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Theory of Connexes. II

By

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Introduction

Here we have a display of the famous game named Hex, where two players White and Black occupy the vertices in the rhombus and who obtains a path between his initially posed pieces wins. It is remarkable that this game always gives a single winner. Regarding the board as the upper half of the sphere, we notice the following statement:

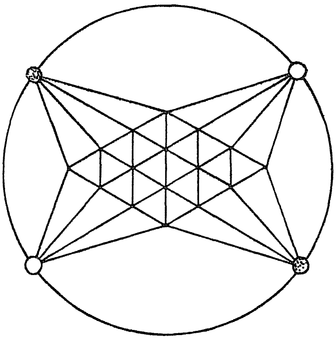

Figure 1

> Suppose there be a simplicial decomposition of the sphere invariant by the antipodal mapping. If two players occupy whole the dipoles of vertices, then there exists strictly one player who obtains in his territory a connected set of vertices invariant under the antipodal action.

Our purpose in this paper is to prove the above statement in more general situation. We have already proved in [3] the converse of the relevant statement, namely, a graph with an action of Z_2 is essentially spherical besides certain exceptions if it admits the unique winner property.

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§1. Preliminary

We fix a set Π of two players $\top$ and $\bot$ and the involution $\hat{}$ of Π namely, $\hat{\top}=\bot$ and $\hat{\bot}=\top$. For any finite set X, we call a mapping $\mathfrak{d}$ from X to Π as a division on X. We consider a compact real 2-dimensional manifold M with an action of a finite group G. We consider also a G-invariant simplicial decomposition $\mathbf{K}=(K^0, K^1, K^2)$ of M.

For $i=0$ or 1, we say two i-simplices to be adjacent if they are distinct and are contained in the boundary of an $i+1$-simplex. The connectivity of a subset of K^i is considered with respect to this adjacency. We assume that the action of G is faithful on K^0 and that any complete subset of K^0 is contained in the boundary of 2-simplex.

For a subset A of K^0, we denote by $[A]$ the subset of M defined as follows:

$$[A]=\{x\in M \mid x \textit{ is a point of a simplex whose vertices are all in } A\}.$$

Let X be a subset of M. Then we denote by $\bar{X}$ the closure of X and define a subgroup $S(X)$ of G as

$$S(X)=\{g\in G \mid X \textit{ is } g\textit{-invariant}\}.$$

Lemma 1. *Let A be a subset of K^0. Then $S([A])$ coincides with $S(A)$. Let B be a connected component of A. Then $[B]$ is a connected component of $[A]$.*

The proof of this lemma is easy and is omitted.

Let $\mathfrak{d}$ be a G-invariant division on K^0. Then we denote by $\mathscr{B}$ the set of connected components of the open set $M-\bigcup_{\pi\in\Pi}[\mathfrak{d}^{-1}(\pi)]$. We fix the division $\mathfrak{d}$ in the rest of this section. We assume that $\mathfrak{d}$ is not constant.

Lemma 2. *Let π be a player, C a connected component of $\mathfrak{d}^{-1}(\pi)$ and $\mathscr{B}_C$ a subset of $\mathscr{B}$ defined as follows:*

$$\mathscr{B}_C=\{\ell\in\mathscr{B} \mid \bar{\ell}\cap\mathfrak{d}^{-1}(\pi)\subset C\}.$$

Assume there be given an element ℓ_0 of $\mathscr{B}_C$. Then

$$S(C)=\{g\in G \mid g\ell_0\in\mathscr{B}_C\}.$$

Especially, $S(C)$ contains $S(\ell_0)$.

This lemma follows immediately the above lemma and the facts that $\mathfrak{d}$ is G-invariant and that C is a connected component.

Lemma 3. *An element of $\mathscr{B}$ is orientable.*

Proof. Let ℓ be an element of $\mathscr{B}$. Then for any 1-simplex in ℓ, there exist exactly two 1-simplices adjacent to it. Touring along the 1-simplices of ℓ, we obtain an orientation with the 0-simplices occupied by $\top$ on the right side.

