}=\psi^{-1}(m_{V,p})$ follows.

In general we have the relation $\psi^{-1}(m_{V,p})\subseteqq\tau_*(\tau^{-1}(\psi^{-1}(m_{V,p})))\subseteqq\mathscr{I}_{Y_\psi}$. Hence the converse is clear. Q. E. D.

Corollary (3.10). *Let $\psi\colon(\tilde{V}, A)\to(V, p)$ be a resolution of a normal two-dimensional singularity (V, p).*

(i) $\psi^{-1}(m_{V,p})$ is $O_{\tilde{V}}$-invertible if and only if the integer $p_a(Y_\psi)$ is the minimum among the set of integers $\{p_a(Y_\phi)\,|\,\phi\colon(V', A')\to(V, p)$ a resolution of singularity $(V, p)\}$.

(ii) Suppose that $\psi^{-1}(m_{V,p})$ is $O_{\tilde{V}}$-invertible and that the equality $p_a(Y_\psi) =p_a(V, p)$ holds. Then $\phi^{-1}(m_{V,p})$ is $O_{V'}$-invertible for any resolution $\phi\colon(V', A')\to(V, p)$.

Proof. (i) is obvious from (3.9). (ii) In this case, the set $\{p_a(Y_\phi)\,|\,\phi\colon(V', A')\to(V, p)$ a resolution of $(V, p)\}$ coincides with $\{p_a(V, p)\}$ by (i) and the definition of $p_a(V, p)$. The assertion follows from (i). Q. E. D.

(3.11) We shall remark the non-negativity of the integer $p_a(Y_\psi)$ below.

Let (V, p) be a normal two-dimensional singularity. In particular $O_{V,p}$ is Cohen-Macaulay. $O_{C,p}$ is also Cohen-Macaulay if and only if h defines a non-zero divisor of $O_{V,p}$. We must remark the following facts on "*the boundedness of number of generators of ideals of Cohen-Macaulay local rings*".

Theorem (3.11.1) (see, e.g., Chapter III [38]). *Let (R, m) be a one-dimensional Cohen-Macaulay local ring with maximal ideal m. Then the following inequalities hold:*

$$\dim_{R/m}(m^k/m^{k+1})\leqq\mathrm{mult}_m R \qquad \textit{for} \quad k\geqq 0.$$

Theorem (3.11.2) (Abhyanker [1], [37]). *Let (R, m) be a Cohen-Macaulay local ring with maximal ideal m. Then the following inequality holds:*

(3.11.2.1) $$\mathrm{embdim}\, R\leqq\mathrm{mult}_m R+\dim R-1.$$

Theorem (3.11.3) (Sally [39]). *If the equality holds in* (3.11.2.1), *then* *the Cohen-Macaulay type of* (R, m) *equals* $\mathrm{mult}_m R-1$. *In particular, if* (R, m) *is Gorenstein of* $\mathrm{mult}_m R \geqq 3$, *the following inequality holds*:

(3.11.3.1) $$\mathrm{embdim}\, R \leqq \mathrm{mult}_m R + \dim R - 2.$$

Corollary (3.12). *Let* $\psi: (\tilde{V}, A) \to (V, p)$ *be a resolution of a normal two-dimensional singularity. Then the inequality* $p_a(Y_\psi) \geqq 0$ *holds. In addition if* (V, p) *is Gorenstein of multiplicity* $\geqq 3$, *the inequality* $p_a(Y_\psi) \geqq 1$ *holds.*

Proof. If $\psi^{-1}(m_{V,p})$ is $O_{\tilde{V}}$-invertible, the inequalities follows from (3.4), (3.11.1), and (3.11.3). Hence by (i) of Corollary (3.10), the assertions follow. Q.E.D.

The reader may expect to have more precise results provided in Section 4.

Chapter II. Studies on the Normal Two-Dimensional Gorenstein Singularities with $p_a = 1$

§4. Decomposition of Zariski's Canonical Resolution

(4.1) Using the results of the previous sections, we shall prove the existence of a resolution with a special condition for the normal two-dimensional Gorenstein singularity with $p_a = 1$. Let (V, p) be a normal two-dimensional singularity. A resolution of the singularity (V, p) is obtained by the following process (due to Zariski, see e.g., [31]);

$\sigma_1: V_1 \to V$ the blowing-up of V at p,
$T_1: \tilde{V}_1 \to V_1$ the normalization of V_1,
$\sigma_2: V_2 \to \tilde{V}_1$ the blowing-up of $\tilde{V}_1$ at a point in the singular locus of $\tilde{V}_1$,
$T_2: \tilde{V}_2 \to V_2$ the normalization of V_2,
and so on.

Moreover this process ends in finite steps.

This resolution is called *Zariski's canonical resolution*. We shall simply refer to it as **Z.C.R.** in this paper. The singularity (V, p) is called "*absolutely isolated*" if all normalizations in **Z.C.R.** are trivial. It is well-known that the normal two-dimensional rational singularity is absolutely isolated (Lipman [30], Tjurina [42]). We shall prove the decomposition theorem of **Z.C.R.** in our situation by a composition of blowing-ups with smooth centers (4.6). First we shall show that (V, p) is Gorenstein of maximal embedding dimension or of multiplicity two. Then the tangent cone of the singularity is Gorenstein by Sally [39]. This fact is basic for our study. This section is the first step toward the theorems in Section 7 (The criterion for absolute isolatedness, etc).

Throughout this section, let us assume that the base field k is an algebraically closed (hence infinite) field.

Theorem (4.2). *Let (V, p) be a normal two-dimensional Gorenstein singularity of multiplicity $\geqq 3$ with $p_a(V, p)=1$.*

(i) (V, p) is Gorenstein of maximal embedding dimension (i.e., the equality holds in (3.11.3.1)), and the equalities $H_{(V,p)}(k)=\rho\cdot k$ for $k\geqq 1$ hold for some integer ρ.

(ii) Let $\psi_1: V_1\to V$ be the blowing-up of V with center $m_{V,p}$, then V_1 is normal Gorenstein.

Proof. (i) Let $\psi: \bar{\tilde{V}}\to\bar{V}$ be a resolution as in (3.2) and (3.3). The condition $p_a(V, p)=1$ implies the equality $p_a(Y_\psi)=1$ by (3.12). Hence by (3.4) and (3.11), we obtain the following equalities:

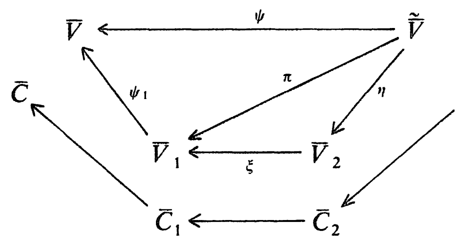
(4.2.1) $\chi(\pi_*(O_{\bar{C}'})/O_{\bar{C}_1})=0,$

(4.2.2) $P_{(\bar{C},p)}(k)=H_{(\bar{C},p)}(k)$ for $k\geqq 2$ and $P_{(\bar{C},p)}(1)=H_{(\bar{C},p)}(1)+1.$

The last equality of (4.2.2) means that $(\bar{C}, p)$ is Gorenstein of maximal embedding dimension. By the condition (3.2.3), (V, p) is also Gorenstein of maximal embedding dimension. The later assertion in (i) is due to Sally [40].

(ii) By the main theorem of Sally [39], $gr_m(O_V)$ is Gorenstein. Hence we can assume that the defining equation h of $\bar{H}$ defines an element $\bar{h}$ of $gr^1_m(O_V)$, which is a non-zero divisor of $gr_m(O_V)$. By (1.11), $\bar{V}$ and $\bar{H}$ intersects tangentially at p, and $\bar{C}_1$ is a locally principal divisor in V_1 with the scheme theoretic relation $\bar{C}_1=\bar{V}_1\cap\bar{H}_1$.

Next we shall show that $\bar{V}_1$ has only isolated singularities. Put $B=|\bar{V}_1\cap\Theta^{(1)}_1|$ (i.e., $|\psi_1^{-1}(p)|$ in $\bar{V}_1$). B is a one-dimensional reduced scheme. By the Bertini theorem over algebraically closed ground field (Theorem (8.18) and Remark (8.18.1) Chapter II [17]), we may assume that $B\cap\bar{H}_1$ is a finite union of reduced zero-dimensional points. Let us blow up $\bar{V}_1$ with center the ideal $\mathscr{I}_B$, which is the defining ideal of B, we shall write it as $\xi: \bar{V}_2\to\bar{V}_1$. Moreover we shall assume that $\pi^{-1}(\mathscr{I}_B)$ is an invertible $O_{\bar{\tilde{V}}}$-ideal. Now we can obtain the following diagram;



Here η is the map from the universality of the blowing-up ξ. The other morphisms are the induced morphisms. We know that $\bar{C}_2$ is also reduced and so that the following equalities hold:

(4.2.3) $$\chi(O_{\bar{C}'})-\chi(O_{\bar{C}_2})=0.$$

(4.2.4) $$\chi(O_{\bar{C}_2})-\chi(O_{\bar{C}_1})=0.$$

Let $|B\cap\bar{C}_1|$ be the set $\{q_1,\dots, q_s\}$. Since $B\cap\bar{C}_1=B\cap\bar{H}_1\cap\bar{V}_1=B\cap\bar{H}_1$, the morphism $\xi|_{\bar{C}_2}\colon\bar{C}_2\to\bar{C}_1$ is the blowing-up of $\bar{C}_1$ with center the product $\prod_{i=1}^{s} m_{q_i}$ of maximal ideals ([21]). By (4.2.4) and the formula (1.3), we obtain the equality $\sum_{i=1}^{s}[\sum_{k\geq 0}\{P_{(\bar{C}_1,q_i)}(k)-H_{(\bar{C}_1,q_i)}(k)\}]=0$. Since $(\bar{C}_1, q_i)$ is one-dimensional reduced (hence is Cohen-Macaulay), the equalities $P_{(\bar{C}_1,q_i)}(k)-H_{(\bar{C}_1,q_i)}(k)=0$ for $k\geqq 0$ follow for $i=1,\dots, s$ by (3.11.1). Hence $(\bar{C}_1, q_i)$ is regular for $i=1,\dots, s$. In general, we have relations $\dim\bar{V}_1=\dim\bar{C}_1+1$ and embdim $\bar{V}_1\leqq$ embdim $\bar{C}_1+1$ at any point of $\bar{C}_1$, since $\bar{C}_1$ is a locally principal divisor of $\bar{V}_1$ (see pp. 41–42 of [38]). Hence $(\bar{V}_1, q_i)$ is also regular for $i=1,\dots, s$. Noting that $\bar{H}_1$ intersects with all irreducible components of B, we may conclude that $\bar{V}_1$ is regular except at finite points over B, so that $\bar{V}_1$ has only isolated singularities.

In addition the Gorenstein-ness of $gr_m(O_V)$ implies the Gorenstein-ness of $\bar{V}_1$ (Theorem (5.1) [39], Lemma (5.1.10) [14]). Therefore $\bar{V}_1$ is normal Gorenstein by Serre's criterion (Theorem 39 [32]). Q. E. D.

Before we proceed to consider the case of multiplicity two, let us note the following:

Theorem (4.3). *Let (V, p) be a normal two-dimensional singularity. Let $\psi\colon(\tilde{V}, A)\to(V, p)$ be a resolution as in* (3.2). *Suppose that the equality $p_a(Y_\psi)=0$ holds. Then the following two statements are true:*

(*i*) *(V, p) is Cohen-Macaulay of maximal embedding dimension* (*i.e., the equality holds in* (3.11.2.1)) *and the equalities $H_{(V,p)}(k)=\rho k+1$ for $k\geqq 0$ hold for some integer ρ.*

(*ii*) *Let $V_1\to V$ be the blowing-up of (V, p); then V_1 is normal.*

Proof. Note, first of all, that the equality $p_a(Y_\mu)=0$ holds for any resolution $\mu: (V'', A'')\to(\bar{V}, p)$ as in (3.2) by (3.9). By (3.4) and (3.11), the equality $p_a(Y_\psi)=0$ implies the following equalities:

(4.3.1) $$\chi(\pi_*(O_{\bar{C}'})/O_{\bar{C}_1})=0,$$

(4.3.2) $$P_{(\bar{C},p)}(k)=H_{(\bar{C},p)}(k) \quad \text{for} \quad k\geqq 1.$$

The last equality means that $(\bar{C}, p)$ is Cohen-Macaulay of maximal embedding dimension. With the aid of Sally's results [39], [40], the remaining parts of proof can be said to be parallel with the proof of (4.2).

