Publ. RIMS, Kyoto Univ.
21 (1985), 541–591

On Central Limit Theorems in Probability Gage Spaces

by

G. O. S. EKHAGUERE*†

Abstract

Some limit theorems in probability gage spaces over semifinite W^*-algebras are proved. In particular, a Levy–Khinchine type of representation for the Fourier transforms of limit probability gages is established. The results are obtained by exploiting some of the algebraic and topological properties of certain sets of operators, called the decomposability algebraic structures, associated with probability gages. This work has a number of points of contact with aspects of (Euclidean) Quantum Field Theory.

Contents

§0. Introduction

The *Centrai Limit Problem* is itself a central problem in Probability Theory. This is the problem of characterizing the limit distributions of sums of triangular arrays of uniformly infinitesimal, not necessarily identically distributed, stochastically independent random variables [1]. In the case of real-valued

Communicated by H. Araki, September 20, 1984.
* Fellow of the Alexander von Humboldt Foundation, Institut für Theoretische Physik, Universität Göttingen D–3400 Göttingen 1, Federal Republic of Germany.
† Permanent address, Department of Mathematics, University of Ibadan, Ibadan, Oyo State, Federal Republic of Nigeria.

random variables, a complete characterization of all nondegenerate such limit distributions was accomplished by Levy [2].

In recent times, attempts have been made to solve the Central Limit Problem in the more general contexts of group-valued [3] and Banach-space-valued [4] random variables. In Ref. [5], Urbanik provides a complete description of a wide class of nondegenerate limit distributions of sums of Banach-space-valued, stochastically independent, random variables by means of a certain semigroup of linear operators associated with each such limit distribution. Our present work extends the results of Ref. [5] to the case where the random variables are densely-defined, stochastically independent, self-adjoint linear operators on a separable Hilbert space.

Other authors [6–10] have also discussed the Central Limit Problem in cases involving various specialized classes of linear operators on Hilbert spaces. In this paper, we provide a fairly general approach to the discussion of the Central Limit Problem for a class of densely-defined, self-adjoint linear operators on separable Hilbert spaces. Our presentation exploits various techniques introduced by Urbanik [5].

The organisation of this paper is as follows. In Section 1, we discuss the fundamentals of a noncommutative integration theory on W^*-algebras of linear operators on separable Hilbert spaces. Our discussion involves tensor algebras over W^*-algebras. A number of concepts and structures, and some of the notation which we require in the sequel, are also introduced there. In particular, we isolate a state on a W^*-algebra which plays the same role as the Dirac point measure in ordinary integration theory. Section 2 deals with the basic notions of the decomposition of a probability gage [11] relative to operators and of the *decomposability algebraic structure* of a probability gage. Some properties of certain operators with respect to which a given probability gage is decomposable are described there. In Section 3, the problem addressed in the rest of the paper is formulated. An analogue of this problem had been formulated and solved by Urbanik [5] in the case of Banach-space-valued random variables. In this section, we also introduce the notion of a *limit pair* and of a *norming sequence* corresponding to such a pair. In Section 4, we describe some of the properties of norming sequences which correspond to certain limit pairs. These are the limit pairs which are nondegenerate in a sense to be found in Section 3. In Section 5, we introduce the notion of an *infinitely divisible pair* (μ, x), where μ is a probability gage on a tensor algebra over a W^*-algebra and x is a self-adjoint operator on the Hilbert space associated with the tensor algebra. We prove that if the decomposability algebraic structure associated with μ contains a certain type of one-parameter semigroup of linear operators, then (μ, x) is an infinitely divisible pair. In this section, we also characterize limit pairs by means of their associated decomposability algebraic structures. In Section 6, we characterize those one-parameter semigroups of linear operators which can be associated with limit pairs. In Section 7, we obtain a Levy-Khinchine type of representation for nondegenerate limit pairs. This is done by means of the Choquet theory of barycentric decomposition on compact, convex spaces.

As is well known [12], Quantum Theory is a noncommutative Probability Theory. Consequently, a number of problems in Quantum Theory have sometimes been discussed [13–17] within the context of noncommutative Probability Theory. In particular, the papers [6–10] discuss the Central Limit Problem for certain types of operators occurring in Quantum Theory. Since the Central Limit Problem is intimately related to the Problem of Infinite Divisibility, it is pertinent to note the references [18–20] which discuss the latter problem by means of certain techniques of Quantum Field Theory. Our own discussion of the Central Limit Problem in this paper has several features in common with aspects of Euclidean Quantum Field Theory [21]. In fact, the transformation Γ introduced in Section 1 is a generalization of the second quantization operator [22] and the positivity-preserving one-parameter semigroup $\{e^{tH}: t \in \mathbb{R}\}$ occurring in Section 5 may easily be interpreted as the evolution operator. Therefore, it appears to us that our presentation and results should be of interest not only to mathematicians and probabilists but also to Quantum Theorists.

§1. Gage Spaces and Some Associated Structures

If $\mathscr{C}_1$ is a W^*-algebra with identity, then in the sequel, $\mathscr{C}_1^s$, $\mathscr{C}_1^+$, $\mathscr{C}_1^*$, $\mathscr{C}_{1*}$, $S(\mathscr{C}_1)$ and $1_{\mathscr{C}_1}$ denote the *self-adjoint* portion, the *positive* portion, the *topological dual*, the *pre-dual*, the *state space*, and the *identity*, respectively, of $\mathscr{C}_1$. For two W^*-algebras $\mathscr{C}_1^{(1)}$ and $\mathscr{C}_1^{(2)}$, the notation $\mathscr{C}_1^{(1)} \hat{\otimes} \mathscr{C}_1^{(2)}$ stands for their *W^*-tensor product* and for any two linear spaces $\mathscr{L}^{(1)}$ and $\mathscr{L}^{(2)}$, the symbol $\mathscr{L}^{(1)} \otimes \mathscr{L}^{(2)}$ denotes their *algebraic tensor product*. We refer to Ref. [23, §1.22] or Ref. [24, Chapter IV] for the notion of *W^*-tensor product* and to

Ref, [34, VI. 3] for the concept of *algebraic tensor product.*

Throughout this paper, X_1 is a fixed complex separable *Hilbert space* and $\mathscr{X}_1$ is a fixed *semi-finite* W^*-algebra of linear operators on X_1.

The spaces X and $\hat{\mathscr{X}}$

For any positive integer n, let X_n (resp. $\mathscr{X}_n$) denote the n-fold W^*-tensor product) of X_1 (resp. $\mathscr{X}_1$) with itself. Then $\hat{\mathscr{X}}_n$ is a semi-finite W^*-algebra [23, Theorem 2.6.6] of linear operators on the Hilbert space X_n. We write $\bigoplus_{n=0}^{\infty} X_n \equiv X$ and $\bigoplus_{n=0}^{\infty} \hat{\mathscr{X}}_n \equiv \hat{\mathscr{X}}$ for the Hilbert space direct sum of $\{X_n: n=0, 1, 2,\dots\}$, and the W^*-direct sum [23] of $\{\hat{\mathscr{X}}_n: n=0, 1, 2,\dots\}$, respectively. Here $X_0 \equiv \mathbb{C}$ and $\hat{X}_0 \equiv \mathbb{C}1\hat{\mathscr{X}}_1$, where $\mathbb{C}$ denotes the complex numbers. A member a of $\hat{\mathscr{X}}$ may be written as follows: $a = \bigoplus_{n=0}^{\infty} a_n$, where a_n lies in $\hat{\mathscr{X}}_n$ and only a finite number of the members of the sequence $\{a_n: n=0, 1, 2,\dots\}$ is nonzero.

The pre-dual $\hat{\mathscr{X}}_*$ of the W^*-algebra $\hat{\mathscr{X}}$ consists of linear functionals ν of the form $\nu = \bigoplus_{n=0}^{\infty} \nu_n$, where ν_n lies in the pre-dual $\hat{\mathscr{X}}_{n*}$ of $\hat{\mathscr{X}}_n$, $\|\nu\|_{\hat{\mathscr{X}}_*} = \sum_{n=0}^{\infty} \|\nu_n\|_{\hat{\mathscr{X}}_*}$, $\|\nu\|_{\hat{\mathscr{X}}_*}$ and $\|\nu\|_{\hat{\mathscr{X}}_{n^*}}$ denote the norms of $\hat{\mathscr{X}}_*$ and $\hat{\mathscr{X}}_{n*}$, respectively, and ν_0 is a scalar multiple of the function on $\hat{\mathscr{X}}_0$ which is identically 1. Moreover, $\hat{\mathscr{X}}_*$ is a Banach subspace of $\hat{\mathscr{X}}^*$ in the norm-topology of $\hat{\mathscr{X}}^*$ [25].

The algebraic tensor algebra $\mathscr{X}$

In the sequel, we write $\mathscr{X}_n$ for the n-fold *algebraic* tensor product of $\mathscr{X}_1$ with itself and $\bigoplus_{n=0}^{\infty} \mathscr{X}_n \equiv \mathscr{X}$ for the *algebraic* tensor algebra [26] over $\mathscr{X}$. Evidently, $\mathscr{X}$ is dense in $\hat{\mathscr{X}}$. The multiplication in $\mathscr{X}$ is defined as follows. For $a = \bigoplus_{n=0}^{\infty} a_n$ and $b = \bigoplus_{n=0}^{\infty} b_n$ in $\mathscr{X}$, with $a_n = \bigotimes_{j=1}^{n} a_{nj}$ and $b_n = \bigotimes_{j=1}^{n} b_{nj}$, then

$$a \cdot b = \bigoplus_{n=0}^{\infty} (a \cdot b)_n$$

where

$$(a \cdot b)_n = \sum_{k=0}^{n} a_{n-k} \otimes b_k, \quad \text{and}$$

$$a_n \otimes b_m = a_{n1} \otimes \cdots \otimes a_{nn} \otimes b_{m1} \otimes \cdots \otimes b_{mm}.$$

Observe that $\hat{\mathscr{X}}_1 = \mathscr{X}_1$.

Noncommutative integration on W^*-algebras

Several versions of a noncommutative integration theory on W^*-algebras have been developed in the literature [11, 27–31]. In this paper, we employ the formulation due to Irving E. Segal [11, 32, 33].

Suppose that μ_n is a faithful, normal, semi-finite trace on $\hat{\mathscr{X}}_n$, $n \geq 0$. Set $\{a_n \in \hat{\mathscr{X}}_n : \mu_n(a_n^* a_n) < \infty\} \equiv \hat{\mathscr{X}}_{\mu_n}$ and $\hat{\mathscr{X}}_{\mu_n} \cdot \hat{\mathscr{X}}_{\mu_n} \equiv \hat{\mathscr{X}}_n^{(\mu_n)}$ $n \geq 0$. Then, there is [23, 24] a unique linear functional, denoted again by μ_n, which coincides with the original μ_n on $\hat{\mathscr{X}}_n^{(\mu_n)} \cap \hat{\mathscr{X}}_n^+$, $n \geq 0$. The linear functional μ_n is called a gage [11] on $\hat{\mathscr{X}}_n$, $n \geq 0$.

We write $G(\hat{\mathscr{X}}_n)$ for the set of all gages on $\hat{\mathscr{X}}_n$, $n \geq 0$. A member μ_n of $G(\hat{\mathscr{X}}_n)$ such that $\mu_n(1_{\hat{\mathscr{X}}_n}) < \infty$ and $\mu_n(1_{\hat{\mathscr{X}}_n}) = 1$ is called a *probability gage*. We denote the set of all probability gages on $\hat{\mathscr{X}}_n$ by $G_1(\hat{\mathscr{X}}_n)$, $n \geq 0$. Furthermore, we define the sets $G(\hat{\mathscr{X}})$ and $G_1(\hat{\mathscr{X}})$ as follows:

$$G(\hat{\mathscr{X}}) \equiv \{\mu = \bigoplus_{n=0}^{\infty} \mu_n : \mu_n \in G(\hat{\mathscr{X}}_n),\ n \geq 0\}, \quad \text{and}$$

$$G_1(\hat{\mathscr{X}}) \equiv \{\mu = \bigoplus_{n=0}^{n} \mu_n \in G(\hat{\mathscr{X}}) : \mu_n(1_{\hat{\mathscr{X}}_n}) < \infty \text{ and } \mu(1_{\hat{\mathscr{X}}}) = 1\}$$

Suppose that $\mu_n \in G(\hat{\mathscr{X}}_n)$. For $1 \leq p < \infty$, let $L^p(X_n, \hat{\mathscr{X}}_n, \mu_n)$ denote the completion of $\hat{\mathscr{X}}_n^{(\mu_n)}$ in the norm-topology given by

$$a_n \longmapsto \|a_n\|_{p,\mu_n} \equiv (\mu_n(|a_n|^p))^{1/p},$$

where $|a_n|$ is the *positive part* of the canonical polar decomposition of $a_n \in \hat{\mathscr{X}}_n^{(\mu_n)}$. We denote the pair $(\hat{\mathscr{X}}_n, \|\cdot\|_{\infty,\mu_n})$, where $\|\cdot\|_{\infty,\mu_n}$ is the operator-norm on $\hat{\mathscr{X}}_n$, by $L^\infty(X_n, \hat{\mathscr{X}}_n, \mu_n)$.

The Banach spaces $L^p(X_n, \hat{\mathscr{X}}_n, \mu_n)$, $\mu_n \in G(\hat{\mathscr{X}}_n)$, $1 \leq p \leq \infty$, have properties which are analogous to those of ordinary L^p-spaces of functions [32, 33]. However, in the present noncommutative setting, each member of $L^p(X_n, \hat{\mathscr{X}}_n, \mu_n)$, $1 \leq p \leq \infty$, $n \geq 0$, is an operator which is affiliated [11] to $\hat{\mathscr{X}}_n$. We remark too that if $\mu_n \in G_1(\hat{\mathscr{X}}_n)$, then $L^q(X_n, \hat{\mathscr{X}}_n, \mu_n)$ may be identified as a Banach subspace of $L^p(X_n, \hat{\mathscr{X}}_n, \mu_n)$, for $q \geq p$, $n \geq 0$. In this case, $L^\infty(X_n, \hat{\mathscr{X}}_n, \mu_n)$ is a Banach subspace of $L^p(X_n, \hat{\mathscr{X}}_n, \mu_n)$, $1 \leq p \leq \infty$, $n \geq 0$.

For $\mu \in G(\hat{\mathscr{X}})$, with $\mu = \bigoplus_{n=0}^{\infty} \mu_n$, we define $L^p(X, \hat{\mathscr{X}}, \mu)$ as the Banach space completion of the algebraic direct sum $\bigoplus_{n=0}^{\infty} L^p(X_n, \hat{\mathscr{X}}_n, \mu_n) \equiv L^p(X, \hat{\mathscr{X}}, \mu)_{alg}$ in the norm-topology furnished by the norm $\|\cdot\|_{p,\mu}$ specified thus:

$$a \longmapsto \|a\|_{p,\mu}^p = \sum_{n=0}^{\infty} \|a_n\|_{p,\mu_n}^p, \quad a = \bigoplus_{n=0}^{\infty} a_n \in L^p(X, \hat{\mathscr{X}}, \mu)_{alg}, \quad 1 \leq p \leq \infty.$$

A member a of $L^p(X, \hat{\mathscr{X}}, \mu)$ will be written as $a = \hat{\bigoplus}_{n \in \mathbb{N}_0} a_n$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

We remark that if $\mathbb{F}(X)$ denotes the completion of X in the norm given by

$$\xi \longmapsto \|\xi\|_{\mathbb{F}(X)}^2 = \sum_{n=0}^{\infty} \|\xi_n\|_{X_n}^2, \quad \xi = \bigoplus_{n=0}^{\infty} \xi_n \in X,$$

then each member of $L^p(X, \hat{\mathscr{X}}, \mu)$ is a densely defined operator on $\mathbb{F}(X)$. Furthermore, $\mathbb{F}(X)$ is analogous to *Fock space*. [22].

Convergence in $G(\hat{\mathscr{X}})$

For $\mu \in G(\hat{\mathscr{X}})$, with $\mu = \bigoplus_{n=0}^{\infty} \mu_n$, denote the self-adjoint portion of $L^p(X, \hat{\mathscr{X}}, \mu)$ by $L^p_s(X, \hat{\mathscr{X}}, \mu)$. In the sequel, we write $L^p_s(X, \hat{\mathscr{X}}, \mu)_{alg}$ for the set of all finite real linear combinations (using the strong sum operation in $L^p(X, \hat{\mathscr{X}}, \mu)$) of members of $L^p_s(X, \hat{\mathscr{X}}, \mu)$ of the form $a = \hat{\bigoplus}_{n \in N_0} a_n$, with $a_n = \bigotimes_{j=1}^{n} a_{nj}$, $n = 0, 1, 2, \dots$. For $\mu_n \in G(\hat{\mathscr{X}}_n)$, $L^p_s(X_n, \hat{\mathscr{X}}_n, \mu_n)$ and $L^p_s(X_n, \hat{\mathscr{X}}_n, \mu_n)_{alg}$ are analogously defined.

For $\mu \in G(\hat{\mathscr{X}})$, $a \in L^p_s(X, \hat{\mathscr{X}}, \mu)_{alg}$, with $a = \hat{\bigoplus}_{n \in N_0} a_n$, $a_n = \bigotimes_{j=1}^{n} a_{nj}$, and any set $\sigma_n = \{\sigma_{nj} : j = 1, 2, \dots, n\}$ contained in $\mathbb{R}$, we define the *exponential* $e^{i\sigma_n \cdot a_n}$ of a_n as follows:

$$e^{i\sigma_n \cdot a_n} = \bigotimes_{j=1}^{n} e^{i\sigma_{nj} a_{nj}} \tag{1.1}$$

(1.2) Definition and Notation: Let $\mu \in G(\hat{\mathscr{X}})$ with $\mu = \bigoplus_{n=0}^{\infty} \mu_n$, and $a \in L^p_s(X, \hat{\mathscr{X}}, \mu)_{alg}$, $1 \le p \le \infty$, with $a = \hat{\bigoplus}_{n \in N_0} \bigotimes_{j=1}^{n} a_{nj}$. Denote the formal sum $\sum_{n=0}^{\infty} \mu_n(e^{i\sigma_n \cdot a_n})$ by $\mu(e^{i\underline{\sigma} \cdot a})$, where $\underline{\sigma} = \{\sigma_{nj} : j = 1, 2, \dots, n;\ n = 0, 1, 2, \dots\}$. Then, we write $G^{(p)}(\hat{\mathscr{X}})$ for the set: $G^{(p)}(\hat{\mathscr{X}}) \equiv \{(\mu, a) \in G(\hat{\mathscr{X}}) \times L^p_s(X, \hat{\mathscr{X}}, \mu)_{alg} : |\mu(e^{i\underline{\sigma} \cdot a})| < \infty$ for all $\underline{\sigma} = \{\sigma_{nj} : j = 1, 2, \dots, n;\ n = 0, 1, 2, \dots\} \subset \mathbb{R}\}$

Using $G_1(\hat{\mathscr{X}})$ in place of $G(\hat{\mathscr{X}})$, we define $G_1^{(p)}(\hat{\mathscr{X}})$ in an analogous fashion.

Remark: In case $\sigma_{nj} = \sigma$ for all $j = 1, 2, \dots, n$; $n = 0, 1, 2, \dots$, then we denote $\mu(e^{i\underline{\sigma} \cdot a})$ simply by $\mu(e^{i(\sigma) \cdot a})$. The notion of convergence in $G(\hat{\mathscr{X}})$ employed by us is induced by the following concept of convergence of pairs.

(1.3) Definition: Let $\mu^0 \in G(\hat{\mathscr{X}})$ and $\mathscr{A}$ be a directed set. Suppose that $\{\mu\} \cup \{\mu^{(\alpha)} : \alpha \in \mathscr{A}\} \subset G(\hat{\mathscr{X}})$ and $\{a\} \cup \{a^{(\alpha)} : \alpha \in \mathscr{A}\} \subset L^p_s(X, \hat{\mathscr{X}}, \mu^0)_{alg}$, with $(\mu, a) \in G^{(p)}(\hat{\mathscr{X}})$ and $(\mu^{(\alpha)}, a^{(\alpha)}) \in G^{(p)}(\hat{\mathscr{X}})$, $\alpha \in \mathscr{A}$. Then, we shall say that the net $\{(\mu^{(\alpha)}, a^{(\alpha)}) : \alpha \in \mathscr{A}\}$ of pairs converges to the pair (μ, a) if, and only if,

$$\mu^{(\alpha)}(e^{i(\sigma) \cdot a^{(\alpha)}}) \text{ converges to } \mu(e^{i(\sigma) \cdot a}), \text{ for each } \sigma \in \mathbb{R}. \tag{1.4}$$

The algebras $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}_n, \mu_n)$ and $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}, \mu)$

Let $\mathscr{D}(\mathbb{R})_1$ denote the complex-valued bounded Borel functions on $\mathbb{R}$. We write $\mathscr{D}(\mathbb{R})_n$ for the n-fold algebraic tensor product of $\mathscr{D}(\mathbb{R})_1$ with itself and set $\bigoplus_{n=0}^{\infty} \mathscr{D}(\mathbb{R})_n \equiv \mathscr{D}(\mathbb{R})$, an *algebraic* direct sum. Here, $\mathscr{D}(\mathbb{R})_0 = \mathbb{C} f_0$, where f_0 is the function on $\mathbb{R}$ which is identically one. Of course, $\mathscr{D}(\mathbb{R})$ is a tensor algebra in a natural way.

Let $\mu \in G_1(\hat{\mathscr{X}})$, with $\mu = \bigoplus_{n=0}^{\infty} \mu_n$ and $(a, g) \in L_s^p(X, \hat{\mathscr{X}}, \mu)_{alg} \times \mathscr{D}(\mathbb{R})$ with $a = \hat{\bigoplus}_{n \in N_0} a_n$, $a_n = \sum_{l=1}^{l'} \lambda_{nl}(\bigotimes_{j=1}^{n} a_{nj}^{(l)})$, $\lambda_{nj} \in \mathbb{R}$, and $g = \bigoplus_{n=0}^{\infty} g_n$, $g_n \in \sum_{k=1}^{k'} \beta_{nk}(\bigotimes_{j=1}^{n} g_{nj}^{(k)})$, $g_{nj}^{(k)} \in \mathscr{D}(\mathbb{R})_1$, $\beta_{nk} \in \mathbb{C}$. Then, define $g_n(a_n)$ and $g(a)$ as follows:

$$g_n(a_n) = \sum_{k=1}^{k'} \sum_{l=1}^{l'} \lambda_{nl}\beta_{nk}(\bigotimes_{j=1}^{n} g_{nj}^{(k)}(a_{nj}^{(l)})), \quad n = 0, 1, 2, \ldots, \quad \text{and}$$

$$g(a) = \bigoplus_{n=0}^{\infty} g_n(a_n).$$

It is evident that the maps $a_n \mapsto g_n(a_n)$, of $L_s^p(X_n, \hat{\mathscr{X}}_n, \mu_n)_{alg}$ into $\mathscr{X}_n$, $n = 0, 1, 2, \ldots$ and $a \mapsto g(a)$, of $L_s^p(X, \hat{\mathscr{X}}, \mu)_{alg}$ into the tensor algebra $\mathscr{X}$, are well-defined. (Functions of self-adjoint operators are defined throughout the paper by means of the spectral theorem [34].)

Next, we define $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}_n, \mu_n)$, $n = 0, 1, 2, \ldots$ and $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}, \mu)$ as follows:

$\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}_n, \mu_n) \equiv$ subalgebra of $\mathscr{X}_n$ generated by $\{g_n(a_n)\colon g_n \in \mathscr{D}(\mathbb{R})_n$ and $a_n \in L_s^p(X_n, \hat{\mathscr{X}}_n, \mu_n)_{alg}\}$

and

$$\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}, \mu) \equiv \bigoplus_{n=0}^{\infty} \mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}_n, \mu_n), \quad 1 \leq p \leq \infty, \quad \mu \in G_1(\hat{\mathscr{X}}).$$

Evidently, $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}, \mu)$ is an algebraic tensor subalgebra of $\mathscr{X}$.

Remark:

(i) In case $\hat{\mathscr{Y}}$ is a W^*-tensor subalgebra of $\hat{\mathscr{X}}$ and $\bigoplus_{n=0}^{\infty} \mu_n$ lies in $G_1(\hat{\mathscr{Y}})$, we define $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{Y}}_n, \mu_n)$, $n = 0, 1, 2, \ldots$, and $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{Y}}, \mu)$, $1 \leq p \leq \infty$, analogously as above.

(ii) Observe that (1.4) implies the following:

$$\mu^{(\alpha)}(g(a^{(\alpha)})) \text{ converges to } \mu(g(a)) \tag{1.5}$$

for all $g = \bigoplus_{n=0}^{\infty} g_n$ in $\mathscr{D}(\mathbb{R})$.

The transformation Γ

Let $\mu_n \in G_1(\hat{\mathscr{X}}_n)$. Then, we denote the Banach algebra of all continuous linear mappings of $L^p(X_n, \hat{\mathscr{X}}_n, \mu_n)$ into itself by $B(L^p(X_n, \hat{\mathscr{X}}_n, \mu_n))$ and write $B(L^p(X_n, \hat{\mathscr{X}}_n, \mu_n))_+$ for the subset of $B(L^p(X_n, \hat{\mathscr{X}}_n, \mu_n))$ consisting of all positive maps, i.e. maps which send positive members of $L^p(X_n, \hat{\mathscr{X}}_n, \mu_n)$ to nonnegative members. The norm of $B(L^p(X_n, \hat{\mathscr{X}}_n, \mu_n))$ will be designated by $|||\cdot|||_{p,\mu_n}$, $n = 0, 1, 2, \ldots$.

For $\mu = \bigoplus_{n=0}^{\infty} \mu_n$ in $G_1(\hat{\mathscr{X}})$, we put

$$\bigoplus_{n=0}^{\infty} B(L^p(X_n, \hat{\mathscr{X}}_n, \mu_n)) \equiv B(L^p(X, \hat{\mathscr{X}}, \mu)), \quad \text{and}$$
$$\bigoplus_{n=0}^{\infty} B(L^p(X_n, \hat{\mathscr{X}}_n, \mu_n))_+ \equiv B(L^p(X, \hat{\mathscr{X}}, \mu))_+ .$$

We denote the norm of $B(L^p(X, \hat{\mathscr{X}}, \mu))$ by $|||\cdot|||_{p,\mu}$, $1 \leq p \leq \infty$.

Corresponding to each $A_n \in B(L^p(X_n, \hat{\mathscr{X}}_n, \mu_n))_+$, we associate an operator

$$\Gamma_n(A_n): \mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}_n, \mu_n) \longrightarrow \mathscr{D}^p(\mathbb{R}, \hat{X}_n, \mu_n)$$

specified by

$$\Gamma_n(A_n)(\sum_{k=1}^{k'} \sum_{l=1}^{l'} \lambda_{kl} g_n^{(1l)}(a_n^{(1l)}) \cdot g_n^{(2l)}(a_n^{(2l)}) \cdots g_n^{(kl)}(a_n^{(kl)}))$$
$$= \sum_{k=1}^{k'} \sum_{l=1}^{l'} \lambda_{kl} g_n^{(1l)}(A_n a^{(1l)}) \cdot g_n^{(2l)}(A_n a_n^{(2l)}) \cdots g_n^{(kl)}(A_n a_n^{(kl)})$$

where $g_n^{(kl)} \in \mathscr{D}(\mathbb{R})_n$, $k=1, 2,\dots, k'$, $l=1, 2,\dots, l'$, $n=0, 1, 2,\dots$

Furthermore, for $\mu = \bigoplus_{n=0}^{\infty} \mu_n \in G_1(\hat{\mathscr{X}})$ and $A = \bigoplus_{n=0}^{\infty} A_n$ in $B(L^p(X, \hat{\mathscr{X}}, \mu))_+$, we define $\Gamma(A)$ by

$$\Gamma(A) = \bigoplus_{n=0}^{\infty} \Gamma_n(A_n).$$

Then, $\Gamma(A)$: $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}, \mu) \longrightarrow \mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}, \mu)$, $1 \leq p \leq \infty$.

It is clear that the operator $\Gamma_n(A_n)$, $n=0, 1, 2,\dots$, and $\Gamma(A)$ are linear on $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}_n, \mu_n)$, $n=0, 1, 2,\dots$, and $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}, \mu)$, respectively, $1 \leq p \leq \infty$. Moreover,

$$\Gamma_n(A_n B_n) = \Gamma_n(A_n)\Gamma_n(B_n), \quad n=0, 1, 2,\dots, \quad \text{and}$$
$$\Gamma(AB) = \Gamma(A)\Gamma(B)$$

for A_n, $B_n \in B(L^p(X_n, \hat{\mathscr{X}}_n, \mu_n))_+$, $n=0, 1, 2,\dots$, and A, B in $B(L^p(X, \hat{\mathscr{X}}, \mu))_+$. Hence, Γ_n (resp. Γ) is a representation of the multiplicative semigroup $B(L^p(X_n, \hat{\mathscr{X}}_n, \mu_n))_+$ (resp. $B(L^p(X, \hat{\mathscr{X}}, \mu))_+$) in the multiplicative semigroup of all bounded linear transformations of $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}_n, \mu_n)$ into $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}_n, \mu_n)$, $n=0, 1, 2,\dots$ (resp. of $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}, \mu)$ into $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}, \mu)$), $1 \leq p \leq \infty$. The operator $\Gamma(A_n)$ (resp. $\Gamma(A)$) may be extended to a map on all of $L^p(X_n, \hat{\mathscr{X}}_n, \mu_n)$ (resp. $L^p(X, \hat{\mathscr{X}}, \mu)$). In the sequel, we always assume that the extension has been done.

The set $D_0^p(\mu, \hat{\mathscr{X}})$, $\mu \in G_1(\hat{\mathscr{X}})$

For each $\mu \in G_1(\hat{\mathscr{X}})$ and nonzero $A \in B(L^p(X, \hat{\mathscr{X}}, \mu)_+$, write μ^A for $\mu \circ \Gamma(A)$. Then, define $D_0^p(\mu, \hat{\mathscr{X}})$ by

$$D_0^p(\mu, \hat{\mathscr{X}}) \equiv \{A \in B(L^p(X, \hat{\mathscr{X}}, \mu))_+ : A \neq 0 \text{ and } \mu^A(1_{\hat{\mathscr{X}}}) = 1\}, \quad 1 \leq p \leq \infty .$$

The degenerate state ε_α

Let $\mathrm{Den}(X_1)_1$ denote the collection of all densely-defined self-adjoint linear operators on X_1. Equipped with the operation of strong sum, $\mathrm{Den}(X_1)_1$ is a real vector space. We write $\mathrm{Den}(X_1)_n$ for the n-fold algebraic tensor product of $\mathrm{Den}(X_1)_1$ with itself and put $\bigoplus_{n=0}^{\infty} \mathrm{Den}(X_1)_n \equiv \mathrm{Den}(X_1)$, and algebraic direct sum. Members of $\mathrm{Den}(X_1)_n$ act on dense domains in X_n; a similar remark is valid about $\mathrm{Den}(X_1)$.