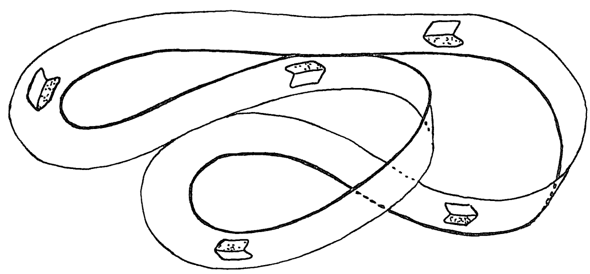

Figure 2

Lemma 4. *Let ℓ be an element of $\mathscr{B}$ and $S_0(\ell)$ the set consisting of the elements of $S(\ell)$ which preserve an orientation of ℓ. Then $S_0(\ell)$ is a cyclic subgroup of $S(\ell)$ of index* 1 *or* 2. *If this index is* 2, *then any element of $S(\ell) - S_0(\ell)$ stabilizes exactly two elements of $K^1 \cup K^2$ contained in ℓ.*

Proof. Let Γ be the graph whose vertices are the 1-simplices contained in ℓ and the adjacency be defined before. Then there is a natural homomorphism from $S(\ell)$ to the automorphism group of Γ, which is injective because G is faithful on K^0. Now our lemma is clear.

Lemma 5. *Let ℓ be an element of $\mathscr{B}$. Then for each player π, $\mathfrak{d}^{-1}(\pi) \cap \bar{\ell}$ is connected and is invariant under $S(\ell)$.*

This lemma is easily verified and its proof is omitted.

Let π be a player and C a connected component of $\mathfrak{d}^{-1}(\pi)$. We define a family $\mathscr{N}_C$ of connected components of $\mathfrak{d}^{-1}(\hat{\pi})$ as follows:

$\mathscr{N}_C = \{E \mid E$ *is a connected components of* $\mathfrak{d}^{-1}(\hat{\pi})$ *such that* $C \cup E$ *is connected*$\}$.

For $E \in \mathscr{N}_C$ we define a subset of G as follows:

$$\begin{pmatrix} C \\ E \end{pmatrix} = \{g \in G \mid gE \in \mathscr{N}_C\}.$$

We define also a new division $\mathfrak{d}_C$ on K^0 as follows:

$$\mathfrak{d}_C^{-1}(\pi) = \mathfrak{d}^{-1}(\pi) - GC.$$

Lemma 6. *Let the assumptions be as above. Assume moreover that $\mathfrak{d}$ is not constant. Then the stabilizer of the connected component C' of $\mathfrak{d}_C^{-1}(\hat{\pi})$ containing C is given as follows*:

$$S(C') = \left\langle \begin{pmatrix} C \\ E \end{pmatrix} \middle| E \in \mathcal{N}_C \right\rangle.$$

If $|\mathcal{N}_C| = 1$, *moreover, then* $S(C') = S(E)$, *where* E *is the element of* $\mathcal{N}_C$.

Proof. It is evident that $S(C')$ contains the relevant group. Let γ be an element of $S(C')$. Then there exists a series $\gamma_1 C$, $g_1 E_1$, $\gamma_2 C$, $g_2 E_2, \ldots, \gamma_{n-1} C$, $g_{n-1} E_{n-1}$, $\gamma_n C$ for a positive integer n where $E_i \in \mathcal{N}_C$, $g_i \in G$, $\gamma_i \in G$, $\gamma_1^{-1}\gamma_n = \gamma$ and the union of each neighbouring two is connected. If $n = 1$, then $\gamma_1^{-1}\gamma_n \in S(C)$. If $n \geqq 2$, then for $1 \leqq i \leqq n-1$

$$\gamma_i^{-1} g_i \quad \text{and} \quad \gamma_{i+1}^{-1} g_i \in \begin{pmatrix} C \\ E_i \end{pmatrix}.$$

Therefore $\gamma = \gamma_1^{-1}\gamma_n$ is an element of the relevant group. The latter part is evident.

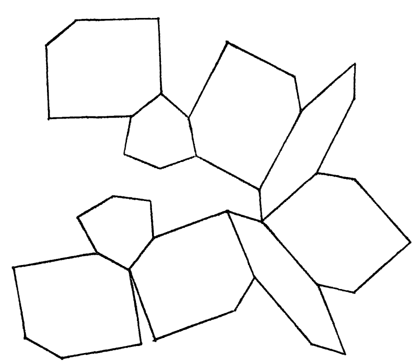

Figure 3

§ 2. Linear Groups on the Unit Sphere (1)

From now on we assume M as the unit sphere in $\boldsymbol{R}^3$. For a positive integer n we define 3×3-matrices $g_-(n)$ and g_+ as follows:

$$g_-(n) = \begin{pmatrix} \cos\frac{\pi}{n} & \sin\frac{\pi}{n} & 0 \\ -\sin\frac{\pi}{n} & \cos\frac{\pi}{n} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$g_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In this section we assume G as one of the linear groups $G_-(n)=\langle g_-(n)\rangle$ and $G_+(n)=\langle g_-(n)^2, g_+\rangle$ with the usual action on M. In case $G=G_+(n)$, we assume $n\geqq 2$. We fix a G-invariant simplicial decomposition $\mathbb{K}=(K^0, K^1, K^2)$ of M.

