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Examples of Absolutely Continuous Schrödinger Operators in Magnetic Fields

By

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§1. Introduction

In this paper we shall consider the two-dimensional Schrödinger operator H which is the self-adjoint realization in $\mathscr{H}=L^2(\boldsymbol{R}^2)$ of the differential operator

$$(1) \qquad L=\left(\frac{1}{i}\frac{\partial}{\partial x}-a\right)^2+\left(\frac{1}{i}\frac{\partial}{\partial y}-b\right)^2$$

where a and b are the operators of multiplication by real-valued C^∞ functions $a(x, y)$ and $b(x, y)$, respectively. The spectral property of H depends not directly on the vector potential (a, b) but on the magnetic field

$$(2) \qquad B(x, y)=\frac{\partial b}{\partial x}(x, y)-\frac{\partial a}{\partial y}(x, y),$$

i.e., all H with (a, b) satisfying (2) with common B are unitarily equivalent to each other (gauge invariance: see, e.g., Leinfelder [5]).

Extensive studies have been made in the case where B is asymptotically constant, that is,

$$B(x, y)\longrightarrow B_0 \quad \text{as} \quad \sqrt{x^2+y^2}\longrightarrow\infty,$$

where B_0 is some constant. In the case where $B_0=0$, the essential spectrum $\sigma_{ess}(H)\equiv\{\lambda\in\boldsymbol{R}|\dim\operatorname{Ran}(E((\lambda-\varepsilon, \lambda+\varepsilon)))=\infty$ for all $\varepsilon>0\}$ of H is $[0, \infty)$ (see [5]), where E denotes the spectral measure associated with H. Moreover, if B is short-range (i.e., $B=O(\sqrt{x^2+y^2}^{\,-1-\delta})$ for some $\delta>0$), H is absolutely continuous, i.e., the subspace of absolute continuity $\mathscr{H}_{ac}\equiv\{u\in\mathscr{H}|\ \|E((-\infty, \lambda])u\|^2$ is absolutely continuous$\}$ for H fills up the whole space $\mathscr{H}$ (see Ikebe and Saitō [2]). In the long-range case, there is an example of H with pure point

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spectrum in the sense that $\mathscr{H}$ is spanned by the eigenvectors of H (see Miller and Simon [7]). On the other hand, in the case where $B_0 \neq 0$, the operator H shows a completely different spectral behavior. Namely, we have shown in [3] that

$$\sigma_{ess}(H) = \{(2k+1)|B_0| \,|\, k \text{ is an integer } \geqq 0\},$$

which implies that H has pure point spectrum (see also Avron, Herbst and Simon [1]). However, as far as we know, the case where B is not asymptotically constant has not been studied very well. We shall consider the case where B satisfies the following property:

(B) $B(x, y)$ depends only on x (i.e., $B(x, y) = B(x)$), $B(x)$ is a C^∞ function such that there exist constants $M_\pm$ satisfying $0 < M_- \leqq B(x) \leqq M_+ < \infty$ for all x.

Let $\dot{H}$ be the operator in $\mathscr{H} = L^2(\boldsymbol{R}^2)$ defined on the space $C_0^\infty(\boldsymbol{R}^2)$ of all C^∞ functions with compact support by

$$\dot{H}u = Lu \qquad \text{for} \quad u \in C_0^\infty(\boldsymbol{R}^2).$$

Let H be the closure of $\dot{H}$. Then $\dot{H}$ is essentially self-adjoint (see Leinfelder and Simader [6]) and hence H is the unique self-adjoint extention of $\dot{H}$. Therefore, the adjoint operator $\dot{H}^*$ of $\dot{H}$ equals to H and, by examining $\dot{H}^*$, one can obtain

$$\begin{cases} D(H) = \{f \in \mathscr{H} | Lf \in \mathscr{H}\}, \\ Hf = Lf \qquad \text{for} \quad f \in D(H), \end{cases} \tag{3}$$

where and in the sequel differentiation is understood in the distribution sense. The aim of the present paper is to show that H is absolutely continuous if either of the following (B1) or (B2) holds:

(B1) In addition to (B), $\limsup_{x \to -\infty} B(x) < \liminf_{x \to +\infty} B(x)$ or $\limsup_{x \to +\infty} B(x) < \liminf_{x \to -\infty} B(x)$.

(B2) In addition to (B), $B(x) = B_0$ for some constant B_0 if $|x|$ is sufficiently large, and there exists a point $\bar{x}$ such that $B'(x) \leqq 0$ for $x \leqq \bar{x}$ and $B'(x) \geqq 0$ for $x \geqq \bar{x}$ (or $B'(x) \geqq 0$ for $x \leqq \bar{x}$ and $B'(x) \leqq 0$ for $x \geqq \bar{x}$) and $B'(x)$ is not identically 0.

Theorem. *Suppose that either* (B1) *or* (B2) *holds. Then H is absolutely continuous.*

§ 2. Reduction to One-Dimensional Hamiltonians

Under the assumption (B), we can take the vector potential (a, b) of the form

$$a=0,\ b=b(x)=\int_0^x B(t)dt \tag{4}$$

in view of (2). Therefore, since we shall assume (B) throughout the paper, we shall henceforth consider the differential operator

$$L=-\frac{\partial^2}{\partial x^2}+\left(\frac{1}{i}\frac{\partial}{\partial y}-b(x)\right)^2. \tag{5}$$

Let H_1 be the operator in $\mathscr{H}$ defined on the Schwartz space $\mathscr{S}(\boldsymbol{R}^2)$ of rapidly decreasing C^∞ functions by

$$H_1u=Lu \qquad \text{for} \quad u\in\mathscr{S}(\boldsymbol{R}^2)$$

(note that $Lu\in\mathscr{H}$ for $u\in\mathscr{S}(\boldsymbol{R}^2)$ by the estimate $|b(x)|\leqq M_+|x|$ obtainable from (4) and (B)). Then $\dot{H}\subset H_1$ and $H_1\subset H$ by (3). Thus H_1 is essentially self-adjoint since $\dot{H}$ is so. Let $\tilde{L}$ be the differential operator

$$(\tilde{L}f)(x, \xi)=\left\{-\frac{\partial^2}{\partial x^2}+(b(x)-\xi)^2\right\}f(x, \xi). \tag{6}$$

Then it is clear that $\tilde{L}\mathscr{F}u=\mathscr{F}Lu$ for $u\in\mathscr{S}(\boldsymbol{R}^2)$ where $\mathscr{F}$ is the partial Fourier transformation

$$\mathscr{F}u(x, \xi)=(2\pi)^{-1/2}\int e^{-iy\xi}u(x, y)dy.$$

Let $\tilde{H}_1$ be the operator in $\tilde{\mathscr{H}}=L^2(\boldsymbol{R}_x\times\boldsymbol{R}_\xi)$ defined by $\tilde{H}_1f=\tilde{L}f$ for $f\in D(\tilde{H}_1)\equiv\mathscr{S}(\boldsymbol{R}_x\times\boldsymbol{R}_\xi)$, and let $\tilde{H}$ be the closure of $\tilde{H}_1$. Then $\tilde{H}_1=\mathscr{F}H_1\mathscr{F}^{-1}$ and hence, since $\mathscr{F}$ is unitary and H_1 is essentially self-adjoint, $\tilde{H}_1$ is essentially self-adjoint. Therefore, $\tilde{H}$ is self-adjoint and $\tilde{H}=\mathscr{F}H\mathscr{F}^{-1}$. Thus we have