In the sequel, $\mathbb{R}(X_1)$ denotes the set of all linear mappings of $\mathrm{Den}(X_1)_1$ into $\mathbb{R}$.

For each n, let $\mathscr{D}_c(\mathbb{R})_n$ be the subalgebra of $\mathscr{D}(\mathbb{R})_n$ consisting of tensor products whose components are complex-valued continuous functions with compact support on $\mathbb{R}$. We put $\bigoplus_{n=0}^{\infty} \mathscr{D}_c(\mathbb{R})_n \equiv \mathscr{D}_c(\mathbb{R})$, an algebraic direct sum.

Fro $g_n \in \mathscr{D}_c(\mathbb{R})_n$ and $a_n \in \mathrm{Den}(X_1)_n$, we define $g_n(a_n)$ in the same way as members of $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}_n, \mu_n)$ were previously defined. We let $\mathrm{Den}(\mathbb{R}, X_1)_n$ denote the subalgebra of $\mathscr{X}_n$ generated by $\{g_n(a_n): g_n \in \mathscr{D}(\mathbb{R})_n$ and $a_n \in \mathrm{Den}(X_1)_n\}$ and put $\bigoplus_{n=0}^{\infty} \mathrm{Den}(\mathbb{R}, X_1)_n \equiv \mathrm{Den}(\mathbb{R}, X_1)$, an algebraic direct sum. We remark that $\mathrm{Den}(\mathbb{R}, X_1)_0 \equiv \mathbb{C}1_{\hat{\mathscr{X}}_1}$.

For $\alpha \in \mathbb{R}(X_1)$, introduce the map $\hat{\alpha}$: $\mathrm{Den}(X_1)_1 \to \mathbb{R}1_{\hat{\mathscr{X}}_1}$ defined by

$$\hat{\alpha}(a_1) = \begin{cases} 1_{\hat{\mathscr{X}}_1}, & \text{if } a_1 = 1_{\hat{\mathscr{X}}_1} \\ \alpha(a_1)1_{\hat{\mathscr{X}}_1}, & \text{otherwise} \end{cases}$$

If $(\lambda, \alpha, \gamma) \in \mathbb{R} \times \mathbb{R}(X_1) \times \mathbb{R}(X_1)$, then we put: $\hat{\alpha} + \hat{\gamma} \equiv \widehat{\alpha+\gamma}$ and $\lambda\hat{\alpha} \equiv \widehat{\lambda\alpha}$. Next, we define the map $\Gamma_1(\hat{\alpha})$: $\mathrm{Den}(\mathbb{R}, X_1) \to \hat{\mathscr{X}}_1$ in precisely the same way that the map $\Gamma_1(A_1)$: $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}_1, \mu_n) \to \mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}_1, \mu_1)$, $A_1 \in B(L^p(X_1, \hat{\mathscr{X}}_1, \mu_n))_+$, was previously defined. More explicitly,

$$\begin{aligned} &\Gamma_1(\hat{\alpha})\Big(\sum_{k=1}^{k'} \sum_{l=1}^{l'} \lambda_{kl} f_1^{(1l)}(a_1^{(1l)}) f_1^{(2l)}(a_1^{(2l)}) \cdots f_1^{(kl)}(a_1^{(kl)})\Big) \\ &= \sum_{k=1}^{k'} \sum_{l=1}^{l'} \lambda_{kl} f_1^{(1l)}(\hat{\alpha}(a_1^{(1l)})) f_1^{(2l)}(\hat{\alpha}(a_1^{(2l)})) \cdots f_1^{(kl)}(\hat{\alpha}(a_1^{(kl)})) \end{aligned}$$

where $\lambda_{kl} \in \mathbb{C}$, $a_1^{(kl)} \in \mathrm{Den}(X_1)_1$, $f_1^{(kl)} \in \mathscr{D}_c(\mathbb{R})_1$, $k = 1, 2, \ldots, k'$, $l = 1, 2, \ldots, l'$, $\alpha \in \mathbb{R}(X_1)$. Again, $\Gamma_1(\hat{\alpha})$ is linear from $\mathrm{Den}(\mathbb{R}_1, X_1)_1$ to $\mathbb{C}1_{\hat{\mathscr{X}}_1}$, for each $\alpha \in \mathbb{R}(X_1)$.

For any $\mu_1 \in G_1(\hat{\mathscr{X}}_1)$, the linear functional $\mu_1 \circ \Gamma_1(\hat{\alpha})$ is a central state on $\mathrm{Den}(\mathbb{R}, X_1)_1$, for each $\alpha \in \mathbb{R}(X_1)$. This state clearly does not depend on which μ_1 in $G_1(\hat{\mathscr{X}}_1)$ is used to define it.

In what follows, we set $\mu\circ\Gamma_1(\hat{\alpha})\equiv\varepsilon_\alpha$, $\alpha\in\mathbb{R}(X_1)$, where μ_1 is an arbitrary member of $G_1(\hat{\mathscr{X}}_1)$, and write $\varepsilon_{\underline{\alpha}_n}$ for the algebraic tensor product $\bigotimes_{j=1}^{n}\varepsilon_{\alpha_{nj}}$, where $\{\alpha_{nj}: j=1,2,\ldots,n\}\equiv\underline{\alpha}_n\subset\mathbb{R}(X_1)$. Then, $\varepsilon_{\underline{\alpha}_n}$ is also a central state on $\mathrm{Den}(\mathbb{R}, X_1)_n$, $n=0, 1, 2,\ldots$. Finally, put $\bigoplus_{n=0}^{\infty}\varepsilon_{\underline{\alpha}_n}\equiv\varepsilon_{\underline{\alpha}}$, where $\underline{\alpha}=\{\alpha_{nj}: j=1, 2,\ldots, n;\ n=0, 1, 2,\ldots\}\subset\mathbb{R}(X_1)$, and $\varepsilon_{\underline{\alpha}_0}$ is the function on $\mathrm{Den}(\mathbb{R}, X_1)_0$ which is identically one. In case $\alpha_{nj}=0$, the zero function on $\mathrm{Den}(X_1)$, for all $j=1, 2,\ldots, n$, $n=0, 1, 2,\ldots$, then we denote $\underline{\alpha}_n$ and $\underline{\alpha}$ by $\underline{0}_n$ and $\underline{0}$, respectively.

Nondegenerate gages in $G_1(\hat{\mathscr{X}})$

Let $\mu\subset G_1(\hat{\mathscr{X}})$, with $\mu=\bigoplus_{n=0}^{\infty}\mu_n$. Then, we say that μ is *nongenerate* provided that μ_n is not of the form $\lambda_n\varepsilon_{\underline{\alpha}_n}$, $\lambda_n\geq 0$, for any $\underline{\alpha}_n=\{\alpha_{nj}: j=1, 2,\ldots, n\}\subset\mathbb{R}(X_1)$, $n=0, 1, 2,\ldots$.

Remark:

In the sequel, we frequently encounter sequences of pairs of gages in $G_1^{(p)}(\hat{\mathscr{X}})$ of the form $\{(\mu^{(n)}\circ\Gamma(A^{(n)}), a^{(n)})\}_{n\geq 1}$. For our purposes, the following sufficient conditions for their convergence are adequate.

(1.6) Proposition:

Let $\mu^0=\bigoplus_{k=0}^{\infty}\mu_k^0\in G_1(\hat{\mathscr{X}})$, *with* $\sup_{k\geq 0}[\mu_k^0(1_{\hat{\mathscr{X}}_k})]^{1/p}<\infty$, *and* $\{\mu\}\cup\{\mu^{(n)}: n=1, 2,\ldots\}\subset G_1(\hat{\mathscr{X}})$, *with* $\{\mu\}\cup\{\mu^{(n)}: n=1, 2,\ldots\}\subset L^p(X, \hat{\mathscr{X}}, \mu^0)^*$, *the topological dual of* $L^p(X, \hat{\mathscr{X}}, \mu^0)$. *Suppose that* $\{a\}\cup\{a^{(n)}: n=1, 2,\ldots\}$ *and* $\{A\}\cup\{A^{(n)}: n=1, 2,\ldots\}$. *are subsets of* $L_s^p(X, \mathscr{X}, \mu^0)_{alg}$ *and* $B(L^p(X, \mathscr{X}, \mu^0))_+$, *respectively, such that* $(\mu\circ\Gamma(A), a)\in G_1^{(p)}(\hat{\mathscr{X}})$, $(\mu^{(n)}\circ\Gamma(A^{(n)}, a^{(n)})\in G_1^{(p)}(\hat{\mathscr{X}})$, $(\mu^{(n)}\circ\Gamma(A^{(n)}), a)\in G_1^{(p)}(\hat{\mathscr{X}})$, $n=1, 2,\ldots$ *and*

$$|\mu_k(e^{i(\sigma)_k\cdot B_k^{(1)}b_k^{(1)}}-e^{i(\sigma)_k\cdot B_k^{(2)}b_k^{(2)}})|\leq C(\sigma)\|B_k^{(1)}b_k^{(1)}-B_k^{(2)}b_k^{(2)}\|_{p,\mu_k^0}^p,$$

$k=0, 1, 2,\ldots$, *where* $C(\sigma)\geq 0$ *depends only on* $\sigma\in\mathbb{R}$, $B^{(j)}$ *denotes either* A *or* $A^{(n)}$, *and* $b^{(j)}$ *denotes either* a *or* $a^{(n)}$, $j=1, 2$; $n=1, 2,\ldots$. *Suppose, moreover, that*

(*i*) $\{a^{(n)}\}_{n\geq 1}$ *converges to* a *in the norm-topology of* $L^p(X, \hat{\mathscr{X}}, \mu^0)$;
(*ii*) $\{\mu^{(n)}\}_{n\geq 1}$ *converges to* μ *in the norm-topology of* $L^p(X, \hat{\mathscr{X}}, \mu^0)^*$;
(*iii*) $\{A^{(n)}\}_{n\geq 1}$ *is contained in a bounded subset of* $B(L^p(X, \hat{\mathscr{X}}), \mu^0))$; *and*
(*iv*) $\{A^{(n)}\}_{n\geq 1}$ *converges to* A *in the norm-topology of* $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$.

Then $\{(\mu^{(n)}\circ\Gamma(A^{(n)}, a^{(n)})\}_{n\geq 1}$ *converges to* $(\mu\circ\Gamma(A), a)$ *in* $G_1^{(p)}(\hat{\mathscr{X}})$.

Note: In the statement of the Proposition, μ_k^0, μ_k and $\mu_k^{(n)}$ are the components of μ^0, μ and $\mu^{(n)}$, respectively, and $B_k^{(j)}$ are the components of $B^{(j)}$, respectively. Furthermore, $(\sigma)_k=\{\sigma_{kj}: j=1, 2,\dots, k\}$, with $\sigma_{kj}=\sigma$ for all k and j.

Proof: By (1.3) and (1.4), we need to show that $(\mu^{(n)}\circ\Gamma(A^{(n)}))\,(e^{i(\sigma)\cdot a^{(n)}})$ converges to $(\mu\circ\Gamma(A))(e^{i(\sigma)\circ a})$, for each $\sigma\in\mathbb{R}$, under the stated hypotheses. But the assertion follows from the trivial estimate:

$$
\begin{aligned}
&|(\mu^{(n)}\circ\Gamma(A^{(n)}))(e^{i(\sigma)\cdot a^{(n)}})-(\mu\circ\Gamma(A))(e^{i(\sigma)\cdot a})|\\
&\quad=|\sum_{k=0}^{\infty}|(\mu_k^{(n)}(e^{i(\sigma)_k\cdot A_k^{(n)}a_k^{(n)}})-\mu_k(e^{i(\sigma)_k\cdot A_k a_k}))|\\
&\quad\le\sum_{k=0}^{\infty}|(\mu_k^{(n)}-\mu_k)(e^{i(\sigma)_k\cdot A_k^{(n)}a_k^{(n)}})|+\sum_{k=0}^{\infty}|\mu_k(e^{i(\sigma)_k\cdot A_k^{(n)}a_k^{(n)}}-e^{i(\sigma)_k\cdot A_k^{(n)}a_k})|\\
&\qquad+\sum_{k=0}^{\infty}|\mu_k(e^{i(\sigma)_k\cdot A_k^{(n)}a_k}-e^{i(\sigma)_k\cdot A_k a_k})|\\
&\quad\le(\sup_{k\ge 0}[\mu_k^0(1_{\hat{\mathscr{X}}_k})]^{1/p})\,\|\mu^{(n)}-\mu\|_{L^p(x,\hat{\mathscr{X}},\mu^0)}+\\
&\qquad\qquad+C(\sigma)(\sup_{n\ge 1}|||A^{(n)}|||_{p,\mu^0}^p)\,\|a^{(n)}-a\|_{p,\mu^0}^p+\\
&\qquad\qquad+C(\sigma)\,\|a\|_{p,\mu^0}^p|||A^{(n)}-A|||_{p,\mu^0}^p. \qquad\square
\end{aligned}
$$

(1.7) *Remark*: In the sequel, if A is the zero operator on $L^p(X, \hat{\mathscr{X}}, \mu)$, we define $\mu\circ\Gamma(A)$ to be $\bigoplus_{n=0}^{\infty}\delta_{0n}\varepsilon_{\underline{0}_n}$ for all $\mu\in G_1(\hat{\mathscr{X}})$, where δ_{jk} is the *Kronecker delta*. In this way, we have $\hat{0}\in D_0^p(\mu, \hat{\mathscr{X}})$, for each $\mu\in G_1(\hat{\mathscr{X}})$.

§2. Decomposability Algebraic Structures of Probability Gages

In this section, we introduce the notion of operator-decomposition of probability gages. Then, we study certain properties of some operators which feature in such decompositions.

We employ the following notation in the rest of this paper.

Notation: Let $\hat{\mathscr{Y}}$ be a W^*-tensor algebra contained in $\hat{\mathscr{X}}$ and Y be the Hilbert space contained in X on which members of $\hat{\mathscr{Y}}$ act. We remark that the identity $1_{\hat{\mathscr{Y}}}$ of $\hat{\mathscr{Y}}$ may not coincide with $1_{\hat{\mathscr{X}}}$, the identity of $\hat{\mathscr{X}}$.

For $\mu\in G_1(\hat{\mathscr{X}})$ and $A\in B(L^p(X, \hat{\mathscr{X}}, \mu))_+$, let $\mu_{\hat{\mathscr{Y}}}$, $A_{\hat{\mathscr{Y}}}$ and $\Gamma_{\hat{\mathscr{Y}}}(A)$ denote the gage induced on $\hat{\mathscr{Y}}$ by μ, the restriction of A to $L^p(Y, \hat{\mathscr{Y}}, \mu_{\hat{\mathscr{Y}}})$ and the restriction of $\Gamma(A)$ to $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{Y}}, \mu_{\hat{\mathscr{Y}}})$, respectively. Furthermore, we denote the subalgebra of functions in $\mathscr{D}(\mathbb{R})$ which are used in generating $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{Y}}, \mu_{\hat{\mathscr{Y}}})$ by $\mathscr{D}(\mathbb{R})_{\hat{\mathscr{Y}}}$. Notice that $\Gamma_{\hat{\mathscr{Y}}}(A)f(a)=f(A_{\hat{\mathscr{Y}}}a)$, for $a\in L_s^p(Y, \hat{\mathscr{Y}}, \mu_{\hat{\mathscr{Y}}})_{alg}$, and $A\in B(L^p(X, \hat{\mathscr{X}}, \mu))_+$.

We shall write $(Y, \hat{\mathscr{Y}})\subset(X, \hat{\mathscr{X}})$, if Y is a Hilbert space contained in X, $\hat{\mathscr{Y}}$ is a

W^*-tensor algebra contained in $\hat{\mathscr{X}}$, and $\hat{\mathscr{Y}}$ acts on Y. We denote $\mu_{\hat{\mathscr{Y}}}\circ\Gamma_{\hat{\mathscr{Y}}}(A)$ by $\mu^A_{\hat{\mathscr{Y}}}$, $A\in B(L^p(X, \hat{\mathscr{X}}, \mu))_+$.

(2.1) Definition: 1. Let $\mu\in G_1(\hat{\mathscr{X}})$ and $A\in B(L^p(X, \hat{\mathscr{X}})\mu))_+$, $1\leq p\leq\infty$. Then, we say that μ is A-decomposable if, and only if, there are $(X^{(1)}_A, \hat{\mathscr{X}}^{(1)}_A)\subset(X, \hat{\mathscr{X}})$, $(X^{(2)}_A, \hat{\mathscr{X}}^{(2)}_A)\subset(X, \hat{\mathscr{X}})$ and $\nu_A\in G_1(\mathscr{X}^{(2)}_A)$ such that

$$\mu^A_{\hat{\mathscr{X}}^{(1)}_A}(1_{\hat{\mathscr{X}}^{(1)}_A})=1 \tag{0}$$

(i) $\Gamma_{\mathscr{X}^{(1)}_A}(A)$ and $\Gamma_{\hat{\mathscr{X}}^{(2)}_A}(A)$ leave $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}^{(1)}_A, \mu_{\mathscr{X}^{(1)}_A})$ and $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}^{(2)}_A, \nu_A)$ invariant, respectively; and

(ii) $\mu((\Gamma_{\mathscr{X}^{(1)}_A}(A)f(a))\cdot g(b))=\mu^A_{\hat{\mathscr{X}}^{(1)}}(f(a))\nu_A(g(b))$, for all $(a, b)\in L^p_s(X^{(1)}_A, \hat{\mathscr{X}}^{(1)}_A, \mu_{\mathscr{X}^{(1)}_A})_{alg}\times L^p_s(X^{(2)}_A, \hat{\mathscr{X}}^{(2)}_A, \nu_A)_{alg}$, and $f\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}^{(1)}_A}$, $g\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}^{(2)}_A}$.

2. We call the probability gages $\mu^A_{\hat{\mathscr{X}}^{(1)}_A}$ and ν_A the *factors* or *components* of μ with respect to A in $B(L^p(X, \hat{\mathscr{X}}, \mu))_+$, $1\leq p\leq\infty$.

Remark:

Equation (2.1) (ii) is equivalent to $L^p(X_A, \hat{\mathscr{X}}_A, \mu_{\hat{\mathscr{X}}_A})=L^p(X^{(1)}_A, \hat{\mathscr{X}}^{(1)}_A, \mu^A_{\hat{\mathscr{X}}^{(1)}_A})\otimes L^p(X^{(2)}_A, \hat{\mathscr{X}}^{(2)}_A, \nu_A)$, where $\hat{\mathscr{X}}_A$ is the linear hull of all elements of the form $(\Gamma_{\hat{\mathscr{X}}^{(1)}_A}(A)f(a))\cdot g(b)$, with a, b, f, g as in (2.1) (ii) and $\hat{\mathscr{X}}_A$ acts on X_A.

Notation: Let $\mu\in G_1(\hat{\mathscr{X}})$. The set of all members of $B(L^p(X, \hat{\mathscr{X}}, \mu))_+$ with respect to which μ is decomposable will be denoted by $D^p(\mu, \hat{\mathscr{X}})$, $1\leq p\leq\infty$.

(2.2) *Remark*:

1. Let $\mu\in G_1(\hat{\mathscr{X}})$. Then $D^p(\mu, \hat{\mathscr{X}})$ contains the zero 0 and the identity I of $B(L^p(X, \hat{\mathscr{X}}, \mu))_+$. Furthermore, $D^p(\mu, \hat{\mathscr{X}})$ is closed in the weak operator topology on $B(L^p(X, \hat{\mathscr{X}}, \mu))$. But unlike the prevailing situation in the Banach space theory described in Ref. [5], $D^p(\mu, \hat{\mathscr{X}})$ is *not* a *semigroup* in the multiplication operation of $B(L^p(X, \hat{\mathscr{X}}, \mu))$. Consequently, some of the techniques developed in Ref. [5] cannot be directly applied here.

2. We shall refer to $D^p(\mu, \hat{\mathscr{X}})$ as the *decomposability algebraic structure* of $\mu\in G_1(\hat{\mathscr{X}})$.

3. In what follows, we study some properties of certain members of $D^p(\mu, \hat{\mathscr{X}})$, $\mu\in G_1(\hat{\mathscr{X}})$, $1\leq p\leq\infty$. First, we note the following straightforward assertion whose proof we omit.

(2.3) Proposition:

Let $\mu\in G_1(\hat{\mathscr{X}})$. Then, the set of all mutually commuting members of D^p. $(\mu, \hat{\mathscr{X}})$ is a semigroup which is closed in the weak-topology of $B(L^p(X, \hat{\mathscr{X}}, \mu))$.

(2.4) **Proposition:**

Let $\mu \in G_1(\hat{\mathscr{X}})$ and P be a projection operator (i.e. $P^2 = P$) lying in $D^p(\mu, \hat{\mathscr{X}})$. Then, $P^\perp \equiv I - P$ also lies in $D^p(\mu, \hat{\mathscr{X}})$ and we have

$$(2.5) \qquad \mu((\Gamma_{\hat{\mathscr{X}}_P}(P)f(a))\cdot(\Gamma_{\hat{\mathscr{X}}_{P^\perp}}(P^\perp)g^{(b)}))) = \mu^P_{\hat{\mathscr{X}}_P}(f(a))\mu^{P^\perp}_{\hat{\mathscr{X}}_{P^\perp}}(g(b))$$

for some $(X_P, \hat{\mathscr{X}}_P) \subset (X, \hat{\mathscr{X}})$, $(X_{P^\perp}, \hat{\mathscr{X}}_{P^\perp}) \subset (X, \hat{\mathscr{X}})$ and all $(a, b) \in L^p_s(X_P, \hat{\mathscr{X}}_P, \mu_{\hat{\mathscr{X}}_P}) \times L^p_s(X_{P^\perp}, \hat{\mathscr{X}}_{P^\perp}, \mu_{\mathscr{X}_{P^\perp}})$ and $(f, g) \in \mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_P} \times \mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_{P^\perp}}$.

Proof: Since P lies in $D^p(\mu, \hat{\mathscr{X}})$, by hypothesis, it follows that there are $(X_P, \hat{\mathscr{X}}_P) \subset (X, \hat{\mathscr{X}})$, $(Y_P, \hat{\mathscr{Y}}_P) \subset (X, \hat{\mathscr{X}})$ and $\nu_P \in G_1(\hat{\mathscr{Y}})$ such that

$$(2.6) \qquad \mu((\Gamma_{\hat{\mathscr{X}}_P}(P)f(a)) \cdot g(b)) = \mu^P_{\hat{\mathscr{X}}_P}(f(a))\nu_P(g(b)),$$

for all $a \in L^p_s(X_P, \hat{\mathscr{X}}_P, \mu_{\hat{\mathscr{X}}_P})_{alg}$, $b \in L^p_s(Y_P, \hat{\mathscr{Y}}_P, \nu_P)_{alg}$, $f \in \mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_P}$, $g \in \mathscr{D}(\mathbb{R})_{\hat{\mathscr{Y}}_P}$. Equation (2.6) implies

$$(2.7) \qquad \begin{aligned} &\mu((\Gamma_{\hat{\mathscr{X}}_P}(P)e^{i\sigma_1 f(a)}) \cdot e^{i\sigma_2 g(b)}) \\ &= \mu^P_{\hat{\mathscr{X}}_P}(e^{i\sigma_1 f(a)})\nu_P(e^{i\sigma_2 g(b)}), \end{aligned}$$

for $(\sigma_1, \sigma_2) \in \mathbb{R}^2$, where for any tensor algebra $\hat{\mathscr{X}}^0$ contained in $\hat{\mathscr{X}}$,

$$e^{i\lambda c} = 1_{\hat{\mathscr{X}}^0} + \sum_{n=1}^{\infty} \frac{(i\lambda)}{n!} c^n, \quad \text{for any} \quad c \in \hat{\mathscr{X}}^0 \subset \hat{\mathscr{X}}.$$

Furthermore, we infer from (2.1) (i) that $\mathscr{D}^P(\mathbb{R}, \hat{\mathscr{X}}_P, \mu_{\hat{\mathscr{X}}_P})$ and $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{Y}}_P, \nu_P)$ are invariant under $\Gamma_{\hat{\mathscr{X}}_P}(P^\perp)$ and $\Gamma_{\hat{\mathscr{Y}}_P}(P^\perp)$, respectively. Hence, (2.7) remains valid for $\sigma_1 = 0$ and $e^{i\sigma_2 g(b)}$ replaced by $\Gamma_{\hat{\mathscr{Y}}_P}(P^\perp)e^{i\sigma_2 g(b)}$. Hence

$$(2.8) \qquad \begin{aligned} \nu_P^{P^\perp}(e^{i\sigma_2 g(b)}) &= \mu((\Gamma_{\hat{\mathscr{X}}_P}(P)1_{\hat{\mathscr{X}}_P}) \cdot (\Gamma_{\hat{\mathscr{Y}}_P}(P^\perp)e^{i\sigma_2 g(b)})) \\ &= \mu^{P^\perp}_{\hat{\mathscr{Y}}_P}(e^{i\sigma_2 g(b)}), \text{ since } \mu \text{ is} \end{aligned}$$

central on $\hat{\mathscr{X}}$, for all $b \in L^p_s(Y_P, \hat{\mathscr{Y}}_P, \nu_P)$, $g \in \mathscr{D}(\mathbb{R})_{\hat{\mathscr{Y}}_P}$, $\sigma_2 \in \mathbb{R}$. By differentiating both sides of (2.8) with respect to σ_2 and evaluating σ_2 at zero, one gets

$$\nu_P^{P^\perp}(g(b)) = \mu^{P^\perp}_{\hat{\mathscr{Y}}_P}(g(b))$$

for all $b \in L^p_s(Y_P, \hat{\mathscr{Y}}_P, \nu_P)_{alg}$ and $g \in \mathscr{D}(\mathbb{R})_{\hat{\mathscr{Y}}_P}$. So, we may identify $\hat{\mathscr{X}}_{P^\perp}$ and $\mu^{P^\perp}_{\hat{\mathscr{X}}_{P^\perp}}$ in (2.5) with $\hat{\mathscr{Y}}_P$ and ν_P, respectively. □

Remark:

The following generalization of Proposition (2.4) will be employed below.

(2.9) **Proposition:**

Let $\mu \in G_1(\hat{\mathscr{X}})$. Let $\{P^{(1)}, P^{(2)}, \ldots, P^{(n)}\}$ be a set of commuting projection *operators contained in* $D^p(\mu, \hat{\mathscr{X}})$ *and satisfying* $P^{(j)}P^{(k)}=\delta_{jk}P^{(k)}$. *Then* $Q^{(n)} \equiv I-\sum_{j=1}^{n} P^{(j)}$ *also lies in* $D^p(\mu, \hat{\mathscr{X}})$ *and we have*

$$
\begin{aligned}
(2.10)\qquad & \mu((\Gamma_{\hat{\mathscr{X}}_{P^{(1)}}}(P^{(1)})f^{(1)}(a^{(1)}))\cdot(\Gamma_{\hat{\mathscr{X}}_{P^{(2)}}}(P^{(2)})f^{(2)}(a^{(2)}))\cdots \\
& \cdots(\Gamma_{\hat{\mathscr{X}}_{P^{(n)}}}(P^{(n)})f^{(n)}(a^{(n)}))\cdot(\Gamma_{\hat{\mathscr{X}}_{Q^{(n)}}}(Q^{(n)})g^{(n)}(b^{(n)}))) \\
& =\mu^{P^{(1)}}_{\hat{\mathscr{X}}_{P^{(1)}}}(f^{(1)}(a^{(1)}))\mu^{P^{(2)}}_{\hat{\mathscr{X}}_{P^{(2)}}}(f^{(2)}(a^{(2)}))\cdots \\
& \cdots\mu^{P^{(n)}}_{\hat{\mathscr{X}}_{P^{(n)}}}(f^{(n)}(a^{(n)}))\mu^{Q^{(n)}}_{\hat{\mathscr{X}}_{Q^{(n)}}}(g^{(n)}(b^{(n)})
\end{aligned}
$$

for some $(X_{P^{(j)}}, \hat{\mathscr{X}}_{P^{(j)}})\subset(X, \hat{\mathscr{X}})$, $j=1, 2,\dots, n$, $(X_{Q^{(n)}}, \hat{\mathscr{X}}_{Q^{(n)}})\subset(X, \hat{\mathscr{X}})$ *and all* $a^{(j)}\in L^p_s(X_{P^{(j)}}, \hat{\mathscr{X}}_{P^{(j)}}, \mu_{\hat{\mathscr{X}}_{P^{(j)}}})_{alg}$, $j=1, 2,\dots, n$, $b^{(n)}\in L^p_s(X_{Q^{(n)}}, \hat{\mathscr{X}}_{Q^{(n)}}, \mu\hat{\mathscr{X}}_{Q^{(n)}})_{alg}$, $f^{(j)}\in \mathscr{D}(\mathbb{R})_{\mathscr{X}_{P^{(j)}}}$, $j=1, 2,\dots$, $g^{(n)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_{Q^{(n)}}}$.

Proof: We prove (2.10) by mathematical induction. To this end, first observe that the case $n=1$ is precisely Proposition (2.4).