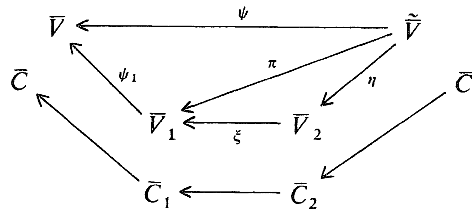
(4.4) Let (V, p) be a normal two-dimensional singularity of $\mathrm{mult}_p V=2$ with $p_a(V, p)=1$. Let $\psi_1: V_1\to V$ be the blowing-up of V with center $m_{V,p}$. Let us assume that V_1 is not normal. (V, p) is a hypersurface of (k^3, o) (by (3.11.2), see also [43]), hence the singular locus $\mathrm{Sing}(V_1)$ of V_1 is seen as a subvariety of $\boldsymbol{P}^2=\Theta_1^{(1)}$. The following fact has been proved for the complex analytic case by the author (in §§1 and 2 of [43]):

Theorem (4.5). *Let the situation be as above. (i) The one-dimensional part of $|\mathrm{Sing}(V_1)|$ is a straight line in $\boldsymbol{P}^2$, call it B, in the same sense. (ii) Let ψ_2: $V_2\to V_1$ be the blowing-up of V_1 with center the defining ideal $\mathscr{I}_B$ of B. Then V_2 is normal and ψ_2 is the normalization of V_1.*

Proof. (i) Let $\psi: \bar{\bar{V}}\to\bar{V}$ be a resolution as in (3.2) and (3.3). By (4.3), we have the inequality $p_a(Y_\psi)\geqq 1$. Hence the condition $p_a=1$ implies the equality $p_a(Y_\psi)=1$. We know the fact that $(\bar{C}, p)$ is a hypersurface of multiplicity two of (k^2, o) and so the equalities $2=P_{(C,p)}(k)=H_{(\bar{C},p)}(k)$ for $k\geqq 1$ hold. By the formula (3.4), we obtain the following equality:

(4.5.1) $$\chi(\pi_*(O_{C'})/O_{C_1})=1.$$

We can assume that $\bar{V}$ and $\bar{H}$ intersect tangentially at p, and that $\bar{C}_1$ is a Cartier divisor on $\bar{V}_1$ with the relation $\bar{C}_1=\bar{V}_1\cap\bar{H}_1$ (by the same argument as (4.2)). Let B be the one-dimensional part of $|\mathrm{Sing}(V_1)|$. Since $\bar{V}_1$ is a non-normal hypersurface, B is non-empty by Serre's criterion. By the Bertini theorem over algebraically closed ground field (Theorem (8.18) and Remark (8.18.1), Chapter II [17]), we can assume that $B\cap\bar{H}_1$ is a finite union of zero-dimensional reduced points. Blow up $\bar{V}_1$ with center the ideal $\mathscr{I}_B$, which is the defining ideal of B, we shall write it as $\xi: \bar{V}_2\to\bar{V}_1$. In addition, assume that $\pi^{-1}(\mathscr{I}_B)$ is an invertible $O_{\bar{\bar{V}}}$-ideal. Now we can obtain the following diagram as same as in (4.2):



the center of ξ is $\mathscr{I}_B$

We know that $\bar{C}_2$ is also reduced and therefore one of the following equalities (A), (B) holds:

(A) $\chi(O_{C'})-\chi(O_{C_2})=1$ and $\chi(O_{\bar{C}_2})-\chi(O_{C_1})=0$.

(B) $\chi(O_{C'})-\chi(O_{C_2})=0$ and $\chi(O_{\bar{C}_2})-\chi(O_{\bar{C}_1})=1$.

However, (A) does not occur (since if (A) holds, we can show that $\bar{V}_1$ is normal by the arguments in the proof of (4.2) (ii)). Hence we obtain (B).

Let $|B\cap\bar{C}_1|$ be the set $\{q_1,\dots,q_s\}$. Since $B\cap\bar{C}_1=B\cap\bar{H}_1\cap\bar{V}_1=B\cap\bar{H}_1$, the morphism $\xi|_{C_2}$: $\bar{C}_2\to\bar{C}_1$ is the blowing-up of $\bar{C}_1$ with center the products $\prod_{i=1}^{s} m_{q_i}$ of maximal ideals ([21]). From the formula (1.3), the equality $\sum_{i=1}^{s}[\sum_{k\geq 0}\{P_{(\bar{C}_1,q_i)}(k)-H_{(\bar{C}_1,q_i)}(k)\}]=1$ is obtained. Since $(\bar{C}_1, q_i)$ for $i=1,\dots,s$ are one-dimensional reduced (hence Cohen-Macaulay), there is one point, say q_1, where the following equalities hold:

$$P_{(\bar{C}_1,q_1)}(0)-H_{(\bar{C}_1,q_1)}(0)=1,\quad P_{(\bar{C}_1,q_1)}(k)=H_{(\bar{C}_1,q_1)}(k)\quad\text{for } k\geqq 1,$$

and $P_{(\bar{C}_1,q_i)}(k)=H_{(\bar{C}_1,q_i)}(k)$ for $k\geqq 0$, $i=2,\dots,s$.

Hence $(\bar{C}_1, q_i)$ for $i=2,\dots,s$ are regular, and $(\bar{V}_1, q_i)$ for $i=2,\dots,s$ are also regular, since $\bar{C}_1$ is a Cartier divisor on $\bar{V}_1$ (by the same arguments in the proof of (4.2)). This means that $s=1$, since B is contained in $\mathrm{Sing}(\bar{V}_1)$. B is one-dimensional reduced and of degree one in $\boldsymbol{P}^2$, hence is a straight line.

(ii) $\bar{V}_2$ is also a local hypersurface. $\bar{V}_1$ is normally flat along B, since $\mathrm{mult}_q\bar{V}_1=$two for any $q\in B$ (Theorem 2 of Chapter 2 [19]). Hence $\dim\psi_2^{-1}(q)$ is constant for $q\in B$. ψ_2 is a finite map. The fact which we have to prove is the normality of $\bar{V}_2$. We shall show that $\bar{V}_2$ has only isolated singularities. Then the normality of $\bar{V}_2$ follows from Serre's criterion.

Here we note the following fact: Since $(\bar{C}_1, q_1)$ is a hypersurface of multiplicity two, the equality $H^{(1)}_{(\bar{C}_1,q_1)}=H^{(0)}_{(\bar{V}_1,q_1)}$ can be easily checked. Hence, by (1.11) and (1.12), $\bar{C}_2$ is a Cartier divisor on $\bar{V}_2$ with the relation $\bar{C}_2=\bar{H}_2\cap\bar{V}_2$, if

$\overline{H}_1$ does not contain B.

Let us separate the proof into the following two cases:

Case 1. There is a fiber f of $\Theta_2^{(2)} \to B$ such that $|E_2 \cap f|$ consists of distinct two points.

Then $E_2 \cap f$ is a union of the reduced two points. Hence E_2 is regular in a neighborhood of $E_2 \cap f$. Since E_2 does not contain any fiber of $\Theta_2^{(2)} \to B$, f intersects with all the irreducible components of E_2. Hence E_2 has only isolated singularities. Since E_2 is a Cartier divisor on $\overline{V}_2$, $\overline{V}_2$ also has only isolated singularities (cf. the proof of (4.2)).

Case 2. $|E_2 \cap f|$ is one point for any fiber f of $\Theta_2^{(2)} \to B$.

Since $||E_2| \cap f| = |E_2 \cap f|$, $|E_2|$ is irreducible. E_2 is written as $E_2 = 2|E_2|$ in $\Theta_2^{(2)}$. Let $|E_2| \cap \overline{H}_2 = \{q'\}$. Since $|E_2| \cap \overline{C}_2 = |E_2| \cap \overline{H}_2 \cap \overline{V}_2 = |E_2| \cap \overline{H}_2$ and $\mathrm{mult}_{q'}(E_2 \cap \overline{H}_2) = 2$, $|E_2| \cap \overline{C}_2$ is a reduced point. Blow up $\overline{V}_2$ with center the ideal $\mathscr{I}_{|E_2|}$, which defines $|E_2|$. We already have the equality $\chi(O_{\overline{C}'}) - \chi(O_{\overline{C}_2}) = 0$ in (B) of the proof of (i).

Now the remaining part of the proof of the normality of V_2 is parallel with the proof of (4.2) (ii); hence we shall omit it.

Combining Theorems above, we obtain the following:

Theorem (4.6). *Let (V, p) be a normal two-dimensional Gorenstein singularity with $p_a(V, p) = 1$. Then* **Z.C.R.** *is obtained by the composition of blowing-ups as follows:*

$$(*)\qquad \begin{array}{ccccccccccc} U = U_0 & \xleftarrow{\psi_1} & U_1 & \xleftarrow{\psi_2} & U_2 & \longleftarrow & \cdots & \xleftarrow{\psi_N} & U_N \\ \cup\!\shortparallel & & \cup\!\shortparallel & & \cup\!\shortparallel & & & & \cup\!\shortparallel \\ V = V_0 & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow & \cdots & \longleftarrow & V_N \end{array}$$

where $V \subset U$ is the minimal embedding, ψ_i the blowing-up of U_{i-1} with smooth center $\Gamma_i \subset V_{i-1}$, and V_i the strict transform of V_{i-1}, $1 \leqq i \leqq N$. Moreover we have: There is an integer M $(\leqq N)$ such that (i) V_i is normal for $i \leqq M$, (ii) ψ_i is a blowing-up with point center p_i such that (V_{i-1}, p_i) is Gorenstein of maximal embedding dimension of multiplicity $\geqq 3$ for $i \leqq M$, (iii) at each stage, in which V_i is normal, there is at most one non-rational singularity, (iv) $\mathrm{mult}_q V_M \leqq 2$ for any point $q \in V_M$, (v) in **Z.C.R.** *for the singularities of V_M, each normalization is trivial or is obtained by one blowing-up along (reduced) $\mathbb{P}^1$.*

Proof. We shall prove only (iii), since the other parts are the combination of the results which we have already proved.

Let us consider the following situation. Let $\psi: (\tilde{V}, A) \to (V, p)$ be a resolution. Suppose there are two effective divisors on $\tilde{V}$, say D_1 and D_2, whose supports $|D_1|$ and $|D_2|$ are contained in A with the condition $|D_1| \cap |D_2| = \phi$. Then there is an effective divisor D_3 such that $|D_1| \cap |D_3|$ and $|D_2| \cap |D_3|$ are both non-empty and discrete, and that $p_a(D_3) \geqq 0$. The inequality $p_a(D_1 + D_2 + D_3) \geqq p_a(D_1) + p_a(D_2)$ can be easily checked. Hence we obtain the inequality $p_a(V, p) \geqq \sum_{q \in V_i} p_a(V_i, q)$ for the normal V_i. Q. E. D.

(4.7) Let us compactify V_i, $i = 1, \dots, N$, of diagram (*) of (4.6) as in (2.1.1). We shall see the behavior of the integers $\chi(\mathcal{O}_{V_i})$, $i = 1, \dots, N$. Let the set of integers $\{j_1 = 1 < \cdots < j_M = M < j_{M+1} < \cdots < j_F\}$ be the subset of $\{1, \dots, N\}$ such that ψ_{j_i} is the blowing-up with point center of multiplicity greater than or equal to three, or the blowing-up with center $\boldsymbol{P}^1$.

Proposition (4.8). *Let (V, p) be a normal two-dimensional Gorenstein singularity such that* ***Z.C.R.*** *for (V, p) is obtained with the diagram (*) of (4.6) (not necessarily $p_a = 1$) and $\{j_1, \dots, j_F\}$, the index set as above. Then the following equalities hold: (i) $\chi(\mathcal{O}_{V_i}) - \chi(\mathcal{O}_{V_{i-1}}) = -1$ for $i = j_1, \dots, j_F$. (ii) $\chi(\mathcal{O}_{V_i}) - \chi(\mathcal{O}_{V_{i-1}}) = 0$ for other index i. Therefore we obtain the equality $F = p_g(V, p)$.*

Proof. The assertion about (ii) follows from (i) of (2.5). The assertion of the cases $i = j_h$, for $h = 1, \dots, M$, has been checked in (ii) of (2.2). Now let us consider the cases $i = j_h$, for $h = M + 1, \dots, F$ (i.e., $\Gamma_i = \boldsymbol{P}^1$). By (2.7), we have the following:

$$\begin{aligned}\chi(\mathcal{O}_{V_i}) - \chi(\mathcal{O}_{V_{i-1}}) &= \frac{1}{6}(\rho_i - 1)(2\rho_i - 1) \sum_{j < i} \frac{\rho_j W_i \cdot W_j}{\rho_i} \\ &\quad + \frac{1}{6}((\rho_i)^2 - 1)(W_i)^2 + \frac{1}{2}(\rho_i - 1)\rho_i \chi(\mathcal{O}_{\Gamma_i}) \\ &= \frac{1}{2} W_{j_h - 1} \cdot W_{j_h} + \frac{1}{2}(W_{j_h})^2 + 1 \quad \text{(by Theorem (4.5))}.\end{aligned}$$

Now we shall show the equality $W_{j_h - 1} = W_{j_h}$. Let us consider the following commutative diagram:

$$\begin{array}{ccccc} & \nearrow & U_{j_h} & \supseteqq & V_{j_h} \\ \Theta_{j_h-1}^{(j_h-1)} = \Theta_{j_h-1}^{(j_h)} & & \downarrow & \psi_{j_h} & \downarrow \\ & \searrow & U_{j_h-1} & \supseteqq & V_{j_h-1}\,. \end{array}$$

We obtain the following equalities:

$$(I_{\Gamma_{j_h}})^2 O_{\Theta_{j_h-1}^{(j_h)}} = I_{V_{j_h-1}} O_{\Theta_{j_h-1}^{(j_h-1)}} = (\psi_{j_h})^{-1}(I_{V_{j_h-1}}) O_{\Theta_{j_h-1}^{(j_h)}} = (I_{\Theta_{j_h}^{(j_h)}})^2 I_{V_{j_h}} \cdot O_{\Theta_{j_h-1}^{(j_h)}}.$$

Since $I_{\Theta_{j_h}^{(j_h)}} \cdot O_{\Theta_{j_h-1}^{(j_h)}} = I_{\Gamma_{j_h}} \cdot O_{\Theta_{j_h-1}^{(j_h-1)}}$, we obtain $I_{V_{j_h}} \cdot O_{\Theta_{j_h-1}^{(j_h)}} = O_{\Theta_{j_h-1}^{(j_h)}}$. This means that $V_{j_h} \cap \Theta_{j_h-1}^{(j_h)}$ is an empty set so $W_{j_h-1} = W_{j_h}$ holds.

By Theorem (2.7) of [46], $(W_{j_h-1})^2 = -2$ holds. The assertion follows from these equalities. Q. E. D.

