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Conditions for Well-Posedness in Gevrey Classes of the Cauchy Problems for Fuchsian Hyperbolic Operators

By

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§1. Introduction

Let us consider the following operator.

$$P=\partial_t^2-t^{2l}\partial_x^2+at^s\partial_x$$

where l and s are non-negative integers and a is non-zero constant. It is well-known that if $s\geqq l-1$, the Cauchy problem for P is well-posed in C^∞ (see Oleinik [11]). Ivrii [6] showed the following. When $0\leqq s<l-1$, the Cauchy problem for P is well-posed in Gevrey class $\gamma_{loc}^{(\kappa)}$ if and only if $1\leqq\kappa<(2l-s)/(l-s-1)$. This simple example shows us a delicate relation among the well-posed class, the order of degeneracy of a principal part and that of a lower order term for non-strictly hyperbolic operators. Hence in this paper we shall consider whether this fact is valid for more general non-strictly hyperbolic operators.

In the case of non-characteristic operators the well-posedness in Gevrey class is studied by Ohya [10], Leray and Ohya [9], Beals [1], Bronstein [3], Ivrii [5], Kajitani [7], Komatsu [8], Steinberg [12], Trepreau [14], Wakabayashi [17] and others. Igari [4] extends Ivrii's example to higher order non-strictly hyperbolic operators with double characteristics under some assumptions on coefficients of the operators.

On the other hand Baouendi and Goulaouic [2] define Fuchsian partial differential operators and discuss Cauchy Kowalevski's type theorem. Tahara

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[13] considers the Cauchy problems for Fuchsian hyperbolic operators in smooth function space. Here we shall note that Fuchsian partial differential operators are the natural extension of non-characteristic operators.

Hence we shall discuss the well-posedness of the Cauchy problems for Fuchsian hyperbolic operators in the Gevrey class. And we shall get a close connection among an admissible class of the Cauchy problem, a principal part and lower order terms.

Let $(t, x) \in [0, T] \times \boldsymbol{R}^n$ and $(D_t, D_x) = (D_t, D_{x_1}, \dots, D_{x_n}) = (-\sqrt{-1}\partial_t, -\sqrt{-1}\partial_{x_1}, \dots, -\sqrt{-1}\partial_{x_n})$. Let us denote by (τ, ξ) the dual variable of (t, x). Next we shall define function spaces used in this paper.

Definition 1. ($\gamma_{loc}^{(\kappa)}$; $\kappa \geqq 1$) We define $\gamma_{loc}^{(\kappa)}$ the set of functions $f(x) \in C^\infty(\boldsymbol{R}^n)$ satisfying the property that for any compact set $K \subset \boldsymbol{R}^n$ there exist constants c, $R > 0$ such that for any multi-indices α

$$|D_x^\alpha f(x)| \leqq c R^{|\alpha|}(|\alpha|!)^\kappa \quad \text{for} \quad x \in K. \tag{1.1}$$

Definition 2. ($\gamma^{(\kappa)}$; $\kappa \geqq 1$) We denote by $\gamma^{(\kappa)}$ the set of functions $f(x) \in C^\infty(\boldsymbol{R}^n)$ with the following property. There exist constants c, $R > 0$ such that for any multi-indices α

$$|D_x^\alpha f(x)| \leqq c R^{|\alpha|}(|\alpha|!)^\kappa \quad \text{for} \quad x \in \boldsymbol{R}^n. \tag{1.2}$$

Definition 3. ($\Gamma^{(\kappa)}$; $\kappa \geqq 1$) We say $f(x) \in H^\infty$ $(\equiv \bigcap_s H^s(\boldsymbol{R}^n))$ belongs to $\Gamma^{(\kappa)}$ if there exist constants c, $R > 0$ such that

$$\|D_x^\alpha f(x)\| \leqq c R^{|\alpha|}(|\alpha|!)^\kappa \tag{1.3}$$

for any multi-indices $\alpha \in \boldsymbol{N}^n$, where $\| \ \|$ is L^2-norm with respect to x.

Now we shall define Fuchsian partial differential operators according to Baouendi-Goulaouic [2]. Let

$$\begin{aligned} P(t, x, D_t, D_x) = t^k D_t^m + P_1(t, x, D_x) t^{k-1} D_t^{m-1} + \cdots \\ + P_k(t, x, D_x) D_t^{m-k} + P_{k+1}(t, x, D_x) D_t^{m-k-1} + \cdots + P_m(t, x, D_x) \end{aligned}$$

be a partial differential operator satisfying the following.

(A–I) $\quad k \in \boldsymbol{Z}, \quad 0 \leqq k \leqq m$

(A–II) $\quad \operatorname{ord} P_j(t, x, D_x) \leqq j$

(A–III) $\quad \operatorname{ord} P_j(0, x, D_x) = 0 \quad$ for $\quad 1 \leqq j \leqq k$

Then P is said to be of Fuchsian type with weight $m-k$ with respect to t. From (A–III) we shall set $P_j(0, x, D_x) = a_j(x)$ for $1 \leqq j \leqq k$. Let $\mathscr{C}(\lambda, x)$ be a characteristic polynomial

$$\mathscr{C}(\lambda, x)=\lambda(\lambda-1)\cdots(\lambda-m+1)+ia_1(x)\lambda(\lambda-1)\cdots(\lambda-m+2)$$
$$+\cdots+i^k a_k(x)\lambda(\lambda-1)\cdots(\lambda-m+k+1).$$

Its roots, called characteristic exponents, are denoted by $\lambda=0, 1,\dots, m-k-1$, $\rho_1(x),\dots, \rho_k(x)$.

(A–IV) there exists a constant $c>0$ such that

$$|(\lambda-\rho_1(x))\cdots(\lambda-\rho_k(x))|\geqq c|\lambda(\lambda-1)\cdots(\lambda-m+k+1)$$

for $\lambda\in \mathbb{Z}$, $\lambda\geqq m-k$.

Under these assumptions, we can consider the following Cauchy problem for P

$$\text{(1.4)}\qquad \begin{cases} P(t, x, D_t, D_x)u(t, x)=f(t, x) \\ D_t^j u(t, x)|_{t=0}=u_j(x), \quad 0\leqq j\leqq m-k-1. \end{cases}$$

Baouendi-Goulaouic [2] study the Cauchy problem for P in the analytic function space and Tahara [13] investigate in C^∞-function space. Since our function space is Gevrey class, we assume that coefficients of P belong to $\mathscr{B}([0, T], \gamma^{(\kappa)})$ i.e.

$$\text{(A–V)}\qquad P_j(t, x, D_x)=\sum_{|\beta|\leqq j} a_{j,\beta}(t, x)D_x^\beta$$

where $a_{j,\beta}(t, x)\in\mathscr{B}([0, T], \gamma^{(\kappa)})$.

Next we shall consider a leading term of P.

$$\text{(A–VI)}\qquad \tau^m+\sum_{j=1}^{k}\sum_{|\beta|=j} a_{j,\beta}(t, x)\tau^{m-j}\xi^\beta+\sum_{j=k+1}^{m}\sum_{|\beta|=j} a_{j,\beta}(t, x)$$
$$\times t^{j-k}\tau^{m-j}\xi^\beta=\prod_{j=1}^{m}(\tau-t^l\lambda_j(t, x, \xi))$$

where $l>0$ is a rational number and $\lambda_j(t, x, \xi)$ are real valued functions with the property:

If $i\neq j$, $\lambda_i\neq\lambda_j$ for any $(t, x)\in[0, T]\times\mathbb{R}^n$, $|\xi|=1$ and for any $b\geqq 0$, α, $\beta\in\mathbb{N}^n$ there exists a constant $c=c_{\alpha,\beta,b}$ such that

$$|D_\xi^\alpha D_t^b D_x^\beta\lambda_j(t, x, \xi)|\leqq c \qquad \text{for} \quad (t, x)\in[0, T]\times\mathbb{R}^n,\ |\xi|=1.$$

Finally we shall assume on lower order terms of P.

(A–VII) For $1\leqq|\beta|\leqq j-1$, $2\leqq j\leqq m$

$$\text{(1.5)}\qquad a_{j,\beta}(t, x)=t^{\gamma(j,\beta)}\hat{a}_{j,\beta}(t, x)$$

where $\gamma(j, \beta)$ is a non-negative integer and $\hat{a}_{j,\beta}(t, x)\in\mathscr{B}([0, T], \gamma^{(\kappa)})$.

We can easily see from (A–III) that $\gamma(j, \beta) \geqq 1$ for $2 \leqq j \leqq k$, $1 \leqq |\beta| \leqq j-1$. Here we shall define a number as follows.

$$(1.6) \qquad \alpha(m-j+|\beta|, \beta) = \begin{cases} \gamma(j, \beta) & \text{if } 2 \leqq j \leqq k \\ \gamma(j, \beta)+j-k & \text{if } k+1 \leqq j \leqq m. \end{cases}$$

Let us note that $\alpha(m-j+|\beta|, \beta) \geqq 1$.
Here we shall define the important number $\sigma \geqq 1$ which determines admissible data classes of the Cauchy problems. For any j $(1 \leqq j \leqq m-1)$ let k_j $(1 \leqq k_j \leqq m-1)$ be the lowest integer such that $\alpha(j, \beta)/l - |\beta| + k_j > 0$ for any β $(1 \leqq |\beta| \leqq j-1)$. Next we set

$$\sigma_j = \max_{1 \leqq |\beta| \leqq j-1} \{|\beta| - \alpha(j, \beta)/l, 0\} \quad \text{and} \quad \nu = \max_{1 \leqq i \leqq m-1} \{\sigma_i/k_i\}.$$

Next we define $\sigma \geqq 1$ such that

$$\sigma = \max_{1 \leqq i \leqq m-1} \{(k_i \nu + m - i)/(m-i)\}.$$

Then we obtain the main theorem.

Theorem 1. *Under the assumptions* (A–I)$\sim$(A–VII), *for any* $u_j(x) \in \gamma_{loc}^{(\kappa)}$ $(0 \leqq j \leqq m-k-1)$ *and for any* $f(t, x) \in \mathscr{B}([0, T], \gamma_{loc}^{(\kappa)})$ *there exists an unique solution* $u(t, x) \in \mathscr{B}([0, T], \gamma_{loc}^{(\kappa)})$ *of the equation*

$$(1.7) \qquad \begin{cases} P(t, x, D_t, D_x)u(t, x) = f(t, x) \\ D_t^j u(t, x)|_{t=0} = u_j(x), \quad 0 \leqq j \leqq m-k-1 \end{cases}$$

for any κ $(1 \leqq \kappa < \sigma/(\sigma-1))$, *i.e. the Cauchy problem* (1.7) *is well-posed in* $\gamma_{loc}^{(\kappa)}$ $(1 \leqq \kappa < \sigma/(\sigma-1))$.

Note. (1) In the case of $\sigma = 1$ the Cauchy problem (1.7) is well-posed in C^∞-function space (see Tahara [13]). (2) From the definition of ν we have $0 \leqq \nu < 1$.

Finally we shall state some examples of Theorem 1.

Example 1. Let P be a second order partial differential operator

$$P = D_t^2 - t^{2l} D_x^2 + a(t, x)D_t + b(t, x)t^s D_x + c(t, x)$$

where l, s are non-negative integers and coefficients a, b, c belong to $\mathscr{B}([0, T], \gamma^{(\kappa)})$. In the case of $s \geqq l-1$ the Cauchy problem for P is well-posed in $\gamma_{loc}^{(\kappa)}$ $(1 \leqq \kappa)$. When $0 \leqq s < l-1$ the Cauchy problem for P is well-posed in $\gamma_{loc}^{(\kappa)}$ $(1 \leqq \kappa < (2l-s)/(l-s-1))$.