Now, suppose that for some $k<n$, we have

$$
\begin{aligned}
(2.11)\qquad & \mu((\Gamma_{\hat{\mathscr{X}}_{P^{(1)}}}(P^{(1)})f^{(1)}(a^{(1)}))\cdot(\Gamma_{\hat{\mathscr{X}}_{P^{(2)}}}(P^{(2)})f^{(2)}(a^{(2)}))\cdots \\
& \cdots(\Gamma_{\hat{\mathscr{X}}_{P^{(k)}}}(P^{(k)})f^{(k)}(a^{(k)}))\cdot(\Gamma_{\hat{\mathscr{X}}_{Q^{(k)}}}(Q^{(k)})g^{(k)}(b^{(k)}))) \\
& =\mu^{P^{(1)}}_{\hat{\mathscr{X}}_{P^{(1)}}}(f^{(1)}(a^{(1)}))\mu^{P^{(2)}}_{\hat{\mathscr{X}}_{P^{(2)}}}(f^{(2)}(a^{(2)}))\cdots \\
& \cdots\mu^{P^{(k)}}_{\hat{\mathscr{X}}_{P^{(k)}}}(f^{(k)}(a^{(k)}))\mu^{Q^{(k)}}_{\hat{\mathscr{X}}_{P^{(k)}}}(g^{(k)}(b^{(k)}))
\end{aligned}
$$

for some $(X_{P^{(j)}}, \hat{\mathscr{X}}_{P^{(j)}})\subset(X, \hat{\mathscr{X}})$, $j=1, 2,\dots, k$, $(X_{Q^{(k)}}, \hat{\mathscr{X}}_{Q^{(k)}})\subset(X, \hat{\mathscr{X}})$ and all $a^{(j)}\in L^p_s(X_{P^{(j)}}, \hat{\mathscr{X}}_{P^{(j)}}, \mu_{\hat{\mathscr{X}}_{P^{(j)}}})_{alg}$, $j=1, 2,\dots, k$, $b^{(k)}\in L^p_s(X_{Q^{(k)}}, \hat{\mathscr{X}}_{Q^{(k)}}, \mu_{\hat{\mathscr{X}}_{Q^{(k)}}})$, $f^{(j)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_{P^{(j)}}}$, $j=1, 2,\dots, k$, $g^{(k)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_{Q^{(k)}}}$, with $Q^{(k)}=I-\sum_{j=1}^{k} P^{(j)}$. Since $P^{(k+1)}\in D^p(\mu, \hat{\mathscr{X}})$, by hypothesis, we also have, by Proposition (2.4), that

$$
\begin{aligned}
(2.12)\qquad & \mu((\Gamma_{\hat{\mathscr{X}}_{P^{(k+1)}}}(P^{(k+1)})f^{(k+1)}(a^{(k+1)}))\cdot(\Gamma_{\hat{\mathscr{X}}_{P^{(k+1)\perp}}}(P^{(k+1)\perp})g^{(k+1)}(b^{(k+1)}))) \\
& =\mu^{P^{(k+1)}}_{\hat{\mathscr{X}}_{P^{(k+1)}}}(f^{(k+1)}(a^{(k+1)}))\mu^{P^{(k+1)\perp}}_{\hat{\mathscr{X}}_{P^{(k+1)\perp}}}(g^{(k+1)}(b^{(k+1)})),
\end{aligned}
$$

for some $(X_{P^{(k+1)}}\hat{\mathscr{X}}_{P^{(k+1)}})\subset(X, \hat{\mathscr{X}})$, and $(X_{P^{(k+1)\perp}}, \hat{\mathscr{X}}_{P^{(k+1)\perp}})\subset(X, \hat{\mathscr{X}})$ and all $a^{(k+1)}\in L^p_s(X_{P^{(k+1)}}, \hat{\mathscr{X}}_{P^{(k+1)}}, \mu_{\hat{\mathscr{X}}_{P^{(k+1)}}})_{alg}$,

$b^{(k)}\in L^p_s(X_{P^{(k+1\perp}}, \hat{\mathscr{X}}_{P^{(k+1)\perp}}, \mu_{\hat{\mathscr{X}}_{P^{(k+1)\perp}}})_{alg}$, $f^{(k+1)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_{P^{(k+1)}}}$, $g^{(k+1)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_{P^{k+1)\perp}}}$. Let $\mathscr{Y}^{(k)}$ be the linear hull of the set $\{(\Gamma_{\hat{\mathscr{X}}_{P^{(k+1)}}}(P^{(k+1)})f^{(k+1)}(a^{(k+1)}))\cdot(\Gamma_{\hat{\mathscr{X}}_{P^{(k+1)\perp}}}(P^{(k+1)^{\perp}})g^{(k+1)}(b^{(k+1)})):$
$:f^{(k+1)}\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_{P^{(k+1)}}}$,
$g^{(k+1)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_{P^{(k+1)\perp}}}$, $a^{(k+1)}\in L^p_s(X_{P^{(k+1)}}, \hat{\mathscr{X}}_{P^{(k+1)}}, \mu_{\hat{\mathscr{X}}_{P^{(k+1)}}})_{alg}$, $b^{(k+1)}\in L^p_s(X_{P^{(k+1)\perp}}, \hat{\mathscr{X}}_{P^{(k+1)\perp}}, \mu_{\hat{\mathscr{X}}_{P^{(k+1)\perp}}})_{alg}$.

Since $P^{(j)}P^{(k)}=P^{(k)}P^{(j)}=\delta_{jk}P^{(k)}$, $j, k=1, 2,\dots, n$, equation (2.12) holds, in particular, for $a^{(k+1)}$ and $b^{(k+1)}$ replaced by $Q^{(k)}_{\hat{\mathscr{X}}_{P^{(k+1)}}}a^{(k+1)}$ and $Q^{(k)}_{\hat{\mathscr{X}}_{P^{(k+1)\perp}}}b^{(k+1)}$, where

$a^{(k+1)}\in L^p_s(X_{P^{(k+1)}}, \hat{\mathscr{X}}_{P^{(k+1)}}, \mu_{\hat{\mathscr{X}}_{P^{(k+1)}}})_{alg}$ and $b^{(k+1)}\in L^p_s(X_{P^{(k+1)\perp}}, \hat{\mathscr{X}}_{P^{(k+1)\perp}}, \mu_{\hat{\mathscr{X}}_{P^{(k+1)\perp}}})$ are arbitrary. Hence

$$(2.13)\quad \mu(\Gamma_{\hat{\mathscr{Y}}^{(k)}}(Q^{(k)})[(\Gamma_{\hat{\mathscr{X}}_{P^{(k+1)}}}(P^{(k+1)})f^{(k+1)}(a^{(k+1)})).$$
$$(\Gamma_{\mathscr{X}_{P^{(k+1)\perp}}}(P^{(k+1)\perp})g^{(k+1}(b^{k+1)}))])$$
$$=\mu((\Gamma_{\hat{\mathscr{X}}_{P^{(k+1)}}}(P^{(k+1)})f^{(k+1)}(a^{(k+1)})\cdot\Gamma(_{\mathscr{X}_{P^{(k+1)\perp}}}(Q^{(k+1)})g^{(k+1)}(b^{(k+1)}))),$$

from the right hand side of (2.12),

$=\mu^{P^{(k+1)}}_{\mathscr{X}_{P^{(k+1)}}}(f^{(k+1)}(a^{(k+1)}))\mu^{Q^{(k+1)}}_{\hat{\mathscr{X}}_{(P^{k+1})\perp}}(g^{(k+1)}(b^{(k+1)}))$, from the left hand side of (2.12), for all $a^{(k+1)}\in L^p_s(X_{P^{(k+1)}}, \hat{\mathscr{X}}_{P^{(k+1)}}, \mu_{\hat{\mathscr{X}}_{P^{(k+1)}}})_{alg}$,

$$b^{(k+1)}\in L^p_s(X_{P^{(k+1)\perp}}, \hat{\mathscr{X}}_{P^{(k+1)\perp}}, \mu_{\hat{\mathscr{X}}_{P^{(k+1)\perp}}}),\ f^{(k+1)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_{P^{(k+1)\perp}}}$$

and $g^{(k+1)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_{P^{(k+1)\perp}}}$ and with $Q^{(k+1)}=I-\sum_{j=1}^{k+1}P^{(j)}$. The right hand side of (2.13) shows that we may identify $\hat{\mathscr{X}}_{P^{(k+1)\perp}}$ with $\hat{\mathscr{X}}_{Q^{(k+1)}}$ and the left hand side of (2.13) shows that $\hat{\mathscr{Y}}^{(k)}\subset\hat{\mathscr{X}}_{Q^{(k)}}$. Finally, one has

$$\mu((\Gamma_{\hat{\mathscr{X}}_{P^{(1)}}}(P^{(1)}(a^{(1)}))\cdot(\Gamma_{\hat{\mathscr{X}}_{P^{(2)}}}(P^{(2)})f^{(2)}(a^{(2)}))\cdots$$
$$\cdots(\Gamma_{\hat{\mathscr{X}}_{P^{(k)}}}(P^{(k)})f^{(k)}(a^{(k)}))\cdot(\Gamma_{\hat{\mathscr{X}}_{P^{(k+1)}}}(P^{(k+1)})f^{(k+1)}(a^{(k+1)})).$$
$$\cdot(\Gamma_{\hat{\mathscr{X}}_{Q^{(k+1)}}}(Q^{k+1)})g^{(k+1)}(b^{(k+1)})))$$
$$=\mu((\Gamma_{\hat{\mathscr{X}}_{P^{(1)}}}(P^{(1)})f^{(1)}(a^{(1)}))\cdot(\Gamma_{\hat{\mathscr{X}}_{P^{(2)}}}(P^{(2)})f^{(2)}(a^{(2)}))\cdots$$
$$\cdots(\Gamma_{\hat{\mathscr{X}}_{P^{(k)}}}(P^{(k)})f^{(k)}(a^{(k)}))\cdot\Gamma_{\hat{\mathscr{X}}_{Q^{(k)}}}(Q^{(k)})[(\Gamma_{\hat{\mathscr{X}}_{P^{(k+1)}}}(P^{(k+1)})f^{(k+1)}(a^{(k+1)})).$$
$$\cdot(\Gamma_{\hat{\mathscr{X}}_{P^{(k+1)\perp}}}(P^{(k+1)\perp})g^{(k+1)}(b^{k+1)}))])$$
$$=\mu^{P^{(1)}}_{\hat{\mathscr{X}}_{P^{(1)}}}(f^{(1)}(a^{(1)}))\mu^{P^{(2)}}_{\hat{\mathscr{X}}_{P^{(2)}}}(f^{(2)}(a^{(2)}))\cdots\mu^{P^{(k)}}_{\hat{\mathscr{X}}_{P^{(k)}}}(f^{(k)}(a^{(k)}))\cdot$$
$$\cdot\mu^{Q^{(k)}}_{\hat{\mathscr{X}}_{Q^{(k)}}}((\Gamma_{\hat{\mathscr{X}}_{P^{(k+1)}}}(P^{(k+1)})f^{(k+1)}(a^{(k+1)})).$$
$$\cdot(\Gamma_{\hat{\mathscr{X}}_{P^{(k+1)\perp}}}(P^{(k+1)\perp})g^{(k+1)}(b^{(k+1)})$$

by (2.11),

$$=\mu^{P^{(1)}}_{\hat{\mathscr{X}}_{P^{(1)}}}(f^{(1)}(a^{(1)}))\mu^{P^{(2)}}_{\hat{\mathscr{X}}_{P^{(2)}}}(f^{(2)}(a^{(2)}))\cdots$$
$$\cdots\mu^{P^{(k)}}_{\hat{\mathscr{X}}_{P^{(k)}}}(f^{(k)}(a^{(k)}))\mu^{P^{(k+1)}}_{\hat{\mathscr{X}}_{P^{(k+1)}}}(f^{(k+1)}(a^{(k+1)}))\cdot\mu^{Q^{(k+1)}}_{\hat{\mathscr{X}}_{Q^{(k+1)}}}(g^{(k+1)}(b^{(k+1)})),$$

by (2.13). This concludes the proof. □

Remark:

We can now prove the following result. In doing so, we employ the notation of Proposition (2.9).

(2.14) Proposition:

Let $\mu\in G_1(\hat{\mathscr{X}})$. *Let* $\{P^{(1)}, P^{(2)},\dots, P^{(n)}\}$ *be a set of commuting projection operators contained in* $D^p(\mu, \hat{\mathscr{X}})$ *and satisfying* $P^{(j)}P^{(k)}=\delta_{jk}P^{(k)}$, $j, k=1, 2,\dots, n$. *Suppose, furthermore, that* $\{A^{(1)}, A^{(2)},\dots, A^{(n)}\}$ *is a set of members of* $D^p(\mu, \hat{\mathscr{X}})$ *satisfying* $A^{(j)}P^{(j)}=P^{(j)}A^{(j)}$, $j=1, 2,\dots, n$. *Put* $\sum_{j=1}^{n} P^{(j)}A^{(j)}\equiv B$. *Then B also lies in* $D^p(\mu, \hat{\mathscr{X}})$.

Proof: Since the set $\{A^{(1)}, A^{(2)},\dots, A^{(n)}\}$ is contained in $D^p(\mu, \hat{\mathscr{X}})$, there are $(X_{A^{(j)}}, \hat{\mathscr{X}}_{A^{(j)}})\subset(X, \hat{\mathscr{X}})$, $(Y_{A^{(j)}}, \hat{\mathscr{Y}}_{A^{(j)}})\subset(X, \hat{\mathscr{X}})$ and $\nu_{A^{(j)}}\in G_1(\hat{\mathscr{Y}}_{A^{(j)}})$, $j=1, 2,\dots, n$, such that

$$\begin{aligned}&\mu((\Gamma_{\hat{\mathscr{X}}_{A^{(j)}}}(A^{(j)})u^{(j)}(x^{(j)}))\cdot(v^{(j)}(y^{(j)})))\\ &\quad=\mu^{A^{(j)}}_{\hat{\mathscr{X}}_{A^{(j)}}}(u^{(j)}(x^{(j)}))\nu_{A^{(j)}}(v^{(j)}(y^{(j)})),\end{aligned}$$

for all $x^{(j)}\in L^p_s(X_{A^{(j)}}, \hat{\mathscr{X}}_{A^{(j)}}, \mu_{\hat{\mathscr{X}}_{A^{(j)}}})_{alg}$, $y^{(j)}\in L^p_s(Y_{A^{(j)}}, \hat{\mathscr{Y}}_{A^{(j)}}, \nu_{A^{(j)}})_{alg}$, $u^{(j)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_{A^{(j)}}}$, $v^{(j)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{Y}}_{A^{(j)}}}$, $j=1, 2,\dots, n$.

Hence, in view of the assumed commutatity of $A^{(j)}$ and $P^{(j)}$, and the idempotency of $P^{(j)}$, $j=1, 2,\dots, n$, the last equation remains valid for $x^{(j)}$ and $y^{(j)}$ replaced by $P^{(j)}_{\hat{\mathscr{X}}_{A^{(j)}}}x^{(j)}$ and $P^{(j)}_{\hat{\mathscr{Y}}_{A^{(j)}}}y^{(j)}$, respcetively, $j=1, 2,\dots, n$. Then we get

$$\begin{aligned}&\mu((\Gamma_{\hat{\mathscr{X}}_{A^{(j)}}}(P^{(j)}A^{(j)})u^{(j)}(x^{(j)}))\cdot(\Gamma_{\hat{\mathscr{Y}}_{A^{(j)}}}(P^{(j)})v^{(j)}(y^{(j)})))\\ &\quad=\mu^{P^{(j)}A^{(j)}}_{\hat{\mathscr{X}}_{A^{(j)}}}(u^{(j)}(x^{(j)}))\nu^{P^{(j)}}_{A^{(j)}}(v^{(j)}(y^{(j)})),\end{aligned}$$

or, equivalently,

$$\begin{aligned}(2.15)\qquad &\mu(\Gamma_{\hat{\mathscr{Z}}^{(j)}}(P^{(j)})[(\Gamma_{\hat{\mathscr{X}}_{A^{(j)}}}(B)u^{(j)}(x^{(j)}))\cdot v^{(j)}(y^{(j)})])\\ &\quad=\mu^{P^{(j)}A^{(j)}}_{\hat{\mathscr{X}}_{A^{(j)}}}(u^{(j)}(x^{(j)}))\nu^{P^{(j)}}_{A^{(j)}}(v^{(j)}(y^{(j)})),\end{aligned}$$

for all $x^{(j)}\in L^p_s(X_{A^{(j)}}, \hat{\mathscr{X}}_{A^{(j)}}, \mu_{\hat{\mathscr{X}}_{A^{(j)}}})_{alg}$, $y^{(j)}\in L^p_s(Y_{A^{(j)}},\hat{\mathscr{Y}}_{A^{(j)}}, \nu_{A^{(j)}})_{alg}$, $u^{(j)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_{A^{(j)}}}$, $v^{(j)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{Y}}_{A^{(j)}}}$, $j=1, 2,\dots, n$, where $\hat{\mathscr{Z}}^{(j)}$ is the linear hull of all elements of the form $(\Gamma_{\hat{\mathscr{X}}_{A^{(j)}}}(B)u^{(j)}(x^{(j)}))\cdot v^{(j)}(y^{(j)})$. It is clear that $\hat{\mathscr{Z}}^{(j)}$ is contained in $\hat{\mathscr{X}}_{P^{(j)}}$, $j=1,2,\dots, n$. Furthermore, since $P^{(j)}A^{(j)}=A^{(j)}P^{(j)}$, we have $\mu^{P^{(j)}A^{(j)}}_{\hat{\mathscr{X}}_{A^{(j)}}}(u^{(j)}(x^{(j)}))=\mu^{P^{(j)}A^{(j)}}_{\hat{\mathscr{X}}_{P^{(j)}}}(u^{(j)}(x^{(j)}))$, for $x^{(j)}\in L^p_s(X_{A^{(j)}}, \hat{\mathscr{X}}_{A^{(j)}}, \mu_{\hat{\mathscr{X}}_{A^{(j)}}})_{alg}$ and $u^{(j)}\in\mathscr{D}(\mathbb{R})_{\mathscr{X}_{A^{(j)}}}$, $j=1, 2,\dots, n$. Hence, replacing $f^{(j)}(a^{(j)})$ in (2.10) by $(\Gamma_{\hat{\mathscr{X}}_{A^{(j)}}}(B)u^{(j)}(x^{(j)}))\cdot v^{(j)}(y^{(j)})$, and taking account of the foregoing remarks, one gets

$$\begin{aligned}(2.16)\quad &\mu([\Gamma_{\hat{\mathscr{X}}_{P^{(1)}}}(P^{(1)})((\Gamma_{\hat{\mathscr{X}}_{A^{(1)}}}(B)u^{(1)}(x^{(1)}))\cdot v^{(1)}(y^{(1)}))]\cdots\\ &\quad\cdots[\Gamma_{\hat{\mathscr{X}}_{P^{(n)}}}(P^{(n)}((\Gamma_{\hat{\mathscr{X}}_{A^{(n)}}}(B)u^{(n)}(x^{(n)}))\cdot v^{(n)}(y^{(n)}))]\cdot\\ &\qquad\cdot(\Gamma_{\hat{\mathscr{X}}_{Q^{(n)}}}(Q^{(n)})g^{(n)}(b^{(n)})))\end{aligned}$$

$$=\mu^{P^{(1)}}_{\hat{\mathscr{X}}_{P^{(1)}}}((\Gamma_{\hat{\mathscr{X}}_{A^{(1)}}}(B)u^{(1)}(x^{(1)}))\cdot v^{(1)}(y^{(1)}))\cdots\mu^{P^{(n)}}_{\hat{\mathscr{X}}_{P^{(n)}}}((\Gamma_{\hat{\mathscr{X}}_{A^{(n)}}}(B)u^{(n)}(x^{(n)}))\cdot \cdot v^{(n)}(y^{(n)}))\cdot\mu^{Q^{(n)}}_{\hat{\mathscr{X}}_{Q^{(n)}}}(g^{(n)}(b^{(n)}))$$

$$=\mu^{P^{(1)}\ A^{(1)}}_{\hat{\mathscr{X}}_{P^{(1)}}}(\mu^{(1)}(x^{(1)}))\cdots\mu^{P^{(n)}\ A^{(n)}}_{\hat{\mathscr{X}}_{P^{(n)}}}(u^{(n)}(x^{(n)}))\nu^{P^{(1)}}_{A^{(1)}}(v^{(1)}(y^{(1)}))\cdots \nu^{P^{(n)}}_{A^{(n)}}(v^{(n)}(y^{(n)}))\mu^{Q^{(n)}}_{\hat{\mathscr{X}}_{Q^{(n)}}}(g^{(n)}(b^{(n)})),\ by\ (2.15),\text{ for all}$$

$$x^{(j)}\in L^p_s(X_{A^{(j)}},\hat{\mathscr{X}}_{A^{(j)}},\mu_{\hat{\mathscr{X}}_{A^{(j)}}})_{alg},$$
$$y^{(j)}\in L^p_s(Y_{A^{(j)}},\hat{\mathscr{Y}}_{A^{(j)}},\nu_{A^{(j)}})_{alg},\ u^{(j)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_{A^{(j)}}},\ v^{(j)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{Y}}_{A^{(j)}}},$$
$$g^{(n)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_{Q^{(n)}}},\ j=1,2,\dots,n,\text{ and } b^{(n)}\in L^p_s(X_{Q^{(n)}},\hat{\mathscr{X}}_{Q^{(n)}},\mu_{\hat{\mathscr{X}}_{Q^{(n)}}})_{alg}.$$

Let $\hat{\mathscr{X}}_B$ be the linear hull of all elements of the form $(\Gamma_{\hat{\mathscr{X}}_{P^{(1)}}}(P^{(1)})u^{(1)}(x^{(1)}))\cdots(\Gamma_{\hat{\mathscr{X}}_{P^{(n)}}}(P^{(n)})u^{(n)}(x^{(n)}))$, with $x^{(j)}\in L^p_s(X_{A^{(j)}},\hat{\mathscr{X}}_{A^{(j)}},\mu_{\hat{\mathscr{X}}_{A^{(j)}}})_{alg}$, $u^{(j)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_{A^{(j)}}}$, $j=1,2,\dots,n$. Then arguing as above, one readily shows that

$$\text{(2.17)}\quad \mu(\Gamma_{\hat{\mathscr{X}}_B}(B)[(\Gamma_{\hat{\mathscr{X}}_{P^{(1)}}}(P^{(1)})u^{(1)}(x^{(1)}))\cdots(\Gamma_{\hat{\mathscr{X}}_{P^{(n)}}}(P^{(n)})u^{(n)}(x^{(n)}))])=\mu^{P^{(1)}A^{(1)}}_{\hat{\mathscr{X}}_{P^{(1)}}}(u^{(1)}(x^{(1)}))\cdots\mu^{P^{(n)}A^{(n)}}_{\hat{\mathscr{X}}_{P^{(n)}}}(u^{(n)}(x^{(n)}))$$

for all $x^{(j)}\in L^p_s(X_{A^{(j)}},\hat{\mathscr{X}}_{A^{(j)}},\mu_{\hat{\mathscr{X}}_{A^{(j)}}})_{alg}$, $u^{(j)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_{A^{(j)}}}$, $j=1,2,\dots,n$.

Using (2.17) in (2.16), one sees that the right hand side of (2.16) may be written as follows:

(2.18) right hand side of (2.16)

$$=\mu^B_{\hat{\mathscr{X}}_B}((\Gamma_{\hat{\mathscr{X}}_{P^{(1)}}}(P^{(1)})u^{(1)}(x^{(1)}))\cdots(\Gamma_{\hat{\mathscr{X}}_{P^{(n)}}}(P^{(n)})u^{(n)}(x^{(n)})))\nu^{P^{(1)}}_{A^{(1)}}(v^{(1)}(y^{(1)}))\cdots \cdots\nu^{P^{(n)}}_{A^{(n)}}(v^{(n)}(y^{(n)}))\mu^{Q^{(n)}}_{\hat{\mathscr{X}}_{Q^{(n)}}}(g^{(n)}(b^{(n)}))$$

This shows that the elements

$$[(\Gamma_{\hat{\mathscr{X}}_{P^{(1)}}}(P^{(1)})u^{(1)}(x^{(1)}))\cdots(\Gamma_{\hat{\mathscr{X}}_{P^{(n)}}}(P^{(n)})u^{(n)}(x^{(n)}))];\ v^{(1)}(y^{(1)});\cdots;\ v^{(n)}(y^{(n)});\ ;\ g^{(n)}(b^{(n)})$$

are stochastically independent [35] for all

$$x^{(j)}\in L^p_s(X_{A^{(j)}},\hat{\mathscr{X}}_{A^{(j)}},\mu_{\hat{\mathscr{X}}_{A^{(j)}}})_{alg},$$
$$y^{(j)}\in L^p_s(Y_{A^{(j)}},\hat{\mathscr{Y}}_{A^{(j)}},\nu_{A^{(j)}})_{alg},\ b^{(n)}\in L^p_s(X_{Q^{(n)}},\hat{\mathscr{X}}_{Q^{(n)}},\mu_{\hat{\mathscr{X}}_{Q^{(n)}}})_{alg},$$
$$u^{(j)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_{A^{(j)}}},\ v^{(j)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{Y}}_{A^{(j)}}},\ g^{(n)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_{Q^{(n)}}},\ j=1,2,\dots,n.$$

Hence, the left hand side of (2.16) may be expressed thus:

(2.19) left hand side of (2.16)

$$=\mu((\Gamma_{\hat{\mathscr{X}}_B}(B)[(\Gamma_{\hat{\mathscr{X}}_{P^{(1)}}}(P^{(1)})u^{(1)}(x^{(1)}))\cdots(\Gamma_{\hat{\mathscr{X}}_{P^{(n)}}}(P^{(n)})u^{(n)}(x^{(n)}))]) (\Gamma_{\hat{\mathscr{Y}}_{A^{(1)}}}(P^{(1)})v^{(1)}(y^{(1)}))\cdots(\Gamma_{\hat{\mathscr{Y}}_{A^{(n)}}}(P^{(n)})v^{(n)}(y^{(n)})\cdot(\Gamma_{\hat{\mathscr{X}}_{Q^{(n)}}}(Q^{(n)})g^{(n)}(b^{(n)})))$$

From (2.18) and (2.19), we conclude that B indeed lies in $D^p(\mu,\hat{\mathscr{X}})$. □

§ 3. Statement of the Problem

In this section, we describe the problem which we tackle in the rest of the paper.

If a_1 is a self-adjoint operator on X_1 affiliated to $\mathscr{X}_1$, we write spec(a_1) for the *spectrum of* a_1 and $a_1=\int_{\mathrm{spec}(a_1)} e_{a_1}(d\lambda)\lambda$ for its *spectral representation.*

We require the following notion.

(3.1) Definition: A triangular array $(\mu_{nj})_{1\leq j\leq k_n, n\geq 1}$ of members of $G_1(\hat{\mathscr{X}}_1)$ will be called *uniformly infinitesimal* with respect to some subset $\{a_{nj}: j=1, 2,\dots, k_n;\ n=1, 2,\dots\}$ of self-adjoint operators on X_1 affiliated to $\mathscr{X}_1$ if, and only if,

(3.2) $$\lim_{n\to\infty}\ \sup_{1\leq j\leq k_n} \mu_{nj}(e_{a_{nj}}(\Lambda'))=0$$

for every neighbourhoud Λ of zero in $\mathbb{R}$ with complement Λ'.

(3.3) *Remark*: Condition (3.2) is easily seen to be equivalent to the following requirement

$$\lim_{n\to\infty}\ \sup_{1\leq j\leq k_n} |\mu_{nj}(e^{i\sigma a_{nj}})-1|=0$$

for each σ contained in a compact subset of $\mathbb{R}$. Hence, a triangular array $(\mu_{nj})_{1\leq j\leq k_n, n\geq 1}\subset G_1(\hat{\mathscr{X}}_1)$ is uniformly infinitesimal with respect to $\{a_{nj}: j=1, 2,\dots, k_n;\ n=1, 2,\dots\}$ if, and only if, the measures $\Lambda\mapsto\mu_{nj_n}(e_{a_{nj_n}}(\Lambda))$ converge weakly to the Dirac measure concentrated at the origin, as $n\to\infty$, for each choice of j_n, with $1\leq j_n\leq k_n$.

(3.4) Notation: We employ the following notation in the sequel.

1. Let $\mu\in G_1(\hat{\mathscr{X}})$, with $\mu=\bigoplus_{n=0}^{\infty}\mu_n$, and $a_m\in L^p(X_m, \hat{\mathscr{X}}_m, \mu_m)$, $m=0, 1, 2,\dots$. Then, we define the imbeddings $\underline{i}_m$ and $\underline{i}_m$ of $L^p(X_m, \hat{\mathscr{X}}_m, \mu_m)$ and $G(\hat{\mathscr{X}}_m)$ in $L^p(X, \hat{\mathscr{X}}, \mu)$ and $G(\hat{\mathscr{X}})$, respectively, by

$$\underline{i}_m(a_m)=\bigoplus_{n=0}^{\infty}\delta_{mn}a_n;\quad\text{and}$$

$$\underline{i}_m(\mu_m)=\bigoplus_{n=0}^{\infty}\delta_{mn}\mu_n,\quad m=0, 1, 2,\dots.$$

2. For $\mu\in G_1(\hat{\mathscr{X}})$ and $\mathscr{B}\subset B(L^p(X, \hat{\mathscr{X}}, \mu))$, we write Sem$(\mathscr{B})$ for the norm-closed, multiplicative subsemigroup of $B(L^p(X, \hat{\mathscr{X}}, \mu))$ generated by $\mathscr{B}$.

3. Let $\mu \in G_1(\hat{\mathscr{X}})$, with $\mu = \bigoplus_{n=0}^{\infty} \mu_n$. Suppose that $A_1 \in B(L^p(X, \hat{\mathscr{X}}, \mu \circ \underline{i}_1))$, is such that $A_1 \otimes A_1 \otimes \cdots \otimes A_1$ (j-fold) lies in $B(L^p(X_j, \hat{\mathscr{X}}_j, \mu_j))$, $j = 1, 2, \ldots$. Then, we denote $A_1 \otimes A_1 \otimes \cdots \otimes A_1$ (j-fold) by $[A_1]_j$ and put $\bigoplus_{j=0}^{\infty} [A_1]_j \equiv [A_1]$. Here, $[A_1]_0$ is the identity operator on $\mathbb{C}1_{\mathscr{X}_1}$. Notice that $[A_1][B_1] = [A_1 B_1]$.

(3.5) **The Problem:** We describe next the problem which we address in the rest of the paper.

