Publ. RIMS, Kyoto Univ.
21 (1985), 411–420

Rotationally-Quasi-Invariant Measures on the Dual of a Hilbert Space

By

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§1. Introduction

Let H be a real Hilbert space equipped with the scalar product $\langle\cdot,\cdot\rangle_H$ and the norm $\|\cdot\|_H$. And let H^a be the algebraic dual space of H. We consider a probability measure μ defined on the σ-field $\mathfrak{B}$ generated by cylinder sets of H^a. That is, $\mathfrak{B}$ is the minimal σ-field with which all functions f_h $(h \in H)$; $x \in H^a \mapsto x(h) \in \boldsymbol{R}$ are measurable. Let $O(H)$ be the group of all orthogonal operators on H. Then for each $U \in O(H)$ its algebraic transpose ${}^tU$ is a measurable transformation on $(H^a, \mathfrak{B})$. So we define μ_U as $\mu_U(B) = \mu({}^tU^{-1}(B))$ for all $B \in \mathfrak{B}$.

Definition

(a) μ is said to be rotationally-invariant, if $\mu_U = \mu$ holds for all $U \in O(H)$.

(b) μ is said to be rotationally-quasi-invariant, if $\mu_U \simeq \mu$ (μ_U and μ are absolutely continuous with each other.) holds for all $U \in O(H)$.

It is well-known that rotationally-invariant measures are characterized as suitable sums of canonical Gaussian measures in terms of the variance parameter. (See, [2].) On the other hand, up to the present time the study of quasi-invariant measures is rather neglected. In this paper, we shall consider such measures and show in Theorem 2 that *for any rotationally-quasi-invariant measure μ, there exists a rotationally-invariant measure which is equivalent with μ.* First in §2 we consider probability measures on $\boldsymbol{R}^\infty$ to discuss the rotational-quasi-invariance, and prove a version of the above statement. The proof of the main theorem will be carried out in §3.

Communicated by S. Matsuura, June 7, 1984.

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§2. Rotationally-Quasi-Invariant Measures on $\boldsymbol{R}^{\infty}$

Let $\boldsymbol{R}^{\infty}$ be the countable direct-product of $\boldsymbol{R}$ and put $\boldsymbol{R}_0^{\infty}=\{x=(x_1,\dots, x_n, \dots)\in \boldsymbol{R}^{\infty}|x_n=0$ except finite numbers of $n\}$. Next, let U be a one-to-one onto linear operator on $\boldsymbol{R}_0^{\infty}$ which is extended to an orthogonal operator on l^2. The group G of all such U's will play an essential role in our discussions. Let $e_n=(0,\dots, 0, \overset{n}{\breve{1}}, 0,\dots)$ $(n=1, 2,\dots)$ be the canonical base on $\boldsymbol{R}_0^{\infty}$. Among subgroups of G, we shall take following important groups $O(n)$, $O(\infty)$ and $O(n)^{\perp}$, $O(n)=\{U\in G|Ue_j=e_j$ for all $j>n\}$, $O(\infty)=\cup_{n=1}^{\infty} O(n)$ and $O(n)^{\perp}=\{U\in O(\infty)| Ue_j=e_j$ for $1\leqq j\leqq n\}$. Clearly, we have $G\supset O(\infty)\supset O(n)^{\perp}\supset O(n+1)^{\perp}$. Now consider a probability measure μ on the usual Borel field $\mathfrak{B}(\boldsymbol{R}^{\infty})$ on $\boldsymbol{R}^{\infty}$. Since for each $U\in G$, its transpose ${}^tU$ (for the duality of $\boldsymbol{R}_0^{\infty}$ and $\boldsymbol{R}^{\infty}$) is a measurable transformation on $\boldsymbol{R}^{\infty}$, so μ_U is defined as before. If $\mu_U\simeq\mu$ holds for all $U\in G$ ($\mu_U=\mu$ holds for all $U\in O(\infty)$), μ is said to be rotationally-G-quasi-invariant (rotationally-$O(\infty)$-invariant), respectively. We begin with a following fundamental lemma.

Lemma 1. *Let μ be rotationally-G-quasi-invariant and put*

$$\varepsilon_n=\sup_{U\in O(n)^{\perp}}\int\left|\frac{d\mu_U}{d\mu}(x)-1\right|d\mu(x).$$

Then we have $\lim_n \varepsilon_n=0$.