Lemma 2.1. *Assume that* (B) *holds. Let $\tilde{H}$ be the self-adjoint operator defined above. Then H is unitarily equivalent to $\tilde{H}$.*

Lemma 2.2. *$C_0^\infty(\boldsymbol{R}_x\times\boldsymbol{R}_\xi)$ is a core for $\tilde{H}$ and*

$$\begin{cases} D(\tilde{H})=\{f\in\tilde{\mathscr{H}}\,|\,\tilde{L}f\in\tilde{\mathscr{H}}\} \\ \tilde{H}f=\tilde{L}f \qquad \textit{for} \quad f\in D(\tilde{H}). \end{cases} \tag{7}$$

Proof. Let $\zeta \in C_0^\infty(\boldsymbol{R}_x \times \boldsymbol{R}_\xi)$ such that $\zeta(x, \xi)=1$ for $\sqrt{x^2+\xi^2} \leqq 1$ and $0 \leqq \zeta \leqq 1$. Then it is not difficult to verify that, for $f \in \mathscr{S}(\boldsymbol{R}_x \times \boldsymbol{R}_\xi)$, $\zeta_n f \to f$, $\tilde{L}(\zeta_n f) \to \tilde{L}f$ strongly in $\tilde{\mathscr{H}}$ as $n \to \infty$, where ζ_n is the operator of multiplication by $\zeta(x/n, \xi/n)$ for $n=1, 2, \ldots$. Hence, since $\mathscr{S}(\boldsymbol{R}_x \times \boldsymbol{R}_\xi)$ is a core for $\tilde{H}$, $C_0^\infty(\boldsymbol{R}_x \times \boldsymbol{R}_\xi)$ is a core for $\tilde{H}$. Moreover, it follows that $\tilde{H}$ coincides with the adjoint operator of $\tilde{H}|_{C_0^\infty(\boldsymbol{R}_x \times \boldsymbol{R}_\xi)}$($|$ denotes the restriction), from which (7) follows. □

$\tilde{L}$ can be written as

$$\tilde{L}f(x, \xi)=(\tilde{L}(\xi)f(\cdot, \xi))(x), \tag{8}$$

where $\tilde{L}(\xi)$ is a second-order ordinary differential operator

$$\tilde{L}(\xi)=-\frac{d^2}{dx^2}+(b(x)-\xi)^2. \tag{9}$$

Let $\dot{\tilde{H}}(\xi)$ be the operator in $L^2(\boldsymbol{R}_x)$ defined by $\dot{\tilde{H}}(\xi)\phi=\tilde{L}(\xi)\phi$ for $\phi \in D(\dot{\tilde{H}}(\xi)) \equiv C_0^\infty(\boldsymbol{R}_x)$, and let $\tilde{H}(\xi)$ be the closure of $\dot{\tilde{H}}(\xi)$. Then, since $\tilde{L}(\xi)$ is in the limit point case at $\pm\infty$ (see, e.g., [8], Appendix to X.1), $\dot{\tilde{H}}(\xi)$ is essentially self-adjoint and hence $\tilde{H}(\xi)$ is self-adjoint.

Lemma 2.3. *Assume that (B) holds. Let ξ be a real number and let $\tilde{H}(\xi)$ be the self-adjoint operator defined above. Then there exists a complete orthonormal system $\{\psi_n(x, \xi)\}_{n=1,2,\ldots}$ in $L^2(\boldsymbol{R}_x)$ of eigenfunctions for $\tilde{H}(\xi)$:*

$$\begin{cases} \tilde{H}(\xi)\psi_n(x, \xi)=\lambda_n(\xi)\psi_n(x, \xi) \\ 0<\lambda_1(\xi)<\lambda_2(\xi)<\lambda_3(\xi)<\cdots \to \infty, \end{cases} \tag{10}$$

so that, for $n=1, 2, \ldots$,

(i) *each $\lambda_n(\xi)$ is non-degenerate,*

(ii) $(2n-1)M_- \leqq \lambda_n(\xi) \leqq (2n-1)M_+$,

(iii) *$\lambda_n(\xi)$ depends analytically on ξ,*

(iv) *$\psi_n(\cdot, \xi) \in D(\tilde{H}(0))$ and depends analytically on ξ with respect to the graph norm $|||u||| \equiv (\|u\|^2+\|\tilde{H}(0)u\|^2)^{1/2}$,*

(v) *$\psi_n(x, \xi)$ is a real-valued continuous function of x and ξ, and, moreover, $\psi_n(x, \xi)$ is infinitely differentiable in x for each ξ and is analytic in ξ for each x.*

Proof. First part ((i) and (ii)): Since $db(x)/dx=B(x) \geqq M_- >0$ by (B) and (4), $b(x)$ is a strictly increasing function of x such that $\lim_{x\to\pm\infty} b(x)=\pm\infty$. Thus the equation $b(x)=\xi$ has a unique solution for each ξ, which we shall denote by x_ξ (i.e., $x_\xi \equiv b^{-1}(\xi)$). Then we have

$$b(x)-\xi=\int_{x_\xi}^{x} B(t)dt\,. \tag{11}$$

Hence we have from the assumption (B)

$$M_-(x-x_\xi)\leqq b(x)-\xi\leqq M_+(x-x_\xi) \qquad \text{for} \quad x\geqq x_\xi\,,$$

$$M_-(x-x_\xi)\geqq b(x)-\xi\geqq M_+(x-x_\xi) \qquad \text{for} \quad x\leqq x_\xi\,,$$

from which we obtain the following inequality for all x:

$$M_-^2(x-x_\xi)^2\leqq(b(x)-\xi)^2\leqq M_+^2(x-x_\xi)^2\,. \tag{12}$$

Since $(b(x)-\xi)^2$ is smooth and tends to ∞ as $|x|\to\infty$ by (12), $\tilde{H}(\xi)$ has compact resolvent and a complete set of eigenfunctions $\{\psi_n(\cdot\,,\,\xi)\}_{n=1,2,\dots}$ with eigenvalues $\{\lambda_n(\xi)\}_{n=1,2,\dots}$ such that $\lambda_1(\xi)\leqq\lambda_2(\xi)\leqq\lambda_3(\xi)\leqq\cdots\to\infty$ (see, e.g., [9], Theorems XIII. 64 and XIII. 67). The proof of the non-degeneracy of $\lambda_n(\xi)$ needs a proposition concerning the eigenfunctions of second order differential equations, which we shall prove in the next section (Proposition 3.1). By Proposition 3.1 with $q(x)=(b(x)-\xi)^2-\lambda_n(\xi)$ and $I=[x_0,\,\infty)$ where $x_0=x_\xi+\sqrt{\lambda_n(\xi)}/M_-$ ($q(x)\geqq0$ on I by (12)), any square integrable real-valued solution of the differential equation

$$\tilde{L}(\xi)u(x)=\lambda_n(\xi)u(x) \tag{13}$$

satisfies the inequality

$$u'(x)u(x)<0 \qquad \left({}'\text{ denotes } \frac{d}{dx}\right) \tag{14}$$

for $x\geqq x_0$. On the other hand, since $\tilde{L}(\xi)$ is of second order, the multiplicity of $\lambda_n(\xi)$ is 1 or 2. If the multiplicity is 2, all the solutions of (13) would belong to $L^2(\boldsymbol{R}_x)$. But this is impossible because we can solve the equation (13) with given initial value of $(u,\,u')$, say, $(1,\,1)$ at the point x_0, which contradicts (14). This implies that the multiplicity of $\lambda_n(\xi)$ is 1, i.e., (i) holds and hence $\lambda_1(\xi)<\lambda_2(\xi)<\lambda_3(\xi)<\cdots$.