Let $\mu^0 \in G_1(\hat{\mathscr{X}})$, with $\mu^0 = \bigoplus_{n=0}^{\infty} \mu_n^0$ and $\sup_{n \geq 0} [\mu_n^0(1_{\hat{\mathscr{X}}_n})]^{1/p} < \infty$, and $\{A_{1n}\}_{n \geq 1} \subset B(L^p(X_1, \hat{\mathscr{X}}_1, \mu^0 \circ \underline{i}_1))_+$, with $[A_{1j}]$ lying in $B(L^p(X, \hat{\mathscr{X}}, \mu^0))_+$, $j = 1, 2, \ldots$. Let $\{y_{1j}\}_{j \geq 1}, \{x_{1j}\}_{j \geq 1} \subset L_s^p(X_1, \hat{\mathscr{X}}_1, \mu^0 \circ \underline{i}_1)$ be such that

$$\mu^0(\underline{i}_{n+1}(\Gamma_1(A_{1n}) f_1^{(1)}(x_{11}) \otimes \Gamma_1(A_{1n}) f_1^{(2)}(x_{12}) \otimes \cdots \otimes \Gamma_1(A_{1n}) f_1^{(n)}(x_{1n}) \otimes \otimes g_1^{(n)}(y_{1n})))$$

$$= \varepsilon_{\alpha_n}(g_1^{(n)}(y_{1n})) \prod_{j=1}^{n} \mu_{1j}^{A_{1n}}(f_1^{(j)}(x_{ij}))$$

for some $\{\mu_{1j}\}_{j \geq 1} \subset G_1(\hat{\mathscr{X}}_1)$, with $A_{1n} \in D_0^p(\mu_{1j}, \hat{\mathscr{X}}_1)$, $j = 1, 2, \ldots, n$, $n = 1, 2, \ldots$, and $\{\alpha_j\} \subset \mathbb{R}(X_1)$, and for all $(f_1^{(j)}, g_1^{(j)}) \in \mathscr{D}(\mathbb{R})_1 \times \mathscr{D}_c(\mathbb{R})_1$, $j = 1, 2, \ldots$. This means that the set $\{x_{1j}, y_{1j}: j = 1, 2, \ldots\}$ consists of stochastically independent members relative to the gage μ^0.

Denote $\underline{i}_{n+1}(y_{1n} \otimes (\bigotimes_{j=1}^{n} x_{ij}))$ and $\underline{i}_{n+1}(\varepsilon_{\alpha_n} \otimes (\bigotimes_{j=1}^{n} \mu_{1j}^{A_{1n}}))$ by $x^{(n)}$ and $\mu^{(n)}$, respectively.

We make the following assumptions.

(3.5.1) A_{1j} is invertible for each $j = 1, 2, \ldots$;

(3.5.2) Sem $(\{[A_{1n}]^{-1}[A_{1m}]: n = 1, 2, \ldots, m; m = 1, 2, \ldots\})$ is compact in the norm topology of $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$;

(3.5.3) the gages $\{\mu_{1j}^{A_{1n}}: j = 1, 2, \ldots, n; n = 1, 2, \ldots\}$ form a uniformly infinitesimal triangular array with respect to $\{x_{1j}: j = 1, 2, \ldots\}$, and there is $(\mu, x) \in G_1^{(p)}(\hat{\mathscr{X}})$ such that the sequence $\{(\mu^{(n)}, x^{(n)})\}_{n \geq 1}$ of pairs in $G_1^{(p)}(\hat{\mathscr{X}})$ converges to the pair (μ, x).

In the sequel, we answer the question: *What are the characteristics of the limit pair* (μ, x)?

Notation: In the sequel, we denote the set of all limit pairs (μ, x), which arise as described in (3.5.3), by $\lim (G_1^{(p)}(\hat{\mathscr{X}}))$.

(3.6) **Definition:** A sequence of operators $\{A_{1j}: j = 1, 2, \ldots\}$ satisfying

(3.5.1) and (3.5.2), and such that (3.5.3) holds, will be called a *norming sequence* corresponding to the limit pair $(\mu, x) \in \lim (G_1^{(p)}(\hat{\mathscr{X}}))$.

(3.7) **Definition**: We say that $(\mu, x) \in \lim (G_1^{(p)}(\hat{\mathscr{X}}))$ is *nondegenerate* provided that μ is nondegenerate and x is not of the form $x = \hat{\bigoplus}_{n \in N_0} \bigotimes_{j=1}^{n} \lambda_{nj} 1_{\hat{\mathscr{X}}_1}$, $\lambda_{nj} \in \boldsymbol{R}$, $j=1, 2,\dots, n$; $n=1, 2,\dots$.

Remarks:

(i) In the sequel, we deal mainly with the nondegenerate members of $\lim (G_1^{(p)} \cdot (\hat{\mathscr{X}}))$, $1 \leq p \leq \infty$.

(ii) An analogue of the problem considered here was solved by Levy in the case of real-valued random variables. He showed that the limit probability measures are the so-called self-decomposable probability measures ([2], p. 195; [36], p. 319). In Ref. [5], Urbanik considers the case of Banach-space-valued random variables and furnishes a characterization of the limit probability measures by means of a certain semigroup of linear operators associated with such probability measures. We also mention the work of N. V. Thu [37] which further generalizes the considerations in Ref. [5].

In this paper, we work within the framework of noncommutative probability theory. Consequently, our random variables are noncommuting self-adjoint operators. In providing an answer to the problem posed in this section, we exploit some of the techniques introduced by Urbanik [5].

The problem considered in this paper is a *Central Limit Problem* [36] in a noncommutative setting. Such a problem (in a noncommutative setting) had previously been considered only in very specialized situations [6–10]. Here, we provide a fairly general formulation and solution. Finally, we refer to Refs. [18–20] which deal with the problem of infinite divisibility.

(iii) The gage $\mu^0 \in G_1(\hat{\mathscr{X}})$ which occurs in (3.5) is called the *common gage* of the set $\{x_{1j}, y_{1j} : j=1, 2,\dots\} \subset L_s^p(X_1, \hat{\mathscr{X}}_1, \mu^0 \circ i_1)$. This gage features repeatedly in the rest of the paper.

§4. Some Properties of Norming Sequences

In this section, we describe some properties associated with the *norming* *sequences* corresponding to the nondegenerate members of $\lim(G_1^{(p)}(\hat{\mathscr{X}}))$.

We employ the notation and assumptions introduced in (3.5), unless we explicitly state otherwise.

(4.1) **Proposition**: *Suppose that* $(\mu, x)\in\lim(G_1^{(p)}(\hat{\mathscr{X}}))$ *is nondegenerate. Let* $\{A_{1j}\}_{j\geq 1}$ *be a norming sequence corresponding to* (μ, x). *Then* $\{[A_{1j}]\}_{j\geq 1}$ *converges to zero in* $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$.

Proof: By (3.5.3), $\mu^{(n)}(e^{i(\sigma)\cdot x^{(n)}})$ converges to $\mu(e^{i(\sigma)\cdot x})$, for each $\sigma\in\mathbb{R}$. Furthermore, it suffices to assume that the assumptions of Proposition (1.6) are fulfilled and that $\{x^{(n)}\}_{n\geq 1}$ converges to x in $L^p(X, \hat{\mathscr{X}}, \mu^0)$, $\{\mu^{(n)}\}_{n\geq 1}$ converges to μ in $L^p(X, \hat{\mathscr{X}}), \mu^0)^*$ and $\{[A_{1n}]\}_{n\geq 1}$ converges to A in $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$. We show that $A=$zero.

Let $n\leq n_k$. Then

$$\begin{aligned}&\mu^{(n_k)}\ (e^{i(\sigma)\cdot x^{(n_k)}})\\&=(\varepsilon_{\alpha_{n_k}}\otimes(\otimes\mu_{1j}^{A_{1n_k}}))(e^{i\sigma y_{1n_k}}\otimes(\bigotimes_{j=1}^{n_k}e^{i\sigma x_{1j}}))\\&=[((\underline{i}_n(\bigotimes_{j=1}^{n}\mu_{1j}))\circ\Gamma([A_{1n_k}]))(\underline{i}_n(\bigotimes_{j=1}^{n}e^{i\sigma x_{1j}}))]\times\\&\quad\times[((\underline{i}_{n_k-n}(\varepsilon_{\alpha_{n_k}}\otimes(\bigotimes_{j=n+1}^{n_k}\mu_{1j})))\circ\Gamma([A_{1n_k}]))(\underline{i}_{n_k-n}(e^{i\sigma y_{1n_k}}\otimes(\bigotimes_{j=n+1}^{n_k}e^{i\sigma x_{1j}})))].\end{aligned}$$

Taking limits of both sides of the last equation as $n_k\to\infty$, we get

$$\text{(4.1.1)}\qquad \mu(e^{i(\sigma)\cdot x})=[((\underline{i}_n(\bigotimes_{j=1}^{n}\mu_{1j}))\circ\Gamma(A))(\underline{i}_n(\bigotimes_{j=1}^{n}e^{i\sigma x_{1j}}))]\times\mu(e^{i(\sigma)\cdot x})$$

Next, by condition (3.52), Sem $(\{[A_{1n_k}]^{-1}A: k=1, 2,\dots\})$ is compact in the norm-topology of $B(L^p\ (X, \hat{\mathscr{X}}, \mu^0))$. Let B be an accumulation point of the sequence $\{[A_{1n_k}]^{-1}A\}_{k\geq 1}$. Passing, if necessary, to a subsequence, we may assume without loss of generality that $[A_{1n_k}]^{-1}A$ converges to B, as $n_k\to\infty$. Hence

$$\text{(4.1.2)}\qquad A=AB$$

From (4.1.1), we get

$$\begin{aligned}&\mu(e^{i(\sigma)\cdot x})\\&=(\mu^{(n)}\circ\Gamma([A_{1n}]^{-1}A))(e^{i(\sigma)\cdot x^{(n)}})\otimes[e^{-\sigma\hat{\alpha}_n(B_{1n}y_{1n})}]\times\mu(e^{i(\sigma)\cdot x})\end{aligned}$$

where $B_{1n}=A_{1n}^{-1}A\circ\underline{i}_{n+1n+1}$.

Taking (3.5.2) and Proposition (1.6) into account and going to the limit as $n\to\infty$ of both sides of the last equation, we get

$$\mu(e^{i(\sigma)\cdot x})=(\mu\circ\Gamma(B))(e^{i(\sigma)\cdot x})\mu(e^{i(\sigma)\cdot x})e^{-i\sigma\gamma_{B,x}},$$

for some $\gamma_{B,x}\in\mathbb{R}$, since $\{e^{-i\sigma\hat{\alpha}_n(B_{1n}y_{1n})}\}_{n\geq 1}$ is precompact in $\mathbb{C}$. Hence

$$(\mu\circ\Gamma(B))(e^{i(\sigma)\cdot x})=e^{i\sigma\gamma_{B,x}}.$$

But $(\mu, x)\in\lim(G_1^{(p)}(\hat{\mathscr{X}}))$ was nondegenerate, by hypothesis. Hence, $\gamma_{B,x}=0$ and $B=0$. By (4.1.2), these give $A=0$. Hence, the claim is established. □

(4.2) Proposition: *Let $n_k\leq m_k$, $k=1, 2,\ldots$ and $n_k\to\infty$. Suppose that $(\mu, x)\in\lim(G_1^{(p)}(\hat{\mathscr{X}}))$ is nondegenerate. Then, for every norming sequence $\{A_{1n}\}_{n\geq 1}$ corresponding to (μ, x), all accumulation points of the sequence $\{[A_{1n_k}]^{-1}[A_{1m_k}]\}_{k\geq 1}$ belong to $D^p(\mu, \hat{\mathscr{X}})$, $1\leq p\leq\infty$.*

Proof: By hypothesis, $\mu^{(n)}(e^{i(\sigma)\cdot x^{(n)}})$ converges to $\mu(e^{i(\sigma)\cdot x})$, for each $\sigma\in\mathbb{R}$. Now,

$$\begin{aligned}&\mu^{(m_k)}(e^{i(\sigma)\cdot x^{(m_k)}})\\&=[(\mu^{(n_k)}\circ\Gamma([A_{1n_k}^{-1}A_{1m_k}]))(e^{i(\sigma)\cdot x^{(n_k)}})]\times\\&\times[\prod_{j=n_k+1}^{m_k}\mu_{1j}^{A_{1m_k}}(e^{i\sigma x_{1j}})]\times[e^{i\sigma(\hat{\alpha}_{m_k}(y_{1m_k})-\hat{\alpha}_{n_k}(A_{1n_k}^{-1}A_{1m_k}y_{1n_k}))}]\end{aligned}$$

where we have used $[A_{1n_k}]^{-1}[A_{1m_k}]=[A_{1n_k}^{-1}A_{1m_k}]$.

Let A be an accumulation point of $\{[A_{1n_k}^{-1}A_{1m_k}]\}_{k\geq 1}$. Then, by (3.5.2), some subsequence of $\{[A_{1n_k}^{-1}A_{1m_k}]\}_{k\geq 1}$, denoted again by $\{[A_{1n_k}^{-1}A_{1m_k}]\}_{k\geq 1}$, converges uniformly to A in $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$, $1\leq p\leq\infty$. Hence, taking limits of both sides of the last equation as $n_k\to\infty$ and invoking Proposition (1.6), we get

$$\mu(e^{i(\sigma)\cdot x})=\mu^A(e^{i(\sigma)\cdot x^{(1)}})\nu_A(e^{i(\sigma)\cdot x^{(2)}}) \tag{4.2.1}$$

for some $x^{(1)}, x^{(2)}\in L_s^p(X, \hat{\mathscr{X}}, \mu^0)_{alg}$ and $\nu_A\in G_1(\hat{\mathscr{X}})$.

Let $(X_A^{(j)}, \hat{\mathscr{X}}_A^{(j)})\subset(X, \hat{\mathscr{X}})$, where $\hat{\mathscr{X}}_A^{(j)}$ is the algebra contained in $\mathscr{X}$ generated by $\{g^{(j)}(x^{(j)})\in\mathscr{D}(\mathbb{R})\}$, $j=1, 2$. Then, using the spectral theorem [34], (4.2.1) may be expressed as follows:

$$\mu((\Gamma_{\hat{\mathscr{X}}_A^{(1)}}(A)g^{(1)}(x^{(1)}))\cdot g^{(2)}(x^{(2)}))=\mu^A_{\hat{\mathscr{X}}_A^{(1)}}(g^{(1)}(x^{(1)}))\nu_A(g^{(2)}(x^{(2)})),$$

for all $g^{(j)}\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_A^{(j)}}$, $j=1, 2$ and $\nu_A\in G_1(\hat{\mathscr{X}}_A^{(2)})$. Hence A lies in $D^p(\mu, \mathscr{X})$, as claimed. □

(4.3) Notation: Henceforth, we denote the set of all accumulation points in $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$ of the sequence $\{[A_{1n}^{-1}A_{1m}]: n=1, 2,\ldots, m;\ m=1, 2,\ldots\}$ by $\mathbb{F}$. If $(\mu, x)\in\lim(G_1^{(p)}(\hat{\mathscr{X}}))$ is nondegenerate, then by Proposition (4.2), $\mathbb{F}\subset D^p(\mu, \hat{\mathscr{X}})$.

(4.4) The semigroup $D_\varepsilon^p(\mu, \hat{\mathscr{X}})$, $\mu \in G_1(\hat{\mathscr{X}})$, $1 \leq p \leq \infty$:

Let $\mu \in G_1(\hat{\mathscr{X}})$. Write $D_{com}^p(\mu, \hat{\mathscr{X}})$ for the maximal set of mutually commuting members of $D^p(\mu, \hat{\mathscr{X}})$, $1 \leq p \leq \infty$. Then, in the sequel, we denote by $D_\varepsilon^p$$(\mu, \hat{\mathscr{X}})$ the subset of $D_{com}^p(\mu, \hat{\mathscr{X}})$, $1 \leq p \leq \infty$, consisting of all A with the property that the factor ν_A of μ with respect to A (using the notation in (2.1) (2)) satisfies $\nu_A \circ \underline{i}_1 = \varepsilon_{\alpha_A}$ for some $\alpha_A \in \mathbb{R}(X_1)$. By Proposition (2.3), $D_\varepsilon^p(\mu, \hat{\mathscr{X}})$ is a semigroup which is closed in the norm-topology of $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$, and the identity I of $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$ lies in $D^p(\mu, \hat{\mathscr{X}})$.

(4.5) Proposition: *For each norming sequence $\{A_{1j}\}_{j\geq 1}$ corresponding to some nondegenerate $(\mu, x) \in \lim(G_1^{(p)}(\hat{\mathscr{X}}))$, the set*

$$D^p(\mu, \hat{\mathscr{X}}) \cap \mathrm{Sem}(\mathbb{F}), \quad 1 \leq p \leq \infty, \tag{4.6}$$

is a compact group containing all the accumulation points of the sequence $\{[A_{1n}^{-1}A_{1n+1}]\}_{n\geq 1}$.

Proof: The set in (4.6) is evidently compact, being a closed subset of a compact subset $\mathrm{Sem}(\mathbb{F})$ of the normed space $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$.

Let A be an accumulation point of $\{[A_{1n}^{-1}A_{1n+1}]\}_{n\geq 1}$. Without loss of generality, assume that $[A_{1n}^{-1}A_{1n+1}]$ converges to A in $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$. Let $(\mu, x) \in \lim(G_1^{(p)}(\hat{\mathscr{X}}))$. Then $\mu^{(n)}(e^{i(\sigma)\cdot x^{(n)}})$ converges to $\mu(e^{i(\sigma)\cdot x})$, $\sigma \in \mathbb{R}$. Now,

$$\begin{aligned}
&\mu^{(n+1)}\,(e^{i(\sigma)x^{(n+1)}})\\
&= e^{i\sigma\hat{\alpha}_{n+1}(y_{1n+1})}\prod_{j=1}^{n+1}\mu_{1j}^{A_{1n+1}}(e^{i(\sigma)\cdot x_{1j}})\\
&= [(\mu^{(n)}\circ\Gamma([A_{1n}^{-1}A_{1n+1}]))(e^{i(\sigma)\cdot x^{(n)}})] \times [\mu_{1n+1}^{A_{1n+1}}(e^{i\sigma x_{1n+1}})] \times\\
&\qquad\qquad \times [e^{i\sigma(\hat{\alpha}_{n+1}(y_{1n+1})-\hat{\alpha}_n(A_{1n}^{-1}A_{1n+1}y_{1n}))}]
\end{aligned}$$

Hence, taking limits of both sides of the last equation, using Proposition (1.6) and the uniform infinitesimality of $\{\mu_{1j}^{A_{1n}}: j=1, 2,\ldots, n\}_{n\geq 1}$ with respect to $\{x_{1j}: j=1, 2,\ldots\}$, we get

$$\mu(e^{i(\sigma)\cdot x)}) = \mu^A(e^{i(\sigma\cdot x^{(1)}})\varepsilon_{\alpha_A}(e^{i\sigma y_1})$$

for some $\alpha_A \in \mathbb{R}(X_1)$, $x^{(1)} \in L_s^p(X, \hat{\mathscr{X}}, \mu^0)_{alg}$, $y_1 \in L_s^p(X_1, \hat{\mathscr{X}}_1, \mu^0\circ\underline{i}_1)$. From the last equation, we readily infer that A lies in $D^p(\mu, \hat{\mathscr{X}}) \cap \mathrm{Sem}(\mathbb{F})$.

To complete the proof, it remains to show that the set in (4.6) is indeed a *group*. To this end, suppose that B is an arbitrary member of the set in (4.6). Since the monothetic semigroup $\mathrm{Sem}(\{B\})$ is compact, the sequence $\{B^n\}_{n\geq 1}$ of the iterates of B forms a group $\mathscr{G}$, say, which coincides with the minimal ideal of $\mathrm{Sem}(\{B\})$ and the unit P, say, of $\mathscr{G}$ is its sole *idempotent* member ([38], Theorem (3.1.1)). Hence, there is $C\in\mathscr{G}$ such that $BC=P=CB$. We want to show that C coincides with B^{-1}, where the inverse is evaluated in $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$.

Observe that C and P lie in the set (4.6). Hence, there are $(X_P, \hat{\mathscr{X}}_P)\subset(X, \hat{\mathscr{X}})$, $(X_{P^\perp}, \hat{\mathscr{X}}_{P^\perp})\subset(X_1, \hat{\mathscr{X}}_1)$ and $\alpha_P\in\mathbb{R}(X_1)$ such that

$$\mu((\Gamma_{\hat{\mathscr{X}}}(P)f(a))\cdot g_1(b_1))=\mu^P_{\hat{\mathscr{X}}_P}(f(a))\varepsilon_{\alpha_P}(g_1(b_1))$$

for all $a\in L^p_s(X_P, \hat{\mathscr{X}}_P, \mu^P_{\hat{\mathscr{X}}_P})_{alg}$, $b_1\in L^p_s(X_{P^\perp}, \hat{\mathscr{X}}_{P^\perp}, \mu^0\circ\underline{i}_1)$, $f\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_P}$ and $g_1\in\mathscr{D}(\mathbb{R})_1$. Since $\mathscr{D}(\mathbb{R})$, $\hat{\mathscr{X}}_{P^\perp}$, ε_{α_P}) is invariant under $\Gamma_{\hat{\mathscr{X}}_{P^\perp}}(P^\perp)$, by (2.1), the last equation gives

$$\begin{aligned}&\mu((\Gamma_{\hat{\mathscr{X}}_P}(P)f(a)\cdot(\Gamma_{\hat{\mathscr{X}}_{P^\perp}}(P^\perp)g_1(b_1)))\\&=\mu^P_{\hat{\mathscr{X}}_P}(f(a))\varepsilon_{\alpha_P}(\Gamma_{\hat{\mathscr{X}}_{P^\perp}}(P^\perp)g_1(b_1)),\end{aligned}$$

which implies,

$$\begin{aligned}&\mu((\Gamma_{\hat{\mathscr{X}}_P}(P)\,e^{i\sigma f(a)})\cdot(\Gamma_{\hat{\mathscr{X}}_{P^\perp}}(P^\perp)e^{i\lambda g_1(b_1)})\\&=\mu^P_{\hat{\mathscr{X}}_P}(e^{i\sigma f(a)})(\varepsilon_{\alpha_P}\circ\Gamma_{\hat{\mathscr{X}}_{P^\perp}}(P^\perp))e^{i\lambda g_1(b_1)})\\&\text{for all}\quad a\in L^p_s\,(X_p, \hat{\mathscr{X}}_P, \mu^P_{\hat{\mathscr{X}}_P}),\ b_1\in L^p_s(X_{P^\perp}, \hat{\mathscr{X}}_{P^\perp}, \mu^0\circ\underline{i}_1),\\&f\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_P}\text{ and } g_1\in\mathscr{D}(\mathbb{R})_1.\end{aligned}$$

Now set $\sigma=0$ in the last equation. Then we get

$$\mu((\Gamma_{\hat{\mathscr{X}}_P}(P)1_{\hat{\mathscr{X}}_P})\cdot(\Gamma_{\hat{\mathscr{X}}_{P^\perp}}(P^\perp)e^{i\lambda g_1(b_1)}))=(\varepsilon_{\alpha_P}\circ\Gamma_{\hat{\mathscr{X}}_{P^\perp}}(P^\perp))(e^{i\lambda g_1(b_1)})$$

or

$$\mu^{P^\perp}_{\hat{\mathscr{X}}_{P^\perp}}(e^{i\lambda g_1(b_1)})=(\varepsilon_{\alpha_P}\circ\Gamma_{\hat{\mathscr{X}}_{P^\perp}}(P^\perp))(e^{i\lambda g_1(b_1)})$$

for all $b_1\in L^p_s(X_{P^\perp}, \hat{\mathscr{X}}_{P^\perp}, \mu^0\circ\underline{i}_1)$ and $g_1\in\mathscr{D}(\mathbb{R})_1$. But (μ, x) in $\lim(G_1^{(p)}(\hat{\mathscr{X}}))$ was nondegenerate, by hypothesis. Hence, we must have $\alpha_P=0$ and $P^\perp=0$. Thus, $P=I$, the identity of $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$, whence $C=B^{-1}$. Hence $\mathscr{G}$ is indeed a group. □

(4.7) Proposition: *Let (μ, x) be a nondegenerate member of $\lim(G_1^{(p)}(\hat{\mathscr{X}}))$. Then, there is a norming sequence $\{A_{1n}\}_{n\geq1}$ corresponding to (μ, x) with the property that $\{[A_{1n}]^{-1}[A_{1n+1}]\}_{n\geq1}$ converges to the identity I of $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$ in the norm-topology.*

Proof: Let $\{B_{1n}\}_{n\geq1}$ be an arbitrary norming sequence corresponding to (μ, x). Then, by (3.5), there are $\{\mu'_{1j}\}_{j\geq1}\subset G_1(\hat{\mathscr{X}}_1)$, $\{x'_{1j}\}_{j\geq1}$, $\{y'_{1j}\}_{j\geq1}\subset L^p_s\cdot(X_1, \hat{\mathscr{X}}_1, \mu^0\circ\underline{i}_1)$ and $\{\alpha'_j\}_{j\geq1}\subset\mathbb{R}(X_1)$, with $\{\mu'^{B_{1n}}_{1j}: j=1, 2,\dots, n;\ n=1, 2,\dots\}$ being uniformly infinitesimal with respect to $\{x'_{1j}: j=1, 2,\dots\}$, such that

$\mu'^{(n)}(e^{i(\sigma)\cdot x'^{(n)}})$ converges to $\mu(e^{i(\sigma)\cdot x})$, where

$$\mu'^{(n)} = \underline{i}_{n+1}(\varepsilon_{\alpha'_n} \otimes (\bigotimes_{j=1}^{n} \mu'^{B_{1n}}_{1j})) \quad \text{and} \quad x'^{(n)} = \underline{i}_{n+1}(y'_{1n} \otimes (\bigotimes_{j=1}^{n} x'_{1j})).$$

Denote $D^p_\varepsilon(\mu, \hat{\mathscr{X}}) \cap \mathrm{Sem}(\mathbb{F})$ by $\mathscr{G}_\varepsilon(\mu, \hat{\mathscr{X}}, \mathbb{F})$. By Proposition (4.5), $\mathscr{G}_\varepsilon(\mu, \hat{\mathscr{X}}, \mathbb{F})$ is a compact group containing all the accumulation points of the sequence $\{[B^{-1}_{1n}B_{1n+1}]\}_{n\geq 1}$. Hence, we can choose a sequence $\{[C_{1n}]\}_{n\geq 1}$ of members of $\mathscr{G}_\varepsilon(\mu, \hat{\mathscr{X}}, \mathbb{F})$ with the property:

(4.8) $\quad [C^{-1}_{1n}] - [B^{-1}_{1n}B_{1n}] \to 0$ in the norm-topology of $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$.

Define $\{A_{1n}\}_{n\geq 1}$ by

$$A_{11} = B_{11},\ A_{12} = B_{12}C_{11}, \ldots,\ A_{1n} = B_{1n}C_{11}C_{12}\ldots C_{1n-1}, \quad n = 2, 3, \ldots.$$

Evidently, A_{1n} is invertible for each n and $\mathrm{Sem}(\{[A^{-1}_{1n}A_{1m}]\colon n=1, 2,\ldots, m;\ m=1, 2,\ldots\})$, being a closed subsemigroup of the norm-compact semigroup $\mathrm{Sem}(\{[B^{-1}_{1n}B_{1m}]\colon n=1, 2,\ldots, m;\ m=1, 2,\ldots\})$, is compact in $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$. Since $[A_{1n}] = [B_{1n}][C_{11}C_{12}\cdots C_{1n-1}]$, it is clear that the sequence $\{[C_{11}C_{12}\cdots \cdots C_{1n}]\}_{n\geq 1}$ is a precompact sequence of members of $\mathscr{G}_\varepsilon(\mu, \hat{\mathscr{X}}, \mathbb{F})$. Furthermore, since $\{\mu'^{B_{1n}}_{1j}\colon j=1, 2,\ldots, n;\ n=1, 2,\ldots\}$ is, by hypothesis, uniformly infinitesimal with respect to $\{x'_{1j}\colon j=1, 2,\ldots\}$, one sees that $\{\mu'^{A_{1n}}_{1j}\colon j=1, 2,\ldots, n;\ n=1, 2,\ldots\}$ is uniformly infinitesimal with respect to $\{x'_{1j}\colon j=1, 2,\ldots\}$

Next, observe that the precompactness of $\{[C_{11}C_{12}\cdots C_{1n}]\}_{n\geq 1}$ implies the precompactness of $\{(\mu'^{(n)} \circ \Gamma([C_{11}C_{12}\cdots C_{1n-1}]))(e^{i(\sigma)\cdot x'^{(n)}})\}_{n\geq 1}$ in $\mathbb{C}$. This is equivalent to the precompactness of the sequence $\{\mu^{(n)}(e^{i(\sigma)\cdot x'^{(n)}})\}_{n\geq 1}$, where $\mu^{(n)} = \underline{i}_{n+1}(\varepsilon_{\alpha_n} \otimes (\bigotimes_{j=1}^{n} \mu'^{A_{1n}}_{1j}))$, $\alpha_n \in \mathbb{R}(X_1)$, and $\hat{\alpha}_n(a_1) = \hat{\alpha}'_n(C_{11}C_{12}\cdots C_{1n-1}a_1)$, $a_1 \in L^p_s(X_1, \hat{\mathscr{X}}_1, \mu^0 \circ \underline{i}_1)$. Furthermore, since $\mu'^{(n)}(e^{i(\sigma)\cdot x'^{(n)}})$ converges to $\mu(e^{i(\sigma)\cdot x})$, it follows, by Proposition (1.6), that the accumulation points of $\{\mu^{(n)}(e^{i(\sigma)\cdot x'^{(n)}})\}_{n\geq 1}$ are of the form $\mu^C(e^{i(\sigma)\cdot x})$, where C is an accumulation point of $\{[C_{11}C_{12}\cdots C_{1n}]\}_{n\geq 1}$. But C lies in $\mathscr{G}_\varepsilon(\mu, \hat{\mathscr{X}}, \mathbb{F})$. Hence, we can choose $\{\beta_n\}_{n\geq 1} \subset \mathbb{R}(X_1)$ such that $\{\mu''^{(n)}(e^{i(\sigma)\cdot x'^{(n)}})\}_{n\geq 1}$ converges to $\mu(e^{i(\sigma)\cdot x})$, where $\mu''^{(n)} = \underline{i}_{n+1}(\varepsilon_{\alpha_n} \otimes (\bigotimes_{j=1}^{n} \mu'^{A_{1n}}_{1j}))$. Consequently, $\{A_{1n}\}_{n\geq 1}$ is a norming sequence corresponding to $(\mu, x) \in \in \lim(G^{(p)}_1(\hat{\mathscr{X}}))$.