Proof. As $\{\varepsilon_n\}$ is monotone decreasing, $\lim_n \varepsilon_n=\varepsilon$ exists at any rate. Assume that $\varepsilon>0$. Then there exists $U_1\in O(1)^{\perp}$ such that $\int\left|\frac{d\mu_{U_1}}{d\mu}(x)-1\right|d\mu(x)>\frac{\varepsilon}{2}$. From the definition of $O(1)^{\perp}$, U_1 belongs to $O(n_2)$ for some n_2. Without loss of generality we can assume that $n_2>1\equiv n_1$. Next replacing $1=n_1$ by n_2 (noting $\varepsilon_{n_2}>\varepsilon/2$), we repeat this procedure, and so on. Then as it is easily seen, a sequence $n_1<\cdots<n_k<\cdots$ and $U_k\in O(n_k)^{\perp}\cap O(n_{k+1})$ are defined inductively such that

$$\int\left|\frac{d\mu_{U_k}}{d\mu}(x)-1\right|d\mu(x)>\frac{\varepsilon}{2}\qquad\text{for all}\quad k. \tag{1}$$

Since $O(n_k)^{\perp}\cap O(n_{k+1})$ is regarded as the orthogonal group on $\boldsymbol{R}^{n_{k+1}-n_k}$, it is compact in the natural topology. Hence the direct-product $K\equiv\prod_{k=1}^{\infty}O(n_k)^{\perp}\cap O(n_{k+1})$ is again a compact group. Naturally, each element $W=(W_1,\dots,$

$W_k,\dots)\in K$ acts on $\boldsymbol{R}_0^\infty$ as $\tilde{W}$; $x=\sum_{j=1}^\infty x_je_j\mapsto x_1e_1+\sum_{k=1}^\infty W_k(x_{n_k+1}e_{n_k+1}+\cdots+x_{n_{k+1}}e_{n_{k+1}})$. It is obvious that $\tilde{W}\in G$ for all $W\in K$. Put $\tilde{\mu}(B)=\int_{W\in K}\mu_{\tilde{W}}(B)dW$, where dW is the normalized Haar measure of K. $\tilde{\mu}$ is invariant under the actions of $\tilde{W}\,(W\in K)$, especially $\tilde{\mu}=(\tilde{\mu})_{U_k}$, and $\tilde{\mu}\simeq\mu$ holds in virtue of G-quasi-invariance of μ. Now $\frac{d\mu_{U_k}}{d\tilde{\mu}}(x)=\frac{d\mu}{d\tilde{\mu}}({}^tU_k^{-1}x)$ holds for $\tilde{\mu}$-a.e.x, because for all $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$ we have $\mu_{U_k}(B)=\int_{{}^tU_k^{-1}(B)}\frac{d\mu}{d\tilde{\mu}}(x)d\tilde{\mu}(x)=\int_B\frac{d\mu}{d\tilde{\mu}}({}^tU_k^{-1}x)d\tilde{\mu}(x)$. It follows from (1) that

$$\int\left|\frac{d\mu}{d\tilde{\mu}}({}^tU_k^{-1}x)-\frac{d\mu}{d\tilde{\mu}}(x)\right|d\tilde{\mu}(x)>\frac{\varepsilon}{2}\qquad\text{for all}\quad k. \tag{2}$$

In this step, we take an $f\in L^1_{\tilde{\mu}}$ such that $\int\left|\frac{d\mu}{d\tilde{\mu}}(x)-f(x)\right|d\tilde{\mu}(x)<\frac{\varepsilon}{8}$ and f depends on only finite numbers of coordinates, say $x_1,\dots,x_s$. Thus if $n_k\geqq s$, we have $f(x)=f({}^tU_k^{-1}x)$. Consequently for $n_k\geqq s$ we have

$$\int\left|\frac{d\mu}{d\tilde{\mu}}({}^tU_k^{-1}x)-\frac{d\mu}{d\tilde{\mu}}(x)\right|d\tilde{\mu}(x)\leqq\int\left|\frac{d\mu}{d\tilde{\mu}}({}^tU_k^{-1}x)-f({}^tU_k^{-1}x)\right|d\tilde{\mu}(x)$$
$$+\int\left|\frac{d\mu}{d\tilde{\mu}}(x)-f(x)\right|d\tilde{\mu}(x)<\frac{\varepsilon}{8}+\frac{\varepsilon}{8}=\frac{\varepsilon}{4}.$$

However it contradicts to (2). Q.E.D.

We note that

$$\int\left|\frac{d\mu_U}{d\mu}(x)-1\right|d\mu(x)=2\sup\{|\mu_U(B)-\mu(B)|\,\big|\,B\in\mathfrak{B}(\boldsymbol{R}^\infty)\}.$$

Now we shall proceed to the definition of the limiting measure μ_ω of μ. Let E, F be Borel sets of $\boldsymbol{R}^m$, $\boldsymbol{R}^l$ respectively, and put P_m; $x\in\boldsymbol{R}^\infty\mapsto(x_1,\dots,x_m)\in\boldsymbol{R}^m$. Applying Lemma 1,

$$\lim_n\mu(x\in\boldsymbol{R}^\infty|(x_1,\dots,x_m)\in E,\,(x_{n+1},\dots,x_{n+l})\in F)\equiv\mu_l(F|P_m^{-1}(E))$$

exists for all E, F. Because for $n'\geq n\geq m$ choose $U_{n,n'}\in O(n)^\perp$ such that $U_{n,n'}e_{n+1}=e_{n'+1},\dots,U_{n,n'}e_{n+l}=e_{n'+l}$. Then we have

$$\mu_{U_{n,n'}}(x|(x_1,\dots,x_m)\in E,\,(x_{n+1},\dots,x_{n+l})\in F)$$
$$=\mu(x|(x_1,\dots,x_m)\in E,\,(x_{n'+1},\dots,x_{n'+l})\in F).$$