Example 2. Let P be a Fuchsian hyperbolic operator satisfying (A–IV)

$$P=t^2D_t^2-t^{2l}D_x^2+a(t, x)tD_t+b(t, x)t^sD_x+c(t, x)$$

where l, s are positive integers and $a, b, c\in\mathscr{B}([0, T], \gamma^{(\kappa)})$. If $s\geqq l$ the Cauchy problem for P is well-posed in $\gamma_{loc}^{(\kappa)}$ for any $\kappa\geqq 1$. In the case of $0\leqq s<l$, $\gamma_{loc}^{(\kappa)}$ $(1\leqq\kappa<(2l-s)/(l-s))$ is admissible data classes of the Cauchy problem for P.

Next example is the generalization of example 1.

Example 3. Let $P=P(t, x, D_t, D_x)$ be an operator of orer m whose coefficients belong to $\mathscr{B}([0, T], \gamma^{(\kappa)})$,

$$P=P_m+P_{m-1}+\cdots+P_0.$$

Its principal symbol $P_m(t, x, \tau, \xi)$ can be factored smoothly in the form;

$$P_m(t, x, \tau, \xi)=\prod_{j=1}^{m}(\tau-t^l\lambda_j(t, x, \xi))$$

where l is non-negative integer, λ_j is real valued function and $\lambda_i\neq\lambda_j$ when $i\neq j$. Furthermore for any $b\geqq 0$ and any multi-indices α, β there exists a constant $c=c_{\alpha,\beta,b}$ such that

$$|D_\xi^\alpha D_t^b D_x^\beta\lambda_j(t, x, \xi)|\leqq c \qquad \text{for} \quad (t, x)\in[0, T]\times \boldsymbol{R}^n,\ |\xi|=1.$$

We assume that each lower order term $P_i(t, x, \tau, \xi)\,(0\leqq i\leqq m-1)$ is represented as follows.

$$P_i(t, x, \tau, \xi)=\sum_{|\beta|=0}^{i} a_{i,\beta}(t, x)t^{s(i,\beta)}\tau^{i-|\beta|}\xi^\beta$$

where $s(i, \beta)$ is a non-negative integer and $a_{i,\beta}(t, x)\in\mathscr{B}([0, T], \gamma^{(\kappa)})$. Then we have easily seen that $\alpha(i, \beta)=s(i, \beta)+m-i+|\beta|$. Therefore applying Theorem 1, we can obtain admissible data classes of the Cauchy problem for P.

§ 2. Sketch of the Proof of Theorem 1

Let us start with the following theorem.

Theorem 2. *Under the assumptions* (A-I)~(A-VII), *assertions* 1° *and* 2 *are realized.*

1° *For any* $u_i(x)\in\Gamma^{(\kappa)}$ $(0\leqq i\leqq m-k-1)$ *and any* $f(t, x)\in\mathscr{B}([0, T], \Gamma^{(\kappa)})$ *there exists a unique solution* $u(t, x)\in\mathscr{B}([0, T], \Gamma^{(\kappa)})$ *of* (1.7) *for any* $1\leqq\kappa<\sigma/(\sigma-1)$.

2° *If* $supp(u_i(x))\subset K$ $(0\leqq i\leqq m-k-1)$ *and* $supp(f(t, x))\subset C_l(K)$ *hold for some compact set* $K\subset \boldsymbol{R}^n$, $u(t, x)$ *also satisfies* $supp(u(t, x))\subset C_l(K)$.
Here we denote by $C_l(K)$

$$C_l(K)=\{(t, x)\in[0, T]\times \boldsymbol{R}^n, \min_{y\in K}|x-y|\leqq\lambda_{\max}|t|^l/l\}$$

where

$$\lambda_{\max}=\max\{|\lambda_i(t, x, \xi)|; (t, x, \xi)\in[0, T]\times \boldsymbol{R}^n\times \boldsymbol{R}^n, |\xi|=1\}.$$

Now we shall show that Theorem 2 implies Theorem 1.

Proof of Theorem 1. We shall begin with the existence of a solution of (1.7). Let $\{\phi_p(x)\}$ be a partition of unity. Namely, $\phi_p(x)$ is compactly supported $\gamma^{(\kappa)}$ functions satisfying (i) $0\leqq\phi_p(x)\leqq 1$, (ii) the summation $\sum\phi_p(x)$ is locally finite and (iii) $\sum\phi_p(x)=1$ on $\boldsymbol{R}^n$. For any $u_i(x)\in\gamma_{loc}^{(\kappa)}$ $(0\leqq i\leqq m-k-1)$ and any $f(t, x)\in\mathscr{B}([0, T], \gamma_{loc}^{(\kappa)})$ we set $u_p^i(x)=\phi_p(x)u_i(x)$, $f_p(t, x)=\phi_p(x)\times f(t, x)$. Then we can easily see $u_p^i(x)\in\Gamma^{(\kappa)}$ and $f_p(t, x)\in\mathscr{B}([0, T], \Gamma^{(\kappa)})$. Therefore from 1° of Theorem 2 we can find a solution $u_p(t, x)\in\mathscr{B}([0, T], \Gamma^{(\kappa)})$ of the equation

$$\begin{cases} P(t, x, D_t, D_x)u_p(t, x)=f_p(t, x) \\ D_t^i u_p(t, x)|_{t=0}=u_p^i(x), \quad 0\leqq i\leqq m-k-1. \end{cases}$$

From Sobolev's lemma we have $\Gamma^{(\kappa)}\subset\gamma^{(\kappa)}$. Therefore solutions $u_p(t, x)\in\mathscr{B}([0, T], \gamma^{(\kappa)})$. Furthermore since the summation $\sum u_p(t, x)$ is locally finite, the function $u(t, x)=\sum u_p(t, x)$ belongs to $\mathscr{B}([0, T], \gamma_{loc}^{(\kappa)})$ and satisfies the equation (1.7).

Secondly we shall consider the uniqueness of the solutions. Let $u(t, x)\in\mathscr{B}([0, T], \gamma_{loc}^{(\kappa)})$ be a solution of the equation

$$\begin{cases} P(t, x, D_t, D_x)u(t, x)=0 \\ D_t^i u(t, x)|_{t=0}=0, \quad 0\leqq i\leqq m-k-1. \end{cases}$$

Following Tahara [13], we shall show $u(t, x)\equiv 0$ for $(t, x)\in[0, T]\times\boldsymbol{R}^n$ by two steps. The first step is to prove that $u(t, x)=0$ in a neighbourhood of $\{0\}\times\boldsymbol{R}^n$. Let $\phi(x)$ be a compactly supported $\gamma_{loc}^{(\kappa)}$-function such that $\phi(x)=1$ in a neighbourhood of some point $x_0\in\boldsymbol{R}^n$. Then $P(t, x, D_t, D_x)\phi(x)u(t, x)\in\mathscr{B}([0, T], \Gamma^{(\kappa)})$. Therefore the solution $w(t, x)$ of

$$\begin{cases} P(t, x, D_t, D_x)w(t, x)=P(t, x, D_t, D_x)\phi(x)u(t, x) \\ D_t^i w(t, x)|_{t=0}=0, \quad 0\leqq i\leqq m-k-1 \end{cases}$$

can be found in $\mathscr{B}([0, T], \Gamma^{(\kappa)})$ and from 2° of Theorem 2 $w(t, x)=0$ in a neighbourhood of $(0, x_0)$. Here put $\tilde{u}(t, x)=\phi(x)u(t, x)-w(t, x)$. Then from the above the function $\tilde{u}(t, x)$ satisfies the equation

$$\begin{cases} P(t, x, D_t, D_x)\tilde{u}(t, x)=0 \\ D_t^i\tilde{u}(t, x)|_{t=0}=0, \quad 0\leqq i\leqq m-k-1 \end{cases}$$

and $\tilde{u}(t, x)=u(t, x)$ in a neighbourhood of $(0, x_0)$. Hence it follows from 1° of Theorem 2 that $\tilde{u}(t, x)=0$ in $[0, T]\times \boldsymbol{R}^n$. Therefore $u(t, x)=0$ in a neighbourhood of $(0, x_0)$. The second step is to show the uniqueness in $[0, T]\times \boldsymbol{R}^n$. Take any $(t_0, x_0)\in(0, T]\times \boldsymbol{R}^n$ and put $K=\overline{D_l(t_0, x_0)}\cap\{t=0\}$ where $D_l(t_0, x_0)=\{(t, x)\in[0, T]\times \boldsymbol{R}^n;\ |x-x_0|<\lambda_{\max}(t_0^l-t^l)/l\}$. From the first step we have $u(t, x)=0$ in a neighbourhood of $[0, \varepsilon]\times K$ for a sufficiently small $\varepsilon>0$. Since P is regularly hyperbolic in $[\varepsilon, T]$ we obtain $u(t, x)=0$ in a neighbourhood of $\overline{D_l(t_0, x_0)}$. Therefore $u(t_0, x_0)=0$. The proof of Theorem 1 is completed.

Q.E.D.

In order to prove Theorem 2 we shall decompose the operator P as follows.

$$P(t, x, D_t, D_x)=Q(t, x, D_t, D_x)+R(t, x, D_t, D_x) \tag{2.1}$$

where

$$\begin{aligned} Q = {} & t^kD_t^m+\sum_{j=1}^{k}\sum_{|\beta|=j}a_{j,\beta}(t, x)t^{k-j}D_t^{m-j}D_x^\beta+\sum_{j=k+1}^{m}\sum_{|\beta|=j}a_{j,\beta}(t, x) \\ & \times D_t^{m-j}D_x^\beta+\sum_{j=1}^{k}a_{j,0}(t, x)t^{k-j}D_t^{m-j}+\sum_{j=k+1}^{m}a_{j,0}(t, x)D_t^{m-j} \end{aligned} \tag{2.2}$$

$$\begin{aligned} R = {} & \sum_{j=2}^{k}\sum_{1\leqq|\beta|\leqq j-1}a_{j,\beta}(t, x)t^{k-j}D_t^{m-j}D_x^\beta \\ & +\sum_{j=k+1}^{m}\sum_{1\leqq|\beta|\leqq j-1}a_{j,\beta}(t, x)D_t^{m-j}D_x^\beta. \end{aligned} \tag{2.3}$$

We shall demonstrate the existence of a solution by method of successive iteration. Hence we consider the following scheme.

$$\begin{cases} Qu_0(t, x)=f(t, x) \\ D_t^iu_0(t, x)|_{t=0}=u_i(x), \quad 0\leqq i\leqq m-k-1, \end{cases} \tag{2.4$_0$}$$

$$\begin{cases} Qu_j(t, x)=-Ru_{j-1}(t, x) \\ D_t^iu_j(t, x)|_{t=0}=0, \quad 0\leqq i\leqq m-k-1, \\ \text{for } j\geqq 1. \end{cases} \tag{2.4$_j$}$$

Here we refer Tahara's result [13].

Proposition 2.1. *Under the assumptions* (A–I)~(A–VI) *assertions* 1° *and* 2° *are realized.*

1° *For any* $u_i(x)\in H^\infty(\boldsymbol{R}^n)$ *and any* $f(t, x)\in\mathscr{B}([0, T], H^\infty(\boldsymbol{R}^n))$ *there exists a unique solution* $u(t, x)\in\mathscr{B}([0, T], H^\infty(\boldsymbol{R}^n))$ *of the equation*

$$(2.5)\qquad \begin{cases} Q(t, x, D_t, D_x)u(t, x)=f(t, x) \\ D_t^i u(t, x)|_{t=0}=u_i(x), \quad 0\leqq i\leqq m-k-1\,. \end{cases}$$

2° *If* $supp\,(u_i(x))\subset K$ $(0\leqq i\leqq m-k-1)$ *and* $supp\,(f(t, x))\subset C_t(K)$ *hold for any compact set* $K\subset\boldsymbol{R}^n$, *then* $u(t, x)$ *also satisfies* $supp\,(u(t, x))\subset C_t(K)$.