Next, from (12) and a comparison theorem based on the min-max principle (see, e.g., [9], p. 270, Lemma), we have (ii) since the n-th eigenvalue of the harmonic oscillator Hamiltonian $-\dfrac{d^2}{dx^2}+M_\pm^2(x-x_\xi)^2$ is $(2n-1)M_\pm$.

For the remainder of the proof of Lemma 2.3, we need a lemma (Lemma 2.4 below). Let $\mathscr{D}$ denote the space $D(\tilde{H}(0))$ equipped with the graph norm $|\!|\!|\cdot|\!|\!|$, $P_n(\xi)$ the projection onto the n-th eigenspace of $\tilde{H}(\xi)$ and $\boldsymbol{B}(X,\,Y)$ the space of bounded operators from X to Y for Banach spaces X and Y.

Lemma 2.4. (a) *The operator b is relatively bounded with relative bound 0 with respect to $\tilde{H}(0)$.*

(b) *$D(\tilde{H}(\xi))$ equipped with its graph norm $\|u\|_{D(\tilde{H}(\xi))} \equiv (\|u\|^2 + \|\tilde{H}(\xi)u\|^2)^{1/2}$ coincides with $\mathscr{D}$ for all $\xi \in \boldsymbol{R}$.*

(c) *$\mathscr{D}$ is a subspace of $C_0(\boldsymbol{R}) \equiv \{f \mid f$ is a continuous function on $\boldsymbol{R}$, $|f(x)| \to 0$ as $|x| \to \infty\}$ and the inclusion map: $\mathscr{D} \to C_0(\boldsymbol{R})$ belongs to $\boldsymbol{B}(\mathscr{D}, C_0(\boldsymbol{R}))$, where $C_0(\boldsymbol{R})$ is equipped with the norm $|f|_\infty \equiv \sup_{x \in \boldsymbol{R}} |f(x)|$.*

(d) *$P_n(\xi)$ is a $\boldsymbol{B}(\mathscr{H}, \mathscr{D})$-valued analytic function of $\xi \in \boldsymbol{R}$.*

Proof. (a) We have in view of (9)

$$\|bu\|^2 = (b^2 u, u) \leqq (\tilde{L}(0)u, u) \leqq \varepsilon \|\tilde{L}(0)u\|^2 + \frac{1}{2\varepsilon} \|u\|^2,$$

for all $u \in C_0^\infty(\boldsymbol{R})$ and for all $\varepsilon > 0$, where (,) denotes the inner product of $L^2(\boldsymbol{R}_x)$. This inequality implies that the operator b is relatively bounded with relative bound 0 with respect to $\dot{\tilde{H}}(0)$ and hence with respect to $\tilde{H}(0) =$ the closure of $\dot{\tilde{H}}(0)$.

(b) We have by (9)

$$\left\{ \begin{array}{l} \tilde{H}(\xi) = \tilde{H}(\xi_0) + V(\xi, \xi_0) \\ V(\xi, \xi_0) = -2(\xi - \xi_0)(b(x) - \xi_0) + (\xi - \xi_0)^2 . \end{array} \right. \tag{15}$$

It is not difficult to verify, by using (a) and (15) with $\xi_0 = 0$, that the norm $|||\cdot|||$ and the norm $\|\cdot\|_{D(\tilde{H}(\xi))}$ are equivalent on $C_0^\infty(\boldsymbol{R})$. Therefore, since $D(\tilde{H}(\xi))$ coincides with the completion of $C_0^\infty(\boldsymbol{R})$ with respect to the norm $\|\cdot\|_{D(\tilde{H}(\xi))}$ by the definition of closure of an operator, we have (b).

(c) Since we have a Sobolev inequality

$$|f|_\infty \leqq C\left(\|f\| + \left\|\frac{d}{dx} f\right\|\right)$$

for $f \in C_0^\infty(\boldsymbol{R})$ where C is a constant and we have in view of (9)

$$\left\|\frac{d}{dx} f\right\|^2 = -\left(\frac{d^2 f}{dx^2}, f\right) \leqq (\tilde{L}(0)f, f) \leqq \frac{1}{2}(\|f\|^2 + \|\tilde{L}(0)f\|^2)$$

for $f \in C_0^\infty(\boldsymbol{R})$, we obtain

$$|f|_\infty \leqq C |||f|||$$

for $f \in C_0^\infty(\boldsymbol{R})$. From this inequality it follows that the identity map on $C_0^\infty(\boldsymbol{R})$ can be extended to a continuous one to one map from the completion $\mathscr{D}$ of

$C_0^\infty(\boldsymbol{R})$ with respect to the norm $|||\cdot|||$ into the completion X of $C_0^\infty(\boldsymbol{R})$ with respect to the norm $|\cdot|_\infty$. Since it is not difficult to check that X is $C_0(\boldsymbol{R})$, we obtain (c).

(d) Fix ξ_0. Then it is clear by (b) that $(\tilde{H}(\xi_0)-\lambda)^{-1}$ is a $\boldsymbol{B}(\mathscr{H}, \mathscr{D})$-valued continuous (analytic) function of $\lambda \notin \sigma(\tilde{H}(\xi_0))$. Let Γ be a circle $\{\lambda | \, |\lambda-\lambda_n(\xi_0)|=\varepsilon\}$ in the complex plane $\boldsymbol{C}$ for some integer n with sufficiently small $\varepsilon>0$ such that $\sigma(\tilde{H}(\xi_0)) \cap \{\lambda \in \boldsymbol{C} | \, |\lambda-\lambda_n(\xi_0)| \leqq 2\varepsilon\}=\{\lambda_n(\xi_0)\}$. Then, there exists a constant $K>0$ such that $\|(\tilde{H}(\xi_0)-\lambda)^{-1}\|_{\boldsymbol{B}(\mathscr{H},\mathscr{D})} \leqq K$ for $\lambda \in \Gamma$. Let $\Delta=\{\xi \in \boldsymbol{C} | \, |\xi-\xi_0|<\delta\}$ with sufficiently small $\delta>0$ such that $\|V(\xi, \xi_0)\|_{\boldsymbol{B}(\mathscr{D},\mathscr{H})} \leqq \frac{1}{2K}$ for $\xi \in \Delta$, where $V(\xi, \xi_0)$ is as in (15). Then the Neumann series