Hence

$$(3)\qquad 2\Big|\mu(x|(x_1,\dots,x_m)\in E,\ (x_{n'+1},\dots,x_{n'+l})\in F)$$
$$-\mu(x|(x_1,\dots,x_m)\in E,\ (x_{n+1},\dots,x_{n+l})\in F)\Big|$$
$$\leqq \sup_{U\in O(n)^{\perp}}\int\Big|\frac{d\mu_U}{d\mu}(x)-1\Big|d\mu(x)=\varepsilon_n \qquad \text{for}\quad n'\geq n\geq m\,.$$

Therefore they form a Cauchy sequence. It is obvious that for $l<l'$, $\mu_{l'}(F\times R^{l'-l}|P_m^{-1}(E))=\mu_l(F|P_m^{-1}(E))$. Consequently $\{\mu_l(\cdot|P_m^{-1}(E))\}_l$ forms a consistent family of measures on $\{R^l\}_l$ by natural projections, and a measure $\mu_\omega(\cdot|P_m^{-1}(E))$ is defined on $\mathfrak{B}(R^\infty)$ such that $\mu_\omega(P_l^{-1}(F)|P_m^{-1}(E))=\mu_l(F|P_m^{-1}(E))$. Especially, we simply write $\mu_\omega(\cdot)$ instead of $\mu_\omega(\cdot|R^\infty)$.

Lemma 2. *For any m and for any Borel set $E\subset R^m$, $\mu_\omega(\cdot|P_m^{-1}(E))$ is $O(\infty)$-invariant.*

Proof. For the proof it is necessary and sufficient to show $\mu_l(\cdot|P_m^{-1}(E))$ is $O(R^l)$-invariant for all l. Let $U\in O(R^l)$. Then for each $n\geq m$ we can take an $U_n\in O(n)^{\perp}\cap O(n+l)$ such that $\mu_{U_n}(x|(x_1,\dots,x_m)\in E,\ (x_{n+1},\dots,x_{n+l})\in F)=\mu(x|(x_1,\dots,x_m)\in E,\ (x_{n+1},\dots,x_{n+l})\in UF)$, for all Borel sets $F\subset R^l$. Hence by Lemma 1, $\Big|\mu(x|(x_1,\dots,x_m)\in E,\ (x_{n+1},\dots,x_{n+l})\in UF)-\mu(x|(x_1,\dots,x_m)\in E,$ $(x_{n+1},\dots,x_{n+l})\in F)\Big|\leqq 2^{-1}\varepsilon_n\to 0\ (n\to\infty)$.

This implies that $\mu_l(UF|P_m^{-1}(E))=\mu_l(F|P_m^{-1}(E))$. Q.E.D.

By the above Lemma, $\mu_\omega(\cdot|P_m^{-1}(E))$ is represented by a suitable sum of canonical Gaussian measures g_v with mean 0 and variance v on $\mathfrak{B}(R^\infty)$. (See, for example [1].) Next let $\mathfrak{B}^n$ be a minimal σ-field on R^∞ with which all the coordinate functions $x_{n+1},\dots,x_{n+k},\dots$ are measurable and put $\mathfrak{B}_\infty=\bigcap_{n=1}^{\infty}\mathfrak{B}^n$.

Lemma 3. *For any m and for any Borel set $E\subset R^m$, $\mu_\omega(B|P_m^{-1}(E))=\mu(B\cap P_m^{-1}(E))$ holds for all $B\in\mathfrak{B}_\infty$.*

Proof. Letting $n'\to\infty$ in (3), we have

$$\Big|\mu_\omega(P_l^{-1}(F)|P_m^{-1}(E))-\mu(x|(x_1,\dots,x_m)\in E,\ (x_{n+1},\dots,x_{n+l})\in F)\Big|\leqq 2^{-1}\varepsilon_n$$

for all $n\geqq m$. By $O(\infty)$-invariance of $\mu_\omega(\cdot|P_m^{-1}(E))$, the above inequality becomes, $|\mu_\omega((x_{n+1},\dots,x_{n+l})\in F|P_m^{-1}(E))-\mu(x|(x_1,\dots,x_m)\in E,\ (x_{n+1},\dots,x_{n+l})\in F)|\leqq 2^{-1}\varepsilon_n$ for all $n\geqq m$. As the right hand does not depend on l, so for all $n\geqq m$ and for all $B\in\mathfrak{B}^n$ we have $|\mu_\omega(B|P_m^{-1}(E))-\mu(P_m^{-1}(E)\cap B)|\leqq 2^{-1}\varepsilon_n$. Especially for any $B\in\mathfrak{B}_\infty$ it holds independently on n. So the proof is complete, letting $n\to\infty$. Q.E.D.