Finally, since the norms of the members of the compact group $\mathscr{G}_\varepsilon(\mu, \hat{\mathscr{X}}, \mathbb{F})$ are uniformly bounded by a positive number k, say, we have

$$\begin{aligned} &\| [A^{-1}_{1n} A_{1n+1}] - I \|_{p,\mu^0} \\ &= \| [C^{-1}_{1n-1}C^{-1}_{1n-2}\cdots C^{-1}_{12}\, C^{-1}_{11}][B^{-1}_{1n}B_{1n+1} - C^{-1}_{1n}][C_{11}C_{12}\cdots C_{1n}] \|_{p,\mu^0}, \end{aligned}$$

using the commutativity of the set $\mathscr{G}_\varepsilon(\mu, \hat{\mathscr{X}}, \mathbb{F})$,

$$\leq k^2 \|| [B_{1n}^{-1} B_{1n+1} - C_{1n}^{-1}] \||_{p,\mu^0},\ 1 \leq p \leq \infty.$$

Hence, $[A_{1n}^{-1} A_{1n+1}] \to I$ in the norm-topology of $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$, $1 \leq p \leq \infty$: thanks to (4.8). □

§5. A Characterization of Nondegenerate Members of $\lim (G_1^{(p)}(\hat{\mathscr{X}}))$

In this section, we characterize the nondegenerate members of $\lim (G_1^{(p)}(\hat{\mathscr{X}}))$ by means of the *decomposability algebraic structures* of their associated probability gages.

Let (μ, x) be a nondegenerate member of $\lim (G_1^{(p)}(\hat{\mathscr{X}}))$. Then, by Proposition (4.7), we may choose a norming sequence $\{A_{1n}\}_{n\geq 1}$ corresponding to (μ, x) such that $[A_{1n}^{-1} A_{1n+1}] \to I$ in the norm-topology of $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$: *we fix this norming sequence throughout this section.*

Let $\mathbb{F}$ be as in (4.3). For each projection operator P in $\mathrm{Sem}(\mathbb{F})$, define $\mathscr{S}_P$ by

$$\mathscr{S}_P \equiv \{A \in \mathrm{Sem}(\mathbb{F}) \colon AP = A = PA\}.$$

Then, $\mathscr{S}_P$ is a compact subsemigroup of $\mathrm{Sem}(\mathbb{F})$. Set

$$\{A \in \mathscr{S}_P \colon A \in D^p(\mu^P_{\hat{\mathscr{X}}_P}, \hat{\mathscr{X}}_P)\} \equiv \mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})$$

where $(X_P, \hat{\mathscr{X}}_P) \subset (X, \hat{\mathscr{X}})$, with $\hat{\mathscr{X}}_P$ being the support of μ^P in $\hat{\mathscr{X}}$.

(5.1) **Proposition:** *$G_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})$ is a compact group with P as its identity.*

Proof. It is clear that $\mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})$ is a closed subsemigroup of $\mathscr{S}_P$: therefore, $\mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})$ is compact. Furthermore, by the definition of $\mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})$, P is the identity of $\mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})$.

Let $A \in \mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})$. Then, $\mathrm{Sem}(\{A\})$ is compact. By ([38], Theorem 3.1.1), $\mathrm{Sem}(\{A\})$ contains a projection operator Q and an operator B such that

$$AB = BA = Q.$$

Since $Q \in \mathscr{S}_P$, we have $PQ = QP = Q$. Hence, Q lies in $\mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})$. We prove next that $Q = P$ whence one concludes that $\mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})$, is a group.

Now, since $Q \in \mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})$, there are $(X_Q, \hat{\mathscr{X}}_Q) \subset (X, \hat{\mathscr{X}})$, $(X_{Q^\perp}, \hat{\mathscr{X}}_{Q^\perp}) \subset (X_1, \hat{\mathscr{X}}_1)$ and $\alpha_Q \in \mathbb{R}(X_1)$ such that

$$\mu^P_{\hat{\mathscr{X}}_P}((\Gamma_{\hat{\mathscr{X}}_Q}(Q) f(a)) \cdot (\Gamma_{\hat{\mathscr{X}}_{Q^\perp}}(Q^\perp)\, g_1(b_1))) = \mu^Q_{\hat{\mathscr{X}}_Q}(f(a))(\varepsilon_{\alpha_Q} \circ \Gamma_{\hat{\mathscr{X}}_{Q^\perp}}(Q^\perp))(g_1(b_1))$$

or, equivalently,

$$\mu((\Gamma_{\hat{\mathscr{X}}_Q}(PQ)f(a))\cdot(\Gamma_{\hat{\mathscr{X}}_{Q^\perp}}(PQ^\perp)g_1(b_1)))=\mu^Q_{\hat{\mathscr{X}}_Q}(f(a))(\varepsilon_{\alpha_Q}\circ\Gamma_{\hat{\mathscr{X}}_{Q^\perp}}(Q^\perp)(g_1(b_1))$$

for all $a\in L^p_s(X_Q, \hat{\mathscr{X}}_Q, \mu^Q_{\hat{\mathscr{X}}_Q})_{alg}$, $b_1\in L^p_s(X_{Q^\perp}, \hat{\mathscr{X}}_{Q^\perp}, \mu^0\circ\underline{i}_1)$, $f\in\mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}_Q}$ and $g_1\in\mathscr{D}(\mathbb{R})_1$. The last equation gives

$$\begin{aligned}&\mu((\Gamma_{\hat{\mathscr{X}}_Q}(Q)e^{i\beta f(a)})\cdot(\Gamma_{\hat{\mathscr{X}}_{Q^\perp}}(P-Q)e^{i\sigma g_1(b_1)}))\\&=\mu^Q_{\hat{\mathscr{X}}_Q}(e^{i\beta f(a)})(\varepsilon_{\alpha_Q}\circ\Gamma_{\hat{\mathscr{X}}_{Q^\perp}}(Q^\perp))\ (e^{i\sigma g_1(b_1)}),\ (\beta, \sigma)\in\mathbb{R}^2,\end{aligned}$$

since $PQ=Q=QP$. Setting $\beta=0$ in the last equation, we get

$$\mu((\Gamma_{\hat{\mathscr{X}}_Q}(Q)1_{\hat{\mathscr{X}}_Q})\cdot(\Gamma_{\hat{\mathscr{X}}_{Q^\perp}}(P-Q)e^{i\sigma g_1(b_1)}))=(\varepsilon_{\alpha_Q}\circ\Gamma_{\hat{\mathscr{X}}_{Q^\perp}}(Q^\perp)(e^{i\sigma g_1(b_1)})$$

for all $\sigma\in\mathbb{R}$, $b_1\in L^p_s(X_{Q^\perp}, \hat{\mathscr{X}}_{Q^\perp}, \mu^0\circ\underline{i}_1)$ and $g_1\in\mathscr{D}(\mathbb{R})_1$. But (μ, x) in $\lim(G_1^{(p)}(\hat{\mathscr{X}}))$ was nondegenerate, by hypothesis. Hence, we must have $\alpha_Q=0$ and $P=Q$. Thus $\mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})$ is indeed a group. □

(5.2) **Proposition:** *If* $A\in\mathscr{S}_P$ *and* $P\in\mathrm{Sem}(\{A\})$, *then* $A\in\mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})$.

Proof: This is straightforward.

Remark: The following result will be employed in the characterization of the nondegenerate pairs in $\lim(G_1^{(p)}(\hat{\mathscr{X}}))$.

(5.3) **Theorem:** *For each nonzero projection operator* $P\in\mathrm{Sem}(\mathbb{F})$, *the semigroup* $\mathscr{S}_P$ *contains a one-parameter semigroup* $\{P\exp tH\colon t\in[0, \infty)\}$, $H\in B(L^p(X, \hat{\mathscr{X}}, \mu^0))$, *with the property* $PH=H=HP$. *Moreover,* $\mathscr{S}_P$ *contains a projection operator* Q *with the properties* $P\neq Q$, $QH=HQ$ *and* $\lim_{t\to\infty}(P-Q)\cdot\exp tH=0$, *where the limit is taken in the norm-topology of* $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$.

Proof: The proof is completely analogous to that of Lemma (4.3) in Ref [5]. Therefore, we only sketch the underlying arguments, for the sake of completeness.

By Proposition (5.1), $\mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})$ is a compact group. Put

$$\lambda_{n,m}=\min\{|||P-K[A^{-1}_{1n}A_{1m}]P|||_{p,\mu^0}\colon K\in\mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})\}.$$

Then

$$\lambda_{n,n}=0,\quad n=1, 2,\ldots \tag{5.3.1}$$

and by Proposition (4.1),

$$\lim_{m\to\infty}\lambda_{n,m}=|||P|||_{p,\mu^0}\geq 1,\quad n=1, 2,\ldots \tag{5.3.2}$$

Using Proposition (4.7) and the compactness of $\mathrm{Sem}(\mathbb{F})$, one shows that for $m\geq n_m$, (5.3.3) $$\limsup_{m\to\infty} (\lambda_{n_m,m+1}-\lambda_{n_m,\ m})=0.$$

Given any number σ satisfying $0<\sigma<1$, there is, as a consequence of (5.3.1) and (5.3.2), an index $m_n(\sigma)\geq n$ such that $a_{n,m_n(\sigma)}<\sigma$ and $a_{n,m_n(\sigma)+1}\leq\sigma$, $n=1, 2,\dots$. Then, applying (5.3.1), (5.3.2) and (5.3.3), as well as the precompactness of the sequence $\{[A_{1n}^{-1}A_{1m_n(\sigma)}]\}_{m_n(\sigma)\geq n,n\geq 1}$ and the compactness of $\mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})$, we can choose an accumulation point $A^{(\sigma)}$ of $\{[A_{1n}^{-1}A_{1m_n(\sigma)}]\}_{m_n(\sigma)\geq n,n\geq 1}$ and $D^{(\sigma)} \in \mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})$ such that

(*) $$|||P-D^{(\sigma)}A^{(\sigma)}P|||_{p,\mu^0}=\sigma=\min\{|||P-CA^{(\sigma)}P|||_{p,\mu^0}\colon C\in\mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})\}.$$

By Proposition (4.2), $A^{(\sigma)}\in\mathrm{Sem}\,(\mathbb{F})$. Hence, defining $B^{(\sigma)}$ by $B^{(\sigma)}=D^{(\sigma)}A^{(\sigma)}P$, it follows that $B^{(\sigma)}\in\mathscr{S}_P$ and

(5.3.4) $$|||P-B^{(\sigma)}|||_{p,\mu^0}=\sigma=\min\{|||P-CB^{(\sigma)}|||_{p,\mu^0}\colon C\in\mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})\},$$

whence

(5.3.5) $$B^{(\sigma)}\in\mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F}).$$

Put

$$\zeta_{n,\sigma}=\min\{|||P-C(B^{(\sigma)})^n|||_{p,\mu^0}\colon C\in\mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})\}.$$

It follows from (5.3.4) that

(5.3.6) $$\zeta_{1,\sigma}=\sigma.$$

By ([38], Theorem 3.1.1), the semigroup $\mathrm{Sem}\,(\{B^{(\sigma)}\})$ contains a projection operator $P^{(\sigma)}$. Furthermore, we have

(5.3.7) $$\limsup_{n\to\infty}\zeta_{n,\sigma}\geq\min\{|||P-CP^{(\sigma)}|||_{p,\mu^0}\colon C\in\mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})\}.$$

Since $P^{(\sigma)}\in\mathscr{S}_P$, it follows that $P-P^{(\sigma)}$ is also a projection operator and, by Proposition (5.2), $P^{(\sigma)}\neq P$. Hence

(5.3.8) $$|||P-P^{(\sigma)}|||_{p,\mu^0}\geq 1.$$

Set

$$\inf\{|||P-CP^{(\sigma)}|||\colon C\in\mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F}),\ 0<\sigma<1\}=\Delta.$$

Then, one can show that

(5.3.9) $$\limsup_{n\to\infty}\zeta_{n,\sigma}\geq\Delta>0,\quad\text{for each}\quad\sigma\in(0, 1),$$

and also that

(5.3.10) $$\limsup_{n\to\infty}(\zeta_{m_n+1,\sigma_n}-\zeta_{m_n,\sigma_n})=0$$

for any sequences $\{m_n\}_{n\geq 1}$ and $\{\sigma_n\}_{n\geq 1}$, with $\sigma_n \to 0$. Given a number Ω satisfying $0<\Omega<\Delta$, then by (5.3.6) and (5.3.9), there is an integer $m_n(\Omega)$ such that $\zeta_{m_n(\Omega),\sigma_n}<\Omega$ and $b_{m_n(\Omega)+1,\sigma_n}\geq \Omega$, where $\{\sigma_n\}_{n\geq 1}$ is any sequence with the property $\sigma_n \to 0$. From (5.3.10), we infer that $\zeta_{m_n(\Omega),\sigma_n}$ converges to Ω. Let $E^{(\Omega)}$ denote an accumulation point of the evidently precompact sequence $\{(B^{(\sigma_n)})^{m_n(\Omega)}\}_{n\geq 1}$ of members of $\mathscr{S}_P$. Then

(5.3.11) $$\min \{|||P-CE^{(\Omega)}|||_{p,\mu^0}: C \in \mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})\} = \Omega,$$

where $0<\Omega<\Delta$, whence

(5.3.12) $$E^{(\Omega)} \in \mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F}), \quad 0<\Omega<\Delta.$$

The net $\{E^{(\Omega)}: 0<\Omega<\Delta\} \subset \mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})$ is precompact. Let $E^{(0)}$ denote its accumulation point as Ω tends to zero. Then using (5.3.11), the compactness of $\mathscr{G}_{\varepsilon,P}(\mu, \hat{\mathscr{X}}, \mathbb{F})$ and ([38], Theorem 3.1.1), one shows that there is an integer q such that

$$|||P-(E^{(0)})|||_{p,\mu^0}<\frac{1}{4}.$$

Put

(5.3.13) $$W=(E^{(\Omega_0)})^q$$

where Ω_0 is a positive number with the property

$$|||(E^{(0)})^q-(E^{(\Omega_0)})^q|||_{p,\mu^0}<\frac{1}{4}.$$

Then

(5.3.14) $$|||P-W|||_{p,\mu^0}<\frac{1}{2}$$

and, from the definition of the operators $E^{(\Omega)}$, it follows that

(5.3.15) $(B^{(\sigma_n)})^{r_n} \to W$ in the norm-topology of $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$

as $r_n \to \infty$. From (*) and (5.3.14), it follows that the operators $B^{(\sigma_n)}$ and W admit representations of the form ([39, Theorem 9.6.1])

(5.3.16) $$B^{(\sigma_n)}=P\exp H^{(n)} \quad \text{and} \quad W=P\exp H \text{ where}$$

$$H^{(n)}, H \in B(L^p(X, \hat{\mathscr{X}}, \mu^0)), PH=HP=H, PH^{(n)}=H^{(n)}P=H^{(n)},$$

(5.3.17) $$WH=HW,$$

and by (5.3.15)

(5.3.18) $r_n H^{(n)} \to H$ in the norm-topology of $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$.

Let t be a positive number. Then, by (5.3.16) and (5.3.18), $(B^{(\sigma_n)})^{[r_n t]} \to P \exp tH$ in the norm-topology of $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$; here $[r_n t]$ denotes the integral part of $r_n t$. Since $B^{(\sigma_n)} \in \mathscr{S}_P$, we infer that $\{P \exp tH: t \geq 0\} \subset \mathscr{S}_P$.

Consider the semigroup Sem $(\{W\})$. By ([38], Theorem 3.1.1), Sem $(\{W\})$ contains a projection operator Q. Using (5.3.17), (5.3.12), Proposition (5.2), and arguing as in [5], we obtain that $\lim_{t\to\infty} (P-Q)\, exp\, tH = 0$, in the norm-topology of $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$. □

Remark: Before continuing the discussion, we introduce some notions and concepts which we use in the sequel.

(5.4) Definition: We call a set $\mathscr{M} \subset G_1(\hat{\mathscr{X}})$ *shift compact* if and only if, for every net $\{\mu^{(\gamma)}: \gamma \in \mathscr{A}\} \subset \mathscr{M}$, there are nets $\{\alpha_\gamma: \gamma \in \mathscr{A}\} \subset \mathbb{R}(X_1)$, $\{b_{1\gamma}: \gamma \in \mathscr{A}\} \subset L_s^p(X, \hat{\mathscr{X}}, \mu^0 \circ \underline{i}_1)$, $\{a^{(\gamma)}: \gamma \in \mathscr{A}\} \subset L_s^p(X, \hat{\mathscr{X}}, \mu^0)_{alg}$, with $(\mu^{(\gamma)}, a^{(\gamma)}) \in G_1^{(p)}(\hat{\mathscr{X}})$, and $(\mu, a) \in G_1^{(p)}(\hat{\mathscr{X}})$ such that a subnet of $\{\mu^{(\gamma)}(e^{i(\sigma)\cdot a^{(\gamma)}}) e^{i\sigma\hat{\alpha}_\gamma(b_{1\gamma})}: \gamma \in \mathscr{A}\}$ converges to $\mu(e^{i(\sigma)\cdot a})$, for each $\sigma \in \mathbb{R}$.

Remark: Using Definition (5.4), one may verify that analogues of the results for shift compact sets of probability measures [3,41] are again valid here.

(5.5) Definition: We call a pair $(\mu, a) \in G_1^{(p)}(\hat{\mathscr{X}})$ *infinitely divisible* if and only if, for each positive integer m, there are $(X^{1/m}, \hat{\mathscr{X}}^{1/m}) \subset (X, \hat{\mathscr{X}})$ and $(\mu^{1/m}, a^{1/m}) \in G_1^{(p)}(\hat{\mathscr{X}}^{1/m})$ such that

$$\mu(e^{i(\sigma)\cdot a}) = (\mu^{1/m}(e^{i(\sigma)\cdot a^{1/m}})^m, \quad \text{for each} \quad \sigma \in \mathbb{R}.$$

The pair $(\mu^{1/m}, a^{1/m})$ will be called a *factor* of (μ, a).

Remark: In the next result, we characterize members of $\lim (G_1^{(p)}(\hat{\mathscr{X}}))$ in terms of their decomposability algebraic structures.

(5.6) Theorem: *Let* $(\mu, x) \in \lim (G_1^{(p)}(\hat{\mathscr{X}}))$. *Suppose that* $D^p(\mu, \hat{\mathscr{X}})$ *contains a one-parameter semigroup* $\{exp\, tH: t \geq 0\}$ *with the property* $\lim_{t\to\infty} exp\, tH = 0$, *where the limit is taken in the norm-topology of* $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$. *Then,* (μ, x) *is an infinitely divisible pair. Furthermore,* $\exp tH_{\hat{\mathscr{X}}^{1/m}} \in D^p(\mu^{1/m}, \hat{\mathscr{X}}^{1/m})$, $t \geq 0$, *for each factor* $(\mu^{1/m}, x^{1/m})$ *of* (μ, x), $m = 1, 2, \ldots$.

Proof: Let $t > 0$ and put $exp\, tH = U(t)$. Then, by hypothesis, there are $(X^{(j)}(t), \hat{\mathscr{X}}^{(j)}(t)) \subset (X, \hat{\mathscr{X}})$, $j = 1, 2$, and $\nu_t \in G_1(\hat{\mathscr{X}}^{(2)}(t))$, such that

(5.6.1) $\mu((\Gamma_{\hat{\mathscr{X}}^{(1)}(t)}(U(t)) f(a)) \cdot g^{(0)}(b^{(0)})) = \mu^{U(t)}_{\hat{\mathscr{X}}^{(1)}(t)}(f(a)) \nu_t(g^{(0)}(b^{(0)}))$, for all $f \in \mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}^{(1)}(t)}$, $g^{(0)} \in \mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}^{(2)}(t)}$, $a \in L_s^p(X^{(1)}(t), \hat{\mathscr{X}}^{(1)}(t), \mu_{\hat{\mathscr{X}}^{(1)}(t)})_{alg}$, and $b^{(0)} \in L_s^p(X^{(2)}(t), \hat{\mathscr{X}}^{(2)}(t), \nu_t)_{alg}$. Since $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}^{(1)}(t), \mu^{U(t)}_{\hat{\mathscr{X}}^{(1)}(t)})$ and $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}^{(2)}(t), \nu_t)$ are invariant under the maps $\Gamma_{\hat{\mathscr{X}}^{(1)}(t)}(U(t))$ and $\Gamma_{\hat{\mathscr{X}}^{(2)}(t)}(U(t))$, respectively, by definition, it follows from (5.6.1) that

$$\mu^{U(t)}_{\hat{\mathscr{X}}^{(1)}(t)}((\Gamma_{\hat{\mathscr{X}}^{(1)}(t)}(U(t))\,f(a))\cdot g^{(0)}(b^{(0)})) = \mu^{U(2t)}_{\hat{\mathscr{X}}^{(1)}(t)}(f(a))\,\nu_t^{U(t)}(g^{(0)}(b^{(0)})),$$

for all $f \in \mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}^{(1)}(t)}$, $g^{(0)} \in \mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}^{(2)}(t)}$, $a \in L_s^p(X^{(1)}(t), \hat{\mathscr{X}}^{(1)}(t), \mu^{U(t)}_{\hat{\mathscr{X}}^{(1)}(t)})_{alg}$, $b^{(0)} \in L_s^p(X^{(2)}(t), \hat{\mathscr{X}}^{(2)}(t), \nu_t)_{alg}$, $t > 0$. Let $\hat{\mathscr{X}}^{(1,t)}$ be the linear hull of all elements of the form $(\Gamma_{\hat{\mathscr{X}}^{(1)}(t)}(U(t))f(a))\cdot g^{(0)}(b^{(0)})$, with $a \in L_s^p(X^{(1)}(t), \hat{\mathscr{X}}^{(1)}(t), \mu^{U(t)}_{\hat{\mathscr{X}}^{(1)}(t)})_{alg}$, $b^{(0)} \in L_s^p(X^{(2)}(t), \hat{\mathscr{X}}^{(2)}(t), \nu_t)_{alg}$, $f \in \mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}^{(1)}(t)}$, $g^{(0)} \in \mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}^{(2)}(t)}$. Then, from (5.6.1), we get

$$\begin{aligned}&\mu((\Gamma_{\hat{\mathscr{X}}^{(1,t)}}(U(t))((\Gamma_{\hat{\mathscr{X}}^{(1)}(t)}(U(t)f(a))g^{(1)}(b^{(1)}))\cdot g^{(0)}(b^{(0)}))\\ &= \mu^{U(2t)}_{\hat{\mathscr{X}}^{(1)}(t)}(f(a))\,\nu_t^{U(t)}(g^{(1)}(b^{(1)}))\,\nu_t(g^{(0)}(b^{(0)})),\end{aligned}$$

for all $f \in \mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}^{(1)}(t)}$, $g^{(0)}, g^{(1)} \in \mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}^{(2)}(t)}$, $a \in L_s^p(X^{(1)}(t), \hat{\mathscr{X}}^{(1)}(t), \mu^{U(2t)}_{\hat{\mathscr{X}}^{(1)}(t)})_{alg}$, $b^{(1)} \in L_s^p(X^{(2)}(t), \hat{\mathscr{X}}^{(2)}(t), \nu_t^{U(t)})_{alg}$ and $b^{(0)} \in L_s^p(X^{(2)}(t), \hat{\mathscr{X}}^{(2)}(t), \nu_t)_{alg}$, $t > 0$.

Iterating the foregoing argument, we obtain

$$\begin{aligned}&\mu((\Gamma_{\hat{\mathscr{X}}^{(1)}(t)}(U(nt))f(a))\cdot(\Gamma_{\hat{\mathscr{X}}^{(2)}(t)}(U((n-1)t))g^{(n-1)}(b^{(n-1)}))\cdots\\ &\qquad\cdots(\Gamma_{\hat{\mathscr{X}}^{(2)}(t)}(U(t))g^{(1)}(b^{(1)}))\cdot g^{(0)}(b^{(0)}))\\ &= \mu^{U(nt)}_{\hat{\mathscr{X}}^{(1)}(t)}(f(a))\prod_{j=0}^{n-1}\nu_t^{U(jt)}(g^{(j)}(b^{(j)})),\end{aligned}$$

for all $f \in \mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}^{(1)}(t)}$, $g^{(j)} \in \mathscr{D}(\mathbb{R})_{\hat{\mathscr{X}}^{(2)}(t)}$, $j = 1, 2, \ldots, n-1$, $a \in L_s^p(X^{(1)}(t), \hat{\mathscr{X}}^{(1)}(t), \mu^{U(nt)}_{\hat{\mathscr{X}}^{(1)}(t)})_{alg}$, $b^{(j)} L_s^p(X^{(2)}(t), \hat{\mathscr{X}}^{(2)}(t), \nu_t^{U(jt)})_{alg}$, $j = 0, 1, 2, \ldots, n-1$, $t > 0$. Hence

$$\begin{aligned}&\mu((\Gamma_{\hat{\mathscr{X}}^{(1)}(t)}(U(t))e^{i(\sigma)\cdot a})\cdot(\Gamma_{\hat{\mathscr{X}}^{(2)}(t)}(U((n-1)t))g^{(n-1)}(b^{(n-1)}))\cdots\\ &\qquad\cdots(\Gamma_{\hat{\mathscr{X}}^{(2)}(t)}(U(t))g^{(1)}(b^{(1)}))\cdot g^{(0)}(b^{(0)}))\\ &= (\mu^{U(nt)}_{\hat{\mathscr{X}}^{(1)}(t)}\otimes(\bigotimes_{j=0}^{n-1}\nu_t^{U(jt)}))(e^{i(\sigma)\cdot a}\otimes(\bigotimes_{j=0}^{n-1}e^{i(\sigma)\cdot b^{(j)}})),\ \sigma \in \mathbb{R},\end{aligned}$$

for all $a \in L_s^p(X^{(1)}(t), \hat{\mathscr{X}}^{(1)}(t), \mu^{U(nt)}_{\hat{\mathscr{X}}^{(1)}(t)})_{alg}$, $b^{(j)} \in L_s^p(X^{(2)}(t), \hat{\mathscr{X}}^{(2)}(t), \nu_t^{U(jt)})_{alg}$, $j = 0, 1, 2, \ldots, n-1$, $t > 0$.

Since $(\mu, x) \in \lim(G_1(\hat{\mathscr{X}}))$, by hypothesis, we may suppose, as we do henceforth, that $\{a, b^{(j)}: j = 0, 1, 2, \ldots, n-1\}_{n \geq 1}$ is such that $(\mu^{U(nt)}_{\hat{\mathscr{X}}^{(1)}(t)}\otimes(\bigotimes_{j=1}^{n-1}\nu_t^{U(jt)}))\cdot(e^{i(\sigma)\cdot a}\otimes(\bigotimes_{j=1}^{n-1}e^{i(\sigma)\cdot b^{(j)}}))$ converges to $\mu(e^{i(\sigma)\cdot x})$, $\sigma \in \mathbb{R}$, as $n \to \infty$. But $\lim_{n\to\infty} U(nt) = 0$, by hypothesis. Hence, $\mu^{U(nt)}_{\hat{\mathscr{X}}^{(1)}(t)}(e^{i(\sigma)\cdot a})$ converges to 1 as $n \to \infty$. Thus

(5.6.2) $\bigotimes_{j=1}^{n-1} v_t^{U(jt)})(\bigotimes_{j=1}^{n-1} e^{i(\sigma)\cdot b^{(j)}})$ converges to $\mu(e^{i(\sigma)\cdot x})$, $\sigma \in \mathbb{R}$.

Next, for each positive integer m, define $\Psi_{(n,t,m)}$ by

$$\Psi_{(n,t,m)} \equiv \bigotimes_{k=0}^{n-1} v_t^{U(kmt)},\ t>0.$$

Then

$$(5.6.3)\qquad \bigotimes_{j=0}^{m-1} \Psi_{(n,t,m)}^{U(jt)} = \bigotimes_{j=0}^{m-1} \bigotimes_{k=0}^{n-1} v_t^{U((j+km)t)} = \bigotimes_{j=0}^{nm-1} v_t^{U(jt)},$$

and it follows from (5.6.2) that $\{\Psi_{(n,t,m)}\}_{n\geq 1}$ is *shift compact*. Hence, there is a sequence $\{\alpha_n\}_{n\geq 1} \subset \mathbb{R}(X_1)$ and $\{a_{1n}\}_{n\geq 1} \subset L_s^p(X_1, \hat{\mathscr{X}}_1, \mu^0 \circ \underline{i}_1)$ such that $\{\Psi_{(n,t,m)}(\bigotimes_{j=0}^{n-1} e^{i(\sigma)\cdot b^{(j)}})e^{i\sigma\hat{\alpha}_n(a_{1n})}\}_{n\geq 1}$ is precompact in $\mathbb{C}$. Let $\Psi_{(t,m)}(e^{i(\sigma)\cdot b'})$ be an accumulation point of the preceding sequence, where $\Psi_{(t,m)} \in G_1(\hat{\mathscr{X}}(t, m))$, for some $(X(t, m), \hat{\mathscr{X}}(t, m)) \subset (X, \hat{\mathscr{X}})$, and $b \in L_s^p(X(r, m), \hat{\mathscr{X}}(t, m), \Psi_{(t,m)})_{alg}$. Then, for some subsequence $n_1 < n_2 < \cdots < n_j < \cdots$, we have the convergence

$$(5.6.4)\qquad \Psi_{(n_j,t,m)}(\bigotimes_{j=0}^{n_j-1} e^{i(\sigma)\cdot b^{(j)}})e^{i\sigma\hat{\alpha}_{n_j}(a_{1n_j})} \to \Psi_{(t,m)}(e^{i(\sigma)\cdot b'}).$$

Now

$$\begin{aligned}&(\bigotimes_{j=0}^{m-1} \Psi_{(n,t,m)}^{U(jt)})(\bigotimes_{j=0}^{m-1}\bigotimes_{k=0}^{n-1} e^{i(\sigma)\cdot b^{(k)}})\\ &= (\prod_{j=0}^{m-1} e^{-\sigma\hat{\alpha}_n(\Gamma_{\hat{\mathscr{X}}_1}(U(jt)a_{1n})})((\bigotimes_{j=0}^{m-1}(\varepsilon_{\alpha_n}\otimes\Psi_{(n,t,m)})^{U(jt)})(\bigotimes_{j=0}^{m-1}(e^{i\sigma a_{1n}}\\ &\qquad \otimes(\bigotimes_{k=0}^{n-1} e^{i(\sigma)\cdot b^{(k)}})))).\end{aligned}$$

Hence, invoking (5.6.2), (5.6.3) and (5.6.4), passing to a subsequence and then taking limits, we get

(5.6.5) $\mu(e^{i(\sigma)\cdot x}) = (\varepsilon_{\alpha(t,m)}\otimes(\bigotimes_{j=0}^{m-1}\Psi_{(t,m)}^{U(jt)}))(e^{i\sigma a_1'}\otimes(\bigotimes_{j=0}^{m-1} e^{i(\sigma)\cdot b'}))$ for some $\alpha_{(t,m)} \in \mathbb{R}(X_1)$, $a_1' \in L_s^p(X_1, \hat{\mathscr{X}}_1, \mu^0 \circ \underline{i}_1)$ and $b^1 \in L_s^p(X(r, m), \hat{\mathscr{X}}(r, m), \Psi_{(t,m)})_{alg}$, $t>0$.