$$(1+V(\xi, \xi_0)(\tilde{H}(\xi_0)-\lambda)^{-1})^{-1}=\sum_{j=0}^{\infty}\{V(\xi, \xi_0)(\tilde{H}(\xi_0)-\lambda)^{-1}\}^j$$

converges in $\boldsymbol{B}(\mathscr{H}, \mathscr{H})$ uniformly for $(\lambda, \xi) \in \Gamma \times \Delta$, and hence becomes a $\boldsymbol{B}(\mathscr{H}, \mathscr{H})$-valued continuous function of $(\lambda, \xi) \in \Gamma \times \Delta$ which is analytic in $\xi \in \Delta$. Hence, we have by (15)

$$(\tilde{H}(\xi)-\lambda)^{-1}=(\tilde{H}(\xi_0)-\lambda)^{-1}(1+V(\xi, \xi_0)(\tilde{H}(\xi_0)-\lambda)^{-1})^{-1},$$

for $(\lambda, \xi) \in \Gamma \times \Delta$, which is a $\boldsymbol{B}(\mathscr{H}, \mathscr{D})$-valued continuous function of $(\lambda, \xi) \in \Gamma \times \Delta$, analytic in ξ. Consequently $P_n(\xi)$ is a $\boldsymbol{B}(\mathscr{H}, \mathscr{D})$-valued analytic function of $\xi \in \Delta$, since we have ([9], Theorem XII. 5) for $\xi \in \Delta$

$$P_n(\xi)=\frac{-1}{2\pi i}\int_\Gamma (\tilde{H}(\xi)-\lambda)^{-1} d\lambda. \tag{16}$$

Thus, since ξ_0 was arbitrary, we obtain (d). □

Proof of Lemma 2.3. Second part ((iii), (iv) and (v)): $D(\tilde{H}(\xi))=D(\tilde{H}(0))$ by (b) of Lemma 2.4 and $\tilde{H}(\xi)u$ is analytic in ξ for all $u \in D(\tilde{H}(0))$. Hence, $\tilde{H}(\xi)$ is an analytic family of type (A) (see [4], p. 345). Thus $\lambda_n(\xi)$ depends analytically on ξ ([4], p. 370, Theorem 1.8) since $\lambda_n(\xi)$ is non-degenerate by (i). This proves (iii).

For (iv) and (v), we first show the existence of an $\mathscr{H}$-valued analytic function $\psi_n(\cdot, \xi)$ of $\xi \in \boldsymbol{R}$ such that $\psi_n(\cdot, \xi)$ is a real and normalized eigenfunction of $\tilde{H}(\xi)$ with eigenvalue $\lambda_n(\xi)$. Since $\tilde{H}(\xi)$ is real (i.e., commutes with the complex conjugation C: $Cu=\bar{u}$), it follows from (16) that $P_n(\xi)$ is real. Moreover, $P_n(\xi)$ is a projection and a $\boldsymbol{B}(\mathscr{H}, \mathscr{H})$-valued analytic function of ξ by (d) of Lemma 2.4 since the inclusion map: $\mathscr{D} \to \mathscr{H}$ is continuous. Thus we can make use of Theorem XII.12 of [9] which guarantees the existence of an analytic family $U_n(\xi)$ of unitary operators such that

$$U_n(\xi)P_n(0)=P_n(\xi)U_n(\xi). \tag{17}$$

By examining the construction of $U_n(\xi)$ (see the proof of Theorem XII. 12 in [9]), it is not difficult to verify that, if $CP_n(\xi)C=P_n(\xi)$ for all real ξ, $CU_n(\xi)C=U_n(\xi)$ for all real ξ. Therefore, if we let $\psi_n(\cdot, 0)$ be a real and normalized element of Ran $(P_n(0))$ and let $\psi_n(\cdot, \xi)\equiv U_n(\xi)\psi_n(\cdot, 0)$, then it is not difficult to check that $\psi_n(\cdot, \xi)$ is real, normalized, analytic in ξ and an eigenfunction of $\tilde{H}(\xi)$ with eigenvalue $\lambda_n(\xi)$ since $\psi_n(\cdot, \xi)$ belongs to Ran$(P_n(\xi))$ by (17). Thus (iv) holds by (d) of Lemma 2.4 since $\psi_n(\cdot, \xi)=P_n(\xi)\psi_n(\cdot, \xi)$.

Finally (v) can be shown as follows: $\psi_n(\cdot, \xi)$ is real-valued as it has been shown in the above. By (iv) and (c) of Lemma 2.4 $\psi_n(\cdot, \xi)$ is a $C_0(\boldsymbol{R})$-valued analytic function of ξ, from which it follows that $\psi_n(x, \xi)$ is continuous in x and ξ, and analytic in ξ for each x. The smoothness of $\psi_n(x, \xi)$ in x for each ξ follows from (10) and the smoothness of $b(x)$. □

Lemma 2.5. *Assume that* (B) *holds. Let* $\tilde{\mathscr{H}}_n$ *be the subspace of* $\tilde{\mathscr{H}}=L^2(\boldsymbol{R}_x\times\boldsymbol{R}_\xi)$ *defined by*

$$\tilde{\mathscr{H}}_n=\{\psi_n(x, \xi)f(\xi)\,|\,f(\xi)\in L^2(\boldsymbol{R}_\xi)\} \qquad (n=1, 2,\dots)$$

where $\psi_n(x, \xi)$ *is as in Lemma* 2.3. *Then we have*:
(i) $\tilde{\mathscr{H}}=\bigoplus_n \tilde{\mathscr{H}}_n$ *(orthogonal sum).*
(ii) $\tilde{H}$ *is reduced by* $\tilde{\mathscr{H}}_n$.
(iii) $\tilde{H}|_{\tilde{\mathscr{H}}_n}$ *(the restriction of* $\tilde{H}$ *to* $\tilde{\mathscr{H}}_n$*) is unitarily equivalent to the operator of multiplication by* $\lambda_n(\xi)$ *on* $L^2(\boldsymbol{R}_\xi)$.

Proof. Since $\{\psi_n(\cdot, \xi)\}$ is a complete orthonormal system in $L^2(\boldsymbol{R}_x)$ by Lemma 2.3, (i) holds.

It follows from (8) and (10) that

$$\tilde{L}\{\psi_n(x, \xi)f(\xi)\}=\lambda_n(\xi)\psi_n(x, \xi)f(\xi). \tag{18}$$

The right-hand side of (18) belongs to $L^2(\boldsymbol{R}_x\times\boldsymbol{R}_\xi)$ since $\lambda_n(\xi)$ is bounded in ξ by Lemma 2.3 (ii). Hence $\tilde{f}\in D(\tilde{H})$ and $\tilde{H}\tilde{f}=\lambda_n(\xi)\tilde{f}$ for $\tilde{f}\in\tilde{\mathscr{H}}_n$ by (7) and (18). Thus we have that $\tilde{\mathscr{H}}_n\subset D(\tilde{H})$ and $\tilde{H}(\tilde{\mathscr{H}}_n)\subset\tilde{\mathscr{H}}_n$. This implies that $\tilde{H}$ is reduced by $\tilde{\mathscr{H}}_n$. Hence we obtain (ii).