In order to observe the explicit form of $\mu_\omega(\cdot|P_m^{-1}(E))$, we use a family of conditional probability measures $\{\mu^x\}_{x\in\boldsymbol{R}^\infty}$ of μ with respect to $\mathfrak{B}_\infty$. For $\{\mu^x\}_{x\in\boldsymbol{R}^\infty}$, it is well-known that

(a) for every $x\in\boldsymbol{R}^\infty$, μ^x is a probability measure on $\mathfrak{B}(\boldsymbol{R}^\infty)$,

(b) for a fixed $E\in\mathfrak{B}(\boldsymbol{R}^\infty)$, $\mu^x(E)$ is a $\mathfrak{B}_\infty$-measurable function of $x\in\boldsymbol{R}^\infty$,

(c) $\mu(E\cap B)=\int_B\mu^x(E)d\mu(x)$ for all $E\in\mathfrak{B}(\boldsymbol{R}^\infty)$ and for all $B\in\mathfrak{B}_\infty$.

Now let $B\in\mathfrak{B}_\infty$. Then by Lemma 3,

$$\mu_\omega(B|P_m^{-1}(E))=\mu(B\cap P_m^{-1}(E))=\int_B\mu^x(P_m^{-1}(E))d\mu(x)=\int_B\mu^x(P_m^{-1}(E))d\mu_\omega(x).$$

By the way, for a fixed E, $\lambda_E(A)=\int_A\mu^x(P_m^{-1}(E))d\mu_\omega(x)$ is an $O(\infty)$-invariant measure in virtue of (b) and of $O(\infty)$-invariance of μ_ω. As the form of $O(\infty)$-invariant measure is completely determined on $\mathfrak{B}_\infty$, (See, [1].) it follows from the above that $\lambda_E(\cdot)=\mu_\omega(\cdot|P_m^{-1}(E))$. Consequently,

Lemma 4. *Let $\{\mu^x\}_{x\in\boldsymbol{R}^\infty}$ be the conditional probability measures of μ with respect to $\mathfrak{B}_\infty$. Then we have*

$$\mu_\omega(B|P_m^{-1}(E))=\int_B\mu^x(P_m^{-1}(E))d\mu_\omega(x),\qquad \textit{for all}\quad B\in\mathfrak{B}(\boldsymbol{R}^\infty).$$

Here we shall add brief results for μ_ω. (The proofs are obvious.)

1. $\mu=\mu_\omega$, if μ is $O(\infty)$-invariant.
2. If μ is a convex sum of two rotationally-G-quasi-invariant measures μ^1 and μ^2, then μ_ω is the same sum of μ_ω^1 and μ_ω^2.

We shall prove $\mu\simeq\mu_\omega$ in the remainder part of this section. Now consider an isometric operator S on l^2, $Se_1=e_2,\dots, Se_n=e_{2n},\dots$. Corresponding to S, we take $U_n\in O(2n)$ for each n such that $U_ne_1=e_2$, $U_ne_2=e_4,\dots, U_ne_n=e_{2n}$, $U_ne_{n+1}=e_1$, $U_ne_{n+2}=e_3,\dots, U_ne_{n+k}=e_{2k-1}$, $\dots$, $U_ne_{2n}=e_{2n-1}$. Since $U_le_j=U_me_j$ $(j=1,\dots,n)$ for $n\leqq l\leqq m$, we have $U_m^{-1}U_l\in O(n)^\perp$. It follows that $\sup\{|\mu({}^tU_m(E))-\mu({}^tU_l(E))|\,\big|\,E\in\mathfrak{B}(\boldsymbol{R}^\infty)\}=\sup\{|\mu({}^tU_m{}^tU_l^{-1}(E))-\mu(E)|\,\big|\,E\in\mathfrak{B}(\boldsymbol{R}^\infty)\}\leqq 2^{-1}\varepsilon_n$. Therefore by Lemma 1, $\lim_n\mu({}^tU_n(E))\equiv\mu_S(E)$ exists for all $E\in\mathfrak{B}(\boldsymbol{R}^\infty)$. It is obvious that $\mu_S\lesssim\mu$. Further putting $\mathfrak{A}_\infty=\{E\in\mathfrak{B}(\boldsymbol{R}^\infty)|{}^tUE=E$ for $^\forall U\in O(\infty)\}$, $\mu_S=\mu$ holds on $\mathfrak{A}_\infty$. In order to observe μ_S, we put p; $x=(x_1,\dots,x_n,\dots)\in\boldsymbol{R}^\infty\mapsto(x_1,x_3,\dots,x_{2n-1},\dots)\in\boldsymbol{R}^\infty$, q; $x=(x_1,\dots,x_n,\dots)\in\boldsymbol{R}^\infty\mapsto(x_2,x_4,\dots,x_{2n},\dots)\in\boldsymbol{R}^\infty$, and T; $x\in\boldsymbol{R}^\infty\mapsto(p(x),q(x))\in\boldsymbol{R}^\infty\times\boldsymbol{R}^\infty$. If E, F are Borel sets of $\boldsymbol{R}^m$, $\mu_S(p^{-1}(P_m^{-1}(F)) \cap q^{-1}(P_m^{-1}(E)) = \mu_S(x|(x_1, x_3,\dots, x_{2m-1}) \in F, (x_2, x_4,\dots, x_{2m}) \in E) = \lim_n \mu(x|(x_1, x_2,\dots, x_m) \in E, (x_{n+1}, x_{n+2},\dots, x_{n+m}) \in F) = \mu_m(F|P_m^{-1}(E)) = \mu_\omega(P_m^{-1}(F)|P_m^{-1}(E)) = \int_{P_m^{-1}(F)} \mu^x(P_m^{-1}(E))d\mu_\omega(x)$. It follows that