Let r be an arbitrary positive integer. Then

$$(5.6.6)\qquad (\bigotimes_{j=1}^{r} v_t^{U(jmt)})\otimes\Psi_{(n,t,m)}^{U(rmt)} = \bigotimes_{j=0}^{r+n-1} v_t^{U(jmt)} = \Psi_{(n,t,m)}\otimes(\bigotimes_{j=n}^{r+n-1} v_t^{U(jmt)}).$$

Thus, the sequence $\{(\bigotimes_{j=n}^{r+n-1} v_t^{U(jmt)})(\bigotimes_{j=n}^{r+n-1} e^{i(\sigma)\cdot b^{(j)}})\}_{n\geq 1}$ must converge to 1. Therefore, the sequence $\{\bigotimes_{j=1}^{r-1} v_t^{U(jmt)}\}_{m\geq 1}$ is shift compact and, hence, for some

$\{\gamma_r\}_{r\geq 1}\subset \mathbb{R}(X_1)$ and $\{h_{1r}\}_{r\geq 1}\subset L_s^p(X_1, \hat{\mathscr{X}}_1, \mu^0\circ \underline{i}_1)$, the sequence $\{(\varepsilon_{\gamma_r}\otimes(\bigotimes_{j=1}^{r-1} \nu_t^{U(jmt)}))(e^{i\sigma h_{1r}}\otimes(\bigotimes_{j=1}^{r-1} e^{i(\sigma)\cdot b^{(j)}}))\}_{m\geq 1}$ is precompact. Let $\omega_{(r,t)}(e^{i(\sigma)\cdot c^{(r)}})$, $\sigma\in\mathbb{R}$, be an accumulation point of the preceding sequence, where $\omega_{(r,t)}\in G_1(\hat{\mathscr{X}}_{[r,t]})$ for some $(X_{[r,t]}, \hat{\mathscr{X}}_{[r,t]})\subset(X, \hat{\mathscr{X}})$ and $c^{(r)}\in L_s^p(X_{[r,t]}, \hat{\mathscr{X}}_{[r,t]}, \omega_{(r,t)})_{alg}$. Then, from (5.6.4) and (5.6.6), we obtain

$$(5.6.7)\qquad \Psi_{(t,m)}(e^{i(\sigma)\cdot b'})=\Psi_{(t,m)}^{U(rmt)}(e^{i(\sigma)\cdot b'})\ \omega_{(r,t)}(e^{i(\sigma)\cdot c^{(r)}}),$$

with $b\in L_s^p(X(t, m), \hat{\mathscr{X}}(t, m), \Psi_{(t,m)})$ and $c^{(r)}$ as previously described.

Let $\{t_k\}_{k\geq 1}$ be a sequence of members converging to zero. From (5.6.5), one sees that there are sets $\{\theta_k\}_{k\geq 1}\subset\mathbb{R}(X_1)$ and $\{d_{1k}\}_{k\geq 1}\subset L_s^p(X_1, \hat{\mathscr{X}}_1, \mu^0\circ\underline{i}_1)$ such that the sequence $\{(\varepsilon_{\theta_k}\otimes\Psi_{(t_k,m)})(e^{i\sigma d_{1k}}\otimes e^{i(\sigma)\cdot b})\}_{k\geq 1}$ converges to some number $\Psi_{[m]}(e^{i(\sigma)\cdot y^{(m)}})$, where $\Psi_{[m]}\in G_1(\hat{\mathscr{X}}_{[m]})$ for some $(X_{[m]}, \hat{\mathscr{X}}_{[m]})\subset(X, \hat{\mathscr{X}})$, and $y^{(m)}\in L_s^p(X_{[m]}, \hat{\mathscr{X}}_{[m]}, \Psi_{[m]})_{alg}$. Letting $t\to 0$, one infers from (5.6.5) that

$$(5.6.8)\quad \mu(e^{i(\sigma)\cdot x}) =(\varepsilon_\theta\otimes\Psi_{[m]}\otimes\Psi_{[m]}\otimes\cdots\otimes\Psi_{[m]})(e^{i\sigma d_1^{(m)}}\otimes e^{i(\sigma)\cdot y^{(m)}}\otimes e^{i(\sigma)\cdot y^{(m)}}\otimes\cdots \cdots\otimes e^{i(\sigma)\cdot y^{(m)}}),$$

for some $\theta\in\mathbb{R}(X_1)$ and $d_1^{(m)}\in L_s^p(X_1, \hat{\mathscr{X}}, \mu^0\circ\underline{i}_1)$. Furthermore, arguing as we did above, (5.6.7) yields

$$(5.6.9)\qquad \Psi_{[m]}(e^{i(\sigma)\cdot y^{(m)}})=\Psi_{[m]}^{U(t)}(e^{i(\sigma)\cdot y^{(m)}})\omega_t(e^{i(\sigma)\cdot y'})$$

for some $\omega_t\in G_1(\hat{\mathscr{X}}_{(t)})$, with $(X_{(t)}, \hat{\mathscr{X}}_{(t)})\subset(X, \hat{\mathscr{X}})$, $y'\in L_s^p(X_{(t)}, \hat{\mathscr{X}}_{(t)}, \omega_t)_{alg}$ and $y^{(m)}\in L_s^p(X_{[m]}, \hat{\mathscr{X}}_{[m]}, \Psi_{[m]})_{alg}$.

Finally, since the right hand side of (5.6.8) may be written thus:

$$(\varepsilon_\theta\otimes\Psi_{[m]}\otimes\Psi_{[m]}\otimes\cdots\otimes\Psi_{[m]})(e^{i\sigma d_1^{(m)}}\otimes e^{i(\sigma)\cdot y^{(m)}}\otimes e^{i(\sigma)\cdot y^{(m)}}\otimes\cdots \cdots\otimes e^{i(\sigma)\cdot y^{(m)}})$$
$$=\varepsilon_\theta(e^{i\sigma d_1^{(m)}})\ (\Psi_{[m]}(e^{i(\sigma)\cdot y^{(m)}}))^m, \text{ then we have}$$
$$\mu(e^{i(\sigma)\cdot x})$$
$$=(\mu^{1/m}(e^{i(\sigma)\cdot x^{1/m}}))^m$$

where $\mu^{1/m}=\varepsilon_{\theta/m}\otimes\Psi_{[m]}$ and $x^{1/m}=d_1^{(m)}\otimes y^{(m)}$. Hence, the pair (μ, x) is infinitely divisible, as claimed. Notice too that from (5.6.9), one readily infers that $U(t)_{\hat{\mathscr{X}}^{1/m}}\in D^p(\mu^{1/m}, \hat{\mathscr{X}}^{1/m})$, $1\leq p\leq\infty$, where $\hat{\mathscr{X}}^{1/m}=\hat{\mathscr{X}}_1\otimes\hat{\mathscr{X}}_{[m]}$. This concludes the proof. □

Remark: The following corollaries, whose proofs we omit, may be readily established.

(5.7) **Corollary**: *Let* $(\mu, x)\in\lim(G_1^{(p)}(\hat{\mathscr{X}}))$. *If* $D^p(\mu, \hat{\mathscr{X}})$ *contains a* *one-parameter semigroup* $\{exp\, tH: t\geq 0\}$, $H\in B(L^p(X,\hat{\mathscr{X}},\mu^0))$, $1\leq p\leq\infty$, *such that* $\lim_{t\to\infty} exp\, tH=0$ *in the norm-topology of* $B(L^p(X,\hat{X},\mu^0))$, *then* $\mu(e^{i(\sigma)\cdot x})\neq 0$, *for all* $\sigma\in\mathbb{R}$.

(5.8) Corollary: *Suppose that* $(\mu, x)\in\lim(G_1^{(p)}(\hat{\mathscr{X}}))$ *satisfies the hypotheses of Theorem* (5.6). *Suppose, moreover, that for some* $(X(t), \hat{\mathscr{X}}(t))\subset(X, \hat{\mathscr{X}})$ *and* $\nu_t\in G_1(\hat{\mathscr{X}}(t))$, *we have*

$$\mu(e^{i(\sigma)\cdot x})=\mu^{U(t)}(e^{i(\sigma)\cdot x})\nu_t(e^{i(\sigma)\cdot x^{(t)}})$$

for some $x^{(t)}\in L_s^p(X(t),\hat{\mathscr{X}}(t),\nu_t)_{alg}$, $\sigma\in\mathbb{R}$, *and* $t\geq 0$. *Then* $(\nu_t, x^{(t)})$ *is an inflnitely divisible pair, for each* $t\geq 0$.

(5.9) Theorem: *Let* $(\mu, x)\in G_1^{(p)}(\hat{\mathscr{X}})$ *be nondegenerate. Then,* (μ, x) *lies in* $\lim(G_1^{(p)}(\hat{\mathscr{X}}))$ *if and only if, the decomposability algebra structure* $D^p(\mu,\hat{\mathscr{X}})$ *of* μ *contains a one-parameter semigroup* $\{exp\, tH: t\geq 0\}$ *with the property that* $\lim_{t\to\infty} exp\, tH=0$ *in the norm-topology of* $B(L^p(X,\hat{\mathscr{X}},\mu^0))$.

Proof: The conditions of the theorem are necessary. To see this, we argue as follows. Suppose that $(\mu, x)\in\lim(G_1^{(p)}(\hat{\mathscr{X}}))$. By Proposition (4.7), there is a norming sequence $\{A_{1n}\}_{n\geq 1}$ corresponding to the nondegenerate pair (μ, x) with the property that $[A_{1n}^{-1}A_{1n+1}]\to I$ in $B(L^p(X,\hat{\mathscr{X}},\mu^0))$. By Proposition (4.2), I lies in $\mathrm{Sem}(F)$. By the repeated use of Theorem (5.3), we obtain a set $\{P^{(0)}=I, P^{(1)},\dots, P^{(r)}\}$ of projection operators and a set $\{H^{(1)}, H^{(2)},\dots, H^{(r)}\}$ of operators with the following properties: $\mathscr{S}_{P^{(j)}}$ contains the one-parameter semigroup

$$\begin{aligned}&P^{(j)}\exp tH^{(j+1)},\ t\geq 0,\ P^{(j)}H^{(j+1)}=H^{(j+1)}P^{(j)}=H^{(j+1)},\\ &P^{(j+1)}\in\mathscr{S}_{P^{(j)}},\ P^{(j+1)}H^{(j+1)}=H^{(j+1)},\ P^{(j)}\neq P^{(j+1)}\quad\text{and}\\ &\lim_{t\to\infty}(P^{(j)}-P^{(j+1)})\exp tH^{(j+1)}=0,\ j=0,1,2,\dots,r-1.\end{aligned}$$

Furthermore, in view of the compactness of $\mathrm{Sem}(F)$, we may assume that $P^{(r)}=0$. Now, the condition $P^{(r-1)}\in\mathscr{S}_{P^{(j)}}$ implies $P^{(j)}P^{(j-1)}=P^{(j-1)}P^{(j)}=P^{(j)}$. Hence, by Proposition (2.4), the projection operator $Q^{(j)}=P^{(j-1)}-P^{(j)}=P^{(j-1)}(I-P^{(j)})$ lies in $D^p(\mu,\hat{\mathscr{X}})$, $1\leq p\leq\infty$. Set $\sum_{j=1}^{r} Q^{(j)}H^{(j)}=H$. Then, $expt\, H=\sum_{j=1} Q^{(j)}\, exp\, tH^{(j)}$, and (again by Proposition (2.4)) $exp\, tH\in D^p(\mu,\hat{\mathscr{X}})$, $t\geq 0$, $1\leq p\leq\infty$. Hence, the conditions are indeed necessary.

The conditions are also sufficient. To see this, assume that $(\mu, x)\in G_1(\hat{\mathscr{X}})\times L^p(X,\hat{\mathscr{X}},\mu^0)$ and that $D^p(\mu,\hat{\mathscr{X}})$ contains $\{exp\, tH: t\geq 0\}$, $H\in B(L^p(X,\hat{\mathscr{X}},$

μ^0)), with $\lim_{t\to\infty} exp\, tH=0$ in $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$. Then, in particular, I lies in $D^p(\mu, \hat{\mathscr{X}})$, $1\le p\le\infty$. We wish to demonstrate that (μ, x) lies in $\lim (G_1^{(p)}(\hat{\mathscr{X}}))$.

Set $\exp(n^{-1}H)\equiv B^{(n)}$, $n=1, 2,\dots$. Since $I\in D^p(\mu, \hat{X})$, it follows that there are $(X_{1j}, \hat{\mathscr{X}}_{1j})\subset(X_1, \hat{\mathscr{X}}_1)$, $j=1, 2,\dots, k$, and $\nu_{1j}\in G_1(\hat{\mathscr{X}}_{1j})$, $j=2, 3,\dots, k$ such that

$$(5.9.1)\quad \mu(\underline{i}_k(\Gamma_{\hat{\mathscr{X}}_{11}}(B^{(1)})e^{i\sigma x_{11}}\otimes\Gamma_{\hat{\mathscr{X}}_{12}}(B^{(2)})e^{i\sigma x_{12}}\otimes\cdots\otimes\Gamma_{\hat{\mathscr{X}}_{1k-1}}(B^{(k-1)})\cdot e^{i\sigma x_{1k-1}}\otimes e^{i\sigma x_{1k}}))$$
$$=\mu_{\hat{\mathscr{X}}_{11}}^{B^{(1)}}(e^{i\sigma x_{11}})\nu_{12}^{B_{22}}(e^{i\sigma x_{12}})\cdots\nu_{1k-1}^{B_{k-1,k-1}}(e^{i\sigma x_{1k-1}})\nu_{1k}(e^{i\sigma x_k}),$$

for all $x_{11}\in L_s^p(X_{11}, \hat{\mathscr{X}}_{11}, \mu_{\hat{\mathscr{X}}_{11}}^{B^{(1)}})$, $x_{1j}\in L_s^p(X_{1j}, \hat{\mathscr{X}}_{1j}, \nu_{1j}^{B_{jj}})$, $j=2, 3,\dots, k$; $\sigma\in\mathbb{R}$, where B_{jj} is the restriction of $B^{(j)}$ to $L^p(X_{1j}, \hat{\mathscr{X}}_{1j}, \nu_{1j})$, $j=2, 3,\dots, k$; $k=2, 3,\dots$. Hence, $\{x_{1j}\}_{j\ge1}$ is a collection of stochastically independent operators. Evidently, we may choose $\{x_{1j}\}_{j\ge1}$ such that $\nu_{1j}(e^{i\sigma x_{1j}})\to1$, as $j\to\infty$, for each $\sigma\in\mathbb{R}$, whence, by definition, $\{\nu_{1j}\}_{j\ge1}$ is uniformly infinitesimal with respect to $\{x_{1j}\}_{j\ge1}$.

Next, put $\exp(\sum_{j=1}^{n} j^{-1}H)\equiv A^{(n)}$, $n=1, 2,\dots$ and set $A_{\hat{\mathscr{X}}_{11}}^{(1)}\equiv A_{11}$, $A_{\hat{\mathscr{X}}_{1n}}^{(n)}\equiv A_{1n}$, $n=2, 3,\dots$. Make the definitions:

$$(5.9.2)\qquad \mu_{\hat{\mathscr{X}}_{11}}\circ\Gamma_1(A_{11}^{-1})\equiv\mu_{11} \text{ and } \nu_{1n}\circ\Gamma_1(A_{1n}^{-1})=\mu_{1n},\ n=2, 3,\dots .$$

One readily checks that

$$\{\exp tH\colon t\ge0\}=\mathrm{Sem}(\{[A_{1n}^{-1}A_{1m}]\colon n=1, 2,\dots\}).$$

Hence $\{A_{1j}\}_{j\ge1}$ satisfies (3.5.1) and (3.5.2). Notice that $A^{(n)}\to0$ in $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$ as $n\to\infty$. Hence, $\mu_{1j_n}^{A_{1n}}(e^{i\sigma x_{1j_n}})\to1$, whenever $\{j_n\}_{n\ge1}$ is bounded. For $j_n\le n$ and $j_n\to\infty$, we have by (5.9.2) that $\mu_{1j_n}^{A_{1n}}=\nu_{1j_n}^{A_{1j_n}^{-1}A_{1n}}$ and hence $\mu_{1j_n}^{A_{1n}}(e^{i\sigma x_{1j_n}})\to1$, since $\{[A_{1j_n}^{-1}A_{1n}]\colon j_n=1, 2,\dots, n\}_{n\ge1}$ is precompact in $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$ and $\nu_{1j}(e^{i\sigma x_{1j}})\to1$. Hence, a part of (3.5.3) is also fulfilled.

Finally, we have

$$(\bigotimes_{j=1}^{n}\mu_{1j}^{A_{1n}})(\bigotimes_{j=1}^{n}e^{i\sigma x_{1j}})$$
$$=\mu_{\hat{\mathscr{X}}_{11}}(e^{i\sigma x_{11}})\nu_{12}(e^{i\sigma A_{12}^{-1}A_{1n}x_{12}})\cdots\nu_{1n-1}(e^{i\sigma A_{1n-1}^{-1}A_{1n}x_{1n-1}})\nu_{1n}(e^{i\sigma x_{1n}})$$
$$=\mu(\underline{i}_n(e^{i\sigma x_{11}}\otimes e^{i\sigma A_{12}^{-1}A_{1n}x_{12}}\otimes\cdots\otimes e^{i\sigma A_{1n-1}^{-1}A_{1n}x_{1n-1}}\otimes e^{i\sigma x_{1n}})),$$

by (5.9.1) and (5.9.2), since $exp\, tH\in D^p(\mu, \hat{\mathscr{X}})$, by hypothesis, for each $t\ge0$.

It is clear that we may assume that the previous choice of $\{x_{1j}\}_{j\ge1}$ assures the convergence of the right hand side of the last equation to $\mu(e^{i(\sigma)\cdot x})$, for each $\sigma\in\mathbb{R}$. This ends the proof. □

§6. A Characterization of Certain Semigroups Contained in $D^p(\mu, \hat{\mathscr{X}})$

Let $H \in B(L^p(X, \hat{\mathscr{X}}, \mu^0))$ and $\{exp\, tH: t \geq 0\}$ be a one-parameter semigroup of operators contained in $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$ with the property that $\lim_{t\to\infty} exp\, tH = 0$ in the norm-topology of $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$. In this section, we answer the question:

When is the semigroup $\{exp\, t\mathrm{H}: t \geq 0\}$ *contained in* $D^p(\mu, \hat{\mathscr{X}})$, $\mu \in G_1(\hat{\mathscr{X}})$, $1 \leq p \leq \infty$?

In answering this query, we supply a characterization of the infinitesimal generator H.

Notation: 1. Let $L^p(X, \hat{\mathscr{X}}, \mu)^*$ denote the topological dual of $L^p(X, \hat{\mathscr{X}}, \mu)$. We write $\langle\cdot,\cdot\rangle_{(\mu)}$ for the canonical duality pairing of $L^p(X, \hat{\mathscr{X}}, \mu)^*$ and $L^p(X, \hat{\mathscr{X}}, \mu)$.

2. The *adjoint* or *dual* of an operator $A \in B(L^p(X, \hat{\mathscr{X}}, \mu))$ relative to $\langle\cdot,\cdot\rangle_{(\mu)}$ will be denoted by $A^\#$, i.e.

$$\langle a_*, Ab\rangle_{(\mu)} = \langle A^\# a_*, b\rangle_{(\mu)}, \text{ for all } (a_*, b) \in L^p(X, \hat{\mathscr{X}}, \mu)^* \times L^P(X, \hat{\mathscr{X}}, \mu).$$

Evidently, $A^\#: L^p(X, \hat{\mathscr{X}}, \mu)^* \to L^p(X, \hat{\mathscr{X}}, \mu)^*$.

(6.1) Definition: Let $(\mu, b) \in G_1^{(p)}(\hat{\mathscr{X}})$. Then, we say that (μ, b) is a *symmetric Gaussian pair* if and only if,

(1) there is a compact operator R: $L^p(X, \hat{\mathscr{X}}, \mu) \to L^P(X, \hat{\mathscr{X}}, \mu)^*$ with the properties:
   (i) $\langle Ra, c\rangle_{(\mu)} = \langle Rc, a\rangle_{(\mu)}$ (symmetry), for all $a, c \in L^p(X, \hat{\mathscr{X}}, \mu)$
   (ii) $\langle Ra, a\rangle_{(\mu)} \geq 0$ (*positivity*), for all $a \in L^P(X, \hat{\mathscr{X}}, \mu)$; and

(2) $\mu(e^{i(\sigma)\cdot b}) = e^{-1/2\sigma^2\langle Rb, b\rangle_{(\mu)}}$, for each $\sigma \in \mathbb{R}$.

Notation: (i) We shall say that the operator R occurring in (6.1) is a *covariance* operator corresponding to the symmetric Gaussian pair $(\mu, b) \in$ $\in G_1^{(p)}(\hat{\mathscr{X}})$.

(ii) We denote the collection of all R, such that R is a covariance operator corresponding to some symmetric Gaussian pair, by $\mathrm{Cov}^{(p)}(\hat{\mathscr{X}})$, $1 \leq p \leq \infty$.

(6.2) Definition: Let $(\mu, b) \in G_1^{(p)}(\hat{\mathscr{X}})$. Then, we say that (μ, b) is a *Poissonian pair* if and only if, there is a normal positive trace [24] π on $\hat{\mathscr{X}}_1$, with $\pi(1_{\hat{\mathscr{X}}_1}) < \infty$, and $(\gamma, b_1, c_1) \in \mathbb{R}(X_1) \times L_s^p(X_1, \hat{\mathscr{X}}_1, \mu \circ i_1) \times L_s^p(X_1, \hat{\mathscr{X}}_1, \mu \circ i_1)$ such that

$$\mu(e^{i(\sigma)\cdot b})=e^{i\sigma\hat{\gamma}(b_1)}e^{\pi(\kappa(\sigma,c_1))}$$

where

$$\kappa(\sigma, c_1)=e^{i\sigma c_1}-1_{\hat{\mathscr{X}}_1}-\frac{i\sigma c_1}{1+|c_1|^2},\ \sigma\in\mathbb{R},\text{ and}$$
$$\pi(e_{c_1}(\{0\}))=0.$$

Remarks: (i) We call the functional π occurring in (6.2) a *Poissonian exponent* corresponding to the Poissonian pair (μ, b). Furthermore, we refer to the triple (γ, b_1, c_1) occurring in (6.2) as the *Poissonian data* for (μ, b).

(ii) We denote the collection of all π, such that π is a Poissonian exponent corresponding to some Poissonian pair, by $\mathrm{Pos}(\hat{\mathscr{X}})$

(iii) The following result is established as in Refs. [40–45].

(6.3) Theorem: *Let $(\mu, b)\in G_1^{(p)}(\hat{\mathscr{X}})$ be an infinitely divisible pair. Then, there are a symmetric Gaussian pair $(\mu_{Ga}, b_{Ga})\in G_1^{(p)}(\hat{\mathscr{X}}_{Ga})$ and Poissonian pair $(\mu_{Po}, b_{Po})\in G_1^{(p)}(\hat{\mathscr{X}}_{Po})$, for some $(X_{Ga}, \hat{\mathscr{X}}_{Ga})\subset(X, \hat{\mathscr{X}})$ and $(X_{Po}, \hat{\mathscr{X}}_{Po})\subset$ $\subset(X, \hat{\mathscr{X}})$ such that*

$$\mu(e^{i(\sigma)\cdot b})=\mu_{Ga}(e^{i(\sigma)\cdot b_{Ga}})\ \mu_{Po}(e^{i(\sigma)\cdot b_{Po}}),\ \sigma\in\mathbb{R}. \tag{6.3.1}$$

(6.4) *Remark*: (i) It follows from (6.3.1) that (6.3.1) may be written thus:

$$\mu(e^{i(\sigma)\cdot b})=\mu(e^{i(\sigma)\cdot b_{Ga}}e^{i(\sigma)\cdot b_{Po}})=\mu_{Ga}(e^{i(\sigma)\cdot b_{Ga}})\mu_{Po}(e^{i(\sigma)\cdot b_{Po}}),\ \sigma\in\mathbb{R}. \tag{6.4.1}$$

(ii) Let $\pi\in\mathrm{Pos}(\hat{\mathscr{X}})$. Then, π extends to a central positive linear functional, denoted again by π, on $\mathscr{X}_1$.

(iii) If $\pi\in\mathrm{Pos}(\hat{\mathscr{X}})$ and $A\in B(L^p(X, \hat{\mathscr{X}}, \mu))_+$, we write π^A for $\pi\circ\Gamma_{\hat{\mathscr{X}}_1}(A)$, where as usual, $\Gamma_{\hat{\mathscr{X}}_1}(A)$ is the restriction of $\Gamma(A)$ to $\mathscr{D}^p(\mathbb{R}, \hat{\mathscr{X}}_1, \mu_{\hat{\mathscr{X}}_1})$.

(6.5) Proposition: *Let $(\mu, b)\in G_1^{(p)}(\hat{\mathscr{X}})$ be an infinitely divisible pair whose symmetric Gaussian pair (μ_{Ga}, b_{Ga}) has covariance R and Poissonian pair (μ_{Po}, b_{Po}) has Poissonian exponent π. Let $A\in D^p(\mu, \hat{\mathscr{X}})$, with*

$$\mu(e^{i(\sigma)\cdot b})=\mu^A(e^{i(\sigma)\cdot b})\nu_A(e^{i(\sigma)\cdot b_A})$$

for some $(X_A, \hat{\mathscr{X}}_A)\subset(X, \hat{\mathscr{X}})$, $\nu_A\in G_1(\hat{\mathscr{X}}_A)$ and $b_A\in L_s^p(X_A, \hat{\mathscr{X}}_A, \nu_A)_{alg}$. Assume, moreover, that (ν_A, b_A) is an infinitely divisible pair. Then, $A_{\hat{\mathscr{X}}_{Ga}}\in D^p(\mu_{Ga}, \hat{\mathscr{X}}_{Ga})$ and $A_{\hat{\mathscr{X}}_{Po}}\in D^p(\mu_{Po}, \hat{\mathscr{X}}_{Po})$, where $\hat{\mathscr{X}}_{Ga}$ (resp. $\hat{\mathscr{X}}_{Po}$) *is the W^*-algebra contained in $\hat{\mathscr{X}}$ generated by the spectral projections of b_{Ga}* (resp. of b_{Po}). *Furthermore, $R-A^{\#}{}_{\hat{\mathscr{X}}_{Ga}}RA_{\hat{\mathscr{X}}_{Ga}}$ lies in* $\mathrm{Cov}^{(p)}(\hat{\mathscr{X}}_{Ga})$ *and $\pi-\pi^A$ lies in* $\mathrm{Pos}(\hat{\mathscr{X}}_{Po})$.

Proof: Let (μ_{AGa}, b_{AGa}) and (μ_{APo}, b_{APo}) denote the symmetric Gaussian pair, with R_{AGa} as its corresponding covariance operator, and the Poissonian pair, with π_{APo} as its corresponding Poissonian pair, respectively, of the pair (ν_A, b_A) which, by hypothesis, is infinitely divisible. Then, by (6.4.1), we have

$$(6.5.1) \qquad \nu_A(e^{i(\sigma)\cdot b_A}) = \nu_A(e^{i(\sigma)\cdot b_{AGa}} e^{i(\sigma)\cdot b_{APo}}) = \mu_{AGa}(e^{i(\sigma)\cdot b_{AGa}})\mu_{APo}(e^{i(\sigma)\cdot b_{APo}}),$$

$\sigma \in \mathbb{R}$. Also, by the assumed infinite divisibility of the pair (μ, b), we have

$$(6.5.2) \quad \mu(e^{i(\sigma)\cdot b}) = \mu(e^{i(\sigma)\cdot b_{Ga}} e^{i(\sigma)\cdot b_{Po}}) = \mu_{Ga}(e^{i(\sigma)\cdot b_{Ga}})\mu_{Po}(e^{i(\sigma)\cdot b_{Po}}),$$

$\sigma \in \mathbb{R}$. Hence

$$(6.5.3) \qquad \mu^A(e^{i(\sigma)\cdot b_{Ga}} e^{i(\sigma)\cdot b_{Po}}) = \mu^A_{Ga}(e^{i(\sigma)\cdot b_{Ga}})\mu^A_{Po}(e^{i(\sigma)\cdot b_{Po}}),\ \sigma \in \mathbb{R},$$

since $\mathscr{D}^p(\mathbb{R}, \hat{\mathfrak{X}}_{Ga}, \mu_{Ga})$ and $\mathscr{D}^p(\mathbb{R}, \hat{\mathfrak{X}}_{P0}, \mu_{P0})$ are invariant, by (2.1), under $\Gamma_{\hat{\mathfrak{X}}_{Ga}}(A)$ and $\Gamma_{\hat{\mathfrak{X}}_{Po}}(A)$, respectively. But

$$\begin{aligned}
&\mu(e^{i(\sigma)\cdot b_{Ga}} e^{i(\sigma)\cdot b_{Po}}) \\
&= \mu_{Ga}(e^{i(\sigma)\cdot b_{Ga}})\mu_{Po}(e^{i(\sigma)\cdot b_{Po}}), \text{ by (6.5.2)} \\
&= \mu^A(e^{i(\sigma)\cdot b})\nu_A(e^{i(\sigma)\cdot b_A}), \text{ by hypothesis} \\
&= \mu^A_{Ga}(e^{i(\sigma)\cdot b_{Ga}})\mu^A_{Po}(e^{i(\sigma)\cdot b_{Po}})\mu_{AGa}(e^{i(\sigma)\cdot b_{AGa}})\mu_{APo}(e^{i(\sigma)\cdot b_{APo}}),
\end{aligned}$$

by (6.5.1) and (6.5.3).