§5. Calculation of the Canonical Divisor

(5.1) The purpose of this section is to establish the adjunction formula (5.4) which gives the following theorem as a corollary:

Theorem (5.2). *Let* (V, p) *be a normal two-dimensional Gorenstein singularity which has a resolution with the conditions in* (4.6) (*not necessarily* $p_a = 1$).

$$\begin{array}{ccccccc} U = U_0 & \xleftarrow{\psi_1} & U_1 & \xleftarrow{\psi_2} & \cdots & \xleftarrow{\psi_N} & U_N \\ \cup\!\shortparallel \quad \cup\!\shortparallel & & \cup\!\shortparallel & & & & \cup\!\shortparallel \\ V = V_0 & \longleftarrow & V_1 & \longleftarrow & \cdots & \longleftarrow & V_N . \end{array}$$

Let the set of integers $\{j_1, \ldots, j_{p_g(V,p)}\}$ *be the subset of* $\{1, \ldots, N\}$ *defined in* (4.7) (*cf.* (4.8)). *Then the canonical divisor* K_{V_N} *in this resolution is written as follows*:

$$K_{V_N} = \sum_{h=1}^{p_g} -W_{j_h}.$$

Comparing this with "*the formula of the canonical divisor by the summation of the elliptic sequence*" (by S. S. -T. Yau [48], see Section 6 of the present paper), we obtain Theorem (7.8) and Corollary (7.9), by which we can deduce some precise results on the resolution process of the maximally elliptic singularity (in [45]).

Theorem (5.2) follows from Theorem (5.4) directly (see (5.5)). In the proof of Theorem (5.4), we shall construct a syzygy of the Rees algebra of the singularity, which has a self-dual property in the T. N. -isomorphy sense. This is done by the following Wahl-Sally's method for "*the lifting of the syzygy from the tangent cone to the original local ring* [47], [40]". Their method is essential in our proof of Theorem (5.4).

(5.3) To make the problem clear, let us fix the situation. Let (V, p) be a normal isolated Gorenstein singularity embedded in a regular scheme U with codimension r. Let us consider the following resolution diagram:

$$(5.3.1)\qquad \begin{array}{ccccccccc} U=U_0 & \xleftarrow{\psi_1} & U_1 & \xleftarrow{\psi_2} & U_2 & \longleftarrow & \cdots & \xleftarrow{\psi_N} & U_N \\ \cup\!\shortparallel & & \cup\!\shortparallel & & \cup\!\shortparallel & & & & \cup\!\shortparallel \\ V=V_0 & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow & \cdots & \longleftarrow & V_N. \end{array}$$

where ψ_i is the blowing-up of U_{i-1} with smooth center Γ_i contained in the singular locus $\mathrm{Sing}(V_{i-1})$ of V_{i-1} and V_i the strict transform of V_{i-1} for $i=1,\dots,N$. Let $\psi\colon U_N\to U$ be the morphism which is the composition of ψ_i's and K_{V_N} the canonical divisor on V_N whose supports are contained in $|\psi^{-1}(p)\cap V_N|$ (cf. §2 of [43] or [18] for the existence of such a divisor). We shall consider in the neighborhood of the inverse image of $\{p\}$ in each stage.

Problem (5.3.2). *Find a divisor D on U_N of the form $D=\sum_{i=1}^{N}\alpha_i\cdot\Theta_i^{(N)}$, where $\alpha_i\in\boldsymbol{Z}$ for $i=1,\dots,N$, which satisfies the condition $D|_{V_N}=K_{V_N}$ (i.e., $O_{U_N}([D])\otimes O_{V_N}([K_{V_N}])$, where $[D]$ denotes the line bundle associated to the divisor D).*

This is equivalent to the following:

Problem (5.3.3). *Find a divisor G on U_N of the form $G=\sum_{i=1}^{N}\beta_i\Theta_i^{(N)}$, where $\beta_i\in\boldsymbol{Z}$ for $i=1,\dots,N$, which satisfies the condition $Ext^r_{O_{U_N}}(O_{V_N}, O_{U_N}([G]))|_{V_N}\cong O_{V_N}$.*

Indeed the equivalence of these problems is seen as follows. Let E be the divisor on U_N defined by the functional determinant of $\psi\colon U_N\to U$. Actually E is written by $E=\sum_{i=1}^{N}((\mathrm{codim}_{\Gamma_i}U_{i-1})-1)\cdot(\psi_{i+1}\circ\cdots\circ\psi_N)^{-1}(\Theta_i^{(i)})$ and satisfies the equality $\Omega^R_{U_N}\cong O_{U_N}([E])$, where R is the dimension of U_N. Here we note the equality $Ext^r_{O_{U_N}}(O_{V_N}, \Omega^R_{U_N})|_{V_N}\cong O_{V_N}([K_{V_N}])$. Hence if (5.3.3) is solved by the divisor G, then the divisor $E-G$ satisfies the condition of (5.3.2). The converse is also clear.

If (V, p) is a hypersurface in U, the answer is well-known. In fact the equality $Ext^1_{O_{U_N}}(O_{V_N}, O_{U_N}([-V_N]))|_{V_N}\cong O_{V_N}$ holds and $\psi^{-1}(I_V)=\prod_{i=1}^{N}(\psi_{i+1}\circ\cdots\circ\psi_N)^{-1}(I_{\Theta_i^{(i)}})^{\rho_i}\cdot I_{V_N}$ is trivial in the neighborhood of $\psi^{-1}(p)$. Such an argument is seen in the proof of Satz 1 [8]. (See (2.6) for notations.).

We have an answer under the following conditions below:

Condition (5.3.4). *There is an integer M' $(\leqq N)$ such that the following conditions are satisfied: (i) V_k is normal isolated Gorenstein at every point of*

V_k *for* $k \leqq M'$. *(ii)* ψ_k *is a blowing-up with point center* Γ_k, *which is also denoted by* p_k, *for* $k \leqq M'$. *(iii)* (V_{k-1}, p_k) *is not a hypersurface for* $k \leqq M'$. *(iv) All singularities of* $V_{M'}$ *are hypersurface singularities.* *(v)* $gr_{m_{V_{k-1},p_k}} \cdot (O_{V_{k-1},p_k})$ *is Gorenstein for* $k \leqq M'$. *(vi) Let* $(V_{k-1}, p_k) \hookrightarrow (W_{k-1}, p_k)$ *be the minimal embedding to a regular scheme* W_{k-1} *at* p_k *for* $k \leqq M'$. *Let* $\bar{S}_k$ *denote* $gr_{m_{W_{k-1},p_k}}(O_{W_{k-1},p_k})$. *The condition (vi) is the existence of a graded free* $\bar{S}_k$*-resolution of* $gr_{m_{V_{k-1},p_k}}(O_{V_{k-1},p_k})$ *of the form*

$$0 \longrightarrow \bar{S}_k(-a_{s_k})^{b_{s_k}} \longrightarrow \bar{S}_k(-a_{s_k-1})^{b_{s_k-1}} \longrightarrow$$
$$\cdots \longrightarrow \bar{S}_k(-a_i)^{b_i} \longrightarrow \bar{S}_k(-a_{i-1})^{b_{i-1}} \longrightarrow$$
$$\cdots \longrightarrow \bar{S}_k(-a_1)^{b_1} \longrightarrow \bar{S}_k \longrightarrow gr_{m_{V_{k-1},p_k}}(O_{V_{k-1},p_k}) \longrightarrow 0,$$

where $0 < a_1 < a_2 < \cdots < a_{s_k}$, *whose localization at* $(m_{W_{k-1},p_k}) \cdot \bar{S}_k$ *is a minimal resolution for* $k \leqq M'$. *Here the integer* s_k *is the codimension of* V_{k-1} *in* W_{k-1} *(by (v)).*

Theorem (5.4). *Assume that the condition* (5.3.4) *is satisfied in* (5.3.1). *Let G be the divisor on* U_N *of the form* $G = \sum_{i=1}^{N} (a_{s_i} + r - s_i) \cdot (\psi_{i+1} \circ \cdots \circ \psi_N)^{-1}(\Theta_i^{(i)})$. *Here, for the case of* $i \geqq M'+1$, *the integer* a_{s_i} *is defined as the multiplicity of* V_{i-1} *at points in a Zariski open set of center* Γ_i *and the integer* s_i *the embedding dimension of* V_{i-1} *at the points in the center* Γ_i *(i.e.,* $s_i = \dim V_{i-1} + 1$*). Then we obtain the equality* $Ext^r_{O_{U_N}}(O_{V_N}, O_{U_N}([G]))|_{V_N} \cong O_{V_N}$.

(5.5) *Proof of* (5.2). By Sally's analysis [39], [40], the conditions of (5.4.3) are satisfied under the assumption of (5.2). The divisor $E-G$ cited after (5.3.3) is written by

$$E - G = \sum_{i=1}^{N} (s_i - \dim \Gamma_i - a_{s_i} - 1)(\psi_{i+1} \circ \cdots \circ \psi_N)^{-1}(\Theta_i^{(i)})$$

by Theorem (5.4). Since (V_{k-1}, p_k) is Gorenstein of maximal embedding dimension, we have the equality $a_{s_k} = \mathrm{mult}_{p_k} V_{k-1}$ from Sally's computation [40]. By noting the equality $s_k = \dim V_{k-1} + \mathrm{mult}_{p_k} V_{k-1} - 2$ from the assumption of such a situation, we can see that (5.2) follows directly.

Q. E. D.

The rest of this section is devoted to the proof of Theorem (5.4).

(5.6) First of all, we shall localize the problem.

Lemma (5.6.1). *Let the integer i satisfy* $1 \leqq i \leqq M'$. *Let* $\mathscr{L}$ *be an invertible*

O_{V_i}-module sheaf. Suppose the trivialization $\mathscr{L}|_{\mathscr{U}_1} \cong O_{V_i}|_{\mathscr{U}_1}$ and $\mathscr{L}|_{\mathscr{U}_2} \cong O_{V_i}|_{\mathscr{U}_2}$ is given for an open covering $\{\mathscr{U}_1, \mathscr{U}_2\}$ of V_i such that $\mathscr{U}_1 = V_i - \Theta_i^{(i)} \cap V_i$ and $\mathscr{U}_2$ an open neighborhood of $V_i \cap \Theta_i^{(i)}$ in V_i. Then we have the isomorphism $\mathscr{L} \cong O_{V_i}$ over V_i.

Proof. There is an open neighborhood $\mathscr{V}_3$ of p_i in V_{i-1} such that $(\psi_i|V_i)^{-1}(\mathscr{V}_3) \subseteqq \mathscr{U}_2$. Since V_{i-1} is normal, we obtain the following:

$$\begin{array}{ccc} \Gamma((\psi_i|_{V_i})^{-1}(\mathscr{V}_3), O_{V_i}) & \longrightarrow & \Gamma(\mathscr{U}_1 \cap (\psi_i|_{V_i})^{-1}(\mathscr{V}_3), O_{V_i}) \\ \wr\| & \curvearrowright & \| \\ \Gamma(\mathscr{V}_3, O_{V_{i-1}}) & \xrightarrow{\cong} & \Gamma(\mathscr{V}_3 - \{p_i\}, O_{V_{i-1}})\,. \end{array}$$

By assumption, there is $h_i \in \Gamma(\mathscr{U}_i, \mathscr{L})$ which is nowhere-vanishing over $\mathscr{U}_i$, for $i=1, 2$. By the diagram above h_1/h_2 and h_2/h_1 extend to the sections g and g' over $(\psi_i|_{V_i})^{-1}(\mathscr{V}_3)$ with the relation $g \cdot g' = 1$. Hence g is nowhere vanishing over $(\psi_i|_{V_i})^{-1}(\mathscr{V}_3)$ and so is $g \cdot h_2$. Then h_1 and $g \cdot h_2$ define a nowhere vanishing section of $\mathscr{L}$ over V_i. Q.E.D.

Lemma (5.6.2). *Let $\mathscr{L}$ be an invertible O_{V_N}-module sheaf. Suppose that the trivialization $\mathscr{L}|_{\mathscr{U}_i} \cong O_{V_N}|_{\mathscr{U}_i}$ is given for an open covering $\{\mathscr{U}_1, \mathscr{U}_2\}$ of V_N such that $\mathscr{U}_1 = V_N - (\bigcup_{i=M'+1}^{N} \Theta_i^{(N)}) \cap V_N$ and $\mathscr{U}_2$ an open neighborhood of $(\bigcup_{i=M'+1}^{N} \Theta_i^{(N)}) \cap V_N$. Then we have the isomorphism $\mathscr{L} \cong O_{V_N}$ over V_N.*

The proof is similar to the proof of (5.6.1), hence we shall omit it.

Suppose F is an invertible $O_{U_{i-1}}$-module sheaf such that $Ext^r_{O_{U_{i-1}}}(O_{V_{i-1}}, F)|_{V_{i-1}} \cong O_{V_{i-1}}$ for some integer i such that $1 \leqq i \leqq M'$. If we prove the isomorphism $Ext^r_{O_{U_i}}(O_{V_i}, O_{U_i}(b[\Theta_i^{(i)}]))|_{V_i} \cong O_{V_i}$ in a neighborhood of $\Theta_i^{(i)} \cap V_i$ for some integer b, we obtain the isomorphism $Ext^r_{O_{U_i}}(O_{V_i}, (\psi_i)^{-1}F \otimes O_{U_i}(b[\Theta_i^{(i)}]))|_{V_i} \cong O_{V_i}$ over V_i by (5.6.1). Furthermore, we have the isomorphism $Ext^r_{O_U}(O_V, O_U)|_V \cong O_V$ by the Gorenstein-ness of V. Hence, by (5.6.1) and (5.6.2), the proof of (5.4) can be deduced to the proof of the following two statements:

(5.6.3). *The isomorphism $Ext^r_{O_{U_i}}(O_{V_i}, O_{U_i}((a_i + r - s_i)[\Theta_i^{(i)}]))|_{V_i} \cong O_{V_i}$ holds in a neighborhood of $V_i \cap \Theta_i^{(i)}$ for $i = 1, \dots, M'$.*

(5.6.4). *Theorem* (5.4) *holds under the assumption* $\operatorname{embdim}_p V = \dim V + 1$.