$$\mu_S(p^{-1}(B_1) \cap q^{-1}(B_2)) = \int_{B_1} \mu^x(B_2)d\mu_\omega(x) \qquad \text{for all} \quad B_1, B_2 \in \mathfrak{B}(\boldsymbol{R}^\infty).$$

Hence

$$\begin{aligned} T\mu_S(B_1 \times B_2) &= \mu_S(p^{-1}(B_1) \cap q^{-1}(B_2)) \\ &= \int \delta_x(B_1)\mu^x(B_2)d\mu_\omega(x) = \int (\delta_x \times \mu^x)(B_1 \times B_2)d\mu_\omega(x). \end{aligned}$$

Consequently, we have

$$(4) \qquad T\mu_S(B) = \int (\delta_x \times \mu^x)(B)d\mu_\omega(x) \qquad \text{for all} \quad B \in \mathfrak{B}(\boldsymbol{R}^\infty \times \boldsymbol{R}^\infty).$$

Here we consider translational-quasi-invariance of μ_S. (For these notions we refer [1], [3] or [4].)

Lemma 5. *If μ_ω is translationally-l^2-quasi-invariant (equivalently the Dirac term of μ_ω is dropped), then for any $h \in l^2$ there exists $B_h \in \mathfrak{B}_\infty$ with $\mu_\omega(B_h) = 1$ such that $\mu^x = \mu^{x+h}$ for all $x \in B_h$.*

Proof. Since $\mu_\omega([B-h] \ominus B) = 0$ for all $B \in \mathfrak{B}_\infty$ (See, [1]), the same holds for μ by Lemma 3. Especially, putting $\mu_h(E) = \mu(E-h)$ for all $E \in \mathfrak{B}(\boldsymbol{R}^\infty)$, $\mu_h = \mu$ holds on $\mathfrak{B}_\infty$. It follows that for any $B \in \mathfrak{B}_\infty$ and for any $E \in \mathfrak{B}(\boldsymbol{R}^\infty)$,

$$\begin{aligned} \mu(E \cap B) &= \int_B \mu^x(E)d\mu(x) = \int_B \mu^x(E)d\mu_h(x) \\ &= \int_{B-h} \mu^{x+h}(E)d\mu(x) = \int_B \mu^{x+h}(E)d\mu(x). \end{aligned}$$

As $\mu^x(E)$ and $\mu^{x+h}(E)$ are both $\mathfrak{B}_\infty$-measurable functions, so $\mu^x(E) = \mu^{x+h}(E)$ holds for μ-a.e.x. Take a countable algebra $\mathscr{F}$ generating $\mathfrak{B}(\boldsymbol{R}^\infty)$ and put $B_h = \bigcap_{S \in \mathscr{F}} \{x|\mu^x(S) = \mu^{x+h}(S)\}$. It is clear that $B_h \in \mathfrak{B}_\infty$, $1 = \mu(B_h) = \mu_\omega(B_h)$ and $\mu^x = \mu^{x+h}$ holds for all $x \in B_h$. Q.E.D.

Lemma 6. *If μ_ω is translationally-l^2-quasi-invariant, then we have $(\mu_S)_{\hat{h}} \simeq \mu_S$ for all $\hat{h} = h_1e_1 + h_2e_3 + \cdots + h_ne_{2n-1} + \cdots$, $\sum_{n=1}^\infty h_n^2 < \infty$.*

Proof. It is enough to show that $T\mu_S$ is translationally-quasi-invariant for all $(h, 0) \in l^2 \times \boldsymbol{R}^\infty$. Using (4) and Lemma 5, it is assured as follows.

$$T\mu_S(B) = 0 \Longleftrightarrow \int (\delta_x \times \mu^x)(B)d\mu_\omega(x) = 0 \Longleftrightarrow$$

$$\int \mu^x (y|(0,y) \in B-(x, 0))d\mu_\omega(x)=0 \Longleftrightarrow$$

$$\int \mu^{x+h}(y|(0, y) \in B-(h, 0)-(x, 0))d\mu_\omega(x)=0 \Longleftrightarrow$$

$$\int_{B_h} \mu^x(y|(0, y) \in B-(h, 0)-(x, 0))d\mu_\omega(x)=0 \Longleftrightarrow T\mu_S(B-(h, 0))=0$$

Q.E.D.

Next we consider the effect of the Dirac term of μ_ω. The following three cases are possible.