If we define T_n: $L^2(\boldsymbol{R}_\xi)\to L^2(\boldsymbol{R}_x\times\boldsymbol{R}_\xi)$ such that $T_nf(x, \xi)=\psi_n(x, \xi)f(\xi)$, then $\tilde{H}T_nf=T_n\{\lambda_nf\}$ by (18) where λ_n denotes the operator of multiplication by $\lambda_n(\xi)$. Thus we have (iii) since T_n is an isometry with the range $\tilde{\mathscr{H}}_n$ and $\tilde{H}G_n=$

$\tilde{H}T_nT_n^* = T_n\lambda_nT_n^*$, where G_n denotes the orthogonal projection onto $\tilde{\mathscr{H}}_n$. This completes the proof of the lemma. □

Lemma 2.6. *Assume that* (B) *holds. Then H is absolutely continuous if no $\lambda_n(\xi)$ is constant, where $\lambda_n(\xi)$ is as in Lemma* 2.3.

Proof. By Lemma 2.1 and Lemma 2.5, it suffices to show that, for each n, λ_n is absolutely continuous if $\lambda_n(\xi)$ is not constant. Since $\lambda_n(\xi)$ is analytic and non-constant, $A \equiv \left\{\xi \mid \frac{d}{d\xi}\lambda_n(\xi)=0\right\}$ is discrete and closed. Hence, if we let $\{I_j\}_{j\in J}$ be the connected components of $\boldsymbol{R}\backslash A$, then I_j are open intervals, J is at most countable, and

$$L^2(\boldsymbol{R}) = \bigoplus_{j\in J} L^2(I_j) \qquad \text{(orthogonal sum)}.$$

Hence it suffices to show that the operator Λ_j of multiplication by $\lambda_n(\xi)$ on $L^2(I_j)$ is absolutely continuous for each $j\in J$. On each interval I_j, $\lambda_n(\xi)$ is strictly monotone, either increasing or decreasing. Consider the case where it is increasing. Let α denote the inverse function of the restriction of $\lambda_n(\xi)$ to I_j. Then α is a strictly increasing smooth function on $\lambda_n(I_j)$. Let E_μ be the spectral measure associated with Λ_j. Then we have

$$\begin{aligned}\|E_\mu u\|^2 &= \int_{\{\xi\in I_j \mid \lambda_n(\xi)<\mu\}} |u(\xi)|^2 d\xi \\ &= \int_a^{\alpha(\mu)} |u(\xi)|^2 d\xi \qquad (a=\inf I_j) \\ &= f_u(\alpha(\mu))\end{aligned}$$

for $u\in L^2(I_j)$, where $f_u(x)=\int_a^x |u(\xi)|^2 d\xi$ and where we extend α to all the real line so that $\alpha(\mu)=\sup I_j$ if $\mu \geqq \sup \lambda_n(I_j)$ and $\alpha(\mu)=a$ if $\mu\leqq \inf \lambda_n(I_j)$. Therefore one can verify without difficulty that the absolute continuity of α (which follows from the smoothness) and the monotonicity of α together with the absolute continuity of f_u imply the absolute continuity of $\|E_\mu u\|^2$. Thus Λ_j is an absolutely continuous operator. □

§3. Properties of $\psi_n(x, \xi)$ and $\lambda_n(\xi)$

Throughout this section, we suppose (B) alone, use the notations in Lemma 2.3 and let n be fixed. In addition, we put

$$Q_{n,\xi}(x) \equiv (b(x)-\xi)^2 - \lambda_n(\xi). \tag{19}$$

Then, by (10), we have

$$\psi_n''(x, \xi) = Q_{n,\xi}(x)\psi_n(x, \xi) \qquad \left({}' \text{ denotes } \frac{d}{dx}\right). \tag{20}$$

In this section we shall invesigate some detailed properties of the eigenfunctions $\psi_n(x, \xi)$ and the eigenvalues $\lambda_n(\xi)$ of $\tilde{H}(\xi)$. The following arguments are similar to those in [10] (p. 110 and p. 165 ff.).

Proposition 3.1. *Let $I=[x_0, \infty)((-\infty, x_0])$. Let u be a real-valued C^2 function and q a non-negative continuous function on I. Suppose that u satisfies*

$$u''(x) = q(x)u(x), \tag{21}$$

$u \in L^2(I)$ and $u \not\equiv 0$. Then $u(x)u'(x) < 0$ for $x \in I$ ($u(x)u'(x) > 0$ for $x \in I$).

Proof. Let $I=[x_0, \infty)$ (the case of $I=(-\infty, x_0]$ can be treated quite similarly). From (21) we have

$$(uu')' = u'^2 + uu'' = u'^2 + qu^2 \geqq 0, \tag{22}$$

and hence $u(x)u'(x)$ increases for $x \geqq x_0$. Suppose that $u(\bar{x})=0$ for some $\bar{x} \geqq x_0$. If $u'(\bar{x})=0$, $u \equiv 0$ by the uniqueness theorem. Thus by the assumption $u'(\bar{x}) \neq 0$. If $u'(\bar{x})>0$, then $u(x)>0$ for x near $\bar{x}$ and larger than $\bar{x}$. This implies $u(x)u'(x) > 0$ for $x > \bar{x}$ and hence $u'(x)$ increases for $x > \bar{x}$ by (21). Thus $u'(x) \geqq u'(\bar{x}) \equiv c > 0$ and $u(x) \geqq c(x-\bar{x})$ for $x \geqq \bar{x}$. But this contradicts the assumption that $u \in L^2(I)$. A similar argument holds in the case where $u'(\bar{x})<0$. Thus we obtain $u(x) \neq 0$ for $x \geqq x_0$. Consider the case where $u(x)>0$ for $x \in I$ (the case where $u(x)<0$ for $x \in I$ can be treated similarly). Suppose that $u'(\bar{x}) \geqq 0$ for some $\bar{x} \in I$. Then $u(x)u'(x) \geqq 0$ for $x \geqq \bar{x}$ by (22), and hence $u'(x) \geqq 0$ for $x \geqq \bar{x}$. But this implies $u(x) \geqq u(\bar{x}) > 0$ for $x \geqq \bar{x}$, which contradicts the assumption $u \in L^2(I)$. Thus $u'(x)$ has to be negative for all $x \in I$, and hence $u(x)u'(x)<0$ for $x \in I$. □

Lemma 3.2. *$\psi_n(x, \xi)\psi_n'(x, \xi)<0$ if $x \geqq x_\xi + L_n$, $\psi_n(x, \xi)\psi_n'(x, \xi)>0$ if $x \leqq x_\xi - L_n$, where $x_\xi = b^{-1}(\xi)$ and $L_n = \sqrt{(2n-1)M_+}/M_-$.*

Proof. By (12), (19) and (ii) of Lemma 2.3, we have

$$Q_{n,\xi}(x) \geqq M_-^2(x-x_\xi)^2 - (2n-1)M_+ \geqq 0 \tag{23}$$

if $|x-x_\xi| \geqq L_n$. Therefore, by (20) and Proposition 3.1, we have the assertion of the lemma since $\psi_n \in L^2(\boldsymbol{R}_x)$. □