Identifying the Gaussian and Poissonian parts of the foregoing decomposition, one gets

$$(6.5.4) \qquad \mu_{Ga}(e^{i(\sigma)\cdot b_{Ga}}) = \mu^A_{Ga}(e^{i(\sigma)\cdot b_{Ga}})\mu_{AGa}(e^{i(\sigma)\cdot b_{AGa}}) \quad \text{and}$$

$$(6.5.5) \qquad \mu_{Po}(e^{i(\sigma)\cdot b_{Po}}) = \mu^A_{Po}(e^{i(\sigma)\cdot b_{Po}})\ \mu_{APO}\ (e^{i(\sigma)\cdot b_{APo}}),\ \sigma \in \mathbb{R}.$$

From

$$\begin{aligned}
&\mu_{Ga}\ (e^{i(\sigma)\cdot b_{Ga}}) = e^{-1/2\sigma^2\langle Rb_{Ga}, b_{Ga}\rangle}(\mu_{G\alpha}), \\
&\mu^A_{Ga}(e^{i(\sigma)\cdot b_{Ga}}) = e^{-1/2\sigma^2\langle RA_{\hat{\mathfrak{X}}_{Ga}} b_{Ga}, A_{\hat{\mathfrak{X}}_{Ga}} b_{Ga}\rangle}(\mu_{G\alpha}), \quad \text{and} \\
&\mu_{AGa}\ (e^{i(\sigma)\cdot b_{AGa}}) = e^{-1/2\sigma^2\langle R_A b_{AGa}, b_{AGa}\rangle}(\mu_{AG\alpha}),
\end{aligned}$$

where R_A is the covariance operator corresponding to the pair (μ_{AGa}, b_{AGa}), one gets $R - A^{\#}_{\hat{\mathfrak{X}}_{Ga}} RA_{\hat{\mathfrak{X}}_{Ga}}$ lies in Cov$^{(p)}(\hat{\mathfrak{X}}_{Ga})$

Finally, since

$$\begin{aligned}
&\mu_{Po}(e^{i(\sigma)\cdot b_{Po}}) \\
&= e^{i\sigma\hat{\gamma}(y_1)} e^{\pi(\kappa(\sigma, c_1))} \\
&= \mu^A_{Po}(e^{i(\sigma)\cdot b_{Po}})\mu_{APo}(e^{i(\sigma)\cdot b_{APo}}),\ \sigma \in \mathbb{R}, \text{ by (6.5.5)}, \\
&= (e^{i\sigma\hat{\gamma}(A_{\hat{\mathfrak{X}}_1} y_1)} e^{\pi^A(\kappa(\sigma, c_1))})(e^{i\sigma\hat{\gamma}_A(y_{A,1})} e^{\pi_A(\kappa(\sigma, c_{A,1}))}), \text{ where}
\end{aligned}$$

we have employed here self-explanatory notation,

$$= e^{i\sigma(\hat{\gamma}(A_{\hat{\mathscr{X}}_1}y_1)+\hat{\gamma}_A(y_{A,1}))}e^{\pi^A(\kappa(\sigma,c_1))+\pi_A(\kappa(\sigma,c_{A,1}))},$$

we have

$$\mu_{APo}(e^{i(\sigma)\cdot b_{APo}}) = e^{i\sigma(\hat{\gamma}(y_1)-\hat{\gamma}(A_{\hat{\mathscr{X}}_1}y_1))}e^{(\pi-\pi^A)(\kappa(\sigma,c_1))}.$$

Hence $\pi-\pi^A \in \mathrm{Pos}\,(\hat{\mathscr{X}}_{Po})$, as claimed. □

Remark: There is now the following answer to the question we asked at the beginning of this section.

(6.6) **Theorem**: *Let* $H \in B(L^p(X, \hat{\mathscr{X}}, \mu^0))$ *and* $\lim_{t\to\infty} \exp tH = 0$ *in the norm-topology of* $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$. *Let* $(\mu, x) \in \lim (G_1^{(p)}(\hat{\mathscr{X}}))$. *Then,* $D^p(\mu, \hat{\mathscr{X}})$ *contains the one-parameter semigroup* $\{\exp tH: t \geq 0\}$ *if and only if,*

(6.6.1) $\mu(e^{i(\sigma)\cdot x}) = \mu(e^{i(\sigma)\cdot x_{Ga}}\, e^{i(\sigma)\cdot x_{Po}}) = \mu_{Ga}(e^{i(\sigma)\cdot x_{Ga}})\mu_{Po}(e^{i(\sigma)\cdot x_{Po}})$,

for some symmetric Gaussian pair (μ_{Ga}, x_{Ga}), *with corresponding covariance operator* R, *and Poissonian pair* (μ_{Po}, x_{Po}), *with corresponding exponent* π, *such that*

(*i*) $H^{\#}_{\hat{\mathscr{X}}_{Ga}}R+RH_{\hat{\mathscr{X}}_{Ga}}$ *is nonpositive in the sense of* (6.1) (*i*) (*ii*); *and*

(*ii*) $\pi \geq \pi\circ\Gamma_{\hat{\mathscr{X}}_1}(e^{tH})$, *for all* $t \geq 0$.

(*Here* $(X_{Ga}, \hat{\mathscr{X}}_{Ga}) \subset (X, \hat{\mathscr{X}})$ *is as in Theorem* (6.5).)

Proof: The conditions are necessary. To see this suppose that $D^p(\mu, \hat{\mathscr{X}})$ contains $\{\exp tH: t \geq 0\}$, with $\lim_{t\to\infty} \exp tH = 0$ in $B(L^p(X, \hat{\mathscr{X}}, \mu))$. Then, by Theorem (5.6), the pair $(\mu, x) \in \lim (G_1^{(p)}(\hat{\mathscr{X}}))$ is infinitely divisible and by Theorem (6.3), the decomposition (6.6.1) holds. Moreover, by Proposition (6.5), R-$(\exp tH^{\#}_{\hat{\mathscr{X}}_{Ga}})R(\exp tH_{\hat{\mathscr{X}}_{Ga}}) \geq 0$ for all $t \geq 0$ (in the sense of (6.1) (i) (ii)) and $\pi - \pi\circ\Gamma_{\hat{\mathscr{X}}_1}(e^{tH})$ is in $\mathrm{Pos}\,(\hat{\mathscr{X}})$ for all $t \geq 0$. Thus, (ii) is already established. To demonstrate (i), notice that for arbitrary $a \in L^p(X_{Ga}, \hat{\mathscr{X}}_{Ga}, \mu_{Ga})$, we have $\langle [R\text{-}\exp tH^{\#}_{\hat{\mathscr{X}}_{Ga}})R(\exp tH_{\hat{\mathscr{X}}_{Ga}}))]a, a\rangle_{(\mu_{Ga})} = \langle [R - \{R + t(H^{\#}_{\hat{\mathscr{X}}_{Ga}}R + RH_{\hat{\mathscr{X}}_{Ga}}) + 0(t^2)\}]a, a\rangle_{(\mu_{Ga})} = \langle -t(H^{\#}_{\hat{\mathscr{X}}_{Ga}}R + RH_{\hat{\mathscr{X}}_{Ga}})a, a\rangle_{(\mu_{Ga})}$, for t close to zero. Since $t \geq 0$, we infer that $H^{\#}_{\hat{\mathscr{X}}_{Ga}}R + RH_{\hat{\mathscr{X}}_{Ga}}$ is nonpositive relative to $\langle \cdot, \cdot \rangle_{(\mu_{Ga})}$, as defined in (6.1) (i) (ii). This establishes (i), whence we conclude that the conditions are indeed necessary.

The conditions are also sufficient. To see this, let $(\mu, x) \in \lim (G_1^{(p)}(\hat{\mathscr{X}}))$. Assume that $\pi \in \mathrm{Pos}\,(\hat{\mathscr{X}}_{Po})$, $\pi^{(t)} = \pi - \pi\circ\Gamma_{\hat{\mathscr{X}}_1}(\exp tH) \geq 0$, for all $t \geq 0$, $R \in \mathrm{Cov}^{(p)}(\hat{\mathscr{X}}_{Ga})$ and $H^{\#}_{\hat{\mathscr{X}}_{Ga}}R + RH_{\hat{\mathscr{X}}_{Ga}}$ is nonpositive using (6.1) (i) (ii), for some

Poissonian pair (μ_{Po}, x_{Po}), with π as its exponent, and Gaussian pair (μ_{Ga}, x_{Ga}), with R as its covariance, and $(X_{Po}, \hat{\mathscr{X}}_{Po}) \subset (X, \hat{\mathscr{X}})$, $(X_{Ga}, \hat{\mathscr{X}}_{Ga}) \subset (X, \hat{\mathscr{X}})$. Then $\pi^{(t)} \in \mathrm{Pos}(\hat{\mathscr{X}}_{Po})$ for all $t \geq 0$ and, moreover,

$$\mu_{Po}(e^{i(\sigma)\cdot x_{Po}}) = \mu_{Po}^{\exp tH}(e^{i(\sigma)\cdot x_{Po}})\mu_{tPo}(e^{i(\sigma)\cdot x_{tPo}}) \tag{6.6.2}$$

where $(\mu_{Po}^{\exp tH}, x_{Po})$ and (μ_{tPo}, x_{tPo}) are Poissonian pairs with corresponding Poissonian exponents $\pi \circ \Gamma_{\hat{\mathscr{X}}_1}(\exp tH)$ and $\pi^{(t)}$, respectively, $t \geq 0$.

Next for arbitrary $a \in L^p(X_{Ga}, \hat{\mathscr{X}}_{Ga}, \mu_{Ga})$, put

$$\xi(t) = \langle [R - (\exp tH^{\#}_{\hat{\mathscr{X}}_{Ga}})R(\exp tH_{\hat{\mathscr{X}}_{Ga}})]a, a\rangle_{(\mu_{Ga})}.$$

Then

$$\begin{aligned}\frac{d\xi}{dt}(t) &= -\langle (\exp tH^{\#}_{\hat{\mathscr{X}}_{Ga}})(H^{\#}_{\hat{\mathscr{X}}_{Ga}}R + RH_{\hat{\mathscr{X}}_{Ga}})(\exp tH_{\hat{\mathscr{X}}_{Ga}})a, a\rangle_{(\mu_{Ga})} \\ &= -\langle (H^{\#}_{\hat{\mathscr{X}}_{Ga}}R + RH_{\hat{\mathscr{X}}_{Ga}})(\exp tH_{\hat{\mathscr{X}}_{Ga}})\, a, (\exp tH_{\hat{\mathscr{X}}_{Ga}})a\rangle_{(\mu_{Ga})}, \ t \geq 0.\end{aligned}$$

Thus, $\frac{d\xi}{dt} \geq 0$, since $H^{\#}_{\hat{\mathscr{X}}_{Ga}}R + RH_{\hat{\mathscr{X}}_{Ga}}$ is, by hypothesis, nonpositive. But $\xi(0) = 0$. Hence, $R - (\exp tH^{\#}_{\hat{\mathscr{X}}_{Ga}})R(\exp tH_{\hat{\mathscr{X}}_{Ga}})$ is nonnegative relative to $\langle \cdot, \cdot \rangle_{(\mu_{Ga})}$. This means that $R(t) = R - (\exp tH^{\#}_{\hat{\mathscr{X}}_{Ga}})R(\exp tH_{\hat{\mathscr{X}}_{Ga}})$ lies in $\mathrm{Cov}^{(p)}(\hat{\mathscr{X}}_{Ga})$, for each $t \geq 0$. Let (μ_{tGa}, x_{tGa}) be the symmetric Gaussian pair with $R(t)$ as its corresponding covariance operator. Then

$$\mu_{Ga}(e^{i(\sigma)\cdot x_{Ga}}) = \mu_{Ga}^{\exp tH}(e^{i(\sigma)\cdot x_{Ga}})\mu_{tGa}(e^{i(\sigma)\cdot x_{tGa}}), \tag{6.6.3}$$

where the symmetric Gaussian pair $(\mu_{Ga}^{\exp tH}, x_{Ga})$ evidently has $(\exp tH^{\#}_{\hat{\mathscr{X}}_{Ga}})R \cdot (\exp tH_{\hat{\mathscr{X}}_{Ga}})$, $t \geq 0$, as its corresponding covariance operator. From (6.6.2) and (6.6.3), we get

$$\begin{aligned}&\mu_{Ga}(e^{i(\sigma)\cdot x_{Ga}})\mu_{Po}(e^{i(\sigma)\cdot x_{Po}}) \\ &= \mu_{Ga}^{\exp tH}(e^{i(\sigma)\cdot x_{Ga}})\mu_{tGa}(e^{i(\sigma)\cdot x_{tGa}})\mu_{Po}^{\exp tH}(e^{i(\sigma)\cdot x_{Po}})\mu_{tPo}(e^{i(\sigma)\cdot x_{tPo}}) \\ &= \mu_{Ga}^{\exp tH}(e^{i(\sigma)\cdot x_{Ga}})\mu_{Po}^{\exp tH}(e^{i(\sigma)\cdot x_{Po}})\mu_{tGa}(e^{i(\sigma)\cdot x_{tGa}})(e^{i(\sigma)\cdot x_{tPo}}).\end{aligned} \tag{6.6.4}$$

But, since $(\mu, x) \in \lim(G_1^{(p)}(\hat{\mathscr{X}}))$ is such that

$$\mu(e^{i(\sigma)\cdot x}) = \mu_{Ga}(e^{i(\sigma)\cdot x_{Ga}})\mu_{Po}(e^{i(\sigma)\cdot x_{Po}}), \ \sigma \in \mathbb{R}, \tag{6.6.1}$$

then, (6.6.4) gives

$$\mu(e^{i(\sigma)\cdot x}) = \mu^{\exp tH}(e^{i(\sigma)\cdot x})\nu_t\,(e^{i(\sigma)\cdot x_t}) \tag{6.6.5}$$

where

$$\nu_t = \mu_{tGa} \otimes \mu_{tPo} \quad \text{and}$$

$$x_t = x_{tGa} \otimes x_{tPo}, \quad t \geq 0.$$

We conclude from (6.6.5) that $\{\exp tH\colon t \geqq 0\} \subset D^p(\mu, \hat{\mathscr{X}})$. Hence, the conditions are indeed also sufficient. □

§7. A Representation for Nondegenerate Limit Parirs

Let $(\mu, x) \in \lim (G_1(\hat{\mathscr{X}}))$ be nondegenerate. It this section, we obtain a representation for $\mu(e^{i(\sigma)\cdot x})$, $\sigma \in \mathbb{R}$. Our representation may be compared with the one obtained by Urbanik [5] in the case of Banach-space-valued random variables.

In the sequel, $H \in B(L^p(X, \hat{\mathscr{X}}, \mu^0))$ and $U(t) = \exp tH$, $t \geq 0$, with $\lim_{t\to\infty} U(t) = 0$ in the norm-topology of $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$. Furthermore, π is a fixed member of $\mathrm{Pos}(\hat{\mathscr{X}})$ throughout the ensuing discussion.

(7.1) **Definition:** A member $h_1 \in \hat{\mathscr{X}}_1^+$ will be called a weight-operator provided that

1. $h_1^+ = \int_{0+}^{\infty} e_{h_1}(d\lambda)\lambda$ is invertible;
2. $h_1 \leq l|h_{10}|^2$ for some positive number l and some $0 \neq h_{10} \in \hat{\mathscr{X}}_1$;
3. h_1 is separating for $\mathrm{Pos}(\hat{\mathscr{X}})$, i.e. if $\pi_1, \pi_2 \in \mathrm{Pos}(\hat{\mathscr{X}})$, with $\pi_1(h_1) = \pi_2(h_1)$, then $\pi_1 = \pi_2$; and
4. $\pi(h_1) < \infty$, for all $\pi \in \mathrm{Pos}(\hat{\mathscr{X}})$.

(7.2) **Notation:** 1. We denote the set of all weight-operators in $\hat{\mathscr{X}}_1^+$ by $W(\hat{\mathscr{X}})$.

2. Let $\mathfrak{A}$ be an arbitrary subset of $\hat{\mathscr{X}}_1$.

Then we define $\hat{\mathscr{X}}_1(\mathfrak{A})$ as follows: $\hat{\mathscr{X}}_1(\mathfrak{A}) \equiv W^*$-subalgebra of $\hat{\mathscr{X}}_1$ generated by $\{\Gamma_{\hat{\mathscr{X}}_1}(U(t))z\colon z \in \mathfrak{A} \text{ and } t \in \mathbb{R}\}$ (Here and hereafter, we use the notation of (6.4) (ii).)

Remark: The following result is employed in the sequel.

(7.3) **Proposition:** *For each $\pi \in \mathrm{Pos}(\hat{\mathscr{X}})$, there exists a sequence $\{\mathfrak{A}_n\}_{n\geq 1}$ of subsets of $\hat{\mathscr{X}}_1$ such that $\hat{\mathscr{X}}_1(\mathfrak{A}_m) \cap \hat{\mathscr{X}}_1(\mathfrak{A}_n) = \{0\}$, if $m \neq n$, and $\pi = \sum_{n=1}^{\infty} \pi \circ P_{\mathfrak{A}_n}$ on $\hat{\mathscr{X}}_1$, where $P_{\mathfrak{A}_n}$ is the projection of $\hat{\mathscr{X}}_1$ onto $\hat{\mathscr{X}}_1(\mathfrak{A}_n)$, $n \geq 1$.*

Proof: Let $h_1 \in W(\hat{\mathscr{X}}_1)$. Since $\pi(h_1) < \infty$, by (7.1) (4), we can find a subset $\mathfrak{A}_1$ of $\hat{\mathscr{X}}_1$ such that $\pi(P_{\mathfrak{A}_1}^{\perp} h_1) < 1$, where $P_{\mathfrak{A}_1}$ is the projection of $\hat{\mathscr{X}}_1$ onto $\hat{\mathscr{X}}_1(\mathfrak{A}_1)$. Now $P_{\mathfrak{A}_1}^{\perp}\hat{\mathscr{X}}_1$ is a W^*-subalgebra of $\hat{\mathscr{X}}_1$. Set $P_{\mathfrak{A}_1}^{\perp}\hat{\mathscr{X}}_1 \equiv \hat{\mathscr{X}}_1(P(\mathfrak{A}_1))$ and

$\pi\circ P_{\mathfrak{A}_1}^{\perp}\equiv\pi_{\mathfrak{A}_1}$, where $P(\mathfrak{A}_1)$ is some subset of $\hat{\mathscr{X}}_1$. Then, $\pi_{\mathfrak{A}_1}(h_1)<\infty$. Hence, there is a subset $\mathfrak{A}_2$ of $P(\mathfrak{A}_1)$ such that $\hat{\mathscr{X}}_1(\mathfrak{A}_2)$ is a W^*-subalgebra of $\hat{\mathscr{X}}(P(\mathfrak{A}_1))$ and $\pi_{\mathfrak{A}_1}(P_{\mathfrak{A}_2}^{\perp}h_1)<1/2$, where $P_{\mathfrak{A}_2}$ is the projection of $\hat{\mathscr{X}}_1(P(\mathfrak{A}_1))$ onto $\hat{\mathscr{X}}_1(\mathfrak{A}_2)$. Since $P_{\mathfrak{A}_1}^{\perp}$ is the identity of $B(\hat{\mathscr{X}}_1(P(\mathfrak{A}_1)))$, the Banach algebra of all mappings of $\hat{\mathscr{X}}_1(P(\mathfrak{A}_1))$ into itself, we have $P_{\mathfrak{A}_2}^{\perp}=P_{\mathfrak{A}_1}^{\perp}-P_{\mathfrak{A}_2}=I-P_{\mathfrak{A}_1}-P_{\mathfrak{A}_2}$. Evidently, $P_{\mathfrak{A}_1}P_{\mathfrak{A}_2}=0=P_{\mathfrak{A}_2}P_{\mathfrak{A}_1}$. Continuing as above, we get a sequence $\{A_n\}_{n\geq 1}$ of subsets of $\hat{\mathscr{X}}_1$ such that $\hat{\mathscr{X}}_1(\mathfrak{A}_m)\cap\hat{\mathscr{X}}_1(\mathfrak{A}_n)=\{0\}$, for $m\neq n$, and $\pi((I-P_{\mathfrak{A}_1}-P_{\mathfrak{A}_2}-\cdots-P_{\mathfrak{A}_n})h_1)<\frac{1}{n}$, where $P_{\mathfrak{A}_n}$ is the projection of $\hat{\mathscr{X}}_1$ onto $\hat{\mathscr{X}}_1(\mathfrak{A}_n)$.

To complete the proof, put $\pi-\sum_{k=1}^{n}\pi\circ P_{\mathfrak{A}_k}\equiv\pi'_n$. Then, $\pi'_n(h_1)<\frac{1}{n}$, implying that $\pi'_n(h_1)\to 0$, as $n\to\infty$. But h_1 is separating for $\mathrm{Pos}(\hat{\mathscr{X}})$. Hence, $\pi=\sum_{k=1}^{\infty}\pi\circ P_{\mathfrak{A}_k}$. □

(7.4) *Remark*: Suppose that $\pi\in\mathrm{Pos}(\hat{\mathscr{X}})$ and $\pi\geq\pi\circ\Gamma_{\hat{\mathscr{X}}_1}(U(t))$, for all $t\geq 0$. Let $\hat{\mathscr{X}}_{10}$ be a W^*-subalgebra of $\hat{\mathscr{X}}_1$. Evidently, if $\Gamma_{\hat{\mathscr{X}}_1}(A)\hat{\mathscr{X}}_{10}\subseteq\hat{\mathscr{X}}_{10}$, $t\geq 0$, then the restriction $\pi_{\hat{\mathscr{X}}_{10}}$ of π to $\hat{\mathscr{X}}_{10}$ belongs to $\mathrm{Pos}(\hat{\mathscr{X}})$ and satisfies $\pi_{\hat{\mathscr{X}}_{10}}\geq\pi_{\hat{\mathscr{X}}_{10}}\circ\Gamma_{\hat{\mathscr{X}}_{10}}(U(t))$, for all $t\geq 0$. Hence, from Proposition (7.3), we obtain the following result.

(7.5) **Proposition**: *Suppose that* $\pi\in\mathrm{Pos}(\hat{\mathscr{X}})$ *and* $\pi\geq\pi\circ\Gamma_{\hat{\mathscr{X}}_1}(U(t))$, *for all* $t\geq 0$. *Then, there is a decomposition* $\pi=\sum_{n=1}^{\infty}\pi_n$, *where* $\pi_n\in\mathrm{Pos}(\hat{\mathscr{X}})$, $\pi_n\geq\pi_n\circ\Gamma_{\hat{\mathscr{X}}_1(\mathfrak{A}_n)}(U(t))$ *for all* $t\geq 0$, *and the supports* $\{\hat{\mathscr{X}}_1(\mathfrak{A}_n)\}_{n\geq 1}$ *of* $\{\pi_n\}_{n\geq 1}$ *are disjoint* W^*-*subalgebras of* $\hat{\mathscr{X}}_1$ *corresponding to some subsets* $\{\mathfrak{A}_n\}_{n\geq 1}$ *of* $\hat{\mathscr{X}}_1$.

(7.6) *Remarks*: 1. Observe that Proposition (7.5) reduces the problem of characterizing the members of $\mathrm{Pos}(\hat{\mathscr{X}})$ satisfying $\pi\geq\pi\circ\Gamma_{\hat{\mathscr{X}}_1}(U(t))$, for all $t\geq 0$, to that of characterizing the central normal positive linear functionals $\pi_{\mathfrak{A}}$ on $\hat{\mathscr{X}}_1(\mathfrak{A})$ satisfying $\pi_{\mathfrak{A}}\geq\pi_{\mathfrak{A}}\circ\Gamma_{\hat{\mathscr{X}}_1(\mathfrak{A})}(U(t))$, for all $t\geq 0$, where $\mathfrak{A}$ is a subset of $\hat{\mathscr{X}}_1$.

2. In the sequel, $\mathscr{L}(\mathfrak{A}, H)$ denotes the set of all positive central normal linear functionals $\pi_{\mathfrak{A}}$ on $\hat{\mathscr{X}}_1(\mathfrak{A})$ such that $\pi_{\mathfrak{A}}\geq\pi_{\mathfrak{A}}\circ\Gamma_{\hat{\mathscr{X}}_1(\mathfrak{A})}(U(t))$, for all $t\geq 0$, where $\mathfrak{A}$ is a subset of $\hat{\mathscr{X}}_1$. We shall characterize $\mathscr{L}(\mathfrak{A}, H)$ by employing the theory of *barycentric decomposition of states* on a W^*-algebra.

3. Let $\mathfrak{A}$ be a subset of $\hat{\mathscr{X}}_1$ and $\hat{\mathscr{X}}_1(\mathfrak{A})$ be as previously defined. Let $\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))$ denote the state space of $\hat{\mathscr{X}}_1(\mathfrak{A})$. Then, $S(\hat{\mathscr{X}}_1(\mathfrak{A}))$ is compact in the $\sigma(\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A})), \hat{\mathscr{X}}_1(\mathfrak{A}))$-topology.

4. Let $R^*=[-\infty, \infty]$ be the usual compactification of R. Let $\mathfrak{S}$ be

a compact subset of $\hat{\mathscr{X}}_1^*$. Then $\mathfrak{S}\times\mathbb{R}$ is, by Tychonov's theorem, a compact space in the induced product topology coming from $\hat{\mathscr{X}}_1^*$ and $\mathbb{R}^*$. Define an equivalence relation in $\mathfrak{S}\times\mathbb{R}^*$ as follows:

$(\varphi_1, t_1)\sim(\varphi_2, t_2)$ for $\varphi_1, \varphi_2\in\mathfrak{S}$ and $t_1, t_2\in\mathbb{R}^*$ if and only if, there exists a real number s such that

$$\varphi_1\circ\Gamma_{\hat{\mathscr{X}}_1}(U(s))=\varphi_2 \quad \text{and} \quad t_2=t_1-s.$$

It is easy to show as in [5] that the relation $\sim$ is continuous. Hence, the quotient space $(\mathfrak{S}\times\mathbb{R}^*)/\sim$, which we denote in the sequel by $\mathfrak{S}^\sim$, is again compact. The coset in $\mathfrak{S}^\sim$ containing $(\varphi, t)\in\mathfrak{S}\times\mathbb{R}^*$ will be denoted by $[\varphi, t]$.

5. For each subset $\mathfrak{A}\subset\hat{\mathscr{X}}_1$ and a compact subset $\mathscr{B}(\mathfrak{A})$ of $\hat{\mathscr{X}}_1(\mathfrak{A})^{*+}$, let $U(\mathscr{B}(\mathfrak{A}))$ denote the set

$$\{\varphi\circ\Gamma_{\hat{\mathscr{X}}_1}(U(t)): \varphi\in\mathscr{B}(\mathfrak{A}) \text{ and } t\in\mathbb{R}\}.$$

Then, the mapping $\varphi\circ\Gamma_{\hat{\mathscr{X}}_1}(U(t))\mapsto[\varphi, t]$, where $\varphi\in\mathscr{B}(\mathfrak{A})$ and $t\in\mathbb{R}$, is an imbedding of $U(\mathscr{B}(\mathfrak{A}))$ into a dense subset of $\mathscr{B}(\mathfrak{A})$. Hence $\mathscr{B}(\mathfrak{A})$ is a compactification of $U(\mathscr{B}(\mathfrak{A}))$. In the sequel, we identify elements $\varphi\circ\Gamma_{\hat{\mathscr{X}}_1}(U(t))$ of $U(\mathscr{B}(\mathfrak{A}))$ and $[\varphi, t]$ of $\mathscr{B}(\mathfrak{A})^\sim$.

(7.7) **Notation:** 1. If $\varphi\in\hat{\mathscr{X}}_1^*$ and $t\geq 0$, we shall denote $\varphi\circ\Gamma_{\hat{\mathscr{X}}_1}(U(t))$ by $\varphi^{U(t)}$ in the sequel.

2. We extend the norm $\|\cdot\|_{\hat{\mathscr{X}}_1^*}$ and the map $\varphi\mapsto\varphi^{U(s)}$, $s\geq 0$, of $U(\mathscr{B}(\mathfrak{A}))$ into $U(\mathscr{B}(\mathfrak{A}))$, onto $\mathscr{B}(\mathfrak{A})^\sim$ by continuity as follows:

$$\|[\varphi, -\infty]\|_{\hat{\mathscr{X}}_1^*}=\infty, \quad \|[\varphi, \infty]\|_{\hat{\mathscr{X}}_1^*}=0,$$

and

$$[\varphi, -\infty]^{U(s)}=[\varphi, -\infty], \quad [\varphi, \infty]^{U(s)}=[\varphi, \infty],$$

where we have denoted the extensions of $\|\cdot\|_{\hat{\mathscr{X}}_1^*}$ and $\varphi\mapsto\varphi^{U(t)}$, $t\geq 0$, again by the same symbols. With these extensions, we get

$$[\varphi, t]^{U(s)}=[\varphi, t+s], \quad (t, s)\in\mathbb{R}^*\times\mathbb{R}^*, \ \varphi\in U(\mathscr{B}(\mathfrak{A})).$$

3. We also specify the actions of $[\varphi\pm\infty]$ on $\hat{\mathscr{X}}_1(\mathfrak{A})$ as follows:

$$([\varphi, \infty])(z)=0 \quad \text{and} \quad ([\varphi, -\infty])(z)=\lim_{\|z\|\to\infty}\varphi^{U(t)}(z).$$

(7.8) **The set $\mathfrak{S}\mathscr{H}(\mathfrak{A}, H)$**

In place of $\mathscr{B}(\mathfrak{A})$, we now consider $\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))$ and its associated sets $U(\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A})))$ and $\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))^\sim$.

For each $\tau\in\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))^\sim$, define π_τ by

(7.8.1) $$\pi_\tau(z)=\tau(h_1^{+-1/2}zh_1^{+-1/2}),\ z\in\hat{\mathscr{X}}_1(\mathfrak{A}),\ h_1\in W(\hat{\mathscr{X}}_1).$$

Then, $\pi_\tau\in\hat{\mathscr{X}}_1(\mathfrak{A})^{*+}$.

Define $\mathscr{H}(\mathfrak{A}, H)$ by

$$\mathscr{H}(\mathfrak{A}, H)=\{\tau\in S(\hat{\mathscr{X}}_1(\mathfrak{A}))^{\sim}\colon \pi_\tau\geq\pi_\tau^{U(t)},\ \text{for all } t\geq 0\}.$$

Evidently, $\mathscr{H}(\mathfrak{A}, H)$ is a closed, convex subset of $\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))^{\sim}$.

Next, for each $\pi_{\mathfrak{A}}\in\mathscr{L}(\mathfrak{A}, H)$, define $\tau_{\mathfrak{A}}^{\pi}$ by

(7.8.2) $$\tau_{\mathfrak{A}}^{\pi}(z)=\pi_{\mathfrak{A}}(h_1^{1/2}zh_1^{1/2}),\ z\in\hat{\mathscr{X}}_1(\mathfrak{A}),\ h_1\in W(\hat{\mathscr{X}}_1).$$

Then, $\tau_{\mathfrak{A}}^{\pi}$ lies in $\hat{\mathscr{X}}_1(\mathfrak{A})^{*+}$. From (7.8.2), we see that

$$\pi_{\mathfrak{A}}(z)=\tau_{\mathfrak{A}}^{\pi}\ (h_1^{+-1/2}zh_1^{+-1/2}),\ z\in\hat{\mathscr{X}}_1(A).$$

Hence, $\pi\in\mathscr{L}(\mathfrak{A}, H)$ if and only if, $\tau_{\mathfrak{A}}^{\pi}$ lies in $\mathscr{H}(\mathfrak{A}, H)$.