(a) $\mu(\{0\})=1$. In this case $\mu=\mu_\omega=\delta_0$, so it is nothing to prove.

(b) $\mu(\{0\})=0$.

(c) $0<\mu(\{0\})<1$. Put $\mu^1(E)=\dfrac{\mu(E \cap \{0\}^c)}{\mu(\{0\}^c)}$ for all $E \in \mathfrak{B}(\boldsymbol{R}^\infty)$.
Then μ^1 is rotationally-G-quasi-invariant. And we have

$$\mu=\mu(\{0\})\delta_0+\mu(\{0\}^c)\mu^1 \quad \text{and} \quad \mu_\omega=\mu(\{0\})\delta_0+\mu(\{0\}^c)\mu^1_\omega.$$

Thus for the proof of $\mu \simeq \mu_\omega$, it is sufficient to consider the case (b). Now let $\mu(\{0\})=0$, and put $N_n=\{x \in \boldsymbol{R}^\infty | x_n=0\}$ for each n. We wish to show $\mu(N_1)=0$, equivalently $\mu(N_n)=0$ for all n. Suppose that it would be false. Since $0=\mu(\{0\})=\lim_n \mu(N_1 \cap N_2 \cap \cdots \cap N_n)$, $\mu(N_1)>\mu(N_1 \cap \cdots \cap N_n)$ holds for sufficiently large n. It follows that $\mu(N_1 \cap N_k^c)>0$ for some $k \geqq 2$, equivalently $\mu(N_1 \cap N_2^c)>0$. Take $U_\theta \in O(2)$, $U_\theta e_1=\cos\theta e_1+\sin\theta e_2$, $U_\theta e_2=-\sin\theta e_1+\cos\theta e_2$. As we have ${}^tU_\theta^{-1}(N_1 \cap N_2^c)=\{x \in \boldsymbol{R}^\infty | x_1 \cos\theta+x_2 \sin\theta=0, \quad -x_1 \sin\theta+x_2 \cos\theta \neq 0\}$, so they are mutually disjoint for different $\theta \in [0, \pi)$. Hence we conclude that $0=\mu({}^tU_\theta^{-1}(N_1 \cap N_2^c))=\mu_{U_\theta}(N_1 \cap N_2^c)$. However it contradicts to $\mu(N_1 \cap N_2^c)>0$. From $\mu(N_n)=0$, it follows that $\mu_\omega(N_1)=\lim_n \mu(N_n)=0$ and therefore $\mu_\omega(\{0\}) \leqq \mu_\omega(N_1)=0$. From these arguments,

Lemma 7. *If μ has no Dirac term, then so is μ_ω.*

Hereafter assume that $\mu(\{0\})=0$. Then we take a probability measure σ on $\mathfrak{B}(\boldsymbol{R}^\infty)$ which is translationally-$\boldsymbol{R}_0^\infty$-quasi-invariant and $\sigma(l^2)=1$. The convolution $\mu*\sigma$ and μ_ω are both translationally-$\boldsymbol{R}_0^\infty$-quasi-invariant, and for any $B \in \mathfrak{B}_\infty$,

$$\mu*\sigma(B)=\int_{l^2} \mu(B-h)d\sigma(h)=\int_{l^2} \mu(B)d\sigma(h)=\mu(B)=\mu_\omega(B).$$

Since the equivalence classes of translationally-$\boldsymbol{R}_0^\infty$-quasi-invariant measures are completely determined on $\mathfrak{B}_\infty$, (See, [1]) we conclude that $\mu*\sigma \simeq \mu_\omega$. Using these results and the following last Lemma we prove that $\mu \simeq \mu_\omega$.

Lemma 8. *Let μ_1 and μ_2 be rotationally-$O(\infty)$-quasi-invariant probability measures. Then for $\mu_1 \lesssim \mu_2$, it is necessary and sufficient that $\mu_1 \lesssim \mu_2$ on $\mathfrak{A}_\infty$.*

Proof. The necessity is obvious. For the sufficiency, put $\mu = \frac{\mu_1 + \mu_2}{2}$. Then there exists some $A \in \mathfrak{B}(\boldsymbol{R}^\infty)$ such that $\mu(A \cap B) = 0 \Leftrightarrow \mu_2(B) = 0$ for any $B \in \mathfrak{B}(\boldsymbol{R}^\infty)$. We claim that A can be taken such as $A \in \mathfrak{A}_\infty$. Since we have $\mu_2(A^c) = 0$, so $\mu_2({}^tUA^c) = 0$ and therefore $\mu(A \cap {}^tUA^c) = 0$. Replacing U by U^{-1}, $\mu(A \ominus {}^tUA) = 0$ holds for all $U \in O(\infty)$. Now using the indicator function χ_A of A and the Haar measure dU of $O(n)$, we put $g_n(x) = \int_{O(n)} \chi_A({}^tUx)dU$. Then g_n is $O(n)$-invariant and $g_n(x) = \chi_A(x)$ holds for μ-a.e.x. Hence putting $\overline{\lim_n}\, g_n(x) = g(x)$, g is $O(\infty)$-invariant and $g(x) = \chi_A(x)$ holds for μ-a.e.x. Finally, put $\hat{A} = \{x \in \boldsymbol{R}^\infty | g(x) = 1\}$. Then it is easily checked that $\hat{A} \in \mathfrak{A}_\infty$ and $\mu(A \ominus \hat{A}) = 0$. Under the above preparation, let $\mu_1 \lesssim \mu_2$ hold on $\mathfrak{A}_\infty$. Then we have $\mu_1(\hat{A}^c) = 0$. And if $\mu_2(E) = 0$ for some $E \in \mathfrak{B}(\boldsymbol{R}^\infty)$, then we have $\mu(E \cap \hat{A}) = 0$ which implies $\mu_1(E \cap \hat{A}) = 0$. It follows that $\mu_1(E) \leqq \mu_1(E \cap \hat{A}) + \mu_1(E \cap \hat{A}^c) = 0$.