Lemma 3.3. *$\psi_n(x, \xi) \to 0$, $\psi_n'(x, \xi) \to 0$ as $x \to \pm\infty$.*

Proof. By Lemma 3.2, $\psi_n(x, \xi)\neq 0$ for $x\geqq x_\xi+L_n$. Suppose that $\psi_n(x, \xi)>0$ for $x\geqq x_\xi+L_n$. Then $\psi'_n(x, \xi)<0$ by Lemma 3.2. This implies that $\lim_{x\to\infty}\psi_n(x, \xi)$ exists and $\geqq 0$. Since $\psi_n\in L^2(\boldsymbol{R}_x)$, this limit must be 0. Moreover, by (20) and (23), $\psi''_n(x, \xi)\geqq 0$ for $x\geqq x_\xi+L_n$. Thus $\lim_{x\to\infty}\psi'_n(x, \xi)$ exists and $\leqq 0$. This limit must be zero since $\psi_n(x, \xi)\to 0$ as $x\to\infty$. The case where $\psi_n(x, \xi)<0$ can be treated similarly. A similar argument shows that the limits of $\psi_n(x, \xi)$ and $\psi'_n(x, \xi)$ as $x\to-\infty$ exist and must be zero. □

Lemma 3.4. *Let*

$$l_n(x, \xi)=\psi'_n(x, \xi)^2-Q_{n,\xi}(x)\psi_n(x, \xi)^2, \tag{24}$$

where $Q_{n,\xi}$ is as in (19) *and let $x_\xi=b^{-1}(\xi)$. Then we have the following*:

(i) *$l_n(x, \xi)$ is strictly decreasing for $x>x_\xi$ and strictly increasing for $x<x_\xi$.*

(ii) *$l_n(x, \xi)\to 0$ as $x\to\pm\infty$.*

(iii) *$l_n(x, \xi)>0$.*

Proof. We have

$$\begin{aligned} l'_n(x, \xi)&=2\psi'_n(x, \xi)\{\psi''_n(x, \xi)-Q_{n,\xi}(x)\psi_n(x, \xi)\}-Q'_{n,\xi}(x)\psi_n(x, \xi)^2 \\ &=-Q'_{n,\xi}(x)\psi_n(x, \xi)^2 \\ &=-2B(x)(b(x)-\xi)\psi_n(x, \xi)^2 \end{aligned} \tag{25}$$

by (20), (19) and (11). Hence, by (11), (B) and the fact that, for each ξ, $\{x|\psi_n(x, \xi)=0\}$ cannot have an accumulation point, we have (i).

By Lemma 3.2, $\psi_n(x, \xi)>0$ for all $x\geqq x_\xi+L_n$, or $\psi_n(x, \xi)<0$ for all $x\geqq x_\xi+L_n$. Consequently, by (20) and (23), we have $\psi''_n(x, \xi)\geqq 0$ for all $x\geqq x_\xi+L_n$, or $\psi''_n(x, \xi)\leqq 0$ for all $x\geqq x_\xi+L_n$. Thus $\psi'_n(x, \xi)$ is increasing or decreasing for $x\geqq x_\xi+L_n$ and tends to 0 as $x\to\infty$ by Lemma 3.3, from which follows that $\psi''_n(x, \xi)\to 0$ as $x\to\infty$. Hence, by (24), (20) and Lemma 3.3, $l_n(x, \xi)=\psi'_n(x, \xi)^2-\psi''_n(x, \xi)\psi_n(x, \xi)\to 0$ as $x\to\infty$. Similarly, we have $l_n(x, \xi)\to 0$ as $x\to-\infty$. Thus, we have obtained (ii). (i) and (ii) imply (iii). □

Lemma 3.5. *$|\psi_n(x, \xi)|\leqq\Phi_n(x-x_\xi)$ where $x_\xi=b^{-1}(\xi)$ and*

$$\Phi_n(x)=\begin{cases} \sqrt{2}\,\lambda_n^{+1/4} & \text{if } |x|\leqq L_n \\ \sqrt{2}\,\lambda_n^{+1/4}\exp\left\{-\dfrac{M_-}{2}(|x|-L_n)^2\right\} & \text{if } |x|\geqq L_n \end{cases}$$

with $\lambda_n^+=(2n-1)M_+$, $L_n=\sqrt{\lambda_n^+}/M_-$.

Proof. First, we have

$$(26) \qquad |\psi_n(x, \xi)| \leqq \sqrt{2}\,\lambda_n^{+1/4} \qquad \text{for all} \quad x.$$

In fact, we have by multiplying (10) by $\psi_n(x, \xi)$ and integrating by parts,

$$\int_{-\infty}^{\infty} \psi_n'(x, \xi)^2 dx + \int_{-\infty}^{\infty} (b(x)-\xi)^2 \psi_n(x, \xi)^2 dx = \lambda_n(\xi) \int_{-\infty}^{\infty} \psi_n(x, \xi)^2 dx \leqq \lambda_n^+$$

where we have used Lemma 3.3, $\|\psi_n(\cdot, \xi)\|_{L^2} = 1$ and (ii) of Lemma 2.3. Therefore, we obtain (26) since

$$\psi_n(x, \xi)^2 = 2\int_{-\infty}^{x} \psi_n'(x, \xi)\psi_n(x, \xi)dx \leqq 2\left(\int_{-\infty}^{x} \psi_n'^2 dx\right)^{1/2}\left(\int_{-\infty}^{x} \psi_n^2 dx\right)^{1/2}.$$

Next, by (iii) of Lemma 3.4, we have $\psi_n'(x, \xi)^2 > Q_{n,\xi}(x)\psi_n(x, \xi)^2$. Hence, we have

$$\frac{\psi_n'(x, \xi)}{\psi_n(x, \xi)} < -\sqrt{Q_{n,\xi}(x)} \qquad \text{for} \quad x \geqq x_\xi + L_n$$

$$\left(\frac{\psi_n'(x, \xi)}{\psi_n(x, \xi)} > \sqrt{Q_{n,\xi}(x)} \qquad \text{for} \quad x \leqq x_\xi - L_n\right)$$

by Lemma 3.2 and (23). Therefore, we obtain

$$(27) \qquad |\psi_n(x, \xi)| \leqq |\psi_n(\bar{x}, \xi)| \exp\left\{-\left|\int_{\bar{x}}^{x} \sqrt{Q_{n,\xi}(t)}dt\right|\right\}$$

if $x \geqq \bar{x} \geqq x_\xi + L_n$ (or if $x \leqq \bar{x} \leqq x_\xi - L_n$). Since we have by (23)

$$Q_{n,\xi}(x) \geqq M_-^2(|x-x_\xi|^2 - L_n^2) \geqq M_-^2(|x-x_\xi| - L_n)^2$$

if $|x-x_\xi| \geqq L_n$, it follows from (26) and (27) that

$$|\psi_n(x, \xi)| \leqq \sqrt{2}\,\lambda_n^{+1/4} \exp\left\{-\frac{M_-}{2}(|x-x_\xi| - L_n)^2\right\}$$

if $|x-x_\xi| \geqq L_n$. This completes the proof of the lemma. □

Lemma 3.6. $\frac{d}{d\xi}\lambda_n(\xi) = \lim_{\substack{A\to\infty \\ A'\to-\infty}} \int_{A'}^{A} \frac{B'(x)}{B(x)^2} l_n(x, \xi)dx$ *where* $l_n(x, \xi)$ *is as in Lemma* 3.4.