(5.7) In this paragraph, we shall prove (5.6.3) under the assumption $W_0 = U$ for $i=1$. We denote W_0 by Z and its strict transform by Z_1 as usual. To construct the desired isomorphism, we use the following:

Lemma (5.7.1) (Artin-Rees-Wahl, Lemma 1.6 [47]). *Let $(O_{Z,p}, m)$ be a local ring with the maximal ideal m and $E \subseteqq F$ finitely generated $O_{Z,p}$-modules. Give F the m-adic filtration and E the induced filtration: $E_n = E \cap m^n \cdot F$ so that $gr\ E \subseteqq gr\ F$. If $gr\ E$ is generated by homogeneous elements of degree q, then $E \cap m^{n+q} F = m^n \cdot E$ for all $n \geqq 0$. The elements $e_1, \dots, e_t$ of E minimally generate E if and only if their initial forms $\bar{e}_1, \dots, \bar{e}_t$ minimally generate $gr\ E$.*

Corollary (5.7.2) ([47], see also [40]). *Let $(O_{Z,p}, m)$ be a local ring as above and let I be an ideal of $O_{Z,p}$ such that $O_{Z,p}/I$ has homological codimension s over $O_{Z,p}$. If*

$$0 \longrightarrow \bar{F}_s \longrightarrow \cdots \longrightarrow \bar{F}_j \xrightarrow{\bar{d}_j} \bar{F}_{j-1} \longrightarrow \cdots$$
$$\longrightarrow \bar{F}_0 = \bar{S} \longrightarrow gr_m(O_{Z,p}/I) \longrightarrow 0$$

is a free resolution for $gr_m(O_{Z,p}/I)$ over $gr_m(O_Z)$ (which we denote by $\bar{S}$,) of the form $\bar{F}_j = \bar{S}(-a_j)^{b_j}$ which satisfies the condition (vi) of (5.3.4), there is a minimal $O_{Z,p}$-free resolution of $O_{Z,p}/I$

$$0 \longrightarrow F_s \longrightarrow \cdots \longrightarrow F_j \xrightarrow{d_j} F_{j-1} \longrightarrow \cdots$$
$$\longrightarrow F_0 = O_{Z,p} \longrightarrow O_{Z,p}/I \longrightarrow 0$$

such that $(\mathrm{Ker}\, d_j) \cap m^{n+a_j} \cdot F_j = m^n \cdot (\mathrm{Ker}\, d_j)$ for $n, j \geqq 0$. This induces the former graded resolution of $gr_m(O_{Z,p}/I)$. (See the proof of Theorem 1.7 of [47].)

Let us apply the Corollary above to our situation as $I = I_V$. Set $S = \bigoplus_{n \geqq 0} m^n$ and $(F_j)_n$ by $(F_j)_n = 0$ for $n < a_j$ and $(F_j)_n = m^{n-a_j} \cdot F_j$ for $n \geqq a_j$. We denote $\bigoplus_{n \geqq 0} (F_j)_n$ by $\boldsymbol{F}_j$. Then by (5.7.2), we obtain the complex of graded S-modules

$$0 \longrightarrow \boldsymbol{F}_s \xrightarrow{\boldsymbol{d}_s} \boldsymbol{F}_{s-1} \longrightarrow \cdots \longrightarrow \boldsymbol{F}_j \xrightarrow{\boldsymbol{d}_j} \boldsymbol{F}_{j-1} \longrightarrow \cdots$$
$$\longrightarrow \boldsymbol{F}_0 = S \longrightarrow \bigoplus_{n \geqq 0} m^n \cdot O_{V,p} \longrightarrow 0 \tag{5.7.3}$$

which is an exact sequence in the sense of T. N.-isomorphism over S. Here the degree preserving graded S-morphism $\boldsymbol{d}_j$ is defined by $\boldsymbol{d}_j = \bigoplus_{n \geqq 0} (d_j)_n$, where $(d_j)_n = d_j|_{(F_j)_n}$. Note the equality $\boldsymbol{F}_j = S(-a_j)$.

We shall consider the dual complex $(\mathrm{Hom}_S(\boldsymbol{F}., S(-a_s)))$.

Assertion (5.7.4). *There is a surjective S-morphism δ: $\mathrm{Hom}_S(\boldsymbol{F}_s, S(-a_s)) \to \bigoplus_{n \geqq 0} m^n \cdot O_{V,p}$ such that*

$$0 \longrightarrow \mathrm{Hom}_S(\boldsymbol{F}_0, S(-a_s)) \longrightarrow \cdots$$
$$\longrightarrow \mathrm{Hom}_S(\boldsymbol{F}_{j-1}, S(-a_s)) \xrightarrow{(\boldsymbol{d}_j)^*} \mathrm{Hom}_S(\boldsymbol{F}_j, S(-a_s)) \longrightarrow \cdots$$
$$\longrightarrow \mathrm{Hom}_S(\boldsymbol{F}_s, S(-a_s)) \xrightarrow{\delta} \bigoplus_{n \geqq 0} m^n \cdot O_{V,p} \longrightarrow 0$$

is exact in the sense of T. N.-isomorphism over S.

Proof of (5.7.4). Note first the following two important facts: The first thing is the symmetry among the integers $\{a_i\}$ such that $a_i + a_{s-i} = a_s$ for $0 \leqq i \leqq s$, where we define a_0 by 0, due to the Gorenstein property of $gr_m(O_{V,p})$ (see p. 179 of [40]). The second is the following fact below due to the Gorenstein property of $O_{V,p}$:

There is a system of $O_{Z,p}$-bilinear non-degenerate pairings $\{\phi_i\}$: ϕ_i: $F_i \otimes_{O_{Z,p}} F_{s-i} \to F_s$ for $0 \leqq i \leqq s$, which induces the isomorphism of complex $\{t_i\}$

$$(5.7.5)\qquad \begin{array}{ccccccccccc}
0 \longrightarrow & F_s & \longrightarrow \cdots \longrightarrow & F_j & \xrightarrow{d_j} & F_{j-1} & \longrightarrow \cdots & \cdots \longrightarrow & F_0 \\
& {\scriptstyle t_s}\downarrow & \curvearrowright & {\scriptstyle t_j}\downarrow & \curvearrowright & {\scriptstyle t_{j-1}}\downarrow & \curvearrowright & \curvearrowright & \downarrow{\scriptstyle t_0} \\
0 \longrightarrow & (F_0)^* & \longrightarrow \cdots \longrightarrow & (F_{s-j})^* & \xrightarrow[(d_{s-j+1})^*]{} & (F_{s-j+1})^* & \longrightarrow \cdots & \cdots \longrightarrow & (F_s)^*
\end{array}$$

by defining t_i as t_i: $F_i \to \mathrm{Hom}_{O_{Z,p}}(F_{s-i}, F_i) = (F_{s-i})^*$; $a \mapsto \phi_i(a, \ )$ (see Theorem 1.5 pp. 454–455 of [9]).

We shall lift $\{t_i\}$ to the correspondence between (5.7.3) and (5.7.4). By the symmetry $a_i + a_{s-i} = a_s$ for $0 \leqq i \leqq s$, we can easily check the following canonical isomorphisms $\mathrm{Hom}_S(\boldsymbol{F}_{s-i}, \boldsymbol{F}_s) \cong \bigoplus_{n \in \mathbf{Z}} \mathrm{Hom}_{O_{Z,p}}(F_{s-i}, (F_s)_n)$ for $0 \leqq i \leqq s$. Let us define the isomorphism $(t_i)_n$: $(F_i)_n \to \mathrm{Hom}_{O_{Z,p}}(F_{s-i}, (F_s)_n)$ by the $O_{Z,p}$-bilinear pairing ϕ_i, which is nothing but the restriction of t_i to $(F_i)_n$. By using those we obtain the S-isomorphisms $\boldsymbol{t}_i$: $\boldsymbol{F}_i \to \mathrm{Hom}_S(\boldsymbol{F}_{s-i}, \boldsymbol{F}_s)$ as $\boldsymbol{t}_i = \bigoplus_{n \in \mathbf{Z}} (t_i)_n$ for $0 \leqq i \leqq s$.

The remaining point which we have to check is the commutativity of the following:

$$\begin{array}{ccc}
0 \longrightarrow & \boldsymbol{F}_s & \longrightarrow \cdots \\
& {\scriptstyle \boldsymbol{t}_s}\downarrow & \\
0 \longrightarrow & \mathrm{Hom}_S(\boldsymbol{F}_0, \boldsymbol{F}_s) & \longrightarrow \cdots
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & \boldsymbol{F}_j & \longrightarrow & \boldsymbol{F}_{j-1} & \longrightarrow \cdots \\
& {\scriptstyle \boldsymbol{t}_j}\downarrow & & {\scriptstyle \boldsymbol{t}_{j-1}}\downarrow & \\
\longrightarrow & \mathrm{Hom}_S(\boldsymbol{F}_{s-j}, \boldsymbol{F}_s) & \longrightarrow & \mathrm{Hom}_S(\boldsymbol{F}_{s-j+1}, \boldsymbol{F}_s) & \longrightarrow \cdots
\end{array}$$

$$\begin{array}{cc}
\cdots \longrightarrow & \boldsymbol{F}_0 = S \\
& {\scriptstyle \boldsymbol{t}_0}\downarrow \\
\cdots \longrightarrow & \mathrm{Hom}_S(\boldsymbol{F}_s, \boldsymbol{F}_s)\,.
\end{array}$$

For any element $\bigoplus_{n \geqq a_i} f_n \in \boldsymbol{F}_i$, where $f_n \in F_i$, we have the relations $\boldsymbol{t}_{i-1}(\boldsymbol{d}_i(\bigoplus_{n \geqq a_i} f_n)) = \boldsymbol{t}_{i-1}(\bigoplus_{n \geqq a_i} d_i(f_n)) = \bigoplus_{n \geqq a_i} \phi_{i-1}(d_i(f_n),\ ) \in \bigoplus_n \mathrm{Hom}_{O_{Z,p}}(F_{s-i+1}, (F_s)_{n+a_{s-i+1}})$ by the definitions of $\boldsymbol{t}_{i-1}$ and $\boldsymbol{d}_i$. Furthermore, we have $\boldsymbol{t}_i(\bigoplus_{n \geqq a_i} f_n) = \bigoplus_{n \geqq a_i} \phi_i(f_n,\ ) \in \bigoplus_n \mathrm{Hom}_{O_{Z,p}}(F_{s-i}, (F_s)_{n+a_{s-i}})$. Looking at the following commutative diagram:

$$
\begin{array}{ccc}
F_{s-i+1} & \xrightarrow{(\boldsymbol{d}_{s-i+1})_{a_{s-i+1}}} & (F_{s-i})_{a_{s-i+1}} \\
 & {\scriptstyle \pi_n} \searrow \quad \swarrow {\scriptstyle \phi_i(f_n,\ )|_{(F_{s-i})_{a_{s-i+1}}}} & \\
 & (F_s)_{n+a_{s-i+1}-a_{s-i}-a_i+a_s} & \\
 & \| \quad \text{by the symmetry of } a_i\text{'s} & \\
 & (F_s)_{n+a_{s-i+1}}, &
\end{array}
$$

we obtain the equality $\bigoplus_{n \geqq a_i} \pi_n = (\boldsymbol{d}_{s-i+1})^*(\boldsymbol{t}_i(\bigoplus_{n \geqq a_i} f_n))$, where π_n is defined by the above diagram and actually is contained in $\mathrm{Hom}_{O_{Z,p}}(F_{s-i+1}, (F_s)_{n+a_{s-i+1}})$. By the commutativity of (5.7.5), we obtain $\pi_n = (d_{s-i+1})^*(\phi_i(f_n,\ )) = \phi_{i-1}(d_i(f_n,\ ))$. This completes the proof of the assertion (5.7.4).

On Proj(S), (5.7.4) induces an isomorphism $Ext^r_{O_{U_1}}(O_{V_1}, O_{Z_1}(a_s[\Theta_1^{(1)}]))|_{V_1} \cong O_{V_1}$ in a neighborhood of $\Theta_1^{(1)} \cap V_1$.

(5.8) Now we shall discuss (5.6.3) for $i=1$ when $V \subset U$ is not necessarily the minimal embedding. We use the notations Z and Z_1 instead of W_0 and its strict transform as in (5.7). We have the following:

$$E_2^{p,q} = Ext^p_{O_{Z_1}}(O_{V_1}, Ext^q_{O_{U_1}}(O_{Z_1}, O_{U_1}(b[\Theta_1^{(1)}]))) \underset{p}{\Longrightarrow} Ext^n_{O_{U_1}}(O_{V_1}, O_{U_1}(b[\Theta_1^{(1)}]))$$

for any integer b ((2.9.2) [2]).

Since V_1 and Z_1 are Cohen-Macaulay, this degenerates and induces the isomorphism

$$Ext^s_{O_{Z_1}}(O_{V_1}, Ext^{r-s}_{O_{U_1}}(O_{Z_1}, O_{U_1}(b[\Theta_1^{(1)}]))) \cong Ext^r_{O_{U_1}}(O_{V_1}, O_{U_1}(b[\Theta_1^{(1)}])).$$

We can show the equality $Ext^{r-s}_{O_{U_1}}(O_{Z_1}, O_{U_1}((r-s)[\Theta_1^{(1)}])) \cong O_{Z_1}$ easily. (Actually this is well-known. We note that this can be shown by the same arguments in (5.7) if we consider the couple Z_1 and U_1 instead of the couple V_1 and Z_1.) By this isomorphism, the proof of (5.6.3) is obtained.