Since $\Gamma^{(\kappa)}\subset H^\infty(\boldsymbol{R}^n)$ $u_0(t, x)$, solution of $(2.4)_0$, belongs to $\mathscr{B}([0, T], H^\infty(\boldsymbol{R}^n))$ by Proposition 2.1. Noting that $R=R(t, x, D_t, D_x)$ is a differential operator, we have $Ru_0(t, x)\in\mathscr{B}([0, T], H^\infty(\boldsymbol{R}^n))$. Therefore it follows from $(2.4)_1$ and Proposition 2.1 that $u_1(t, x)$ also belongs to $\mathscr{B}([0, T], H^\infty(\boldsymbol{R}^n))$. Successive use of these steps brings us to $u_j(t, x)\in\mathscr{B}([0, T], H^\infty(\boldsymbol{R}^n))$ for any $j\geqq 0$. A formal solution of (1.7) is given in the form

$$(2.6)\qquad u(t, x)=\sum_{j=0}^{\infty}u_j(t, x)\,.$$

Accordingly we must show that the summation (2.6) is convergent in some sense.

Our plan is as follows. In §3 we shall get an energy inequality of the equation

$$(2.7)\qquad \begin{cases} \hat{Q}(t, x, D_t, D_x)v(t, x)=g(t, x) \\ D_t^i v(t, x)|_{t=0}=0, \quad 0\leqq i\leqq s-1 \end{cases}$$

where $\hat{Q}=t^{m-k}Q(t, x, D_t, D_x)$, s is a sufficiently large integer and $g(t, x)\in\mathscr{B}([0, T], \Gamma^{(\kappa')})$. In §4 we shall estimate derivatives of a solution of (2.7) and in §5 we obtain estimates of $\hat{R}(t, x, D_t, D_x)v(t, x)$ where $\hat{R}=t^{m-k}R(t, x, D_t, D_x)$. By the consideration of §4 and 5 we shall prove Theorem 2 in §6.

§3. Energy Estimates for Solutions of (2.7)

First we shall define symbol classes of pseudo-differential operators used in this section.

Definition. 1° For real m S^m is the symbol class of classical pseudo-differential operators.

2° For positive integer ν and real m $\mathscr{B}_\nu([0, T], S^m)$ is the set of functions

$a(t, x, \xi)$ which are represented in the form

$$a(t, x, \xi)=\sum_{i=1}^{q} t^{\nu_i} a_i(t, x, \xi)$$

where $\nu_i=\nu_i'/\nu$ (ν_i' is a non-negative integer), $q \in N$ and $a_i(t, x, \xi) \in \mathscr{B}([0, T], S^m)$.

The purpose of this section is to show the following lemma.

Lemma 3.1. *Let $\Phi(t)$ be*

$$\Phi(t)=\sum_{j=0}^{m-1} \sum_{i=0}^{m-1-j} t^{il+j} \| \Lambda^i D_t^j v \|$$

where Λ is the pseudo-differential operator with symbol $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$. Then there exist constants $c_1, c_2>0$ such that

$$\Phi(t) \leqq c_1 \int_0^t t^{c_2} \tau^{-c_2-1} \|\hat{Q} v\| d\tau \tag{3.1}$$

for $v(t, x) \in \mathscr{B}([0, T], H^\infty(\boldsymbol{R}^n))$, $D_t^j v(t, x)|_{t=0}=0$ $0 \leqq j \leqq s-1$ where $s \geqq N_1=c_2^*+1$ and c_2^* is the lowest integer greater than or equal to c_2.

Here we whall note the properties of operator $\hat{Q}(t, x, D_t, D_x)$.

Lemma 3.2. 1° *The partial differential operator $\hat{Q}(t, x, D_t, D_x)=t^{m-k} Q(t, x, D_t, D_x)$ is decomposed into the sum of $\hat{Q}_1(t, x, D_t, D_x)$ and $\hat{Q}_2(t, x, D_t, D_x)$ where*

$$\begin{aligned} \hat{Q}_1= & t^m D_t^m+\sum_{j=1}^{k} \sum_{|\beta|=j} a_{j,\beta}(t, x) t^{m-j} D_t^{m-j} D_x^\beta \\ & +\sum_{j=k+1}^{m} \sum_{|\beta|=j} a_{j,\beta}(t, x) t^{j-k} t^{m-j} D_t^{m-j} D_x^\beta, \\ \hat{Q}_2= & \sum_{j=1}^{k} a_{j,0}(t, x) t^{m-j} D_t^{m-j}+\sum_{j=k+1}^{m} a_{j,0}(t, x) t^{j-k} t^{m-j} D_t^{m-j} \end{aligned}$$

2° *The functions $a_{j,\beta}(t, x)(1 \leqq j \leqq k)$ and $t^{j-k} a_{j,\beta}(t, x)$ $(k+1 \leqq j \leqq m)$ are represented by*

$$a_{j,\beta}(t, x)=t^{l|\beta|} \hat{a}_{j,\beta}(t, x) \qquad \text{for} \quad 1 \leqq j \leqq k,$$

$$t^{j-k} a_{j,\beta}(t, x)=t^{l|\beta|} \hat{a}_{j,\beta}(t, x) \qquad \text{for} \quad k+1 \leqq j \leqq m,$$

where $\hat{a}_{j,\beta}(t, x) \in \mathscr{B}([0, T], \gamma^{(\kappa)})$, $1 \leqq j \leqq m$.

3° *$\hat{Q}_1(t, x, \tau, \xi)$ has the following form.*

$$\hat{Q}_1(t, x, \tau, \xi)=\prod_{j=1}^{m}(t\tau-t^l \lambda_j(t, x, \xi))$$

Proof. Multiplying t^{m-k} by Q we can easily obtain 1° from the equality (2.2). 3° is a direct conclusion of the assumpsion (A–VI). We have 2° by expanding the right hand side of 3°.

Following Uryu [15], [16] we shall prove Lemma 3.1. From assumption (A–VI) if $i\neq j$, $\lambda_i\neq\lambda_j$ for any $(t, x)\in[0, T]\times \boldsymbol{R}^n$, $|\xi|=1$. Here modifying $\lambda_i(t, x, \xi)$ near $\xi=0$, we may suppose that for any i, j $(i\neq j)$ there exists a constant c such that

$$|(\lambda_i-\lambda_j)(t, x, \xi)|\geqq c\langle\xi\rangle \tag{3.2}$$

where $\lambda_i(t, x, \xi)\in\mathscr{B}([0, T], S^1)$. Furthermore let us note the following. Since l is a positive rational number, l can be written in the form of irreducible fraction $l=\nu'/\nu$, $\nu, \nu'\in \boldsymbol{N}$. Let $\partial_j=tD_t-t^l\lambda_j(t, x, D_x)$ where

$$\lambda_j(t, x, D_x)u(t, x)=(2\pi)^{-n}\int e^{ix\cdot\xi}\lambda_j(t, x, \xi)\hat{u}(t, \xi)d\xi\,.$$

We define the modules W_μ $(0\leqq\mu\leqq m-1)$ over the ring of pseudo-differential operators in x of order zero. Π_m is the operator in the form of $\Pi_m=\partial_1\partial_2\cdots\partial_m$. Let W_{m-1} be the module generated by the momomial operators $\Pi_m/\partial_i=\partial_1\partial_2\cdots\check{\partial}_i\cdots\partial_m$ of order $m-1$ and let W_{m-2} be the module generated by the operators $\Pi_m/\partial_i\partial_j$ $(i\neq j)$ and so on.

In order to prove Lemma 3.1 we prepare several lemmas.

Lemma 3.3. *For any i, j there exist pseudo-differential operators $A_{i,j}$, $B_{i,j}$, $C_{i,j}\in\mathscr{B}_\nu([0, T], S^0)$ such that*

$$[\partial_i, \partial_j]=A_{i,j}\partial_i+B_{i,j}\partial_j+C_{i,j} \tag{3.3}$$

where [,] is commutator.

Proof. Let $\sigma_0([\partial_i, \partial_j])$ be the principal symbol of $[\partial_i, \partial_j]$. Then, by the formula of product of pseudo-differential operators, we obtain

$$\begin{aligned}\sigma_0([\partial_i, \partial_j])=&\sum_{\alpha=0}^{n}\{D_{\xi_\alpha}(t\xi_0-t^l\lambda_i)\partial_{x_\alpha}(t\xi_0-t^l\lambda_j)\\ &\qquad-D_{\xi_\alpha}(t\xi_0-t^l\lambda_j)\partial_{x_\alpha}(t\xi_0-t^l\lambda_i)\}\\ =&\,t^lD_{i,j}(t, x, \xi)\end{aligned}$$

where $D_{i,j}(t, x, \xi)\in\mathscr{B}_\nu([0, T], S^1)$. Here we used the notations $x_0=t$, $\xi_0=\tau$. If we define functions $A_{i,j}(t, x, \xi)$ and $B_{i,j}(t, x, \xi)$ by $A_{i,j}(t, x, \xi)=D_{i,j}(t, x, \xi)/(\lambda_j-\lambda_i)$ and $B_{i,j}(t, x, \xi)=D_{i,j}(t, x, \xi)/(\lambda_i-\lambda_j)$ respectively, then $A_{i,j}$, $B_{i,j}\in\mathscr{B}_\nu([0, T], S^0)$ and the equality

$$A_{i,j}(t, x, \xi)(t\xi_0 - t^l\lambda_i) + B_{i,j}(t, x, \xi)(t\xi_0 - t^l\lambda_j) = t^l D_{i,j}(t, x, \xi)$$

holds. Then we obtain

$$[\partial_i, \partial_j] = A_{i,j}\partial_i + B_{i,j}\partial_j + C_{i,j}$$

for some $C_{i,j}(t, x, \xi) \in \mathscr{B}_\nu([0, T], S^0)$. Q.E.D.

Lemma 3.4. *For any monomial* $\omega_\mu^\alpha \in W_\mu$ $(0 \leqq \mu \leqq m-1)$ *there exist* ∂_i *and* $\omega_{\mu+1}^\beta \in W_{\mu+1}$ *such that*

$$\partial_i \omega_\mu^\alpha = \omega_{\mu+1}^\beta + \sum_{j=1}^{\mu+1} \sum_\gamma c_{\gamma,j}\omega_{\mu+1-j}^\gamma \tag{3.4}$$

where $c_{\gamma,j}(t, x, \xi) \in \mathscr{B}_\nu([0, T], S^0)$, $\omega_{\mu+1-j}^\gamma \in W_{\mu+1-j}$.

Proof. For any $\omega_\mu^\alpha = \partial_{j_1} \cdots \partial_{j_\mu}$ $(j_1 < j_2 < \cdots < j_\mu)$ there exists some $j \notin \{j_1, \dots, j_\mu\}$ $1 \leqq j \leqq m$. Since $[\partial_i, \partial_j] = A_{i,j}\partial_i + B_{i,j}\partial_j + C_{i,j}$ by Lemma 3.3 we have immediately (3.4). Q.E.D.

Lemma 3.5. *Let* $\Psi(t)$ *be*

$$\Psi(t) = \sum_{j=1}^m \sum_\alpha \|\omega_{m-j}^\alpha v\| \qquad \text{for} \quad v(t, x) \in C^\infty([0, T] \times \mathbb{R}^n).$$

Then we have the following energy inequality.

$$t\frac{d}{dt}\Psi(t) \leqq \text{const. } \Psi(t) + \|\Pi_m v\|$$

Proof. By Lemma 3.4

$$\partial_i \omega_\mu^\alpha v = \omega_{\mu+1}^\beta v + \sum_{j=1}^{\mu+1} \sum_\gamma c_{\gamma,j}\omega_{\mu+1-j}^\gamma v. \tag{3.5}$$

Using u for $\omega_\mu^\alpha v$ and g for the right hand side of (3.5), we obtain a first order hyperbolic equation $\partial_i u = g$. Then

$$\begin{aligned} t\frac{d}{dt}\|u\|^2 &= 2\,\mathrm{Re}\left(t\frac{d}{dt}u, u\right) \\ &= 2\,\mathrm{Re}\,(\sqrt{-1}t^l\lambda_i(t, x, D_x)u + \sqrt{-1}g, u) \\ &\leqq \text{const. } \|u\|^2 + 2\|g\| \times \|u\|. \end{aligned}$$

Therefore we can easily obtain the following inequality.

$$t\frac{d}{dt}\|\omega_\mu^\alpha v\| \leqq \text{const. } \{\|\omega_\mu^\alpha v\| + \sum_{j=1}^{\mu+1} \sum_\gamma \|\omega_{\mu+1-j}^\gamma v\|\} + \|\omega_{\mu+1}^\beta v\| \tag{3.6}$$

By the definition of $\Psi(t)$, the desired inequality holds. Q.E.D.