Proof. Since $\frac{d}{d\xi}\lambda_n(\xi) = \left(\frac{dH(\xi)}{d\xi}\psi_n(\cdot, \xi), \psi_n(\cdot, \xi)\right)$, we have

$$\begin{aligned}
\frac{d}{d\xi}\lambda_n(\xi) &= 2\int_{-\infty}^{\infty} (\xi - b(x))\,\psi_n(x, \xi)^2 dx \\
&= 2\int_{-\infty}^{\infty} -\frac{(b(x)-\xi)B(x)\psi_n(x, \xi)^2}{B(x)}\,dx \\
&= \lim_{\substack{A\to\infty \\ A'\to-\infty}} \int_{A'}^{A} \frac{1}{B(x)} l_n'(x, \xi)dx,
\end{aligned}$$

where we used (25). Hence we obtain the desired equality by integration by parts with the aid of Lemma 3.4 (ii). $\square$

§4. Proof of the Theorem

We start the proof of the theorem by the following two lemmas, which assert that the asymptotic behavior of $\lambda_n(\xi)$ as $\xi \to \pm\infty$ is determined by that of $B(x)$ as $x \to \pm\infty$:

Lemma 4.1. *Suppose that $B_1(x)$ and $B_2(x)$ satisfy the assumption* (B). *Let $\tilde{H}_j(\xi)$ be the operator $\tilde{H}(\xi)$ in Lemma 2.3 with B replaced by B_j and let $\lambda_n^{(j)}(\xi)$ be the n-th eigenvalue of $\tilde{H}_j(\xi)$ $(j=1, 2)$. Assume that $B_1(t)-B_2(t) \to 0$ as $t \to \infty$ $(t \to -\infty)$. Then, for each n, $\lambda_n^{(1)}(b_1(s))-\lambda_n^{(2)}(b_2(s)) \to 0$ as $s \to \infty$ $(s \to -\infty$, respectively), where $b_j(x)=\int_0^x B_j(t)dt$ $(j=1, 2)$.*

Proof. We show that the Lemma holds under the assumption $B_1(t)-B_2(t) \to 0$ as $t \to \infty$. The case where $B_1(t)-B_2(t) \to 0$ as $t \to -\infty$ can be treated similarly.

Let n be fixed and let $j, k \leqq n$. Let

$$\alpha_{jk}(s) \equiv (\tilde{H}_2(b_2(s))\psi_j^{(1)}(\cdot\,, b_1(s)), \psi_k^{(1)}(\cdot\,, b_1(s))) \tag{28}$$

where $\psi_j^{(1)}(x, \xi)$ is the eigenfunction of $\tilde{H}_1(\xi)$ with the eigenvalue $\lambda_j^{(1)}(\xi)$ as in Lemma 2.3. Then, since

$$(\tilde{H}_1(b_1(s))\psi_j^{(1)}(\cdot\,, b_1(s)), \psi_k^{(1)}(\cdot\,, b_1(s)))=\lambda_j^{(1)}(b_1(s))\delta_{jk}$$

where $\delta_{jk}=1$ $(j=k)$ and 0 $(j \neq k)$, we have

$$\begin{aligned} &|\alpha_{jk}(s)-\lambda_j^{(1)}(b_1(s))\delta_{jk}| \\ &\quad \leqq \int_{-\infty}^{\infty} |(b_2(x)-b_2(s))^2-(b_1(x)-b_1(s))^2||\psi_j^{(1)}(x, b_1(s))||\psi_k^{(1)}(x, b_1(s))|dx\,. \end{aligned} \tag{29}$$

On the other hand, we have by (B) with $B=B_1$ or B_2,

$$\begin{aligned} &|(b_2(x)-b_2(s))^2-(b_1(x)-b_1(s))^2| \\ &\quad = \left|\left(\int_s^x (B_1(t)+B_2(t))dt\right)\left(\int_s^x (B_1(t)-B_2(t))dt\right)\right| \\ &\quad \leqq 2M_+|x-s|\left|\int_s^x (B_1(t)-B_2(t))dt\right|\,. \end{aligned} \tag{30}$$

Moreover, we have, by Lemma 3.5 and by noting that $\Phi_1(x)<\Phi_2(x)<\cdots<\Phi_n(x)$,

$$|\psi_j^{(1)}(x, b_1(s))| \leqq \Phi_n(x-s) \qquad (j=1, 2,\ldots, n)\,. \tag{31}$$

Thus, from (29), (30) and (31), we obtain for $j,\ k\leqq n$

$$|\alpha_{jk}(s)-\lambda_j^{(1)}(b_1(s))\delta_{jk}| \leqq 2M_+\int_{-\infty}^{\infty}\left\{|x-s|\left|\int_s^x (B_1(t)-B_2(t))dt\right|\Phi_n(x-s)^2\right\}dx. \tag{32}$$

Let $\varepsilon>0$. Then there exists a real number R such that $\sup_{R\leqq t}|B_1(t)-B_2(t)|\leqq\varepsilon$ by the assumption. Hence, by (32) and (B), we have for $s\geqq R$,

$$\begin{aligned}|\alpha_{jk}(s)-\lambda_j^{(1)}(b_1(s))\delta_{jk}| &\leqq 4M_+^2\int_{-\infty}^{R}|x-s|^2\Phi_n(x-s)^2dx+2M_+\varepsilon\int_R^{\infty}|x-s|^2\Phi_n(x-s)^2dx\\ &\leqq 4M_+^2\int_{-\infty}^{R-s}x^2\Phi_n(x)^2dx+2M_+\varepsilon\int_{-\infty}^{\infty}x^2\Phi_n(x)^2dx.\end{aligned} \tag{33}$$

Hence, by noting that $\Phi_n(x)$ depends only on n, M_- and M_+, and $x^2\Phi_n(x)^2$ is integrable on $\boldsymbol{R}$, we have by (33)

$$|\alpha_{jk}(s)-\lambda_j^{(1)}(b_1(s))\delta_{jk}|\leqq C\varepsilon \tag{34}$$

for sufficiently large s and for $j,\ k\leqq n$, where C is a constant dependent only on n, M_- and M_+. Let $V_n(s)$ be the linear subspace of $L^2(\boldsymbol{R}_x)$ spanned by $\{\psi_j^{(1)}(\cdot, b_1(s))\}_{j=1,\dots,n}$ and let $R_n(s)$ be the orthogonal projection onto $V_n(s)$. Then we have by (28)

$$\left\{\begin{array}{l}(\alpha_{jk}(s)) \text{ is the Hermitian symmetric matrix of}\\ R_n(s)\tilde{H}_2(b_2(s))R_n(s)|_{V_n(s)} \text{ with respect to the basis}\\ \{\psi_j^{(1)}(\cdot, b_1(s))\}_{j=1,\dots,n}.\end{array}\right. \tag{35}$$