(5.9) *Proof of* (5.6.4). We have already seen (5.6.4) for the case where V is a hypersurface in U (5.3). Here we shall discuss about (5.6.4) when $V \subset U$ is not necessarily the minimal embedding. We shall use the notations Z and Z_i

for $i=1,\dots,N$, instead of W_0 and its strict transform in U_i for $i=1,\dots,N$. Similar to (5.8), we have the following:

$$E_2^{p,q}=Ext^p_{O_{Z_N}}(O_{V_N}, Ext^q_{O_{U_N}}(O_{Z_N}, O_{U_N}([G']))) \underset{p}{\Longrightarrow} Ext^n_{O_{U_N}}(O_{V_N}, O_{U_N}([G']))$$

for the divisor G' on U_N. ((2.9.2) [2]). This degenerates too. Hence, for the proof of (5.6.4), we have to show the isomorphism $Ext^{r-s}_{O_{U_N}}(O_{Z_N}, O_{U_N}([G'])) \cong O_{Z_N}$ for the divisor $G'=\sum_{i=1}^{N}(r-s)(\psi_{i+1}\circ\cdots\circ\psi_N)^{-1}(\Theta_i^{(i)})$. Here $s=\dim V+1$. For the proof of the above isomorphism, we localize the arguments. Actually by noting the equalities $H^j_{\Gamma_i}(O_{Z_{i-1}})=0$ for $j=0, 1$ and for $i=1,\dots,N$, we can prove a localization lemma which is similar to (5.6.1) by a way similar to the proof of (5.6.1). The proof at each step after the localization which is corresponding to (5.7), is easy. Details are left to the reader.

This completes the proof of Theorem (5.4).

§6. On Yau's Elliptic Sequence

(6.1) For the study of the singularity satisfying the condition $p_a=1$, the elliptic sequence introduced by S. S.-T. Yau [48] is very effective. The elliptic sequence is originally the set of divisors on the minimal good resolution with some special properties (Definition (3.3) of [48]). In this section, we shall extend the definition of elliptic sequence to all resolution of the singularity with the condition $p_a=1$, and characterize them among all the effective divisors by some numerical conditions (Theorem (6.4)). This characterization will play an important role in our studies.

(6.2) Let (V, p) be a normal two-dimensional singularity and $\psi: (\tilde{V}, A) \to (V, p)$ a resolution of (V, p). Let B be a reduced connected divisor of $\tilde{V}$ such that $B \subseteqq A$. We denote the Artin's fundamental cycle on B by $\boldsymbol{Z}_B$. Before we proceed to define the elliptic sequence, let us note the following:

Proposition (6.2.1). *Let $\{C_i;\ i=1,\dots,r\}$ be the set of reduced connected divisors on $\tilde{V}$ such that $C_i \subseteqq A$ and that C_i and C_j have no common irreducible component for any couple of integers (i, j) with $i \neq j$. Then the following inequality holds*:

$$\sum_{i=1}^{r} p_a(\boldsymbol{Z}_{C_i}) \leqq p_a(\boldsymbol{Z}_0)\,.$$

This easily follows from "*Laufer's computation sequence methods*" (see

[28], [48]). Here we shall omit the proof of (6.2.1).

Definition (6.3) (the elliptic sequence). Let the situation be as in (6.2). Let us assume the condition $p_a(Z_0)=1$ (cf. Remark (6.5)). We shall define the set of reduced connected divisors $\{B_i\}$ on $\tilde{V}$ by the following canonical inductive procedure. (i) Set B_1 by $A=B_1$. (ii) Define B_{i+1} from B_i as follows: Put $\tilde{B}_i$ by $\tilde{B}_i=\bigcup_{A_j\subseteqq B_i \text{ such that } Z_{B_i}\cdot A_j=0} A_j$. Decompose $\tilde{B}_i$ into the connected components as $\tilde{B}_i=\bigcup_{j=1}^{m_i} D_{ij}$. If the condition $p_a(Z_{D_{ij}})\leqq 0$, $j=1,\dots, m_i$, hold, stop. The set of divisors $\{Z_{B_h};\ h=1,\dots, i\}$ is called the *elliptic sequence.* If there exists a component, say D_{i1}, such that $p_a(Z_{D_{i1}})=1$, set B_{i+1} by $B_{i+1}=D_{i1}$. Here note that there is at most one component as above by (6.2.1).

Clearly B_{h+1} is properly contained in B_h for any h. Therefore, the process above stops in finite steps.

The following statement gives an intrinsic characterization of the elliptic sequence:

Theorem (6.4). *Let the situation be as in* (6.3). *Let* $\{Z_{B_k};\ k=1,\dots, l\}$ *be the elliptic sequence on* $(\tilde{V}, A)$. *Then the following equality holds*:

$$\{D;\ \textit{non-zero effective divisor on } \tilde{V} \textit{ such that } (i)\ |D|\subseteqq A$$
$$(ii)\ D\cdot A_j\leqq 0 \textit{ for } j=1,\dots, m\quad (iii)\ p_a(D)=p_a(V, p)\}=\{\sum_{i=1}^{k} Z_{B_i};\ k=1,\dots, l\}.$$

The divisor $\sum_{i=1}^{l} Z_{B_i}$ *is characterized as the unique maximal element among the set* $\{D;\ p_a(D)=1,\ D>0$ *and* $|D|\subseteqq A\}$.

Proof (cf. the proof of Theorem (3.7) of [48]). First we shall show the inclusion relation that the former set $\subseteqq$ the latter set. Let F be a non-zero effective divisor on $\tilde{V}$ which belongs to the former set. By the definition of fundamental cycle Z_0, the divisor $F-Z_0$ is effective. If $Z_0-F=0$, this agrees with our assertion. Hence we may consider the case of $F-Z_0>0$. We have the following:

$$\begin{aligned} p_a(V, p) &= p_a(F) \qquad \text{(by the assumption of } F)\\ &= p_a(F-Z_0)+p_a(Z_0)+(F-Z_0)\cdot Z_0-1\\ &= p_a(F-Z_0)+(F-Z_0)\cdot Z_0 \qquad (\text{by } p_a(Z_0)=1)\\ &\leqq p_a(V, p)+(F-Z_0)\cdot Z_0 \qquad (\text{by the definition of } p_a).\end{aligned}$$

Since $(F-Z_0)\cdot Z_0\leqq 0$, we obtain the relations $p_a(F-Z_0)=p_a(V, p)$ and $Z_0\cdot A_j=0$

for any $A_j \subseteqq |F-\boldsymbol{Z}_0|$, i.e., $|F-\boldsymbol{Z}_0| \subseteqq \tilde{B}_1$.

Next we shall show the equality $F-\boldsymbol{Z}_0=B_2$. Let B' be any reduced connected divisor on $\tilde{V}$ such that $B' \subseteqq A$. Then we have the following:

$$\begin{aligned} p_a(V, p) &\geqq p_a(F-\boldsymbol{Z}_0+B') \\ &= p_a(F-\boldsymbol{Z}_0)+p_a(B')+(F-\boldsymbol{Z}_0)\cdot B'-1 \\ &\geqq p_a(V, p)+(F-\boldsymbol{Z}_0)\cdot B'-1 \quad (\text{by } p_a(B')\geqq 0). \end{aligned}$$

This means the inequality $(F-\boldsymbol{Z}_0)\cdot B' \leqq 1$. Hence $|F-\boldsymbol{Z}_0|$ is connected. Therefore $|F-\boldsymbol{Z}_0|$ is contained in B_2 by the definition of the elliptic sequence. Let A_j be any irreducible component of B_2. Then we have the following:

$$(F-\boldsymbol{Z}_0)\cdot A_j = F\cdot A_j - \boldsymbol{Z}_0\cdot A_j = F\cdot A_j \leqq 0.$$

Hence we have $|F-\boldsymbol{Z}_0|=B_2$. Therefore we have checked that $F-\boldsymbol{Z}_0$ belongs to the set $\{D$; non-zero effective divisor on $\tilde{V}$ such that (i) $|D| \subseteqq B_2$ (ii) $D\cdot A_j \leqq 0$ for any $A_j \subseteqq B_2$ (iii) $p_a(D)=p_a(V, p)\}$.

Repeating the arguments above, we can obtain the desired inclusion relation.

Now we shall show the converse inclusion relation. We already know that $p_a(V, p)=1$ above and $p_a(\sum_{i=1}^{k} \boldsymbol{Z}_{B_i})=1$ for $k=1,\dots, l$. Hence, the only thing which we must show is the fact that $\sum_{i=1}^{k} \boldsymbol{Z}_{B_i}\cdot A_j \leqq 0$ for $j=1,\dots, m$, $k=1,\dots, l$. Before we proceed to prove it, we must note the following fact (see [46] p. 443):

Fact (6.4.1). *Let (W, p) be the normal two-dimensional singularity and $\psi: (\tilde{W}, A)\to(W, p)$ a resolution of (W, p) with the decomposition into the irreducible components of the exceptional set $A=\bigcup_{j=1}^{m} A_j$. Then there is an effective divisor D_0 on $\tilde{W}$ whose support is contained in A such that $p_a(D_0)=p_a(W, p)$ and $p_a(D_0+E)<p_a(W, p)$ for any non-zero effective divisor E whose support is contained in A. Moreover for such a divisor D_0, we can easily check the relation $D_0\cdot A_j \leqq 0$ for $j=1,\dots, m$.*

We take D_0 as in (6.4.1) in our situation. Then D_0 is written as $D_0=\sum_{i=1}^{k_0} \boldsymbol{Z}_{B_i}$ for some integer $k_0 \leqq l$ by the former arguments. In fact k_0 is equal to l as follows: If $k_0 \neq l$, the equality $p_a(D_0+\boldsymbol{Z}_{B_{k_0+1}})=p_a(D_0)$ contradicts the condition in (6.4.1). Hence we obtain the relation $\sum_{i=1}^{l} \boldsymbol{Z}_{B_i}\cdot A_j \leqq 0$ for $j=1,\dots, m$.

Let us discuss the divisor $\sum_{i=1}^{k} \boldsymbol{Z}_{B_i}$ for $k<l$. If A_j is contained in B_{k+1}, then $\sum_{i=1}^{k} \boldsymbol{Z}_{B_i}\cdot A_j=0$. If A_j is not contained in B_{k+1}, we have $\sum_{i=k+1}^{l} \boldsymbol{Z}_{B_i}\cdot A_j \geqq 0$. Hence we obtain

$$\sum_{i=1}^{k} Z_{B_i}\cdot A_j=\sum_{i=1}^{l} Z_{B_i}\cdot A_j-\sum_{i=k+1}^{l} Z_{B_i}\cdot A_j\leqq 0\,.$$

The remaining assertion is clear from the arguments above. Q.E.D.

Remark (6.5). As we mentioned in the proof, we have checked the statement "$p_a(V, p)=1$ *if* $p_a(Z_0)=1$", which is originally stated in [46] and proved by many authors (cf. Remark (2.2) of [43]).

Remark (6.6). With the aid of our theorem, we can change the procedure from B_i to B_{i+1} in (6.3) in the following way: Assume that we have already defined $\{B_k;\ k=1,\dots, i\}$. Put $\tilde{L}_i$ by $\tilde{L}_i=\bigcup_{A_j\subseteqq A \text{ such that } \sum_{k=1}^{i} Z_{B_k}\cdot A_j=0} A_j$. Choose the connected component, say L_{i+1}, of $\tilde{L}_i$ such that $p_a(Z_{L_{i+1}})=1$. Suppose that such a divisor L_{i+1} exists. Clearly B_{i+1} is contained in L_{i+1}. However we can check the equality $B_{i+1}=L_{i+1}$ as follows. We have the following equality:

$$\begin{aligned} p_a\Big(\sum_{k=1}^{i} Z_{B_k}+Z_{L_{i+1}}\Big) &= p_a\Big(\sum_{k=1}^{i} Z_{B_k}\Big)+p_a(Z_{L_{i+1}})+\Big(\sum_{k=1}^{i} Z_{B_k}\Big)\cdot Z_{L_{i+1}}-1 \\ &= p_a\Big(\sum_{k=1}^{i} Z_{B_k}\Big) \\ &=1\,. \end{aligned}$$

Hence we obtain $\sum_{k=1}^{i} Z_{B_k}+Z_{L_{i+1}}\leqq\sum_{k=1}^{l} Z_{B_k}$ by (6.4). Hence $Z_{L_{i+1}}\leqq\sum_{k=i+1}^{l} Z_{B_k}$. Therefore we obtain the relation about the supports of each side of this inequality $L_{i+1}\subseteqq B_{i+1}$. Hence $L_{i+1}=B_{i+1}$. This equality means that the Laufer sequence in [51] is nothing but the elliptic sequence. (This fact was already proved by J. Stevens in the appendix to §1 of [57].)

For the arguments in the next section, let us note the following statements: Even if the statement below seems differ from the original in the reference, it is only an easy modification of the original. Hence we shall only give the outline of the proof of it here.

Corollary (6.6.1) (Proposition (2.1) [49]). *Let (V, p) be a normal two-dimensional singularity with $p_a=1$ and $\psi\colon(\tilde{V}, A)\to(V, p)$ a resolution. Let $\{Z_{B_i};\ i=1,\dots, l\}$ be the elliptic sequence on $(\tilde{V}, A)$. Then the inequalities $(Z_{B_i})^2\leqq(Z_{B_{i+1}})^2$ for $i=1,\dots, l-1$, hold.*

Proof. $\sum_{i=1}^{j+1} Z_{B_i}\cdot A_h\leqq 0$ for any irreducible component of A. Hence

$$(\boldsymbol{Z}_{B_i})^2-(\boldsymbol{Z}_{B_{i+1}})^2=(\sum_{j=1}^{i+1}\boldsymbol{Z}_{B_j})\cdot(\boldsymbol{Z}_{B_i}-\boldsymbol{Z}_{B_{i+1}})\leqq 0.$$ Q. E. D.

