Publ. RIMS, Kyoto Univ.
21 (1985), 421–423

A Remark to 'Global Regularity and Spectra of Laplace-Beltrami Operators on Pseudoconvex Domains'

By

Kensho TAKEGOSHI*

Let $D \Subset M$ be a pseudoconvex domain with smooth boundary ∂D on a complex manifold M and let $\boldsymbol{B} \to M$ be a positive holomorphic line bundle on M. In the previous paper [6], we proved the following statement which is inspired by Kohn's work [3] (cf. [6] Theorem N_s and Corollary).

For every non-negative integer s, there exists a positive integer m(s) such that if $m \geqq m(s)$, *for every* $v \in C^{p,q}_{\infty}(\bar{D}, \boldsymbol{B}^{\otimes m})$ *with* $\bar{\partial} v = 0$, *there exists* $u \in C^{p,q-1}_{s}(\bar{D}, \boldsymbol{B}^{\otimes m})$ *with* $\bar{\partial} u = v$.

Here $C^{p,q}_{s}(\bar{D}, \boldsymbol{B}^{\otimes m})$ denotes the space of $\boldsymbol{B}^{\otimes m}$-valued differential forms of type (p, q) and of class C^s up to boundary $(0 \leqq s \leqq \infty)$. With respect to this statement, it is natural to ask *whether we need to take the tensor product of* $\boldsymbol{B}$ *so many times actually*. In this connection, we give here the following example as a partial answer to this question.

Assertion. *There exist a pseudoconvex domain* $D \Subset \boldsymbol{L}$ *with smooth boundary* ∂D *on a complex manifold* $\boldsymbol{L}$ *and a positive holomorphic line bundle* $\boldsymbol{B} \to \boldsymbol{L}$ *satisfying the following properties*:

i) *D is Stein,*

ii) *there exists* $v \in C^{0,1}_{\infty}(\bar{D}, \boldsymbol{B})$ *with* $\bar{\partial} v = 0$ *such that any solution* $u \in C^{0,0}_{\infty}(D, \boldsymbol{B})$ *of* $\bar{\partial} u = v$ *satisfies* sing. supp. $(u) \neq \phi$.

Here sing. supp. (u) denotes the singular support of u with respect to the closed domain $\bar{D}$. More precisely, the complement of sing. supp. (u) consists of all points $x \in \bar{D}$ such that x has a neighborhood U with the property that the

Communicated by S. Nakano, July 13, 1984.
* Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan

restriction of u to $U \cap \bar{D}$ is in $C_0^{0,0}(U \cap \bar{D}, \boldsymbol{B})$. *The above example tells us that even when* $s=0$, *we need to take the tensor product of* $\boldsymbol{B}$ *sufficiently many times in order to gain the boundary regularity of the* $\bar{\partial}$*-solution.* On the other hand, we do not know whether the integer $m(s)$ can be taken bounded or not as s tends to infinity. With respect to the propagation of singularities for the $\bar{\partial}$-operator, the reader is also referred to [2], [3] and [4].

The author thanks to Professor S. Nakano and Dr. T. Ohsawa for their kind advices.

Construction of $D \Subset \boldsymbol{L}$ *and* $\boldsymbol{B} \rightarrow \boldsymbol{L}$.

According to [1], Appendix, we construct $D \Subset \boldsymbol{L}$. Let A be a non-singular compact curve whose genus is $\geqq 2$ and let $\boldsymbol{L} \xrightarrow{p} A$ be a holomorphic line bundle with $\deg(\boldsymbol{L})=0$. Let $\{a_{ij}\}$ be a system of transition functions of $\boldsymbol{L}$ with respect to a trivializing covering $\{V_i\}$. Then we can find non-constant harmonic functions h_i on V_i such that $a_{ij}=\exp(-\sqrt{-1}(h_i-h_j))$ on $V_i \cap V_j$. Let $w_i=0$ be the local defining equation of the zero section in $p^{-1}(V_i)$ such that $w_i=a_{ij}^{-1}w_j$ on $p^{-1}(V_i \cap V_j)$. Then D is defined as follows:

$$D \cap p^{-1}(V_i)=\{(z_i, w_i)\colon |w_i|^2+\mathrm{Re}(w_i \exp(-\sqrt{-1}h_i))<0\}\,.$$

Then D is a pseudoconvex domain with smooth real analytic boundary ∂D which contains the zero section of $\boldsymbol{L}$ and ∂D is strongly pseudoconvex outside the zero section of $\boldsymbol{L}$.

Next we take a holomorphic line bundle $\boldsymbol{E} \rightarrow A$ with $\deg(\boldsymbol{E})=1$. Using the global function $\Phi=|w_i|^2$ on $\boldsymbol{L}$, we can assume that the pull back of $\boldsymbol{E}$ by the mapping $p\colon \boldsymbol{L} \rightarrow A$ is a positive line bundle on $\boldsymbol{L}$. We set $\boldsymbol{B}=p^*\boldsymbol{E}$.

Proof of i) *and* ii).

Since D does not contain any compact curve, by [1], D is Stein. By the choice of A and $\boldsymbol{E}$, we obtain $\dim_{\boldsymbol{C}} H^1(A, \mathcal{O}(\boldsymbol{E})) \geqq 1$. Hence we can take a $\bar{\partial}$-closed $\boldsymbol{E}$-valued differential form of type (0, 1) and of class C^∞ which is not $\bar{\partial}$-exact, say f. We set $v=p^*f$. Then it is clear that $v \in C_\infty^{0,1}(\bar{D}, \boldsymbol{B})$ and $\bar{\partial}v=0$. Since D is Stein, there exists $u \in C_\infty^{0,0}(D, \boldsymbol{B})$ with $\bar{\partial}u=v$ on D. If sing supp. (u) $=\phi$ in the above sense, then we can restrict u to the zero section of $\boldsymbol{L}$ and so f is $\bar{\partial}$-exact. This contradicts to the choice of f. This means that any solution $u \in C_\infty^{0,0}(D, \boldsymbol{B})$ of $\bar{\partial}u=v$ satisfies sing. supp. $(u) \neq \phi$.

References

[1] Diederich, K., Ohsawa, T., A Levi problem on two dimensional complex manifolds, *Math. Ann.*, **261** (1982), 255–261.

[2] Diederich, K., Plug, P., Necessary conditions for hypoellipticity of the $\bar{\partial}$-problem, *Ann. of Math. Studies*, **100**, P. U. Press, (1981), 151–154.

[3] Kohn, J. J., Global regularity for $\bar{\partial}$ on weakly pseudoconvex manifolds, *Trans. Amer. Math. Soc.*, **181** (1973), 273–292.

[4] ———, Subellipticity of the $\bar{\partial}$-Neumann problem on pseudoconvex domains: sufficient conditions, *Acta Math.*, **142** (1979), 79–122.

[5] ———, Boundary regularity of $\bar{\partial}$, *Ann. of Math. Studies*, **100**, P. U. Press, (1981), 243–260.

[6] Takegoshi, K., Global regularity and spectra of Laplace-Beltrami operators on pseudoconvex domains, *Publ. RIMS, Kyoto Univ.*, **19** (1983), 275–304.