Lemma 3.6. *Under the assumptions of Theorem 2, there exist symbols of* *pseudo-differential operators* $c_{\alpha,j}(t, x, \xi) \in \mathscr{B}_\nu([0, T], S^0)$ *and monomial operators* $\omega^\alpha_{m-j} \in W_{m-j}$ *such that*

$$\hat{Q} - \Pi_m = \sum_{j=1}^{m} \sum_{\alpha} c_{\alpha,j}\omega^\alpha_{m-j}. \tag{3.7}$$

Proof. We shall show (3.7) by two steps. The first step is to show the following. Let $\Pi_\mu = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_\mu}$ $(1 \leqq i_1 < i_2 < \cdots < i_\mu \leqq m)$. Then $\sigma(\Pi_\mu)$, the symbol of Π_μ, can be written in the form:

$$\sigma(\Pi_\mu) = \prod_{j=1}^{\mu} (t\tau - t^l\lambda_{i_j}) + R_{\mu-1} + \cdots + R_0 \tag{3.8}$$

where $R_{\mu-j}(t, x, \tau, \xi) = \sum_{\beta=0}^{\mu-j} b_{\beta,j}(t, x, \xi)t^{l\beta}(t\tau)^{\mu-j-\beta}$ and $b_{\beta,j}(t, x, \xi) \in \mathscr{B}_\nu([0, T], S^\beta)$.

We carry out the proof of (3.8) by induction on μ. When $\mu = 1$ (3.8) is trivial. Assume that (3.8) is valid for μ. Since $\Pi_{\mu+1} = \Pi_\mu \partial_{i_{\mu+1}}$ we have the following by the product formula for two pseudo-differential operators.

$$\sigma(\Pi_{\mu+1}) = \sigma(\Pi_\mu)\ (t\tau - t^l\lambda_{i_{\mu+1}}) + \sum_{\alpha \neq 0} D^\alpha_\xi \sigma(\Pi_\mu)\ \partial^\alpha_x(t\tau - t^l\lambda_{i_{\mu+1}}).$$

Therefore by the assumption of induction we can easily get (3.8) with $\mu + 1$.

The second step is to show (3.7). From (3.8) with $\mu = m$

$$\sigma(\hat{Q} - \Pi_m) = \sum_{j=1}^{m} \sum_{i=0}^{m-j} c_{i,j}(t, x, \xi)t^{il}(t\tau)^{m-i-j}$$

where $c_{i,j}(t, x, \xi) \in \mathscr{B}_\nu([0, T], S^i)$. Let the principal symbol of $\hat{Q} - \Pi_m$ be

$$\hat{P}_{m-1}(t, x, t\tau, \xi) = \sum_{i=0}^{m-1} c_{i,1}(t, x, \xi)t^{il}(t\tau)^{m-1-i}.$$

We want to determine $A_j(t, x, \xi) \in \mathscr{B}_\nu([0, T], S^0)$ so that

$$\hat{P}_{m-1}(t, x, t\tau, \xi) = \sum_{j=1}^{m-1} A_j(t, x, \xi) \prod_{i \neq j} (t\tau - t^l\lambda_i(t, x, \xi)). \tag{3.9}$$

Since $\hat{P}_{m-1}(t, x, t^l\lambda_j(t, x, \xi), \xi) = t^{l(m-1)}K_j(t, x, \xi)$ where $K_j(t, x, \xi) \in \mathscr{B}_\nu([0, T], S^{m-1})$, the equality (3.9) gives $t^{l(m-1)}K_j(t, x, \xi) = A_j(t, x, \xi)t^{l(m-1)} \prod_{i \neq j} (\lambda_j - \lambda_i)$. Then we can find

$$A_j(t, x, \xi) = \{\prod_{i \neq j} (\lambda_j - \lambda_i)\}^{-1} K_j(t, x, \xi)$$

in $\mathscr{B}_\nu([0, T], S^0)$. Applying (3.8) for $\mu = m - 1$, we have

$$\sigma(\hat{Q} - \Pi_m - \sum_{j=1}^{m} A_j \prod_{i \neq j} \partial_i) = \sum_{j=2}^{m} \sum_{i=0}^{m-j} d_{i,j}(t, x, \xi)t^{il}(t\tau)^{m-i-j}$$

where $d_{i,j}(t, x, \xi) \in \mathscr{B}_\nu([0, T], S^i)$. Repeating these steps we arrive at the relation (3.7). Q.E.D.

Lemma 3.7. *There exists a constant $c_1 > 0$ such that*

$$\Phi(t) \leqq c_1 \Psi(t). \tag{3.10}$$

Proof. It is sufficient to show

$$t^{il+j} \|\Lambda^i D_t^j v\| \leqq \text{const.}\ \Psi(t) \qquad \text{for} \quad 0 \leqq j \leqq m-1, \ 0 \leqq i \leqq m-1-j.$$

Since the symbol of $t^{il+j}\Lambda^i D_t^j$ is $t^{il+j}\langle\xi\rangle^i \tau^j$, by the same method of the proof of Lemma 3.6 we have

$$t^{il+j}\Lambda^i D_t^j = \sum_{i=1}^{m} \sum_{\alpha} d_{\alpha,i}\, \omega_{m-i}^{\alpha}$$

where $d_{\alpha,i}(t, x, \xi) \in \mathscr{B}_\nu([0, T], S^0)$, $\omega_{m-i}^\alpha \in W_{m-i}$. Hence $t^{il+j}\|\Lambda^i D_t^j v\| \leqq \text{const.} \times \Psi(t)$. Q.E.D.

Now we proceed to prove Lemma 3.1. It can be easily seen from Lemma 3.6 that

$$\begin{aligned} \|\Pi_m v\| &= \|(\Pi_m - \hat{Q})v + \hat{Q}v\| \\ &\leqq \|(\Pi_m - \hat{Q})v\| + \|\hat{Q}v\| \\ &\leqq \text{const.}\ \Psi(t) + \|\hat{Q}v\|. \end{aligned}$$

This inequality combined with Lemma 3.5 directly shows

$$t\frac{d}{dt}\Psi(t) \leqq c_2 \Psi(t) + \|\hat{Q}v\| \qquad \text{for some} \quad c_2 > 0.$$

From this inequality

$$\frac{d}{dt} t^{-c_2}\Psi(t) \leqq t^{-c_2-1}\|\hat{Q}v\|. \tag{3.13}$$

Let us note $D_t^j v(t, x)|_{t=0} = 0$ for $0 \leqq j \leqq s-1$, $s \geqq c_2^* + 1$. Therefore we can get the following by integration of both sides of (3.13) from 0 to t.

$$\Psi(t) \leqq \int_0^t t^{c_2}\tau^{-c_2-1}\|\hat{Q}v\|\, d\tau$$

Finally using Lemma 3.7 we complete the proof of Lemma 3.1. Q.E.D.

§ 4. Estimate for Solutions of (2.7)

Assume the existence of solutions of (2.7)

$$(2.7)\qquad \begin{cases} \hat{Q}v(t, x) = g(t, x) \\ D_t^i v(t, x)|_{t=0} = 0 \qquad \text{for} \quad 0 \leqq i \leqq s-1 \end{cases}$$

where $g(t, x) \in \mathscr{B}([0, T], \Gamma^{(\kappa)})$. Then we have

$$(4.1)\qquad D_t^i g(t, x)|_{t=0} = 0 \qquad \text{for} \quad 0 \leqq i \leqq s-1\,.$$

Therefore we may assume the following on $g(t, x)$. For any $r \geqq 0$ there exist constants c, R, $K > 0$ such that

$$(4.2)\qquad \|\Lambda^r g(t, x)\| \leqq cR^r r!^\kappa t^s \exp(Kr^* t^l)$$

where $r! = \Gamma(r+1)$ and r^* is the lowest integer greater than or equal to r. For simplification we use the notation $w_r(s, t, R) = R^r r!^\kappa t^s \exp(Kr^* t^l)$.
Now we shall prove the basic lemma of this section.

Lemma 4.1. *Let* $\Phi_r(t)$ *be*

$$(4.3)\qquad \Phi_r(t) = \sum_{j=0}^{m-1} \sum_{i=0}^{m-1-j} t^{il+j} \|\Lambda^{r+i} D_t^j v\|\,.$$

Then for any $r \geqq 0$ *there exist constants* A, $s_0 > 0$ *such that for sufficiently large* R, K *and for* $s \geqq s_0$

$$(4.4)\qquad \Phi_r(t) \leqq Acs^{-1} w_r(s, t, R)\,.$$

Proof. We carry out the proof by induction on r. When $r = 0$, it follows from Lemma 3.1 and (4.2) that for any $s \geqq N_1$

$$\Phi_0(t) \leqq c_1 \int_0^t t^{c_2} \tau^{-c_2-1} c w_0(s, \tau, R) d\tau$$
$$\leqq c_1 (s - c_2)^{-1} c w_0(s, t, R)\,.$$

Here we make s sufficiently large such that $s - c_2 \geqq s/2$, then we have

$$\Phi_0(t) \leqq Acs^{-1} w_0(s, t, R)$$

where $A \geqq 2c_1$, $s \geqq s_0 = \max(2c_2, N_1)$. Assuming that (4.4) is valid for any $0 \leqq r \leqq n$, we shall demonstrate that (4.4) is valid also for $n < r \leqq n+1$. For $r > 0$ operating the pseudo-differential operator Λ^r on both sides of $\hat{Q}v(t, x) = g(t, x)$, we get

$$\hat{Q}\Lambda^r v(t, x) = \Lambda^r g(t, x) + [\hat{Q}, \Lambda^r] v(t, x)\,.$$

We shall estimate the commutator $[\hat{Q}, \Lambda^r] v(t, x)$. From 1° and 2° of Lemma 3.2

$$\hat{Q}_1 = t^m D_t^m + \sum_{j=1}^{m} \sum_{|\beta|=j} \hat{a}_{j,\beta}(t, x) t^{l|\beta|+m-j} D_t^{m-j} D_x^\beta$$

$$\hat{Q}_2 = \sum_{j=1}^{m} \hat{a}_{j,0}(t, x) t^{m-j} D_t^{m-j} .$$

Therefore

$$[\hat{Q}, \Lambda^r] = \sum_{j=1}^{m} \sum_{|\beta|=j} [\hat{a}_{j,\beta} D_x^\beta, \Lambda^r] t^{l|\beta|+m-j} D_t^{m-j} + \sum_{j=1}^{m} [\hat{a}_{j,0}, \Lambda^r] t^{m-j} D_t^{m-j} .$$

By use of a product formula of pseudo-differential operators (Lemma A–1 in Appendix) we obtain for $0 \leqq |\beta| \leqq m$

$$\sigma([\hat{a}_{j,\beta} D_x^\beta, \Lambda^r]) = -\sum_{i=1}^{N-1} g_i^\beta(t, x, \xi) - r_N^\beta(t, x, \xi)$$

where $N = r^* + |\beta|$ and

$$g_i^\beta(t, x, \xi) = \sum_{|\alpha|=i} \frac{1}{\alpha !} \partial_\xi^\alpha \langle \xi \rangle^r D_x^\alpha \hat{a}_{j,\beta}(t, x) \xi^\beta .$$

Then from Lemma A–3 there exist constants $\hat{c}$, $\hat{R} > 0$ such that

$$\|g_i^\beta(t, x, D_x)u\| \leqq \hat{c}\hat{R}^{i-|\beta|}(i-|\beta|)!^\kappa \binom{r^*}{i} \|\Lambda^{r+|\beta|-i}u\| \quad \text{for} \quad i = 1, 2, \dots, r^*,$$

$$\|g_i^\beta(t, x, D_x)u\| \leqq \hat{c}\hat{R}^{i-|\beta|}(i-|\beta|)!^\kappa \|\Lambda^{r+|\beta|-i}u\| \quad \text{for} \quad i = r^*+1, \dots, N-1 \quad \text{and}$$

$$\|r_N^\beta(t, x, D_x)u\| \leqq \hat{c}\hat{R}^r r!^\kappa \|u\|$$

where $u \in \mathscr{B}([0, T], H^\infty(\mathbf{R}^n))$.