Theorem. *Let the assumptions be as above. Let $\mathfrak{d}$ be a G-invariant division on K^0. We regard a player π as a winner if $\mathfrak{d}^{-1}(\pi)$ has a G-invariant connected component. Then there exists a unique winner.*

Proof. This statement is obvious if $\mathfrak{d}$ is a constant mapping. Suppose it to be false and let $\mathfrak{d}$ be a counter example minimal with respect to the number $|\mathscr{B}|$. We choose a pair $(\ell, [C])$ of an element ℓ of $\mathscr{B}$ and a connected component $[C]$ of $M-\ell$ such that $[C]$ is minimal. Then, by the Jordan curve theorem, ℓ is the only element of $\mathscr{B}$ whose closure intersects with C. Lemma 2 tells us $S(C)=S(\ell)$. If $G=G_+(n)$ for $n\geqq 2$, then

$$S(C)=S(\ell)=S_0(\ell),$$

and if $G=G_-(n)$, then by Lemma 4

$$S(C)=S(\ell)\not\ni g_-(n).$$

In any way, we have $S(C)\neq G$.

Now we consider a division $\mathfrak{d}_C$ with respect to the player $\pi=\mathfrak{d}(C)$. Then $\mathfrak{d}_C^{-1}(\pi)\subset\mathfrak{d}^{-1}(\pi)$. We have seen above that there is no G-invariant connected component of $\mathfrak{d}^{-1}(\pi)$ besides the ones of $\mathfrak{d}_C^{-1}(\pi)$. On the other hand, by Lemma 6, every G-invariant connected component $\mathfrak{d}_C^{-1}(\hat{\pi})$ remains a connected component even if it is restricted to $\mathfrak{d}^{-1}(\hat{\pi})$. This contradicts the minimality of $\mathfrak{d}$.

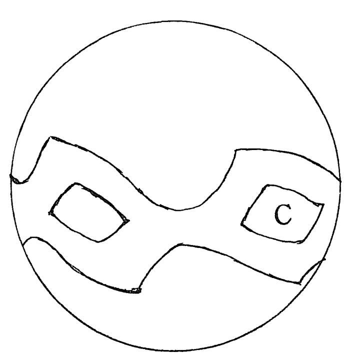

Figure 4

§3. Linear Groups on the Unit Sphere (2)

In the last section we have studied the action of $G_{-}(n)$ and $G_{+}(n)$ on the unit sphere. We know that the finite linear group of degree 3 is conjugate in SL(3, $\mathbf{R}$) to a subgroup of $\langle g_{-}(n), g_{+}\rangle$ for a positive integer n or of polyhedral groups. Then it is still possible that the simplicial decomposition in the last section admits an action of larger groups. We give here an example.

Let G be the group generated by the reflections on xy-, yz- and zx-planes, which contains $G_{-}(1)$. Let $\mathbf{K}=(K^0, K^1, K^2)$ be a G-invariant simplicial decomposition of M. Let $\mathfrak{d}$ be a G-invariant division on K^0. Then one of $\mathfrak{d}^{-1}(\top)$ and $\mathfrak{d}^{-1}(\perp)$ has a $G_{-}(1)$ invariant connected component by our theorem, which is G-invariant. This causes the following proposition.

Proposition. *Let* $\mathbf{K}=(K^0, K^1, K^2)$ *be a simplicial decomposition of a triangle and* $\mathfrak{d}$ *a division on* K^0. *Then exactly one of* $\mathfrak{d}^{-1}(\top)$ *and* $\mathfrak{d}^{-1}(\perp)$ *contains a connected components which intersects each edge of the initial triangle.*

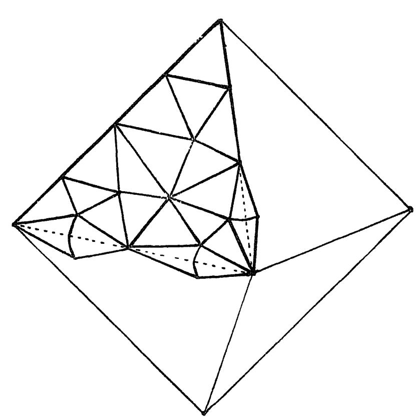

Figure 5

If the initial triangle is on a plane and any 1-simplex is parallel to an edge of the previons triangle, then this example is equivalent to what Komiya [1] calls trinitrix, which was announced to the author by his friend Mr. Tsujino.

Bibliography

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