Q.E.D.

Theorem 1. *For a rotationally-G-quasi-invariant measure μ, we have $\mu \simeq \mu_\omega$.*

Proof. By the preceding arguments, it may be assumed that $\mu(\{0\}) = 0$. First we shall show $\mu \lesssim \mu_\omega$. Let $A \in \mathfrak{A}_\infty$ and $\mu_\omega(A) = 0$ which is equivalent to $\mu * \sigma(A) = 0$. It implies that $\mu(A - h) = 0$ for some $h = h_1e_1 + \cdots + h_ne_n + \cdots \in l^2$. Take θ_n for each n such that ${}^tU^{-1}h = \sqrt{h_1^2 + h_2^2}\,e_1 + \cdots + \sqrt{h_{2n-1}^2 + h_{2n}^2}\,e_{2n-1} + \cdots$ holds for $U \in G$ defined by $Ue_{2n-1} = \cos\theta_n e_{2n-1} + \sin\theta_n e_{2n}$, $Ue_{2n} = -\sin\theta_n e_{2n-1} + \cos\theta_n e_{2n}$ $(n = 1, 2, \dots)$. Since for any $U_n \in O(n)$, we have $W_n = U^{-1}U_nU \in O(n+1)$, so ${}^tU_n^{-1}\,{}^tU^{-1}(A) = {}^tU^{-1}\,{}^tW_n^{-1}(A) = {}^tU^{-1}(A)$ and therefore ${}^tU^{-1}(A) \in \mathfrak{A}_\infty$. It follows from $\mu_U(A - h) = 0$ that $\mu({}^tU^{-1}(A) - {}^tU^{-1}h) = 0$ which implies $\mu_S({}^tU^{-1}(A) - {}^tU^{-1}h) = 0$. By Lemma 6, we have $\mu_S({}^tU^{-1}(A)) = 0$. As $\mu = \mu_S$ holds on $\mathfrak{A}_\infty$, so it holds $\mu({}^tU^{-1}(A)) = 0$, equivalently $\mu(A) = 0$.

Next we shall show $\mu_\omega \lesssim \mu$. We use a representation of μ_ω by a probability measure P on $(0, \infty)$, $\mu_\omega(B) = \int_{(0,\infty)} g_v(B)dP(v)$ for all $B \in \mathfrak{B}(\boldsymbol{R}^\infty)$. Put $r(x) = \overline{\lim_N} \frac{1}{N} \sum_{n=1}^N x_n^2$. Then $r(x)$ is a $\mathfrak{B}_\infty$-measurable function and $g_v(r^{-1}(v)) = 1$ by the law of large numbers. It follows that $\mu_\omega(x | r(x) \in (\alpha, \beta]) = P((\alpha, \beta])$. By the way, g_v takes only the values 1 or 0 on $\mathfrak{A}_\infty$, because g_v is $O(\infty)$-ergodic. Now we put $B_A = r^{-1}\{v|g_v(A)=1\}$ for each $A \in \mathfrak{A}_\infty$. Then $g_v(A)=1$ implies $B_A \supset r^{-1}(v)$ and therefore $g_v(B_A)=1$. While, $g_v(A)=0$ implies $B_A \cap r^{-1}(v)=\emptyset$ and therefore $g_v(B_A)=0$. Consequently, $g_v(A \ominus B_A)=0$ for all v, hence we have $\mu_\omega(A \ominus B_A)=0$. We note the same holds for μ, since we have seen $\mu_\omega \gtrsim \mu$. Now the proof follows from these arguments and Lemma 3. Let $A \in \mathfrak{A}_\infty$ and $\mu(A)=0$. Taking $B_A \in \mathfrak{B}_\infty$ as above, we have $0=\mu(B_A)=\mu_\omega(B_A)=\mu_\omega(A)$.