It follows from this estimate that

$$\begin{aligned} I_{j,0} &= \|[\hat{a}_{j,0}, \Lambda^r](t^{m-j} D_t^{m-j} v)\| \\ &\leqq \sum_{i=1}^{r^*-1} \hat{c}\hat{R}^i i!^\kappa \binom{r^*}{i} t^{m-j} \|\Lambda^{r-i} D_t^{m-j} v\| \\ &\quad + \hat{c}\hat{R}^r r!^\kappa t^{m-j} \|D_t^{m-j} v\| \end{aligned}$$

and for $|\beta| = j \geqq 1$

$$\begin{aligned} I_{j,\beta} &= \|[\hat{a}_{j,\beta} D_x^\beta, \Lambda^r](t^{l|\beta|+m-j} D_t^{m-j} v)\| \\ &\leqq \sum_{i=1}^{r^*} \hat{c}\hat{R}^{i-j}(i-j)!^\kappa \binom{r^*}{i} t^{lj+m-j} \|\Lambda^{r+j-i} D_t^{m-j} v\| \\ &\quad + \sum_{i=r^*+1}^{j+r^*-1} \hat{c}\hat{R}^{i-j}(i-j)!^\kappa t^{lj+m-j} \|\Lambda^{r+j-i} D_t^{m-j} v\| \\ &\quad + \hat{c}\hat{R}^r r!^\kappa t^{lj+m-j} \|D_t^{m-j} v\| . \end{aligned}$$

From the assumption of induction we have that

$$(4.6)\quad \begin{cases} t^{l(j-1)+m-j}\|\Lambda^{r+j-i}D_t^{m-j}v\| \leqq Acs^{-1}w_{r-i+1}(s,\ t,\ R) \\ \qquad \text{for } i=2,\ 3,\dots,\ r^*\ , \\ t^{l(j-1)+m-j}\|\Lambda^{r+j-i}D_t^{m-j}v\| \leqq Acs^{-1}w_0(s,t,\ R) \\ \qquad \text{for } i=r^*+1,\dots,\ j+r^*-1\ , \\ t^{m-j}\|\Lambda^{r-i}D_t^{m-j}v\| \leqq Acs^{-1}w_{r-i}(s,\ t,R) \\ \qquad \text{for } i=1,\ 2,\dots,\ r^*-1 \text{ and} \\ t^{m-j}\|D_t^{m-j}v\| \leqq Acs^{-1}w_0(s,\ t,\ R)\,. \end{cases}$$

Hence it follows from (4.6) that

$$\begin{cases} I_{j,0} \leqq \hat{c}cAs^{-1}\{\sum_{i=1}^{r^*-1}\hat{R}^i i!^\kappa\binom{r^*}{i}w_{r-i}(s,t,R)+\hat{R}^r r!^\kappa w_0(s,\ t,\ R)\}\ , \\ I_{j,\beta} \leqq \hat{c}\hat{R}r^* t^{lj+m-j}\|\Lambda^{r+j-1}D_t^{m-j}v\| \end{cases}$$

$$+\hat{c}cAs^{-1}t^l\{\sum_{i=2}^{r^*}\hat{R}^{i-j}(i-j)!^\kappa\binom{r^*}{i}w_{r-i+1}(s,\ t,\ R)$$

$$+\sum_{i=r^*+1}^{j+r^*-1}\hat{R}^{i-j}(i-j)!^\kappa w_0(s\ t,\ R)+\hat{R}^r r!^\kappa w_0(s,\ t,\ R)\}..$$

Here let us calculate $I=\sum_{i=1}^{r^*-1}\hat{R}^i i!^\kappa\binom{r^*}{i}w_{r-i}(s,\ t,\ R)$.

$$I \leqq \sum_{i=1}^{r^*-1}(\hat{R}/R)^i\binom{r^*}{i}i!^\kappa(r-i)!^\kappa r!^{-\kappa}w_r(s,\ t,\ R)$$

$$=\sum_{i=1}^{r^*-1}(\hat{R}/R)^i\binom{r^*}{i}\binom{r}{i}^{-\kappa}w_r(s,\ t,\ R)\,.$$

Since $\kappa \geqq 1$ if we make $R \geqq 2\hat{R}$, then we obtain $I \leqq$ const. $w_r(s,\ t,\ R)$. Therefore $I_{j,0} \leqq$ const. $cAs^{-1}w_r(s,\ t,\ R)$.

Next we shall calculate $J=\sum_{i=2}^{r^*}\hat{R}^{i-j}(i-j)!^\kappa\binom{r^*}{i}w_{r-i+1}(s,\ t,\ R)\,.$

$$J \leqq \sum_{i=2}^{r^*}\hat{R}^{-j}R(\hat{R}/R)^i\binom{r^*}{i}(i-j)!^\kappa(r-i+1)!^\kappa r!^{-\kappa}w_r(s,\ t,\ R)$$

$$\leqq r^*\sum_{i=2}^{r^*}\hat{R}^{-j}R(\hat{R}/R)^i\binom{r^*}{i-1}\binom{r}{i-1}^{-\kappa}w_r(s,\ t,\ R)\,.$$

Let $R \geqq 2\hat{R}$, then we get $J \leqq$ const. $r^* w_r(s,\ t,\ R)$. Hence the following estimate holds.

$$I_{j,\beta} \leqq \hat{c}\hat{R}r^* t^{lj+m-j}\|\Lambda^{r+j-1}D_t^{m-j}v\|$$
$$+\text{const. } cAs^{-1}t^l r^* w_r(s,\ t,\ R)\,.$$

Noting that

$$(4.7)\qquad \|[\hat{Q},\ \Lambda^r]v\| \leqq \sum_{j=1}^{m}\{I_{j,0}+\sum_{|\beta|=j}I_{j,\beta}\}\,,$$

there exist constants c_3, $\tilde{c}>0$ such that

$$\|[\hat{Q}, \Lambda^r]v\| \leqq c_3 r^* t^l \Phi_r(t) + \tilde{c}cAs^{-1}r^*t^l w_r(s, t, R) + \tilde{c}cAs^{-1}w_r(s, t, R). \tag{4.8}$$

By the way, since $\hat{Q}\Lambda^r v = \Lambda^r g + [Q, \Lambda^r]v$ we can see from Lemma 3.1 that

$$\Phi_r(t) \leqq c_1 \int_0^t t^{c_2}\tau^{-c_2-1}\{\|\Lambda^r g\| + \|[\hat{Q}, \Lambda^r]v\|\}d\tau.$$

Let $f(t)$ be

$$f(t) = c_1 c \int_0^t t^{c_2}\tau^{-c_2-1}\{w_r(s, \tau, R) + \tilde{c}As^{-1}r^*\tau^l w_r(s, \tau, R) + \tilde{c}As^{-1}w_r(s, \tau, R)\}d\tau,$$

then from (4.8)

$$\Phi_r(t) \leqq f(t) + c_1 c_3 r^* \int_0^t t^{c_2}\tau^{-c_2-1+l}\Phi_r(\tau)d\tau. \tag{4.9}$$

Here it follows from Lemma A–4 in appendix that

$$\Phi_r(t) \leqq f(t) + \bar{c}r^* \int_0^t t^{c_2}\tau^{-c_2-1+l}\exp\left\{\frac{\bar{c}r^*}{l}(t^l - \tau^l)\right\} f(\tau)d\tau \tag{4.10}$$

where $\bar{c} = c_1 c_3$. Noting that $w_r(s, t, R) = R^r r!^\kappa t^s \exp(Kr^* t^l)$, we have

$$f(t) \leqq c_1 c\{(s-c_2)^{-1}w_r(s, t, R) + \tilde{c}As^{-1}r^*(Kr^*l)^{-1}w_r(s, t, R) + \tilde{c}As^{-1}(s-c_2)^{-1}w_r(s, t, R)\}.$$

Here we make K and s sufficiently large, then

$$f(t) \leqq \frac{cA}{2}s^{-1}w_r(s, t, R).$$

Therefore from (4.10)

$$\Phi_r(t) \leqq \frac{cA}{2}s^{-1}w_r(s, t, R) + \bar{c}r^*\{(Kl-\bar{c})r^*\}^{-1}\frac{cA}{2}s^{-1}w_r(s, t, R).$$

Let K be sufficiently large a number such that $\bar{c}(Kl-\bar{c})^{-1} \leqq 1$, then $\Phi_r(t) \leqq cAs^{-1}w_r(s, t, R)$. Q. E. D.

Lemma 4.2. *For any $r \geqq 0$ and $0 \leqq i+j \leqq m-1$ there exist constants $\bar{A}$, $s_1 > 0$ such that for sufficiently large R, K and for any $s \geqq s_1$*

$$t^{il+j}\|\Lambda^{r+i}D_t^j v\| \leqq c\bar{A}s^{-(m-i-j)}w_r(s, t, R). \tag{4.11}$$

Proof. It follows from Lemma 4.1 that for any $s \geqq s_0$

$$t^{il+m-1-i}\|\Lambda^{r+i}D_t^{m-1-i}v\| \leqq c\bar{A}s^{-1}w_r(s, t, R)\,. \tag{4.12}$$

For any integer $p \geqq 1$ we can see

$$\|w\| \leqq \int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_p}\|D_\tau^p w\|d\tau dt_p dt_{p-1}\cdots dt_2$$

where $t_1 = t$. Here let $w(t, x)$ and p be $w(t, x) = \Lambda^{r+i}D_t^j v(t, x)$ and $p = m-1-i-j$ respectively, then we have from (4.12)

$$\begin{aligned}\|\Lambda^{r+i}D_t^j v\| &\leqq \int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_p} cAs^{-1}w_r(s-p', \tau, R)d\tau dt_p\cdots dt_2\\ &\leqq cAs^{-1}(s/2)^{-p}w_r(s-p'+p, t, R)\end{aligned}$$

where $p' = il+m-1-i$ and s is sufficiently large such that $s-p' \geqq s/2$. Therefore we conclude that

$$t^{il+j}\|\Lambda^{r+i}D_t^j v\| \leqq c(2^p A)s^{-(m-i-j)}w_r(s, t, R)\,.$$

Q.E.D.

Now we shall state the main proposition of this section.

Proposition 4.3. *For any integers $1 \leqq j \leqq m-1$, $0 \leqq i \leqq j$, $1 \leqq \hat{k} \leqq j$ and any real $0 \leqq q \leqq 1$ there exist constants A and s_1 such that*

$$t^{j-i}\|\Lambda^{r+i}D_t^{j-i}v\| \leqq c\bar{A}s^{-(m-j+\hat{k}q)}w_{r+\hat{k}q}(s-il+\hat{k}ql, t, R) \tag{4.13}$$

where $s \geqq s_1$ and R, K are sufficiently large numbers.

For the proof of this proposition we need the following lemma of Igari [4].