Let $\mu_1(s)\leqq\cdots\leqq\mu_n(s)$ be the eigenvalues of $(\alpha_{jk}(s))$. Then we have by (35) and by applying the min-max principle to the operator $\tilde{H}_2(b_2(s))$ (see [9], p. 270, Lemma)

$$\lambda_j^{(2)}(b_2(s))\leqq\mu_j(s)\qquad (j=1,\dots,n). \tag{36}$$

On the other hand we have by (34)

$$|\mu_j(s)-\lambda_j^{(1)}(b_1(s))|\leqq C'\varepsilon\qquad (C'=nC,\ j=1,\dots,n) \tag{37}$$

for sufficiently large s. Thus for any $\varepsilon>0$, the following inequality holds for sufficiently large s by (36) and (37):

$$\lambda_j^{(2)}(b_2(s))\leqq\lambda_j^{(1)}(b_1(s))+C'\varepsilon\qquad (j=1,\dots,n), \tag{38}$$

where C' is a constant dependent only on n, M_- and M_+. By interchanging the subscripts 1 and 2 in the above argument, we have

$$\lambda_j^{(1)}(b_1(s)) \leqq \lambda_j^{(2)}(b_2(s)) + C'\varepsilon \qquad (j=1,\dots, n) \tag{39}$$

for sufficiently large s. By (38) and (39), for each n,

$$\lambda_n^{(1)}(b_1(s)) - \lambda_n^{(2)}(b_2(s)) \longrightarrow 0 \text{ as } s \longrightarrow \infty .$$

This proves the lemma. □

Lemma 4.2. $(2n-1)\liminf_{x\to\pm\infty} B(x) \leqq \liminf_{\xi\to\pm\infty} \lambda_n(\xi)$ *and* $\limsup_{\xi\to\pm\infty} \lambda_n(\xi) \leqq (2n-1) \limsup_{x\to\pm\infty} B(x)$.

Proof. Let $B_- \equiv \liminf_{x\to\infty} B(x)$ and let $\varepsilon > 0$. Then, there exists a real number R such that $B(x) \geqq B_- - \varepsilon$ for $x \geqq R$ and hence there exists a C^∞ function $B_1(x)$ satisfying the assumption (B) and such that $B_1(x) = B(x)$ for $x \geqq R$ and $B_1(x) \geqq B_- - 2\varepsilon$ for all x. Then, by applying Lemma 4.1 to this $B_1(x)$ and $B_2(x) \equiv B(x)$, we have that, for each n,

$$\lambda_n^{(1)}(b_1(s)) - \lambda_n^{(2)}(b_2(s)) \longrightarrow 0 \text{ as } s \longrightarrow \infty . \tag{40}$$

On the other hand, Lemma 2.3 (ii) applies to $B_1(x)$ with M_- replaced by $B_- - 2\varepsilon$ since $B_1(x) \geqq B_- - 2\varepsilon$ for all x. Thus we have

$$\lambda_n^{(1)}(\xi) \geqq (2n-1)(B_- - 2\varepsilon) \tag{41}$$

for all $\xi \in \boldsymbol{R}$ and for $n = 1, 2, \dots$. It follows from (40) and (41) that

$$\liminf_{s\to\infty} \lambda_n^{(2)}(b_2(s)) \geqq (2n-1)(B_- - 2\varepsilon) .$$

Thus, since ε was arbitrary and since $\lambda_n^{(2)}(\xi) = \lambda_n(\xi)$ and $b_2(s) \to \infty$ as $s \to \infty$, we have

$$\liminf_{\xi\to\infty} \lambda_n(\xi) \geqq (2n-1)B_- = (2n-1) \liminf_{x\to\infty} B(x) ,$$

which proves the first inequality where the double sign is $+$. We can obtain the remaining inequalities by an argument similar to the above. □

Proof of the Theorem. Note that, by Lemma 2.6, it suffices to show that, for each n, $\lambda_n(\xi)$ is not constant.

In the case where the assumption (B1) holds, $\lambda_n(\xi)$ is not constant since $\limsup_{\xi\to\mp\infty} \lambda_n(\xi) < \liminf_{\xi\to\pm\infty} \lambda_n(\xi)$ by Lemma 4.2 and (B1).

Let us consider the case where (B2) holds. Let $B'(x) \leqq 0$ for $x \leqq \bar{x}$, $B'(x) \geqq 0$ for $x \geqq \bar{x}$, and let R be a constant such that $B(x) = B_0$ for $|x| \geqq R$. Then we have

$$\int_{-\infty}^{\infty} \frac{B'(x)}{B(x)^2}\,dx = \int_{-R}^{R} \frac{B'(x)}{B(x)^2}\,dx = \frac{1}{B(-R)} - \frac{1}{B(R)} = 0\,.$$

Hence we obtain by Lemma 3.6

$$\frac{d}{d\xi}\lambda_n(\xi) = \int_{-R}^{R} \frac{B'(x)}{B(x)^2}\,(l_n(x, \xi) - l_n(\bar{x}, \xi))\,dx\,. \tag{42}$$

On the other hand, if we let ξ be so large that $x_\xi > R$ (note that, since $b(x) = \int_0^x B(t)dt \to \infty$ as $x \to \infty$, $x_\xi = b^{-1}(\xi) \to \infty$ as $\xi \to \infty$), then $l_n(x, \xi)$ is strictly increasing in $[-R, R]$ by (i) of Lemma 3.4. Thus we have that $l_n(x, \xi) - l_n(\bar{x}, \xi) < 0$ for $x < \bar{x}$, $l_n(x, \xi) - l_n(\bar{x}, \xi) > 0$ for $x > \bar{x}$ and hence $B'(x)(l_n(x, \xi) - l_n(\bar{x}, \xi))$ is non-negative and does not vanish identically for $x \in [-R, R]$ by the assumption (B2). Consequently, we have by (42) that $\frac{d}{d\xi}\lambda_n(\xi) > 0$. Thus $\lambda_n(\xi)$ is non-constant for all n. The case where $B'(x) \geqq 0$ for $x < \bar{x}$, $B'(x) \leqq 0$ for $x > \bar{x}$ can be treated similarly. This completes the proof of the theorem. $\square$

Finally, we remark that, by examining the above argument, one is able to determine the spectrum of H in the case where $B(x)$ is assumed to be monotone in addition to (B). In fact, if we let, e.g., $B(x)$ be increasing, then Lemma 2.3 (ii) holds with $M_\pm$ replaced by $B_\pm \equiv \lim_{x \to \pm\infty} B(x)$ and hence $\sup_\xi \lambda_n(\xi) = (2n-1)B_+$, $\inf_\xi \lambda_n(\xi) = (2n-1)B_-$ by Lemma 4.2. Thus we have by Lemma 2.1 and Lemma 2.5

$$\sigma(H) = \bigcup_{n=1}^{\infty} [(2n-1)B_-, (2n-1)B_+]\,.$$

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