Put $$\{\tau\in S(\hat{\mathscr{X}}_1(\mathfrak{A}))^{\sim}\colon \tau(1_{\hat{\mathscr{X}}_1(\mathfrak{A})})=1\}\equiv S_1(\hat{\mathscr{X}}_1(\mathfrak{A}))^{\sim},$$

and denote $\mathscr{H}(\mathfrak{A}, H)\cap S_1(\hat{\mathscr{X}}_1(\mathfrak{A}))^{\sim}$ by $\mathfrak{S}\mathscr{H}(\mathfrak{A}, H)$. Then $\mathfrak{S}\mathscr{H}(\mathfrak{A}, H)$ is a convex, compact subset of $\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))^{\sim}$. We proceed to determine the extreme points of $\mathfrak{S}\mathscr{H}(\mathfrak{A}, H)$.

The extreme points of $S\mathscr{H}(\mathfrak{A}, H)$

For each $\eta\in\hat{\mathscr{X}}_1(\mathfrak{A})^{*+}$, the integral

(7.8.3) $$\int_0^\infty dt\ \eta^{U(t)}\ (h_1)\equiv[c(h_1, H, \eta)]^{-1}$$

is convergent for each $h_1\in W(\hat{\mathscr{X}}_1)$. This observation follows from (7.1) (2) and the compactness of $\{\exp tH\colon t\geq 0\}\cup\{0\}$ in the norm topology of $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$.

Now, for each $\varphi\in U(\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A})))$, define τ_φ as follows:

(7.8.4) $$\tau_\varphi(z)=c(h_1, H, \varphi)\int_0^\infty dt([\varphi, t])(h_1^{1/2}zh_1^{1/2}),\quad z\in\hat{\mathscr{X}}_1(\mathfrak{A}).$$

Then, τ_φ is a state on $\hat{\mathscr{X}}_1(\mathfrak{A})$ for $\varphi\in U(\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A})))$, i.e. $\tau_\varphi\in\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))^{\sim}$. Substituting τ_φ for τ in (7.8.1), we get

$$\begin{aligned}\pi_{\tau_\varphi}(z)&=\tau_\varphi(h_1^{+-1/2}zh_1^{+-1/2})\\&=c(h_1, H, \varphi)\int_0^\infty dt([\varphi, t])(h_1^{1/2}(h_1^{+-1/2}zh_1^{+-1/2})h_1^{1/2})\\&=c(h_1, H, \varphi)\int_0^\infty dt([\varphi, t])(z),\ z\in\hat{\mathscr{X}}_1(\mathfrak{A}).\end{aligned}$$

Analogously,

$$\pi_{\tau_\varphi}^{U(s)}(z)=c(h_1, H, \varphi)\int_s^\infty dt([\varphi, t])(z), \quad s\geq 0, \quad z\in\hat{\mathscr{X}}_1(\mathfrak{A}), \quad \varphi\in U(\mathfrak{S}(\hat{\mathscr{X}}(\mathfrak{A}))).$$

Hence, for each $s\geq 0$, $z\geq 0$,

$$\pi_{\tau_\varphi}(z)-\pi_{\tau_\varphi}^{U(s)}(z)=\int_0^s dt([\varphi, t])(z)\geq 0, \quad \text{for all} \quad \varphi\in U(\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))).$$

We extend the definition of τ_φ to φ in $\mathbb{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))^\sim\backslash U(\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A})))$ by assuming that $\tau_\varphi=\varepsilon_\varphi$. Then, τ_φ again lies in $\mathfrak{S}H(\mathfrak{A}, H)$. Furthermore, the map $\varphi\mapsto\tau_\varphi$ of $\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))^\sim$ into $\mathfrak{S}H(\mathfrak{A}, H)$ is continuous and injective. Hence, this map is a homeomorphism of $\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))^\sim$ onto $\mathfrak{S}\mathscr{H}(\mathfrak{A}, H)$.

Let $\mathrm{Ext}(\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A})))$ denote the set of extreme points of $\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))$. In the sequel $[\mathfrak{S}\mathscr{H}(\mathfrak{A}, H)]$ denotes the set

$$(\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))^\sim\backslash U(\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))))\cup(\mathbb{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))).$$

The set $\mathrm{Ext}(\mathfrak{S}\mathscr{H}(\mathfrak{A}, H))$ of extreme points of $\mathfrak{S}\mathscr{H}(\mathfrak{A}, H)$ admits the following description:

(7.9) **Proposition:** The set $\mathrm{Ext}(\mathfrak{S}\mathscr{H}(\mathfrak{A}, H))$ coincides with the set $\{\tau_\varphi\colon \varphi\in[\mathfrak{S}\mathscr{H}(\mathfrak{A}, H)]\}$.

Proof: Let $\mathfrak{S}$ be a subset of $\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))$. Then, the sets $U(S)$, $\{[\varphi, -\infty]$ $: \varphi\in U(\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A})))\}$ and $\{[\varphi, \infty]\colon \varphi\in U(\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A})))\}$ are invariant under the maps $\varphi\mapsto\varphi^{U(t)}$, $t\in\mathbb{R}$, of $\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))^\sim$ into itself. Hence, the members of $\mathrm{Ext}(\mathfrak{S}\mathscr{H}(\mathfrak{A}, H))$ must be either of the forms $\{[\varphi, -\infty]\}$ and $\{[\varphi, \infty]\}$, $\varphi\in U(\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A})))$, or be contained in sets of the form $U(\{\psi\})$, $\psi\in\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))$. But the positive linear functionals τ_φ, with $\varphi\in\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))^\sim\backslash U(\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A})))$ are all extreme points of $\mathfrak{S}\mathscr{H}(\mathfrak{A}, H)$. Hence, we need now only determine the extreme points of sets of the form $U(\{\psi\})$, $\psi\in\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))$.

For any interval $I\subseteq\mathbb{R}$, let $\hat{\mathscr{X}}_1(\mathfrak{A}, I)$ be the W^*-subalgebra of $\hat{\mathscr{X}}_1(\mathfrak{A})$ generated by $\{\Gamma_{\hat{\mathscr{X}}_1}(U(t))z\colon z\in\mathfrak{A} \text{ and } t\in I\}$. Write P_I for the conditional expectation of $\hat{\mathscr{X}}_1(\mathfrak{A})$ given $\hat{\mathscr{X}}_1(\mathfrak{A}, I)$; denote $P_{(-\infty,t]}$ simply by $P_{\leq t}$, $t\in\mathbb{R}$.

Suppose now that $\tau\in U(\{\psi\})$, $\psi\in\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))$. Then, one readily sees that $\tau\in\mathfrak{S}\mathscr{H}(\mathfrak{A}, H)$ if and only if,

$$\pi_\tau\circ P_{(t_1,t_2)}\geq\pi_\tau^{U(s)}\circ P_{(t_1,t_2)}$$

or, equivalently,

$$\pi_\tau\circ P_{(t_1,t_2)}\geq\pi_\tau\circ P_{(t_1,t_2-s)}$$

for all $s \geq 0$, all $t_1, t_2 \in \boldsymbol{R}$, $t_1 < t_2$ and with π_τ as defined in (7.8.1). Since $P_{(t_1,t_2)} = P_{<t_2} - P_{t_1}$, for all $t_1, t_2 \in \boldsymbol{R}$, $t_1 < t_2$, the last inequality is equivalent to

$$(*) \qquad \pi_\tau \circ (P_{<t_2} - P_{<t_1}) \geq \pi_\tau \circ (P_{<t_2-s} - P_{<t_1-s})$$

for all $s \geq 0$, all $t_1, t_2 \in \boldsymbol{R}$, $t_1 < t_2$. In particular, setting $t_1 = t$ and $t_2 = t+s$ in (*), we get

$$\pi_\tau \circ P_{<t} \leq \frac{1}{2}(\pi_\tau \circ P_{<t+s} + \pi_\tau \circ P_{<t-s})$$

for all $t \in \boldsymbol{R}$ and $s \geq 0$. Hence, the function $t \mapsto \pi_\tau(P_{<t}a_1)$ is convex for all $a_1 \in \hat{\mathscr{X}}(\mathfrak{A})^+$. Consequently, there is a nonnegative, monotone nondecreasing function $t \mapsto \tau_t(a_1)$, $a_1 \in \hat{\mathscr{X}}_1(\mathfrak{A})^+$, such that

$$(**) \qquad \pi_\tau(P_{<s}a_1) = \int_{-\infty}^{s} dt\, \tau_t(a_1),\ s \in \boldsymbol{R},\ a_1 \in \hat{\mathscr{X}}_1(\mathfrak{A})^+.$$

Assuming, as we may, that the function $t \mapsto \pi_\tau(P_{<t}a_1)$ is continuous from the left, for each $a_1 \in \hat{\mathscr{X}}_1(\mathfrak{A})^+$, then $t \mapsto \tau_t(a_1)$ is uniquely determined by τ, for all $a_1 \in \hat{\mathscr{X}}_1(\mathfrak{A})^+$. On the other hand, there is evidently a unique, nonnegative function g_ψ^τ on $\boldsymbol{R}$ such that

$$\tau_t(a_1) = g_\psi^\tau(t)([\psi, t])(a_1),$$

for all $a_1 \in \hat{\mathscr{X}}_1(\mathfrak{A})^+$ and $t \in \boldsymbol{R}$. Hence, (**) may be expressed thus

$$(7.9.1) \qquad \pi_\tau(P_{<s}a_1) = \int_{-\infty}^{s} dt g_\psi^\tau(t)([\psi, t])(a_1),\ a_1 \in \hat{\mathscr{X}}_1(\mathfrak{A})^+,\ s \in \boldsymbol{R},$$

$\tau \in U(\{\psi\})$, $\psi \in \boldsymbol{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))$. It follows now from (7.8.1) that

$$(7.9.2) \quad \tau(a_1) = \pi_\tau(h_1^{1/2}a_1h_1^{1/2}) = \int_{-\infty}^{\infty} dt\, g_\psi^\tau(t)\ ([\psi, t])(h_1^{1/2}\, a_1 h_1^{1/2}),\ h_1 \in \boldsymbol{W}(\hat{\mathscr{X}}_1)$$

$a_1 \in \hat{\mathscr{X}}_1(\mathfrak{A})$, whence

$$(7.9.3) \qquad \tau(1_{\hat{\mathscr{X}}_1}) = 1 = \int_{-\infty}^{\infty} dt g_\psi^\tau(t)([\psi, t])(h_1), \quad \text{if} \quad \tau \in \mathfrak{S}\mathscr{H}(\mathfrak{A}, H).$$

Conversely, any pair (g_ψ^τ, ψ), with $g_\psi^\tau\colon \boldsymbol{R} \to [0, \infty)$ and $\psi \in \mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))$, such that (7.9.3) holds determines a state τ on $\hat{\mathscr{X}}_1(\mathfrak{A})$ by (7.9.2). Moreover, $\pi_\tau(P_{<s}a_1)$, $(s, a_1) \in \boldsymbol{R} \times \hat{\mathscr{X}}_1(\mathfrak{A})^+$, defined as in (7.9.1) satisfies (*), indicating that τ lies in $\mathfrak{S}\mathscr{H}(\mathfrak{A}, H)$. Hence, since ψ is the only extreme point of $\{\psi\} \subset S(\hat{\mathscr{X}}_1(\mathfrak{A}))$ τ in $U(\{\psi\})$ is pure if and only if, g_ψ^τ cannot be expressed as a nontrivial convex combination of two nonnegative functions. But this is possible only in the case $g_\psi^\tau(t) = 0$, if $t \leq t_0$, and $g_\psi^\tau(t) = d(\psi)$, if $t > t_0$, for some

t_0, and $d(\psi)$ in $\mathbb{R}$, depending only on ψ. By (7.9.3), we see that

$$d(\psi)=\int_{t_0}^{\infty} dt([\psi, t])(h_1), \quad h_1 \in W(\hat{\mathscr{X}}_1).$$

From the definition of τ_φ in (7.8.4), it follows that the members of $\mathrm{Ext}(\mathfrak{S}\mathscr{H}(\mathfrak{A}, H))$ are of the form τ_ψ, with $\psi \in [\mathfrak{S}\mathscr{H}(\mathfrak{A}, H)]$. This concludes the proof. □

Remark: Each member of $\mathscr{H}(\mathfrak{A}, H)$ is a positive scalar multiple of a member of $\mathfrak{S}\mathscr{H}(\mathfrak{A}, H)$. Hence, applying the Choquet theory of barycentric decomposition on compact convex sets [46, 47], we get the following result.

(7.10) Proposition: *A member $\omega_{\mathfrak{A}}$ of $\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))^{\sim}$ belongs to $\mathscr{H}(\mathfrak{A}, H)$ if and only if, there exists a probability measure $\Theta_{\omega_{\mathfrak{A}}}$ on $\mathfrak{S}\mathscr{H}(\mathfrak{A}, H)$ such that*

$$\omega_{\mathfrak{A}}(z)=\int_{[\mathfrak{S}\mathscr{H}(\mathfrak{A}, H)]} \Theta_{\omega_{\mathfrak{A}}}(d\varphi)\tau_\varphi(z), \; z \in \hat{\mathscr{X}}_1(\mathfrak{A}).$$

If $\omega_{\mathfrak{A}} \in U(\hat{\mathscr{X}}_1(\mathfrak{A}))$, then $\Theta_{\omega_{\mathfrak{A}}}$ is concentrated on $\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))$.

Remark: From (7.8.2) and (7.8.3), and the considerations at the beginning of (7.8), we get

$$\begin{aligned}\pi_{\mathfrak{A}}(z) &= \tau_A^\pi(h_1^{+-1/2} z h_1^{+-1/2}) \\ &= \int_{\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))} \Theta_{\tau_A^\pi}(d\varphi)\tau_\varphi(h_1^{+-1/2} z h_1^{+-1/2}), \text{ since } \tau_{\mathfrak{A}}^\pi \in \mathscr{H}(\mathfrak{A}, H) \\ &= \int_{\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))} \Theta_{\tau_A^\pi}(d\varphi)c(h_1, H, \varphi)\int_0^\infty dt\, \varphi^{U(t)}(z), \; z \in \hat{\mathscr{X}}_1(\mathfrak{A}),\end{aligned}$$

since we identify $[\varphi, t]$ with $\varphi^{U(t)}$, for $\varphi \in U(\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A})))$. Hence, we have the following assertion:

(7.11) Corollary: *Let $\pi_{\mathfrak{A}}$ be a member of $\mathrm{Pos}(\hat{\mathscr{X}}_1)$, with $\pi \in \hat{\mathscr{X}}_1(\mathfrak{A})_*^+$. Then, $\pi \in \mathscr{L}(\mathfrak{A}, H)$ if and only if, there is a probability measure $\Theta_{\tau_{\mathfrak{A}}^\pi}$ on $\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}))$ such that*

$$\pi_{\mathfrak{A}}(z)=\int_{S(\hat{\mathscr{X}}_1(\mathfrak{A}))} \Theta_{\tau_A^\pi}(d\varphi)c(h_1, H, \varphi)\int_0^\infty dt\, \varphi^{U(t)}(z), \; z \in \hat{\mathscr{X}}_1(\mathfrak{A}),$$

where $c(h_1, H, \varphi)$ is as defined in (7.8.3).

Remark: Let us now characterize those members of $\mathrm{Pos}(\hat{\mathscr{X}}_1)$ which satisfy the condition $\pi \geq \pi^{U(t)}$, for all $t \geq 0$.

By Proposition (7.3), we have a decomposition of the form $\pi=\sum_{n=1}^{\infty} \pi_n$, where

$\pi_n \in \mathrm{Pos}(\hat{\mathscr{X}}_1)$, $\pi_n \geq \pi_n^{U(t)}$, for all $t \geq 0$, the support of π_n is $\hat{\mathscr{X}}_1(\mathfrak{A}_n)$, $\hat{\mathscr{X}}_1(\mathfrak{A}_n) \cap \hat{\mathscr{X}}_1(\mathfrak{A}_m) = \{0\}$, $n \neq m$, and $\{\mathfrak{A}_n\}_{n\geq 1} \subset \hat{\mathscr{X}}_1$. By Corollary (7.11), there is a probability measure $\Theta_{\tau_n^\pi}$ on $\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}_n))$ such that

$$\pi_n(z) = \int_{\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}_n))} \Theta_{\tau_n^\pi}(d\varphi) c(h_1, H, \varphi) \int_0^\infty dt\, \varphi^{U(t)}(z),\ z \in \hat{\mathscr{X}}_1(\mathfrak{A}_n), \quad n \geq 1.$$

Hence

$$\begin{aligned}\pi(z) &= \sum_{n=1}^\infty \pi_n(z) \\ &= \sum_{n=1}^\infty \int_{\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}_n))} \Theta_{\tau_n^\pi}(d\varphi) c(h_1, H, \varphi) \int_0^\infty dt\, \varphi^{U(t)}(z),\ z \in \hat{\mathscr{X}}_1.\end{aligned}$$

Consequently, for $h_1 \in \mathbb{W}(\hat{\mathscr{X}}_1)$, we have

$$\begin{aligned}\pi(h_1) &= \sum_{n=1}^\infty \int_{\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}_n))} \Theta_{\tau_n^\pi}(d\varphi)\ c(h_1, H, \varphi) c(h_1, H, \varphi)^{-1} \quad \text{by (7.8.3)}, \\ &= \sum_{n=1}^\infty \Theta_{\tau_n^\pi}(\mathfrak{S}(\hat{\mathscr{X}}_1(\mathfrak{A}_n))) < \infty, \quad \text{since} \quad h_1 \in \mathbb{W}(\hat{\mathscr{X}}_1).\end{aligned}$$

Thus, setting $\sum_{n=1}^\infty \Theta_{\tau_n^\pi} \equiv \Theta_\pi$, we get a finite Borel measure on $\mathfrak{S}(\hat{\mathscr{X}}_1)$ such that

$$\pi(z) = \int_{\mathfrak{S}(\hat{\mathscr{X}}_1)} \Theta_\pi(d\varphi) c(h_1, H, \varphi) \int_0^\infty dt\, \varphi^{U(t)}(z),\ z \in \hat{\mathscr{X}}_1.$$

Notation: For each $\varphi \in \mathfrak{S}(\hat{\mathscr{X}}_1)$, define $\pi_{h_1, H, \varphi}$ by

(7.12) $\quad \pi_{h_1, H, \varphi}(z) = c(h_1, H, \varphi) \tau_\varphi(h_1^{+-1/2} z h_1^{+-1/2}),\ z \in \hat{\mathscr{X}}_1,\ h_1 \in \mathbb{W}(\hat{\mathscr{X}}_1).$

Then

$$\pi(\kappa(\sigma, c_1)) = \int_{\mathfrak{S}(\hat{\mathscr{X}}_1)} \Theta_\pi(d\varphi) \pi_{h_1, H, \varphi}(\kappa(\sigma, c_1)),\ \sigma \in \mathbb{R},$$

where c_1 is as in (6.2).

From Theorem (6.6) and the foregoing considerations, we now have the following assertion.

(7.13) **Theorem**: *Let $h_1 \in \mathbb{W}(\hat{\mathscr{X}}_1)$, $H \in B(L^p(X, \hat{\mathscr{X}}, \mu^0))$ and $\lim_{t\to\infty} \exp tH = 0$ in the norm topology of $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$. Let $(\mu, x) \in \lim(G_1^{(p)}(\hat{\mathscr{X}}))$. Then, $D^p(\mu, \hat{\mathscr{X}})$ contains $\{\exp tH: t \geq 0\}$ if and only if, there exist a Gaussian pair (μ_{Ga}, x_{Ga}) with a covariance operator R for which $H^\#_{\hat{\mathscr{X}}_{Ga}} R + RH_{\hat{\mathscr{X}}_{Ga}}$ is non-positive in the sense of (6.1) (1) (ii), a Poissonian pair (μ_{Po}, x_{Po}) with Poissonian exponent π and Poissonian data (γ, y_1, c_1), and a finite measure Θ_π on $\mathfrak{S}(\mathscr{X}_1)$, such that*

$$\mu(e^{i(\sigma)\cdot x}) = \exp\left(i\sigma\hat{\gamma}(y_1) - \frac{1}{2}\sigma^2\langle Rx_{G_a}, x_{G_a}\rangle_{(\mu_{Ga})} + \int_{\mathfrak{S}(\mathscr{X}_1)} \Theta_\pi(d\varphi)\pi_{h_1,H,\varphi}(\kappa(\sigma, c_1))\right)$$

where $\pi_{h_1,H,\varphi}$ *is as defined in* (7.12), *for each* $\sigma\in\mathbb{R}$.

Remark: Combining Theorems (5.9) and (6.6), we obtain the following solution of the problem described in Section 3.

(7.14) **Theorem**: *Let* $h_1\in W(\hat{\mathscr{X}}_1)$. *Then, a pair* $(\mu, x)\in G_1^{(p)}(\hat{\mathscr{X}})$, *with* μ *nondegenerate, is a member of* $\lim(G_1^{(p)}(\hat{\mathscr{X}}))$ *if and only if, there exists an operator* H *in* $B(L^p(X, \hat{\mathscr{X}}, \mu^0))$, *with* $\lim_{t\to\infty}\exp tH=0$, *a Gaussian pair* (μ_{Ga}, x_{G_a}) *with a covariance operator* R *for which* $H^\sharp_{\hat{\mathscr{X}}_{Ga}}R+RH_{\hat{\mathscr{X}}_{Ga}}$ *is nonpositive in the sense of* (6.1) (1) (*ii*), *a Poissonian pair* (μ_{Po}, x_{Po}) *with Poissonian exponent* π *and Poissonian data* (γ, y_1, c_1), *and a finite measure* Θ_π *on* $\mathfrak{S}(\hat{\mathscr{X}}_1)$ *such that*

$$\mu(e^{i(\sigma)\cdot x}) = \exp\left(i\sigma\hat{\gamma}(y_1) - \frac{1}{2}\sigma^2\langle Rx_{Ga}, x_{Ga}\rangle_{(\mu_{Ga})} + \int_{\mathfrak{S}(\hat{\mathscr{X}}_1)} \Theta_\pi(d\varphi)_{\pi_{h1,H,\varphi}}(\kappa(\sigma, c_1))\right)$$

for each $\sigma\in\mathbb{R}$.

(7.15) *Remarks*: Results similar to Theorems (7.13) and (7.14) have been obtained by Urbanik [5] in the case of random variables with values in a real Banach space. See also [37].

Acknowledgement

I am grateful to my host Professor H.-J. Borchers for making the facilities of the Institut für Theoretische Physik der Universität Göttingen, West Germany, available to me. Part of this work was done at the Zentrum für interdisziplinäre Forschung (ZiF) der Universität Bielefeld, West Germany. I would therefore like to thank Professor L. Streit for the warm hospitality he offered me at ZiF. Finally, I would like to gratefully acknowledge the financial support of the Alexander von Humboldt Stiftung.

References

[1] Billingsley, P., *Probability and Measure*, John Wiley & Sons, Chichester, 1979.
[2] Levy, P., *Théorie de l'Addition des Variables Aléatoires*, Gauthier–Villars, Paris, 1954.

[3] Heyer, H., *Probability Measures on Locally Compact Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **94**, Springer–Verlag, Berlin, Heidelberg, New York, 1977.

[4] de Acosta, A. and Giné, E., Convergence of Moments and Related Functionals in the General Central Limit Theorem in Banach Spaces, *z. Wahrscheinlichkeitstheorie verw. Geb.*, **48** (1979), 213–231.

[5] Urbanik, K., Lévy's Probability Measures on Banach Spaces, *Studia Mathematica*, **LXIII** (1978), 283–308.

[6] Cushen, C. D. and Hudson, R. L., A Quantum-mechanical Central Limit Theorem, *J. Appl. Probability*, **8** (1971), 454–469.

[7] Hudson, R., A Quantum-mechanical Central Limit Theorem for Anticommuting Observables, *J. Appl. Probability*, **10** (1973), 502–509.

[8] Jajte, R., Convergence of Observables, *In*: *Lecture Notes in Mathematics*, **472** (*Probability–Winter School Proceedings*), Edited by Z. Ciesielski, K. Urbanik & W. A. Woyczyński), 57–68, Springer–Verlag Berlin, Heidelberg, New York (1975).

[9] Ladogórski, C., A Limit Theorem for Triangular Arrays of Representations of Canonical Anticommutation Relations, *ibid.*, 95–106.

[10] Urbanik, K., Stable Symmetric Probability Laws in Quantum Mechanics, *ibid.*, 195–206.

[11] Segal, I. E., A Noncommutative Extension of Abstract Integration, *Ann. of Math.*, **57** (1953), 401–457; *correction*: **58** (1953), 595–596.

[12] Mackey, G. W., *The Mathematical Foundations of Quantum Mechanics*, W. A. Benjamin, New York, 1963.

[13] Davies, E. B., *Quantum Theory of Open Systems*, Academic Press, London, 1976.

[14] Ekhaguere, G. O. S., Markov Fields in Noncommutative Probability Theory on W^*-algebras, *J. Math. Phys.*, **20** (8) (1979), 1679–1683.

[15] ———, On the Existence and Uniqueness of Noncommutative Stochastic Processes, *J. Phys. A., Math. Gen.*, **15** (1982), 3453–3463.

[16] Gross, L., Existence and Uniqueness of Physical Ground States, *J. Funct. Anal.*, **10** (1972), 52–109.

[17] Wilde, I. F., The Free Fermion Field as a Markov Field, *J. Funct. Anal.*, **15** (1974), 12–21.

[18] Hegerfeldt, G. C., Prime Field Decompositions and Infinitely Divisible States on Borchers' Tensor Algebra, *Commun. Math. Phys.*, **45** (1975), 137–151.

[19] Streater, R. F., Infinitely Divisible Representations of Lie Algebras, *z. Wahrscheinlichkeitstheorie verw. Geb.*, **19** (1971), 67–80.

[20] Mathon, D. and Streater, R. F., Infinitely Divisible Representations of Clifford Algebras, *z. Wahrscheinlichkeitstheorie verw. Geb.*, **20** (1971), 308–316.

[21] Simon, B., *The* $P(\Phi)_2$ *Euclidean* (*Quantum*) *Field Theory*, Princeton University Press, Princeton, New Jersey, 1974.

[22] Cook, J., The Mathematics of Second Quantization, *Trans. Amer. Math. Soc.* **74** (1953), 222–245.

[23] Sakai, S., *C*-algebras and W*-Algebras*, Springer–Verlag, Berlin, Heidelberg, New York, 1971.

[24] Takesaki, M., *Theory of Operator Algebras* I, Springer–Verlag, Berlin, Heidelberg, New York, 1979.

[25] Pedersen, G. K., C^*-Algebras And Their Automorphism Groups, Academic Press, London, New York, San Francisco, 1979.

[26] Diximier, J., *Enveloping Algebras*, North-Holland Publishing Company, Amsterdam New York, Oxford, 1977.

[27] Araki, H. and Masuda, T., Positive Cones and L_p-Spaces for von Neumann Algebras. *Publ. RIMS, Kyoto Univ.*, **18** (1982), 339–411.

[28] Masuda, T., L_p-Spaces for von Neumann Algebra with Reference to a Faithful Normal Semifinite Weight, *Publ. RIMS, Kyoto Univ.*, **19** (1983), 673–727.

[29] Gudder, S. P. and Hudson, R. L., A Noncommutative Probability Theory, *Trans. Amer. Math. Soc.*, **245** (1978), 1–41.

[30] Hilsum, H., Les Espaces L_p d'Une Algebre de von Neumann, *J. Funct. Anal.*, **40** (1981), 151–169.

[31] Kosaki, H., Positive cones And L^p-Spaces Associated With A von Neumann Algebra, *J. Operator Theory*, **6** (1981), 13–23.

[32] Nelson, E., Notes on Non-Commutative Integration, *J. Junct. Anal.* **15** (1974), 103–116.

[33] Kunze, R. A., Fourier Transforms on Locally Compact Unimodular Groups, *Trans. Amer. Math. Soc.*, **89** (1958), 519–540.

[34] Gaal, S. A., *Linear Analysis and Representation Theory*, Springer–Verlag, Berlin, Heidelberg, New York, 1973.

[35] Driessler, W. and Wilde, I. F., Stochastic Independence in Noncommutative Probability Theory, *Math. Proc. Cambridge Philos. Soc.*, **86** (1979), 103–114.

[36] Loéve, M., *Probability Theory*, John Wiley, New York, 1950.

[37] Thu, N. V., Limit Theorems for Random Fields, *Dissertationes Mathematicae* **Vol. CLXXXVI**, Państwowe Wydawnictwo Naukowe, Warsaw, 1981.

[38] Paalman-de Miranda, A. B., *Topological Semigroup*, Amsterdam, 1964.

[39] Hille, E. and Phillips, R. S., *Functional Analysis And Semigroups*, Amer. math. Soc. Coll. Publ. **31**, Providence, 1957,.

[40] Dettweiler, E., *Grenzwertsätze für Wahrscheinlichkeitsmasse auf Badrikianschen Räumen*, Thesis, Eberhard–Karls Universität Tübingen, 1974.

[41] Parthasarathy, K. R., *Probability Measures on Metric Spaces*, Academic Press, New York, London, 1967.

[42] Gnedenko, V. B. and Kolmogorov, A. N., *Limit Distributions for Sums of Independent Random Variables*, Cambridge, 1954,.

[43] Guichadet, A., *Symmetric Hilbert Spaces and Related Topics*, Lecture Notes in Mathematics, **261**, Springer–Verlag, Berlin, Heidelberg, New York, 1972.

[44] Varadhan, S. R. S., Limit Theorems for Sums of Independent Random Variables with Values in a Hilbert Space, *Sankhya, The Indian Journal of Statistics*, **24** (1962), 213–238.

45. Tortrat, A., Sur la Structure des Lois Indéfiniment Divisibles dans Les Espaces Vectoriels, *z. Wahrscheinlichkeitstheorie verw. Geb.*, **11** (1969), 311–326.

[46] Bishop, E. and de Leeuw, K., The Representation of Linear Functionals by Means on The Set of Extreme Points, *Ann. Inst. Fourier (Grenoble)*, **9** (1959), 305–331.

[47] Choquet, G., Le Théorem de Représentation Intégrale dans Les Ensemble Convexes Compact, *Ann. Inst. Fourier*, **10** (1960), 333–344.