Lemma 4.4. *If $p \geqq 0$, $q \geqq 0$ and $p+q=1$, then*

$$\|\Lambda^r u\| \leqq \|\Lambda^{r-p}u\|^q\|\Lambda^{r+q}u\|^p\,.$$

Proof of Proposition 4.3. Let the left hand side of (4.13) be $T_r(t)$, then it follows from Lemma 4.2 that

$$\begin{cases} (4.14)_1 & T_r(t) \leqq c\bar{A}s^{-(m-j+\hat{k})}w_{r+\hat{k}}(s-(i-\hat{k})l, t, R)\\ (4.14)_2 & T_r(t) \leqq c\bar{A}s^{-(m-j+\hat{k}-1)}w_{r+\hat{k}-1}(s-(i-\hat{k}+1)l, t, R)\\ & \vdots\\ (4.14)_{\hat{k}+1} & T_r(t) \leqq c\bar{A}s^{-(m-j)}w_r(s-il, t, R)\,.\end{cases}$$

From Lemma 4.4 we immediately have

$$T_r(t) \leqq T_{r-p}(t)^q T_{r+q}(t)^p\,.$$

Combining two inequalities $(4.14)_1$ and $(4.14)_2$, we can verify that

$$(4.14)'_1 \qquad T_r(t) \leqq c\bar{A}s^{-(m-j+\hat{k}-p)} w_{r+\hat{k}-p}(s-(i-\hat{k})l-pl,\ t,\ R).$$

In the same way, combining $(4.14)_n$ and $(4.14)_{n+1}$ for $n=2,\dots,\hat{k}$, we obtain

$$(4.14)'_2 \qquad T_r(t) \leqq c\bar{A}s^{-(m-j+\hat{k}-1-p)} w_{r+\hat{k}-1-p}(s-(i-\hat{k}+1)l-pl,\ t,\ R)$$

$$\vdots$$

$$(4.14)'_{\hat{k}} \qquad T_r(t) \leqq c\bar{A}s^{-(m-j+1-p)} w_{r+1-p}(s-(i-1)l-pl,\ t,\ R).$$

Next we apply the above process to inequalities $(4.14)'_1,\dots,(4.14)'_{\hat{k}}$, then

$$\begin{cases} (4.14)''_1 \quad T_r(t) \leqq c\bar{A}s^{-(m-j+\hat{k}-2p)} w_{r+\hat{k}-2p}(s-(i-\hat{k})l-2pl,\ t,\ R) \\ (4.14)''_2 \quad T_r(t) \leqq c\bar{A}s^{-(m-j+\hat{k}-1-2p)} w_{r+\hat{k}-1-2p}(s-(i-\hat{k}+1)l-2pl,\ t,\ R) \\ \qquad \vdots \\ (4.14)''_{\hat{k}-1} \quad T_r(t) \leqq c\bar{A}s^{-(m-j+2-2p)}\ w_{r+2-2p}(s-(i-2)l-2pl,\ t,\ R). \end{cases}$$

Repeating these steps, we finally attain to the only one inequality as follows.

$$\begin{aligned} T_r(t) &\leqq c\bar{A}s^{-(m-j+\hat{k}-\hat{k}p)} w_{r+\hat{k}-\hat{k}p}(s-(i-\hat{k})l-\hat{k}pl,\ t,\ R) \\ &= c\bar{A}s^{-(m-j+\hat{k}q)} w_{r+\hat{k}q}(s-il+\hat{k}ql,\ t,\ R). \end{aligned}$$

Q. E. D.

§ 5. Estimate for $\hat{R}v$

We begin with the following lemma.

Lemma 5.1. *The partial differential operator* $\hat{R}=t^{m-k}R(t, x, D_t, D_x)$ *is represented in the form*:

$$\hat{R}(t, x, D_t, D_x)=\sum_{j=1}^{m-1}\sum_{|\beta|=1}^{j} \hat{a}_{m+|\beta|-j,\beta}(t, x)t^{\alpha(j,\beta)+j-|\beta|}D_t^{j-|\beta|}D_x^{\beta}.$$

Proof. From (2.3), (1.5) and (1.6) we have

$$\begin{aligned} \hat{R} &= \sum_{j=2}^{k}\sum_{|\beta|=1}^{j-1} \hat{a}_{j,\beta}(t, x)\, t^{\gamma(j,\beta)+m-j}D_t^{m-j}D_x^{\beta} \\ &\quad + \sum_{j=k+1}^{m}\sum_{|\beta|=1}^{j-1} \hat{a}_{j,\beta}(t, x)t^{\gamma(j,\beta)+j-k+m-j}D_t^{m-j}D_x^{\beta} \\ &= \sum_{j=2}^{m}\sum_{|\beta|=1}^{j-1} \hat{a}_{j,\beta}(t, x)t^{\alpha(m-j+|\beta|,\beta)+m-j}D_t^{m-j}D_x^{\beta}. \end{aligned}$$

Let us replace $m-j+|\beta|$ with j. Hence we can get the desired result. Q. E. D.

It follows from Lemma 4.3 that for any integer $1\leqq k_j\leqq j$ and real $0\leqq q\leqq 1$

$$\begin{aligned} t^{\alpha(j,\beta)+j-|\beta|}\|\Lambda^r D_t^{j-|\beta|}D_x^{\beta}v\| &\leqq c\bar{A}s^{-(m-j+k_jq)} \\ &\quad \times w_{r+k_jq}(s-|\beta|l+k_jql+\alpha(j,\ \beta),\ t,\ R) \end{aligned}$$

Then owing to Lemma 5.1, we have

$$\|\Lambda^r\hat{R}v\| \leqq c\bar{A}B\sum_{j=1}^{m-1}\sum_{|\beta|=1}^{j} s^{-(m-j+k_jq)} \times w_{r+k_jq}(s-|\beta|l+k_jql+\alpha(j,\beta), t, R) \tag{5.1}$$

where $B>0$ is independent of r. Here let $1\leqq k_j\leqq j$ be the smallest integer satisfying the inequalities:

$$\alpha(j,\beta)/l-|\beta|+k_j>0 \qquad \text{for any} \quad \beta \ (1\leqq|\beta|\leqq j-1). \tag{5.2}$$

Next we shall remember the definition of σ_j and ν.

$$\sigma_j=\max_{1\leqq|\beta|\leqq j-1}\{|\beta|-\alpha(j,\beta)/l, 0\}, \quad \nu=\max_{1\leqq j\leqq m-1}\{\sigma_j/k_j\}.$$

Then we can verify that

$$-|\beta|l+k_jql+\alpha(j,\beta)\geqq(q-\nu)l. \tag{5.3}$$

Since $0\leqq\nu<1$ we make $0\leqq q\leqq 1$ satisfying $q>\nu$. Hence it follows from (5.1) and (5.3) that

$$\|\Lambda^r\hat{R}v\| \leqq c\bar{A}B\sum_{j=1-1}^{m-1} s^{-(m-j+k_jq)}w_{r+k_jq}(s+(q-\nu)l, t, R). \tag{5.4}$$

Furthermore let θ be a positive number so that for $1\leqq j\leqq m-1$

$$k_jq\theta\leqq m-j+k_jq \tag{5.5}$$

then

$$\|\Lambda^r\hat{R}v\| \leqq c\bar{A}B\sum_{j=1}^{m-1} s^{-k_jq\theta}w_{r+k_jq}(s+(q-\nu)l, t, R). \tag{5.6}$$

Let us summarize the above results.

Lemma 5.2. *Let $v(t,x)$ be the solution of the equation*

$$\begin{cases} \hat{Q}v(t,x)=g(t,x) \\ D_t^i v(t,x)|_{t=0}=0 \quad 0\leqq i\leqq s-1 \end{cases} \tag{2.7}$$

where $s\geqq s_1$ and $g(t,x)$ satisfies the following inequality.

$$\|\Lambda^r g\|\leqq cw_r(s,t,R). \tag{4.2}$$

Then there exist constants $\bar{A}$, $B>0$ which are independent of r such that for sufficiently large R and K

$$\|\Lambda^r v\|\leqq c\bar{A}s^{-m}w_r(s,t,R)\leqq c\bar{A}w_r(s,t,R), \tag{5.7}$$

(5.8) $$\|\Lambda^r \hat{R}v\| \leqq c\bar{A}B \sum_{j=1}^{m-1} s^{-k_j q\theta} w_{r+k_j q}(s+(q-\nu)l, t, R),$$

where $q>\nu$, $0\leqq q\leqq 1$ *and* $\theta(>0)$ *satisfies the property that for* $1\leqq j\leqq m-1$ $k_j q\theta \leqq m-j+k_j q$.

§6. Proof of Theorem 2

We shall first prepare the following basic lemma.

Lemma 6.1. *Under the assumptions* (A–I)$\sim$(A–VI), *the assertions* 1° *and* 2° *hold.*

1° *For any* $u^i(x)\in\Gamma^{(\kappa)}$ *and any* $\hat{f}(t, x)\in\mathscr{B}([0, T], \Gamma^{(\kappa)})$ *satisfying* $D_t^i\hat{f}(t, x)|_{t=0}=0$ $(0\leqq i\leqq m-k-1)$ *there exists a unique solution* $u(t, x)\in\mathscr{B}([0, T], \Gamma^{(\kappa)})$ *of the equation*

(6.1) $$\begin{cases} \hat{Q}(t, x, D_t, D_x)u(t, x)=\hat{f}(t, x) \\ D_t^i u(t, x)|_{t=0}=u^i(x), \quad 0\leqq i\leqq m-k-1. \end{cases}$$

2° *Especially if* $u^i(x)\equiv 0$ *for* $0\leqq i\leqq m-k-1$ *and* $D_t^i\hat{f}(t, x)|_{t=0}=0$ *for* $0\leqq i\leqq m-k-1+\hat{s}$ *we obtain that* $D_t^i u(t, x)|_{t=0}=0$ *for* $0\leqq i\leqq m-k-1+\hat{s}$ *where* $\hat{s}$ *is a non-negative integer.*

Proof. 1° From the assumption $\hat{f}(t, x)=t^{m-k}h(t, x)$ where $h(t, x)$ belongs to $\mathscr{B}([0, T], \Gamma^{(\kappa)})$. Therefore we have the equation equivalent to (6.1)

(6.2) $$\begin{cases} Q(t, x, D_t, D_x)u(t, x)=h(t, x) \\ D_t^i u(t, x)|_{t=0}=u^i(x), \quad 0\leqq i\leqq m-k-1. \end{cases}$$

Applying Proposition 2.1, we know the unique existence of the solution $u(t, x)\in\mathscr{B}([0, T], H^{\gamma}(R^n))$ of (6.2). Hence let us show that $u(t, x)$ belongs to $\mathscr{B}([0, T], \Gamma^{(\kappa)})$. It follows from (A–IV) that we can calculate the derivatives of $u(t, x)$ at $t=0$ and each derivative belongs to $\Gamma^{(\kappa)}$. Here for any integer $s\geqq 1$ let $u_s(t, x)$ be

$$u_s(t, x)=u(t, x)-\sum_{i=0}^{s-1} t^i/i!\partial_t^i u(t, x)|_{t=0},$$

then $u_s(t, x)$ satisfies the equation

$$\hat{Q}u_s(t, x)=\hat{f}(t, x)-\hat{Q}(\sum_{i=1}^{s-1} t^i/i!\partial_t^i u(t, x)|_{t=0})\equiv f_s(t, x).$$

Hence we have that $f_s(t, x)\in\mathscr{B}([0, T], \Gamma^{(\kappa)})$ and

$$\begin{cases} D_t^j u_s(t, x)|_{t=0}=0 & \text{for} \quad 0 \leqq j \leqq s-1, \\ D_t^j f_s(t, x)|_{t=0}=0 & \text{for} \quad 0 \leqq j \leqq s-1. \end{cases}$$

We can reduce (6.1) to the equation

$$(6.3) \qquad \begin{cases} \hat{Q}u_s(t, x)=f_s(t, x) \\ D_t^j u_s(t, x)|_{t=0}=0 & \text{for} \quad 0 \leqq j \leqq s-1. \end{cases}$$

From Lemma 5.2 we can see that $u_s(t, x) \in \mathscr{B}([0, T], \Gamma^{(\kappa)})$. Consequently $u(t, x)$ belongs to $\mathscr{B}([0, T], \Gamma^{(\kappa)})$. Assertion 2° is clear from (A–IV).