Proposition (6.7). *Let (V, p) be a normal two-dimensional singularity with $p_a(V, p)=1$. Let us consider the resolution diagram as in* (3.7).

$$\begin{array}{ccc} (V, p) & \stackrel{\psi}{\longleftarrow} & (\tilde{V}, A) \\ & \nwarrow^{\phi} \quad \nearrow_{\tau} & \\ & (\tilde{V}', A') & \end{array}\quad .$$

Let $\{\boldsymbol{Z}_{B_i};\ i=1,\dots, l\}$ (resp. $\{\boldsymbol{Z}_{B'_i};\ i=1,\dots, l'\}$) be the elliptic sequence on $(\tilde{V}, A)$ (resp. on (V', A')). Then the following equalities hold. $l=l'$ and $\tau^{-1}(\boldsymbol{Z}_{B_i})=\boldsymbol{Z}_{B'_i}$ for $i=1,\dots, l$.

Proof. First of all, $\tau^{-1}(\boldsymbol{Z}_{B_1})=\boldsymbol{Z}_{B'_1}$ is well-known by Proposition (2.9) of [46]. Hence $\boldsymbol{Z}_{B_1}\cdot A_j=\boldsymbol{Z}_{B'_1}\cdot\tau^{-1}(A_j)$ holds for any irreducible component A_j of A. Then we can easily check the relation $\tau^{-1}(\tilde{B}_1)=\tilde{B}'_1$. Therefore the procedure to define $\{B_i;\ i=1,\dots, l\}$ is parallel with the procedure for $\{B'_i;\ i=1,\dots, l'\}$. Q. E. D.

Here recall that (V, p) is called numerically Gorenstein if there exists a divisor K' of the form $K'=\sum_{j=1}^{m} d_j\cdot A_j$, where $d_j\in\boldsymbol{Z}$ for $j=1,\dots, m$, such that $K'\cdot A_j=\Omega^2_{\tilde{V}}\cdot A_j$ holds for $j=1,\dots, m$ for some (hence for every) resolution ψ: $(\tilde{V}, A)\to(V, p)$ (in the usual notations).

Theorem (6.8) (Theorem 3.10 of [48]). *Let (V, p) be a normal two-dimensional numerical Gorenstein singularity with $p_a(V, p)=1$. Let ψ: $(\tilde{V}, A)\to(V, p)$ be the minimal resolution and $\{\boldsymbol{Z}_{B_i};\ i=1,\dots, l\}$ the elliptic sequence over $(\tilde{V}, A)$. Then $-K'=\sum_{i=1}^{l}\boldsymbol{Z}_{B_i}$ holds and $\boldsymbol{Z}_{B_1}$ is equal to the minimal elliptic cycle $\boldsymbol{E}$ on $(\tilde{V}, A)$.*

Outline of the proof. It is well-known that $-K'$ is contained in the former set of Theorem (6.4). Hence there is an integer k_0 such that $-K'=\sum_{i=1}^{k_0}\boldsymbol{Z}_{B_i}$ holds. In fact $k_0=l$ as follows. If $k_0\neq l$, the equality

$$p_a(\boldsymbol{Z}_{B_{k_0+1}})=\frac{1}{2}\boldsymbol{Z}_{B_{k_0+1}}\cdot(K'+\boldsymbol{Z}_{B_{k_0+1}})+1=\frac{1}{2}(\boldsymbol{Z}_{B_{k_0+1}})^2+1$$

contradicts the fact $p_a(\boldsymbol{Z}_{B_{k_0+1}})=1$. The remaining part of the assertion is cheked as follows: By the uniqueness of the minimal elliptic cycle $\boldsymbol{E}$, we have the relation $\boldsymbol{E}\leqq\boldsymbol{Z}_{B_1}$. With the aid of Lemma (3.3) of [28], we can easily check that $-K'-E$ is contained in the former set of Theorem (6.4). Hence there is an integer k_1 such that $-K'-E=\sum_{i=1}^{k_1} Z_{B_i}$. Then k_1 must be $l-1$ from the above equality for K'. Q.E.D.

Theorem (6.9) (Theorem 3.9 of [48], Proposition 2.2 of [51]). *Let* (V, p) *be a normal two-dimensional singularity with* $p_a(V, p)=1$ *and* $\{Z_{B_i}; i=1,\dots, l\}$ *the elliptic sequence on some resolution of* (V, p). *Then the inequality* $p_g(V, p)\leqq$ *the length of the elliptic sequence* l *holds.*

This is nothing but Proposition (2.2) of [51] by Remark (6.6).

According to S. S.-T. Yau, we introduce the following:

Definition (6.10) (Definition 3.10 of [48]). A normal two-dimensional singularity (V, p) with $p_a(V, p)=1$ is called *maximally elliptic* if (V, p) is a numerical Gorenstein singularity with the condition $p_g(V, p)=$the length of the elliptic sequence.

Theorem (6.11) (Theorem 3.11 of [48]). *Every maximally elliptic singularity is Gorenstein.*

§7. The Correspondence of Zariski's Canonical Resolution and the Minimal Resolution

(7.1) Based on the results of the previous sections, we discuss the relation between Zariski's canonical resolution and the minimal resolution of the normal two-dimensional Gorenstein singularity with $p_a=1$.

(7.2) Let (V, p) be a normal two-dimensional Gorenstein singularity with $p_a=1$. Let us consider the diagram $(*)$ of Theorem (4.6)

$$\text{(7.2.1)}\qquad \begin{array}{ccccccccccc} U= U_0 & \xleftarrow{\psi_1} & U_1 & \longleftarrow & \cdots & \xleftarrow{\psi_M} & U_M & \longleftarrow & \cdots & \xleftarrow{\psi_N} & U_N \\ \cup\!\!\shortparallel & & \cup\!\!\shortparallel & & & & \cup\!\!\shortparallel & & & & \cup\!\!\shortparallel \\ V= V_0 & \longleftarrow & V_1 & \longleftarrow & \cdots & \longleftarrow & V_M & \longleftarrow & \cdots & \longleftarrow & V_N \end{array}$$

which is the decomposition of Zariski's canonical resolution of (V, p) into the composition of the blowing-ups with smooth centers. (cf. Theorem (4.6) for the various conditions and the notations about this diagram.)

The first problem in this section is "*When does this resolution give the minimal resolution of* (V, p)?". Then we also discuss the absolute isolatedness of (V, p). We note the following well-known fact:

Theorem (7.3). *Let (V, p) be a normal two-dimensional singularity of multiplicity two. Then (V, p) is absolutely isolated if and only if (V, p) is rational.*

This is proved by many authors (D. Kirby [23], E. Brieskorn [8], G. N. Tjurina [42], J. Lipman [30], M. Reid [35], E. Horikawa [22]). Here we note that this follows also from Theorem (4.3) and our p_g-formula of Section 2.

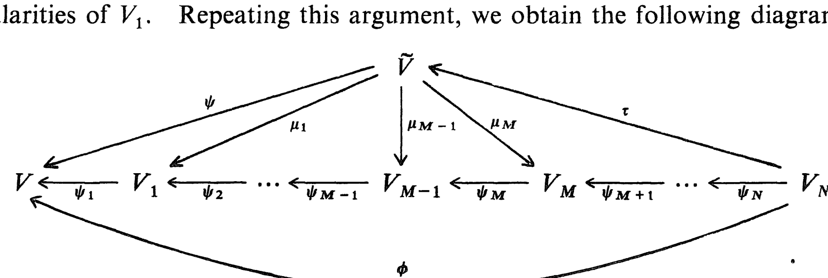
Hence the normal two-dimensional Gorenstein singularity with $p_a=1$ is absolutely isolated if and only if the singularities of V_M are rational in diagram $(*)$ of Theorem (4.6).

Theorem (7.4). *Let (V, p) be a normal two-dimensional Gorenstein singularity with $p_a=1$, $\psi: (\tilde{V}, A)\to(V, p)$ the minimal resolution of (V, p) and $\boldsymbol{E}$ the minimal elliptic cycle on $(\tilde{V}, A)$. Then the following numerical criteria hold:*

(*i*) *(V, p) is absolutely isolated if and only if the inequality $(\boldsymbol{E})^2\leqq -3$ holds.*

(*ii*) *Zariski's canonical resolution of (V, p) gives the minimal resolution of (V, p) if and only if the inequality $(\boldsymbol{E})^2\leqq -2$ holds.*

Proof. Let $\phi: V_N\to V$ be the **Z.C.R.** which is obtained in (4.6). First of all, we shall show the fact that $\tilde{V}$ dominates V_M as follows: If $M=0$ (i.e., $V_M=V$), this is trivial. If $M\geqq 1$ (i.e., $\mathrm{mult}_p V\geqq 3$), the equalities $p_a(Y_\phi)=p_a(V, p)=1$ are obtained in the proof of (4.6). Hence by (3.10), $\psi^{-1}(m_{V,p})$ is an invertible $O_{\tilde{V}}$-ideal sheaf. We obtain the morphism $\mu_1: \tilde{V}\to V_1$ such that the relation $\psi=\mu_1\circ\psi_1$ holds. Indeed μ_1 gives the minimal resolution of the singularities of V_1. Repeating this argument, we obtain the following diagram:



Here μ_i is the morphism which commutes the above diagram and gives the minimal resolution of the singularities of V_i for $i=1,\dots, M$.

(i) Suppose (V, p) is not absolutely isolated. There is a singularity

p_{M+1} on V_M with $p_a(V_M, p_{M+1})=1$. Then by the uniqueness of the minimal elliptic cycle E on $(\tilde{V}, A)$, we obtain the relation of supports $|E|\subseteqq|\mu_M^{-1}(p_{M+1})|$. Hence the inequalities $-(E)^2\leqq-(Z_{|\mu_M^{-1}(p_{M+1})|})^2\leqq\text{mult}_p V$ follow by (6.6.1), (6.8) and Theorem (2.7) of [46].

Conversely suppose there is no non-rational point on V_M. Then $p_g(V_{M-1}, p_M)=1$ by (4.8), so that (V_{M-1}, p_M) is a minimally elliptic singularity. In general, **Z.C.R.** of the rational singularity gives the minimal resolution ([30]). Hence ϕ gives the minimal resolution of (V_{M-1}, p_M). By Theorem (5.2), the canonical line bundle on V_N is written as $[-W_M]$ in the neighborhood of $|(\psi_M\circ\cdots\circ\psi_N)^{-1}(p_M)|$. By Theorem (3.4) of [28], the divisor W_M coincides with the minimal elliptic cycle on $(V_N, |(\psi_M\circ\cdots\circ\psi_N)^{-1}(p_M)|)$. Hence the uniqueness of the minimally elliptic cycle on $V_N=\tilde{V}$ implies the equality $E=W_M$. Therefore the equalities $-(E)^2=-(W_M)^2=-\text{mult}_{p_M} V_{M-1}\geqq 3$ hold by Theorem (2.7) of [46].

(ii) We have already seen that **Z.C.R.** ϕ of (V, p) gives the minimal resolution of (V, p) if and only if ϕ gives the minimal resolution of the singularities of V_M in the above diagram. Hence the problem is in the case where there is a singular point p_{M+1} in V_M with $p_a(V_M, p_{M+1})=1$. Then the relations $2=\text{mult}_{p_{M+1}} V_M\geqq-(Z_{|\mu_M^{-1}(p_{M+1})|})^2\geqq-(E)^2$ hold by the uniqueness of E over $(\tilde{V}, A)$.

Suppose the inequality $(E)^2\leqq-2$ holds. Then we obtain $\text{mult}_{p_{M+1}} V_M=-(Z_{|\mu_M^{-1}(p_{M+1})|})^2$. By Theorem (2.7) of [46], $\mu_M^{-1}(m_{V_M,p_{M+1}})$ is an invertible $O_{\tilde{V}}$-ideal in the neighborhood of $|\mu_M^{-1}(p_{M+1})|$. Therefore $\tilde{V}$ dominates the normalization of V_{M+1}. Repeating this argument, the equality $V_N=\tilde{V}$ follows.

Conversely suppose the equality $V_N=\tilde{V}$ holds. Let us assume that $\psi_{j_{p_g}}$ is the final blowing-up with the center P^1. The natural morphism $\tilde{V}=V_N\to V_{j_{p_g}-2}$ induces the minimal resolution of the singularity $(V_{j_{p_g}-2}, p_{j_{p_g}-1})$. By the same argument in the later half of proof of (i) the equality $E=W_{j_{p_g}}$ follows. On the other hand the equality $W_{j_{p_g}-1}=W_{j_{p_g}}$ is proved in (4.8). Therefore we obtain $(E)^2=(W_{j_{p_g}-1})^2=-\text{mult}_{p_{j_{p_g}-1}} V_{j_{p_g}-2}=-2$ by Theorem (2.7) of [46].

Q. E. D.

Remark (7.5). (i) of Theorem (7.4) extends the results in [28] and [50] about the absolute isolatedness in the case of $p_g\leqq 2$.