Q. E. D.

§3. Rotationally-Quasi-Invariant Measures on H^a

In this section we prove the result announced in the introduction. Let μ be a rotationally-quasi-invariant probability measure on $(H^a, \mathfrak{B})$. Take an arbitrary countably-infinite orthonormal system $f_1, f_2, \dots, f_n, \dots$ and put $\mu^f = T_f \mu$ by a map T_f ; $x \in H^a \mapsto (x(f_1), \dots, x(f_n), \dots) \in \boldsymbol{R}^\infty$. Then μ^f is a rotationally-G-quasi-invariant measure on $\mathfrak{B}(\boldsymbol{R}^\infty)$. Because taking an $\hat{U} \in O(H)$ for each $U \in G$ such that $\langle \hat{U} f_k, f_j \rangle_H = \langle U e_k, e_j \rangle_{l^2}$ $(k, j=1, 2, \dots)$, we can assure that $T_f {}^t\hat{U} = {}^tU T_f$. It follows that $\mu^f = T_f \mu \simeq T_f {}^t\hat{U}\mu = {}^tU T_f \mu = {}^tU \mu^f$. Hence the limiting measure μ^f_ω is defined, $\mu^f_\omega(B) = \int_{[0,\infty)} g_v(B) dP^f(v)$ for all $B \in \mathfrak{B}(\boldsymbol{R}^\infty)$, and it holds $\mu^f \simeq \mu^f_\omega$ by Theorem 1. In order to observe that P^f does not depend on the choice of $f_1, \dots, f_n, \dots$, we take another system $f'_1, \dots, f'_n, \dots$. We perform Schmidt's orthogonalization process for $f_1, f'_1, \dots, f_n, f'_n, \dots$ to obtain an orthonormal system $h_1, \dots, h_n, \dots$. It is clear that f_n and f'_n are finite linear combinations of $h_1, \dots, h_n, \dots,$. We wish to show $P^f = P^h$. Now consider an operator T on $\boldsymbol{R}^\infty_0$ such that $Te_n = \sum_{k=1}^\infty \langle f_n, h_k \rangle_H e_k$ for each n. (Actually it is a finite sum.) T preserves l^2-norm as easily seen. Hence we have ${}^tT\mu^h_\omega = \mu^h_\omega$. Further noting that $\{x \in \boldsymbol{R}^\infty | r({}^tTx) \in (\alpha, \beta]\} \in \mathfrak{B}_\infty$ for $\alpha, \beta \in \boldsymbol{R}$, it follows that $P^h((\alpha, \beta]) = \mu^h_\omega(x \in \boldsymbol{R}^\infty | r(x) \in (\alpha, \beta]) = \mu^h_\omega(x \in \boldsymbol{R}^\infty | r({}^tTx) \in (\alpha, \beta]) = \mu^h(x \in \boldsymbol{R}^\infty | r({}^tTx) \in (\alpha, \beta]) = \mu(x \in H^a | r({}^tTT_h x) \in (\alpha, \beta]) = \mu(x \in H^a | r(T_f x) \in (\alpha, \beta]) = \mu^f(x \in \boldsymbol{R}^\infty | r(x) \in (\alpha, \beta]) = P^f((\alpha, \beta])$. Similarly we have $P^h = P^{f'}$. So putting $P = P^f = P^{f'}$, a rotationally-invariant probability measure ν is defined on $(H^a, \mathfrak{B})$, $\nu(B) = \int_{[0,\infty)} G_v(B) dP(v)$, for all $B \in \mathfrak{B}$, where G_v is a canonical Gaussian measure on $(H^a, \mathfrak{B})$ with mean 0 and variace v. We show that $\mu \simeq \nu$. In fact, first we note that $T_f \nu = \mu^f_\omega$. Next, for any $A \in \mathfrak{B}$, there exist some countably-infinite orthonormal system $f_1, f_2, \dots, f_n, \dots$ and $\tilde{A} \in \mathfrak{B}(\boldsymbol{R}^\infty)$ such that $\chi_A(x) = \chi_{\tilde{A}}((x(f_1), \dots, x(f_n), \dots))$ for all $x \in H^a$. It follows that $\nu(A)=0 \Leftrightarrow \mu^f_\omega(\tilde{A})=0 \Leftrightarrow \mu^f(\tilde{A})=0 \Leftrightarrow \mu(A)=0$. Thus,

Theorem 2. *For any rotationally-quasi-invariant probability measure* μ *on* $(H^a, \mathfrak{B})$, *there exists a rotationally-invariant probability measure* ν *such that* $\mu \simeq \nu$. *The explicit form of* ν *is as follows.* $\nu(B) = \int_{[0,\infty)} G_v(B) dP(v)$ *for all* $B \in \mathfrak{B}$, *where* G_v *is a canonical Gaussian measure on* $(H^a, \mathfrak{B})$ *with mean* 0 *and variance* v, *and* P *is a probability measure on* $[0, \infty)$ *defined by* $P(E) = \mu(x \in H^a | \overline{\lim_N} \frac{1}{N} \sum_{n=1}^N x(f_n)^2 \in E)$ *for Borel sets* $E \subset \boldsymbol{R}$, *using a countably-infinite orthonormal system* $f_1, f_2, \ldots, f_n, \ldots$ *on* H.

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