Q. E. D.

Let us consider the scheme $(2.4)_j$. Then $(2.4)_j$ is equivalent to the following scheme.

$$(6.4)_0 \qquad \begin{cases} \hat{Q}u_0(t, x)=t^{m-k}f(t, x) \\ D_t^i u_0(t, x)|_{t=0}=u^i(x), \quad 0 \leqq i \leqq m-k-1, \end{cases}$$

$$(6.4)_j \qquad \begin{cases} \hat{Q}u_j(t, x)=-\hat{R}u_{j-1}(t, x) \equiv f_{j-1}(t, x) \\ D_t^i u_j(t, x)|_{t=0}=0, \quad 0 \leqq i \leqq m-k-1 & \text{for} \quad j \geqq 1. \end{cases}$$

Lemma 6.2. *Let $u_j(t, x)$ be the solution of $(6.4)_j$, then $u_j(t, x) \in \mathscr{B}([0, T], \Gamma^{(\kappa)})$ for $j \geqq 0$ and there exists an integer $\tilde{s} \geqq 1$ such that for $j \geqq 1$ $D_t^i u_j(t, x)|_{t=0}=0$ for $0 \leqq i \leqq m-k-1+\tilde{s}(j-1)$.*

Proof. From Lemma 6.1 we have $u_0(t, x) \in \mathscr{B}([0, T], \Gamma^{(\kappa)})$. Since $\hat{R}=t^{m-k}R(t, x, D_t, D_x)$, $f_0(t, x)$ satisfies that $f_0(t, x) \in \mathscr{B}([0, T], \Gamma^{(\kappa)})$ and for $0 \leqq i \leqq m-k-1$ $D_t^i f_0(t, x)|_{t=0}=0$. Therefore it follows from Lemma 6.1 that $u_1(t, x) \in \mathscr{B}([0, T], \Gamma^{(\kappa)})$. Repeating these steps, $u_j(t, x) \in \mathscr{B}([0, T], \Gamma^{(\kappa)})$ for $j \geqq 0$.

Let us consider the second assertion. From 2° of Lemma 6.1 we can verify that $D_t^i u_1(t, x)|_{t=0}=0$ for $0 \leqq i \leqq m-k-1$. Put $\tilde{s}=\min\{\alpha(j, \beta);\ 1 \leqq j \leqq m-1, 1 \leqq |\beta| \leqq j\} \geqq 1$. Then it follows from Lemma 5.1 that $D_t^i f_1(t, x)|_{t=0}=0$ for $0 \leqq i \leqq m-k-1+\tilde{s}$. By Lemma 6.1 we obtain $D_t^i u_2(t, x)|_{t=0}=0$ for $0 \leqq i \leqq m-k-1+\tilde{s}$. Successive use of these steps brings us to

$$D_t^i u_j(t, x)|_{t=0}=0 \qquad \text{for} \quad 0 \leqq i \leqq m-k-1+\tilde{s}(j-1).$$

Q. E. D.

The following lemma is the direct consequence of Lemma 6.2.

Lemma 6.3. *For any $s \geqq 0$ there exists $N=N(s) \in \boldsymbol{Z}$ such that for any $j \geqq N-1$ $D_t^i u_j(t, x)|_{t=0}=0$, $0 \leqq i \leqq s-1$.*

Now let us demonstrate the proof of Theorem 2. By Lemma 6.2 and Lemma 6.3 we may assume that for any $r \geqq 0$

$$\|A^r \hat{R} u_{N-1}\| \leqq c w_r(s, t, R) \tag{6.5}$$

where c and R are positive constants. Hence the following lemma holds.

Lemma 6.4. *Under* (6.5) *if* $\kappa < \theta$ *there exist constants* $\tilde{A}$, $\tilde{B}$, $\tilde{R}$ *which are independent of* r *such that*

$$\|A^r u_{N+n}\| \leqq c\tilde{B}\tilde{A}^n n^{n(\kappa-\theta)q} w_r(s+n(q-\nu)l, t, \tilde{R}) \tag{6.6}$$

for $n=0, 1, \ldots$, *where* q, ν, θ *are the same as in Lemma* 5.2.

Proof. It follows from Lemma 5.2 and (6.5) that

$$\begin{cases} \|A^r u_N\| \leqq c\bar{A} w_r(s, t, R) \\ \|A^r \hat{R} u_N\| \leqq c\bar{A}B \sum_{j=1}^{m-1} s^{-k_j q\theta} w_{r+k_j q}(s+(q-\nu)l, t, R). \end{cases}$$

Next applying Lemma 5.2 to $(6.4)_{N+1}$,

$$\begin{cases} \|A^r u_{N+1}\| \leqq c\bar{A}^2 B \sum_{j=1}^{m-1} s^{-k_j q\theta} w_{r+k_j q}(s+(q-\nu)l, t, R) \\ \|A^r \hat{R} u_{N+1}\| \leqq c\bar{A}^2 B^2 \sum_{i,j=1}^{m-1} \{s+(q-\nu)l\}^{-k_i q\theta} s^{-k_j q\theta} \\ \qquad \times w_{r+(k_i+k_j)q}(s+2(q-\nu)l, t, R). \end{cases}$$

Inductively we obtain that for any $n \geqq 0$

$$\|A^r u_{N+n}\| \leqq c\bar{A}^{n+1} B^n \sum_{i_1=1}^{m-1} \cdots \sum_{i_n=1}^{m-1} e_{k_{i_1},\ldots,k_{i_n}} \times w_{r+(k_{i_1}+\cdots+k_{i_n})q}(s+n(q-\nu)l, t, R) \tag{6.7}$$

where $e_{k_{i_1},\ldots,k_{i_n}} = \{s+(q-\nu)l(n-1)\}^{-k_{i_n} q\theta} \times \cdots \times s^{-k_{i_1} q\theta}$.

Let us make s sufficiently large, then $e_{k_{i_1},\ldots,k_{i_n}}$ is estimated as follows.

$$e_{k_{i_1},\ldots,k_{i_n}} \leqq D^n n^{-k_{i_n} q\theta} (n-1)^{-k_{i_{n-1}} q\theta} \times \cdots \times 1^{-k_{i_1} q\theta}$$

for some constant $D > 0$. Furthermore from Lemma A–5 in Appendix

$$e_{k_{i_1},\ldots,k_{i_n}} \leqq A_1 R_1^n D^n n^{-(k_{i_1}+\cdots+k_{i_n})q\theta} \tag{6.8}$$

Using Lemma A–7, the estimates (6.7) and (6.8) imply that

$$\|A^r u_{N+n}\| \leqq cA_1 A_3 \bar{A}^{n+1} B^n D^n R_1^n R_3^n \times \sum_{i_1=1}^{m-1} \cdots \sum_{i_n=1}^{m-1} n^{(k_{i_1}+\cdots+k_{i_n})(\kappa-\theta)q} w_r(s+n(q-\nu)l, t, R').$$

Here let us make $\kappa < \theta$, then it follows from the above inequality that

$$\|\Lambda^r u_{N+n}\| \leqq c\tilde{B}\tilde{A}^n n^{(\kappa-\theta)nq} w_r(s+n(q-\nu)l,\ t,\ \tilde{R})$$

for some constants $\tilde{A}$, $\tilde{B}$, $\tilde{R}$. Q. E. D.

Now we shall prove the convergence of summation $\sum_{j=0}^{\infty} u_j(t,x)$.

Lemma 6.5. *If κ satisfies $1 \leqq \kappa < \sigma/(\sigma-1)$, the series $u(t, x) = \sum_{j=0}^{\infty} u_j(t, x)$ is convergent in $\mathscr{B}([0, T], \Gamma^{(\kappa)})$. Hence $u(t, x)$ belongs to $\mathscr{B}([0, T], \Gamma^{(\kappa)})$.*

Proof. First of all we shall show that if $1 \leqq \kappa < \sigma/(\sigma-1)$ there exist constants q, θ satisfying (6.9)~(6.11).

(6.9) $\quad q > \nu,\ 0 \leqq q \leqq 1$

(6.10) $\quad \kappa < \theta$

(6.11) $\quad$ For any $1 \leqq j \leqq m-1$, $k_j q\theta \leqq m-j+k_j q$.

If $1 \leqq \kappa < \sigma/(\sigma-1)$ we have

$$\kappa < \sigma/(\sigma-1) \leqq (k_j\nu + m - j)/k_j\nu \quad (\equiv k_{j,\nu}) \qquad \text{for} \quad 1 \leqq j \leqq m-1.$$

Then for any $\kappa < k_{j,\nu}$ there exist constants $q > \nu$, $\theta > \kappa$ so as $\theta \leqq k_{j,q}$. These q and θ satisfy (6.9)~(6.11). Therefore we can apply Lemma 6.4. Here let us decompose $u(t, x)$ by

$$\begin{aligned} u(t, x) &= \sum_{j=0}^{N-1} u_j(t, x) + \sum_{j=N}^{\infty} u_j(t, x) \\ &= u_N^1(t, x) + u_N^2(t, x). \end{aligned}$$

From Lemma 6.4 the series $u_N^2(t, x)$ is convergent in $\mathscr{B}([0, T], \Gamma^{(\kappa)})$ and $u_N^2(t, x)$ belongs to $\mathscr{B}([0, T], \Gamma^{(\kappa)})$. Since $u_j(t, x) \in \mathscr{B}([0, T], \Gamma^{(\kappa)})$ for $1 \leqq j \leqq N-1$ we can see $u(t, x) \in \mathscr{B}([0, T], \Gamma^{(\kappa)})$. Q. E. D.

Consequently we have established the existence of a solution in Theorem 2. Next we shall show the uniqueness of solutions.

Lemma 6.6. *Under the assumptions of Theorem 2, let $u(t, x) \in \mathscr{B}([0, T], \Gamma^{(\kappa)})$ be a solution of a homogeneous equation;*

$$(6.12) \qquad \begin{cases} P(t, x, D_t, D_x)u(t, x) = 0 \\ D_t^i u(t, x)|_{t=0} = 0 \qquad \text{for} \quad 0 \leqq i \leqq m-k-1 \end{cases}$$

where $1 \leqq \kappa < \sigma/(\sigma-1)$. Then $u(t, x)$ vanishes identically.

Proof. From the assumption (A–VI) $u(t, x)$ is flat at $t=0$. Hence $u(t, x)$ satisfies the following.

$$(6.13)\qquad \begin{cases} \hat{Q}u(t,x) = -\hat{R}u(t,x) \\ D_t^i u(t,x)|_{t=0} = 0 \qquad \text{for any} \quad i \geqq 0. \end{cases}$$

We may assume that for sufficiently large s there exist constants c and R such that

$$\|\Lambda^r u\| \leqq c w_r(s, t, R) \qquad \text{for any} \quad r \geqq 0.$$

By Lemma 6.4 we can get from the above estimate that

$$\|\Lambda^r u\| \leqq c\tilde{B}\tilde{A}^n n^{n(\kappa-\theta)q} w_r(s+n(q-v)l, t, \tilde{R}).$$

Let n be infinity, then we conclude $u(t, x) \equiv 0$. Q. E. D.

Finally we shall demonstrate that 2° of Theorem 2 is realized by convergence of (2.6) and 2° in Proposition 2.1. From Proposition 2.1 and $(2.4)_0$ if supp $(u^i(x)) \subset K$ and supp $(f(t, x)) \subset C_l(K)$ for some compact set $K \subset \boldsymbol{R}^n$, supp $(u_0(t, x)) \subset C_l(K)$. Since $R(t, x, D_t, D_x)$ is a differential operator, we get supp $(Ru_0(t, x)) \subset C_l(K)$. Therefore it follows from $(2.4)_1$ and Proposition 2.1 that supp $(u_1(t, x))$ belongs to $C_l(K)$. Repeating these steps we have supp $(u_j(t, x)) \subset C_l(K)$ for any $j \geqq 0$. From the convergence of (2.6) we conclude supp $(u(t, x)) \subset C_l(K)$. The proof of Theorem 2 is completed.

Appendix

Following Igari [4], we introduce a certain class of pseudo-differential operators.