(7.6) Second, we consider the relation of the **Z.C.R.** and the minimal resolution without the condition on the integer $(\boldsymbol{E})^2$ in the minimal resolution. We shall represent the relation in correspondence with the inverse images $\{W_i;\ i=1,\dots,N\}$ of the center of the blowing-ups on V_N and the elliptic sequence $\{\boldsymbol{Z}_{B_i};\ i=1,\dots,l\}$ on the minimal resolution $(\tilde{V}, A)$. To state our result, we need the following terminologies. Let us consider the diagram $(*)$ of (4.6). Let the set of integers $\{j_1,\dots,j_{p_q}\}$ be the subset of $\{1,\dots,N\}$ defined in (4.7) (cf. (4.8)). In particular V_{j_h} is normal for $h=1,\dots,p_g$. Then we shall call the following $p_g(V, p)$ normal points:

Definition (7.7) (Definition 7 of [44]). In diagram $(*)$ of (4.6), the normal point q is a *starting point* if q is one of the following points.

(i) (V, p). (ii) Point q on V_{j_h} which satisfies the condition $p_a(V_{j_h}, q)=1$.

Let the set of integers $\{i_1,\dots,i_{p_g}\}$ be the subset of $\{1,\dots,N\}$ such that ψ_{i_h} is the blowing-up with the center "*a starting point*", for $h=1,\dots,p_g$, and that the inequality $i_h<i_{h+1}$ holds for all h. In particular the relations $i_1=1$ and $i_h=j_{h-1}+1$ for $h=2,\dots,p_q$, follow from the definition of the starting points.

Combining with (ii) of Theorem (7.4), the following statements extend the results about the description of the maximal ideal cycle by S. S. -T. Yau (Theorem 3.15 of [48], Theorem 2.1 of [50]):

Theorem (7.8). *Let (V, p) be a normal two-dimensional Gorenstein singularity over* $\boldsymbol{C}$ *with $p_a=1$. Let $\psi\colon(\tilde{V}, A)\to(V, p)$ be the minimal resolution of (V, p) and $\{\boldsymbol{Z}_{B_i};\ i=1,\dots,l\}$ the elliptic sequence on $(\tilde{V}, A)$. Let us consider Zariski's canonical resolution $\phi\colon V_N\to V$ and the following commutative diagram*:

$$\begin{array}{ccc} V & \xleftarrow{\quad\psi\quad} & \tilde{V} \\ & {\scriptstyle\phi}\nwarrow \quad \nearrow{\scriptstyle\tau} & \\ & V_N\ . & \end{array}$$

(*i*) *There is a sequence of the integers $1\leqq k_1<\cdots<k_{p_a}=l$ such that the following equalities hold*:

$$\sum_{i=1}^{k_1} \boldsymbol{Z}_{B_i}=\tau_*(W_{i_1})=\cdots=\tau_*(W_{j_1}),$$
$$\vdots$$
$$\sum_{i=k_{h-1}+1}^{k_h} \boldsymbol{Z}_{B_i}=\tau_*(W_{i_h})=\cdots=\tau_*(W_{j_h}),$$
$$\vdots$$

$$\sum_{i=k_{p_g-1}+1}^{k_{p_g}} Z_{B_i}=\tau_*(W_{i_{p_g}})=\cdots=\tau_*(W_{j_{p_g}}).$$

(ii) The equality $k_{p_g-1}=l-1$ holds. Hence the divisor $\tau_*(W_{i_{p_g}})(=\cdots=\tau_*(W_{j_{p_g}})$ by (i)) is the minimal elliptic cycle on $(\tilde{V}, A)$ (cf. (6.8)).

Corollary (7.9). *Let (V, p) be a maximally elliptic singularity over $\mathbf{C}$ and $\psi: (\tilde{V}, A)\to(V, p)$ the minimal resolution of (V, p). Then the maximal ideal cycle Y_ψ coincides with the fundamental cycle Z_0.*

Proof of (7.9). By the definition of maximally elliptic singularity (6.10), we have the relation $l=p_g(V, p)$ in (7.8). Hence the equalities $k_i=i$, for $i=1,\dots, p_g$, follow. Q. E. D.

The rest of this paper is devoted to the proof of Theorem (7.8). The proof of (i) is divided into three parts as follows. First of all, we shall show the following claim: In the proof of this, we shall use the assumption that the singularity is defined over $\mathbf{C}$.

Claim (7.10). *The equalities $\tau_*(W_{i_1})=\cdots=\tau_*(W_{j_1})$ hold.*

Proof of Claim. If the multiplicity of V at p is greater than or equal to three, the equality $i_1=j_1$ holds by the definition of the indexes $\{i_h\}$ and $\{j_h\}$. We shall consider the case of multiplicity two. As we have seen in (4.8), the equality $W_{j_1-1}=W_{j_1}$ holds. Hence if V_1 is non-normal (i.e., $j_1-1=1$ holds), the equality $W_1=W_{j_1}$ induces our claim.

In the rest of this paragraph, we shall discuss the case that V_1 is normal. Let us represent the singularity (V, p) in the following form:

$$(V, p)=(\{(x, y, z)\in \mathbf{C}|z^2-g(x, y)=0\}, o).$$

(Cf. §1 of [43] about this description in detail.) We shall employ the notations of §1 of [43].

By using the projection π from V to (x, y)- plane H, which is defined by $H=\{z=0\}$, the following diagram is induced from **Z.C.R.** of (V, p).

$$\begin{array}{ccccccc}
U & \xleftarrow{\psi} & U_1 & \xleftarrow{\psi_1} & \cdots & \longleftarrow & U_{T(n)+\gamma_n} \\
\cup\!\parallel & & \cup\!\parallel & & & & \cup\!\parallel \\
V & \longleftarrow & V_1 & \longleftarrow & \cdots & \longleftarrow & V_{T(n)+\gamma_n} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_1} & & & & \downarrow{\scriptstyle \pi_{T(n)+\gamma_n}} \\
H & \longleftarrow & H_1 & \longleftarrow & \cdots & \longleftarrow & H_{T(n)+\gamma_n}. \\
\parallel & & & & & & \\
\{z=0\} & & & & & &
\end{array}
\qquad (T(n)+\gamma_n=N)$$

Let us denote the discriminant locus $\{g=0\}$ in H of π by D. Here $\mathrm{mult}_p D = \mathit{three}$ by the assumption that V_1 is normal. The discriminant locus for π_1 is $(D)_1 \cup E_1^{(1)}$, where $(D)_1$ denotes the strict transform of D by ψ_1 and $E_1^{(1)}$ the exceptional locus of ψ_1 in H_1 (i.e., $\Theta_1^{(1)} \cap H_1 = E_1^{(1)}$). The computations by E. Horikawa and D. Kirby (Lemma 5 of [22], §§2.7–2.8 of [23]) say "*If the condition* $\mathrm{mult}_q (D)_1 \cup E_1^{(1)} \leqq 3$ *holds for all points of* $(D)_1 \cap E_1^{(1)}$, *then* (V, p) *is an absolutely isolated singularity* (*hence a rational singularity*)." Since $p_a(V, p)=1$ holds by our assumption, there is a point of $(D)_1 \cap E_1^{(1)}$, say q_0, such that $\mathrm{mult}_{q_0} (D)_1 \cup E_1^{(1)} \geqq 4$ holds. In fact the conditions $\mathrm{mult}_{q_0} (D)_1 = 3$, $(D)_1 \cap E_1^{(1)} = \{q_0\}$ and $\mathrm{mult}_{q_0} (D)_1 \cup E_1^{(1)} = 4$ hold. In particular the equality $q_0 = p_2$ holds. Blow up V_1 at p_2, say $\psi_2 \colon V_2 \to V_1$. Then V_2 has the singularity along $\boldsymbol{P}^1$. The normalization of V_2 is obtained by the blowing-up of V_2 along this $\boldsymbol{P}^1$, say $\psi_3 \colon V_3 \to V_2$, as in (4.5). This means that j_1 is three. Hence we have the following diagram:

$$\begin{array}{ccccccc}
U & \longleftarrow & U_1 & \longleftarrow & U_2 & \longleftarrow & U_3 \\
\cup\!\shortparallel & \psi_1 & \cup\!\shortparallel & \psi_2 & \cup\!\shortparallel & \psi_3 & \cup\!\shortparallel \\
V & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow & V_3 \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_1} & & {\scriptstyle \pi_2}\searrow & & \swarrow{\scriptstyle \pi_3} \\
H & \longleftarrow & H_1 & \longleftarrow & & H_2 . &
\end{array}$$

The discriminant locus of π_2 is $(D)_2 \cup E_1^{(2)} \cup 2 \cdot E_2^{(2)}$. Here $(D)_2$ (resp. $E_1^{(2)}$) denotes the strict transform of $(D)_1$ (resp. $E_1^{(1)}$) by ψ_2 and $E_2^{(2)}$ the exceptional locus on H_2 of ψ_2 (i.e., $E_2^{(2)} = H_2 \cap \Theta_2^{(2)}$). Then the equality $\Theta_1^{(2)} \cap H_2 = E_1^{(2)}$ holds in this diagram (see §1 of [43]). The discriminant locus of π_3 is $(D)_2 \cup E_1^{(2)}$. As we have seen in (4.8), the intersection $\Theta_2^{(3)} \cap V_3$ is the empty set. Applying the arguments of (4.8) to the data (H, E, D), we can see that the emptyness of $E_1^{(2)} \cap (D)_2$ follows from the equality $\mathrm{mult}_p D = \mathrm{mult}_{p_2} (D)_1$. Hence V_3 is non-singular in $\Theta_1^{(3)} \cup \Theta_2^{(3)}$. Therefore we obtain the relation

$$W_1 = F + W_3 \quad \text{on} \quad V_N,$$

where $F = \Theta_1^{(N)} \cap V_N$. Hence the supports $|F|$ and $|W_3|$ have no common irreducible component.

Furthermore, by Theorem (5.2), we have the relation

$$-K_{V_N} = \sum_{h=1}^{p_g} W_{j_h} \quad \text{on} \quad V_N .$$

The adjunction formula induces the equalities

$$-K_{\tilde{V}}=\tau_*(-K_{V_N})=\sum_{h=1}^{p_g}\tau_*(W_{j_h}) \quad \text{on} \quad \tilde{V}.$$

Then the equalities of the supports $A=|-K_{\tilde{V}}|=|\tau_*(W_{j_1})|$ on $\tilde{V}$ means that $|F|$ is contracted by τ. Hence the equality $\tau_*(W_1)=\tau_*(W_3)$ follows.

Finally our claim is established.

(7.11) By using the claim above, we shall show the existence of the integer k_1 of (i).

First we note the following well-known relation:

$\tau_*(W_{i_1})\cdot A_j=Y_\psi\cdot A_j\leqq 0$ for any irreducible component A_j of A.

By Lemma (3.8), the equality

$p_a(W_{j_1})=p_a(\tau_*(W_{j_1}))-$(non-negative integer)

holds. We already have checked the equality $p_a(W_{j_1})=1$ in Section 4. Hence the equality $p_a(\tau_*(W_{j_1}))=1$ holds. Since the equality $\tau_*(W_{i_1})=\tau_*(W_{j_1})$ holds, there is an integer k_1 such that $\tau_*(W_{i_1})=\sum_{i=1}^{k_1} Z_{B_i}$ holds by Theorem (6.4).

(7.12) We shall discuss the divisors $\tau_*(W_{j_h})$ for $h\geqq 2$. As we mentioned in (7.10), the equality

$$-K_{\tilde{V}}=\sum_{h=1}^{p_g}\tau_*(W_{j_h}) \quad \text{on} \quad \tilde{V}$$

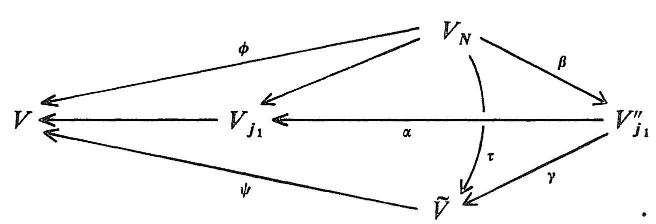
holds. Furthermore by a theorem of Yau ((6.8) in the present paper), the following equality holds.

$$-K_{\tilde{V}}=\sum_{i=1}^{l} Z_{B_i} \quad \text{on} \quad \tilde{V}.$$

Hence by the result of (7.11), we obtain the equality $\sum_{i=k_1+1}^{l} Z_{B_i}=\sum_{h=2}^{p_g}\tau_*(W_{j_h})$ on $\tilde{V}$. In particular we obtain the equality of supports

(7.12.1) $$B_{k_1+1}=|\tau_*(W_{j_2})| \quad \text{on} \quad \tilde{V}.$$

Let α: $V''_{j_1}\to V_{j_1}$ be the proper modification of V_{j_1} which is induced from the minimal resolution of the singularities of V_{j_1}. Let us consider the following commutative diagram:



.

Here β and γ are canonically induced.

Clearly the equality $\tau_*(W_{i_2})=\gamma_*(\beta_*(W_{i_2}))$ holds. In fact $\beta_*(W_{i_2})$ is the maximal ideal cycle of the singularity (V_{j_1}, p_{i_2}) with respect to the resolution α. Then we have the relations of supports

$$|\gamma^{-1}(|\tau_*(W_{i_2})|)| \supseteqq |\beta_*(W_{i_2})| = |\alpha^{-1}(p_{i_2})| \quad \text{on} \quad V''_{j_1}.$$

We shall show that these three sets are indeed the same as the one below. By (7.12.1) and the definition of the elliptic sequence (6.3), we have the relations

$$\tau_*(W_{j_1})\cdot\tau_*(W_{j_h})=0 \qquad \text{for} \quad h\geqq 2, \quad \text{on} \quad \tilde{V}.$$

Moreover, by Claim (7.10) they imply the relations

$$\tau_*(W_i)\cdot\tau_*(W_{j_h})=0 \qquad \text{for} \quad h\geqq 2,\ 1\leqq i\leqq j_1 \quad \text{on} \quad \tilde{V}.$$

On V_N, the divisor $\tau^{-1}(\tau_*(W_i))$ is written in the following form:

$$\tau^{-1}(\tau_*(W_i))=W_i+F_i \quad \text{on} \quad V_N,$$

where the divisor F_i is not necessarily effective but the support $|F_i|$ is contracted by τ. Hence the equality $F_i\cdot\tau^{-1}(\tau_*(W_{j_h}))=0$ follows for $h\geqq 1$ and for all i. We obtain the equalities

(7.12.2) $\qquad W_i\cdot\tau^{-1}(\tau_*(W_{j_h}))=0 \qquad \text{for} \quad h\geqq 2,\ 1\leqq i\leqq j_1 \quad \text{on} \quad V_N.$

Lemma (7.12.3). *Let (X, p) be a normal two-dimensional singularity and $\psi\colon(\tilde{X}, A)\to(X, p)$ a resolution of (X, p) such that $\psi^{-1}(m_{X,p})$ is $O_{\tilde{X}}$-invertible. Let us consider the following diagram*:

$$\begin{array}{ccc} & \swarrow^{\psi} & \tilde{X} \\ & & \downarrow{\scriptstyle v} \\ X & \xleftarrow[\psi_1]{} & X_1 \end{array} .$$

Here $\psi_1\colon X_1\to X$ is the blowing-up of X with center $m_{X,p}$ followed by the normalization, and v the induced natural morphism.

Then an irreducible component A_j of A is contracted to a single point of X_1 by v if and only if the equality $A_j\cdot Y_\psi=0$ holds.

This is well-known (cf. [42]).

In general we have the relation $W_1\cdot A_j\leqq 0$ for any irreducible component A_j of A in our situation. Hence (7.12.2) for $i=1$ induces the relations

$W_1\cdot A_j=0$ for any irreducible component A_j of $|\tau^{-1}(\tau_*(W_{j_2}))|$.

By (7.12.3) and the connectedness of $|\tau^{-1}(\tau_*(W_{j_2}))|$, $|\tau^{-1}(\tau_*(W_{j_2}))|$ is contracted to a point of V_1. By the same argument as above, the support $|\tau^{-1}(\tau_*(W_{j_2}))|$ is contracted to a point of V_i for $2\leqq i\leqq j_1$. Hence we obtain the equality

(7.12.4) $$|\gamma^{-1}(|\tau_*(W_{j_2})|)|=|\alpha^{-1}(p_{i_2})|.$$

Since the neighborhood system of $|\alpha^{-1}(p_{i_2})|$ in V''_{j_1} blows down to the singularity (V_{j_1}, p_{i_2}), there exists the elliptic sequence $\{Z_{B''_i}; i=1,..., l''\}$ on $(V''_{j_1}, |\alpha^{-1}(p_{i_2})|)$ such that the condition $B''_1=|\alpha^{-1}(p_{i_2})|$ holds (cf. Definition (6.3)). Then the elliptic sequence on $(V''_{j_1}, |\gamma^{-1}(A)|)$ is the set of the divisors of the form $\{\gamma^{-1}(Z_{B_i}); i=1,..., l\}$ by Proposition (6.7). Hence by the relation (7.12.4) and the definition of the elliptic sequence (6.3), we obtain the equalities

$$\gamma^{-1}(Z_{B_{k_1+i}})=Z_{B''_i} \quad \text{for} \quad 1\leqq i\leqq l'' \quad \text{on} \quad V''_{j_1}.$$

We apply the arguments in (7.10) and (7.11) to the resolution α of the singularity (V_{j_1}, p_{i_2}). Then we obtain the equality $\beta_*(W_{i_2})=\beta_*(W_{j_2})$ and the integer k_2 such that the relation $\beta_*(W_{i_2})=\sum_{i=k_1+1}^{k_2}\gamma^{-1}(Z_{B_i})$ holds. Therefore we obtain the equalities $\tau_*(W_{i_2})=\gamma_*(\beta_*(W_{i_2}))=\gamma_*(\beta_*(W_{j_2}))=\tau_*(W_{j_2})$ and

$$\tau_*(W_{i_2})=\gamma_*(\beta_*(W_{i_2}))=\sum_{i=k_1+1}^{k_2}\gamma_*(\gamma^{-1}(Z_{B_i}))=\sum_{i=k_1+1}^{k_2}Z_{B_i}.$$

Repeating the arguments in this paragragh (7.12), the assertion of (i) follows.

(7.13) *Proof of (ii).* Let ϕ: $V_N\to V$ be the **Z.C.R.** of (V, p) and E_{V_N} the minimal elliptic cycle on $(V_N, |\phi^{-1}(p)|)$. By the same arguments as the later half of proof of (i) of Theorem (7.4), we can obtain the equality $W_{j_{p_g}-1}=W_{j_{p_g}}=E_{V_N}$ on V_N. The equality $p_a(\tau^{-1}(E))=1$ implies the relation $\tau^{-1}(E)\geqq E_{V_N}$ on V_N by the definition of minimal elliptic cycle. Hence the following relations hold:

$$E=\tau_*(\tau^{-1}(E))\geqq\tau_*(E_{V_N})=\tau_*(W_{j_{p_g}}) \quad \text{on} \quad \tilde{V}.$$

Therefore the integer k_{p_g-1} should equal to the integer $l-1$ by the equality $E=Z_{B_l}$ (6.8) and (i) of Theorem (7.8).

This completes the proof of Theorem (7.8).

References

[1] Abhyanker, S. S., Local rings of high embedding dimension, *Amer. J. Math.*, **89** (1967), 1073–1077.

[2] Altman, A. and Kleiman, S., *Introduction to Grothendieck duality theory*, Lecture Note in Math., **146**, Springer, 1970.

[3] Artin, M., Some numerical criteria for contractability of curves on algebraic surfaces, *Amer. J. Math.*, **84** (1962), 485–496.

[4] ———, On isolated rational singularities of surface, *Amer. J. Math.*, **88** (1966), 129–136.

[5] ———, Algebraic approximation of structures over complete local ring, *Publ. I. H. E. S.*, **36** (1969), 23–58.

[6] Banica, C. and Stanasila, O., *Algebraic methods in the global theory of complex spaces*, EDITURA ACADEMIEI., John Wiley And Sons, 1976.

[7] Banica, C. and Ueno, K., On the Hilbert-Samuel polynomial in complex analytic geometry, *J. Math. Kyoto Univ.*, **20** (1980), 381–389.

[8] Brieskorn, E., Über die Auflösung gewisser Singularitäten von holomorphen Abbildung, *Math. Ann.*, **166** (1966), 76–102.

[9] Buchsbaum, D. A. and Eisenbud, D., Algebraic structures for finite free resolutions and some structure theorems for ideals of codimension 3, *Amer. J. Math.*, **99** (1977) 447–485.

[10] A'Campo, N., Le function zêta d'une monodromie, *Comment. Math. Helve.*, **50** (1975), 233–248.

[11] Fisher, G., *Complex analytic geometry*, Lecture Note in Math., **538**, Springer, 1976.

[12] Flenner, H., Die Sätze von Bertini für lokale Ringe, *Math. Ann.*, **229** (1977), 97–111.

[13] Fujiki, A., On the minimal models of complex manifolds, *Math. Ann.*, **253** (1980), 111–128.

[14] Goto, S. and Watanabe Kei-ichi, On graded rings I, *J. Math. Soc. Japan.*, **30** (1978), 179–213.

[15] Grothendieck, A. and Dieudonne, J, *Elements de Geometri Algebrique II, III,* Publ. I. H. E. S., **8**, **11**, **17**.

[16] Grothendieck, A., *Local cohomology*, Lecture Note in Math., **41**, Springer, 1966.

[17] Hartshorne, R., *Algebraic geometry*, Graduate Texts in Math., **52**, Springer, 1977.

[18] Hartshorne, R. and Ogus, A., On the factoriality of local rings of small embedding codimension, *Comm. Algebra*, **1** (5) (1974), 415–437.

[19] Hironaka, H., Resolution of singularities of an algebraic variety over a field of characteristic zero I, II, *Ann. of Math.*, **79** (1964), 109–326.

[20] ———, Certain numerical characters of singularities, *J. Math. Kyoto Univ.*, **10** (1970), 151–187.

[21] Hironaka, H. and Rossi, H., On the equivalence of imbeddings of exceptional complex space, *Math. Ann.*, **156** (1964), 313–333.

[22] Horikawa, E., On deformation of quintic surfaces, *Invent. Math.*, **31** (1975), 43–85.

[23] Kirby, D., The structure of an isolated multiple point of a surface II, *Proc. London Math. Soc.*, (3) VII (1957), 1–18.

[24] ———, The genus of an algebraic curve, *Quart. J. Math. Oxford.*, (2) **12** (1961), 217–226.

[25] Kleiman, S. L., Toward a numerical theory of ampleness, *Ann. of Math.*, **84** (1966), 293–344.

[26] Laufer, H. B., *Normal two-dimensional singularities*, Ann. Math. Studies, **71**, Princeton Univ. Press, 1971.

[27] ———, On μ for surface singularities, *Proc. Symposia Pure Math.*, **30**, "*Several Complex Variables*" Amer. Math. Soc., (1977), 37–44.

[28] ———, On minimally elliptic singularities, *Amer. J. Math.*, **99** (1977), 1257–1295.

[29] ———, Simultaneous resolutions of some families of isolated surface singularities, *Announcement in Arcata*, 1981.

[30] Lipman, J., Rational singularities, with applications to algebraic surfaces and unique factorization, *Publ. I. H. E. S.*, **36** (1969), 195–279.
[31] ————, Introductions to resolution of singularities, *Proc. Symposia Pure Math.*, **29** (1975), 187–230.
[32] Matsumura, H., *Commutative algebra*, Math. Lecture Note, Second edition, Benjamin, 1980.
[33] Northcott, D. G., A note on the coefficients of the abstract Hilbert function. *J. London Math. Soc.*, **35**, (1960), 209–214.
[34] Ohyanagi, S., Yoshinaga, E. and Hiura, M., Remarks on maximally elliptic singularities, *Science reports of Yokohama National Univ. Sec. II.*, **29** (1982), 1–11.
[35] Reid, M., Hyperelliptic linear systems on a *K3* surface, *J. London Math. Soc.* (2), **13** (1976), 427–437.
[36] ————, Canonical 3-folds, *Journces de geometrie algebrique d'Anges* 1979, "*Algebraic geometry*" edited by A. Beauville, Sijthoff and Noordhoff, (1980), 273–310.
[37] Sally, J. D., On the associated graded ring of a local Cohen-Macaulay ring, *J. Math. Kyoto Univ.*, **17** (1977), 19–21.
[38] ————, *Numbers of generators of ideals in local rings*, Lecture Notes in Pure and Applied Math., **35**, Marcel Dekker, 1978.
[39] ————, Tangent cones at Gorenstein singularities, *Compositio Math.*, **40** (1980), 167–175.
[40] ————, Cohen-Macaulay local rings of maximal embedding dimension, *J. Algebra*, **56** (1979), 168–183.
[41] Serre, J. P., *Algebre locale multiplicites*, Third ed., Lecture Notes in Math., **11**, Springer, 1975.
[42] Tjurina, G. N., Absolute isolatedness of rational singularities and triple points, *Func. Anal. its Appl.*, **2** (1968), 324–333 (English translation).
[43] Tomari, M., A geometric characterization of normal two-dimensional singularities of multiplicity two with $p_a \leqq 1$, *Publ. R. I. M. S. Kyoto Univ.*, **20** (1984), 1–20.
[44] ————, On the resolution process of normal Gorenstein surface singularity with $p_a \leqq 1$, *Proc. Japan Academy*, **59** (1983), 211–213.
[45] ————, The singularity satisfying the relation $L(V, p) = p_g(V, p)$ (undecided title) *in preparation*.
[46] Wagreich, P., Elliptic singularities of surfaces, *Amer. J. Math.*, **92** (1970), 419–454.
[47] Wahl, J. M., Equations defining rational singularities, *Ann. Scient. Ec. Norm. Sup.*, **10** (1977), 231–264.
[48] Yau, S. S.-T., On maximally elliptic singularities, *Trans. Amer. Math. Soc.*, **257** (1980) 269–329.
[49] ————, Hypersurface weighted dual graphs of normal singularities of surfaces, *Amer. J. Math.*, **101** (1979), 761–812.
[50] ————, Gorenstein singularities with geometric genus equal to two, *Amer. J. Math.*, **101** (1979), 813–854.
[51] ————, On strongly elliptic singularities, *Amer. J. Math.*, **101** (1979), 855–884.
[52] ————, Sheaf cohomology on 1-convex manifolds, "*Recent Developments in Several Complex Variables*" *Ann. of Math. Studies*, Princeton Univ., **100** (1981), 429–452.
[53] Giraud, J., Surfaces d'Hilbert-Blumenthal III, in "*Surface Algébriques*" *Séminaire de Géometrie Algébrique Orsay* 1976–1978., Lect. Notes in Math., **868**, Springer 35–57.
[54] Morales, M., Calcul de quelques invariants des singularités de surface normale. in "*Nœuds, tresses et singularités*" *Comptes rendus du Séminaire tenu aux Plans-sur-Bex* (*Suisse*)., — Monographie de L'Enseignement Mathématique., **31**, Universite de Genéve,

(1983), 191–203.

[55] ———, Polynome d'Hilbert-Samuel des clotures integrales des puissances d'un ideal m-primaire, *preprint*.

[56] ———, Polyedre de Newton et genre geometrique d'une singularite intersection complete, *preprint*.

[57] Stevens, J., Elliptic surface singularities and smoothings of curves, 1983, *preprint*.

Mathematical Institute, University of Leiden, The Netherlands.