Definition. 1) *For any $m \in \boldsymbol{R}^1$ and $\kappa > 1$ we denote by $S^m(\kappa)$ the set of functions $h(x, \xi) \in C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ satisfying the property that for any multi-indices α, β, there exist constants c_α and R such that*

$$|\partial_\xi^\alpha D_x^\beta h(x, \xi)| \leqq c_\alpha R^{|\beta|}(|\beta|!)^\kappa \langle\xi\rangle^{m-|\alpha|} \qquad \textit{for} \quad (x, \xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n.$$

2) *For any $h(x, \xi) \in S^m(\kappa)$ we shall define semi-norms of $h(x, \xi)$ such that for any integer $l \geqq 0$*

$$|h(x, \xi)|_l = \max_{|\alpha+\beta| \leqq l} \sup_{(x,\xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n} |\partial_\xi^\alpha D_x^\beta h(x, \xi)| \langle\xi\rangle^{-m+|\alpha|}.$$

We define a pseudo-differential operator with a symbol $h(x, \xi) \in S^m(\kappa)$ by

$$H(x, D_x)u = (2\pi)^{-n} \int e^{ix\cdot\xi} h(x, \xi) \hat{u}(\xi) d\xi.$$

Hence we have a composition formula of pseudo-differential operators.

Lemma A–1. (*See Igari* (4)). *Let* $h(x, \xi) \in S^m(\kappa)$ *and* $r \geqq 0$. *Then*

$$\sigma(\Lambda^r H) = \sum_{i=1}^{N-1} \sum_{|\gamma|=i} 1/\gamma! \partial_\xi^\gamma \langle\xi\rangle^r h_\gamma(x, \xi) + r_N(x, \xi)$$

where $N = r^* + m$ *and* $h_\gamma(x, \xi) = D_x^\gamma h(x, \xi)$. *Then for any integer* $l \geqq 0$ *there exist constants* c_l, $R > 0$ *such that*

$$|h_\gamma(x, \xi)\langle\xi\rangle^{-m}|_l \leqq c_l R^{|\gamma|-m}(|\gamma|-m)!^\kappa \tag{A.1}$$

$$|r_N(x, \xi)|_l \leqq c_l R^r r!^\kappa . \tag{A.2}$$

The following lemma is well known.

Lemma A–2. *For any* $h(x, \xi) \in S^0$ *there exists constant* c *and non-negative integer* l *dependent only on dimension* n *such that*

$$\|H(x, D_x)u\| \leqq c|h(x, \xi)|_l \|u\| . \tag{A.3}$$

Lemma A–3. *Under the assumptions of Lemma* A–1 *we denote* $h_i(x, \xi)$ *by*

$$h_i(x, \xi) = \sum_{|\gamma|=i} 1/\gamma! \partial_\xi^\gamma \langle\xi\rangle^r h_\gamma(x, \xi) .$$

Then there exist $\hat{c}$, $\hat{R} > 0$ *such that*

$$\begin{cases} \|H_i(x, D_x)u\| \leqq \hat{c}\hat{R}^{i-m}(i-m)!^\kappa \binom{r^*}{i} \|\Lambda^{m+r-i}u\| & \textit{for} \quad 1 \leqq i \leqq r^* \\ \|H_i(x, D_x)u\| \leqq \hat{c}\hat{R}^{i-m}(i-m)!^\kappa \|\Lambda^{m+r-i}u\| & \textit{for} \quad r^*+1 \leqq i \leqq N-1 , \end{cases} \tag{A.4}$$

$$\|R_N(x, D_x)u\| \leqq \hat{c}\hat{R}^r r!^\kappa \|u\| . \tag{A.5}$$

Proof. (A.5) is a direct consequence of (A.2) and Lemma A–2. For the proof of (A.4) it is sufficient to show the following inequalities.

$$\begin{cases} |h_i(x, \xi)\langle\xi\rangle^{-m-r+i}|_l \leqq \tilde{c}\hat{R}^{i-m}(i-m)!^\kappa \binom{r^*}{i} & \text{for} \quad 1 \leqq i \leqq r^* , \\ |h_i(x, \xi)\langle\xi\rangle^{-m-r+i}|_l \leqq \tilde{c}\hat{R}^{i-m}(i-m)!^\kappa & \text{for} \quad r^*+1 \leqq i \leqq N-1 \end{cases} \tag{A.6}$$

for some constant $\tilde{c}$.

We can easily see that for any $|\alpha'| \leqq l$

$$\sum_{|\gamma|=i} 1/\gamma! |\partial_\xi^{\alpha'} \{\langle\xi\rangle^{-r+i} \partial_\xi^\gamma \langle\xi\rangle^r\}| \leqq \begin{cases} A^i \binom{r^*}{i} \langle\xi\rangle^{-|\alpha'|} & \text{for} \quad 1 \leqq i \leqq r^* , \\ A^i \langle\xi\rangle^{-|\alpha'|} & \text{for} \quad r^*+1 \leqq i \leqq N-1 , \end{cases} \tag{A.7}$$

where A is independent of r and i.

Now we shall estimate the absolute value of

$$I_{\alpha,\beta}=\partial_\xi^\alpha D_x^\beta\{h_i(x, \xi)\langle\xi\rangle^{-m-r+i}\} \qquad \text{for} \quad |\alpha+\beta|\leqq l.$$

$$\begin{aligned}|I_{\alpha,\beta}|&=|\sum_{|\gamma|=i\ \alpha'\leqq\alpha} 1/\gamma!\binom{\alpha}{\alpha'}\partial_\xi^{\alpha'}\{\langle\xi\rangle^{-r+i}\partial_\xi^\gamma\langle\xi\rangle^r\}\\ &\quad\times\partial_\xi^{\alpha-\alpha'}D_x^\beta\{\langle\xi\rangle^{-m}h_\gamma(x, \xi)\}|\\ &\leqq\sum_{\alpha'\leqq\alpha}\binom{\alpha}{\alpha'}\sum_{|\gamma|=i} 1/\gamma!|\partial_\xi^{\alpha'}\{\langle\xi\rangle^{-r+i}\partial_\xi^\gamma\langle\xi\rangle^r\}\\ &\quad\times|h_\gamma(x, \xi)\langle\xi\rangle^{-m}|_l\langle\xi\rangle^{-|\alpha-\alpha'|}.\end{aligned}$$

It follows from (A.1) and (A.7)

$$|I_{\alpha,\beta}|\leqq\begin{cases} BA^iR^{i-m}\binom{r^*}{i}(i-m)!^\kappa\langle\xi\rangle^{-|\alpha|} & i=1, 2,\dots, r^*,\\ BA^iR^{i-m}(i-m)!^\kappa\langle\xi\rangle^{-|\alpha|} & i=r^*+1,\dots, N-1\end{cases}$$

for some constant B. The proof is completed. Q. E. D.

Lemma A-4. *Let $\phi(t)$ and $\psi(t)\in C^0([0, T])$. Assume that the following integral inequality is satisfied.*

$$\phi(t)\leqq\psi(t)+c\int_0^t \tau^{l-1}\phi(\tau)d\tau$$

where c and l are positive constants. Then we obtain

$$\phi(t)\leqq\psi(t)+c\int_0^t \tau^{l-1}\psi(\tau)\exp\left\{\frac{c}{l}(t^l-\tau^l)\right\}d\tau. \tag{A.8}$$

Proof. Let $\Phi(t)$ be $\Phi(t)=\int_0^t \tau^{l-1}\phi(\tau)d\tau$, then

$$\frac{d}{dt}\Phi(t)-ct^{l-1}\Phi(t)\leqq t^{l-1}\psi(t).$$

Hence we can easily see (A.8). Q. E. D.

Lemma A-5. *Let $i_1,\dots, i_n$ $(n=1, 2,\dots)$ be elements of $\{1, 2,\dots, m-1\}$. Then there exist constants A_1, R_1 such that*

$$n^{i_1+\cdots+i_n}\leqq A_1R_1^n\, 1^{i_1}2^{i_2}\cdots n^{i_n}. \tag{A.9}$$

Proof. Put $S=n^{i_1+\cdots+i_n}/1^{i_1}2^{i_2}\cdots n^{i_n}$. Then

$$\begin{aligned}S&=\left(\frac{n}{1}\right)^{i_1}\left(\frac{n}{2}\right)^{i_2}\times\cdots\times\left(\frac{n}{n}\right)^{i_n}\\ &\leqq\left(\frac{n}{1}\right)^{m-1}\left(\frac{n}{2}\right)^{m-1}\times\cdots\times\left(\frac{n}{n}\right)^{m-1}=\left(\frac{n^n}{n!}\right)^{m-1}.\end{aligned}$$

Stirling's formula yields

$$n! \sim \sqrt{2}\, n^{n+1/2} e^{-n} \quad \text{as} \quad n \to \infty .$$

Therefore there exist some constants A_1, R_1 such that $S \leqq A_1 R_1^n$.

Q. E. D.

Lemma A–6. *Let* $i_1, \dots, i_n \in \{1, 2, \dots, m-1\}$ *for* $n = 1, 2, \dots$ *and* $0 \leqq q \leqq 1$. *Then the following inequality holds.*

(A.10) $$\{q(i_1 + \cdots + i_n)\}! \leqq A_2 R_2^n n^{q(i_1 + \cdots + i_n)}$$

for some constants A_2, $R_2 > 0$ *which are independent of* n.

Proof. There exists $\hat{A}_2 > 0$ so as $x! \leqq \hat{A}_2 x^x$ for any $x > 0$. Therefore

$$\begin{aligned}\{q(i_1 + \cdots + i_n)\}! &\leqq \hat{A}_2 \{q(i_1 + \cdots + i_n)\}^{(i_1 + \cdots + i_n)q} \\ &\leqq \hat{A}_2 \{q(m-1)n\}^{(i_1 + \cdots + i_n)q} \\ &\leqq \hat{A}_2 \{q(m-1)\}^{(m-1)nq} n^{(i_1 + \cdots + i_n)q}.\end{aligned}$$

Let $A_2 = \hat{A}_2$, $R_2 = \{q(m-1)\}^{(m-1)q}$, then we have (A–10). Q. E. D.

Lemma A–7. *Let* $i_1, \dots, i_n$ $(n = 1, 2, \dots)$ *be elements of* $\{1, 2, \dots, m-1\}$ *and* $0 \leqq q \leqq 1$. *Then there exists* A_3, R_3, R' *which are independent of* $r \geqq 0$ *such that*

(A.11) $$w_{r+(i_1 + \cdots + i_n)}(s, t, R) \leqq A_3 R_3^n \, n^{(i_1 + \cdots + i_n)q\kappa} w_r(s, t, R') .$$

Proof. From the definition in §4,

$$\begin{aligned} w_{r+(i_1 + \cdots + i_n)q}(s, t, R) = R^{r+(i_1 + \cdots + i_n)q} &\{r + q(i_1 + \cdots + i_n)\}!^{\kappa} \\ &\times t^s \exp(K\{r + (i_1 + \cdots + i_n)q\}^* t^l) .\end{aligned}$$

Let us note the following facts.

(A.12) $$\{r + (i_1 + \cdots + i_n)q\}! \leqq 2^{r + q(i_1 + \cdots + i_n)} \{q(i_1 + \cdots + i_n)\}! r!$$

(A.13) $$(i_1 + \cdots + i_n)q \leqq (m-1)n$$

(A.14) $$\{r + (i_1 + \cdots + i_n)q\}^* \leqq r^* + (m-1)n$$

Then it follows from Lemma A–4 that

(A.15) $$\{q(i_1 + \cdots + i_n)\}! \leqq A_2 R_2^n n^{(i_1 + \cdots + i_n)q} .$$

Therefore we obtain

$$\begin{aligned} w_{r+(i_1 + \cdots + i_n)}(s, t, R) \leqq A_2^{\kappa} &\{2^{(m-1)\kappa} R_2^{\kappa} R^{(m-1)} e^{K(m-1)t^l}\}^n \\ &\times n^{(i_1 + \cdots + i_n)q\kappa} w_r(s, t, 2^{\kappa} R) .\end{aligned}$$

Q. E. D.

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