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On the Simultaneous Transformation of Density Operators by Means of a Completely Positive, Unity Preserving Linear Map

by

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Abstract

Let $\{\omega_1,\ldots,\omega_n\}$ and $\{\sigma_1,\ldots,\sigma_n\}$ be finite sets of normal states over the bounded linear operators acting over some infinite-dimensional separable Hilbert-space. In the paper mutually equivalent sets of necessary and sufficient conditions are derived that there exists a completely positive, unity preserving linear map T transforming $\sigma_1,\ldots,\sigma_n$ simultaneously into $\omega_1\ldots,\omega_n$: $\omega_k=\sigma_k\circ T$, $k=1,\ldots,n$. Of particular interest is this "n-tuple-problem" in case of pairs of density operators (normal states), i.e. for $n=2$. In this situation possible connections with the notion of "*generalized transition probability*" are analyzed, and at least in case of normal states over the bounded operators a characterization of functionals of this type is proposed and applied.

§1. Main Results, Examples

Let $\mathscr{A}$ be a unital C^*-algebra, with unit 1, topological dual $\mathscr{A}^*$, and group of unitary elements $\mathscr{U}(\mathscr{A})$. To each $V\in\mathscr{U}(\mathscr{A})$ let us associate a bounded linear operator T_V over the dual by $T_V(\omega)(X)=\omega(VXV^*)=\omega^V(X)$, for all $X\in\mathscr{A}$ and any $\omega\in\mathscr{A}^*$. On the set of bounded linear operators $\mathscr{B}(\mathscr{A}^*)$ over $\mathscr{A}^*$ we may introduce the *weak operator topology* (w-topology) which is characterized by the system of semi-norms $\{q_{\omega,X}:\omega\in\mathscr{A}^*,\ X\in\mathscr{A}\}$ given by $q_{\omega,X}(T)=|T(\omega)(X)|$, with $T\in\mathscr{B}(\mathscr{A}^*)$. Then, by $\mathscr{C}_u(\mathscr{A})$ the weak closure of the convex hull of the operators T_V will be denoted: $\mathscr{C}_u(\mathscr{A})=\overline{\mathrm{conv}\{T_V: V\in\mathscr{U}(\mathscr{A})\}}^W$. In this paper as an essential result the following theorem is proved (proofs of most of the assertions of this part are given in §§3–5):

1.1 Theorem. *Let $\mathscr{H}$ be an infinite-dimensional separable complex*

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Hilbert-space. Let $\omega_1,\dots,\omega_n$ *and* $\sigma_1,\dots,\sigma_n$ *be finite sets of normal states over the bounded linear operators* $\mathscr{B}(\mathscr{H})$ *over* $\mathscr{H}$. *The following conditions are equivalent*:

(i) *there exists a completely positive* (*c.p.*), *unity-preserving* (*unital*) *linear map* T *over* $\mathscr{B}(\mathscr{H})$ *such that*

$$\omega_k=\sigma_k\circ T \qquad \text{for} \quad k=1,\dots,n;$$

(ii) *there is an irreducible* UHF-*algebra* $\mathscr{A}$ *of bounded linear operators on* $\mathscr{H}$ *such that*

$$\omega_{k/\mathscr{A}}=\phi(\sigma_{k/\mathscr{A}}),\ \forall_k,$$

for some $\phi\in\mathscr{C}_u(\mathscr{A})$, *where* $\omega/_{\mathscr{A}}$ *indicates the restriction onto the subalgebra* $\mathscr{A}$.

Moreover, if one of the conditions holds true then (*ii*) *remains true for every irreducible UHF-algebra* $\mathscr{A}$ *over* $\mathscr{H}$.

Throughout this paper, a unital C^*-subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{H})$ is referred to as a UHF-algebra if there exists an ascending sequence $\mathscr{A}_1\subset\mathscr{A}_2\subset\cdots\subset\mathscr{A}$ of *finite* type-I-factors $\mathscr{A}_k$ over $\mathscr{H}$ such that $\mathscr{A}$ is the norm-closure of $\bigcup_k^{\infty}\mathscr{A}_k$.

Let $\mathscr{S}(\mathscr{A})$ denote the convex set of all states over a unital C^*-algebra $\mathscr{A}$. By $\mathscr{S}(\mathscr{A})^n$ the n-fold product $\mathscr{S}(\mathscr{A})\times\cdots\times\mathscr{S}(\mathscr{A})$ of $\mathscr{S}(\mathscr{A})$ will be meant. Let $f:\mathscr{S}(\mathscr{A})^n\ni(\omega_1,\dots,\omega_n)\mapsto f(\omega_1,\dots,\omega_n)\in R^1$ be a realvalued function. Let us define a subset $\mathfrak{M}(f;c)$ of $\mathscr{S}(\mathscr{A})^n$ by

$$\mathfrak{M}(f;c)=\{(\omega_1,\dots,\omega_n)\in\mathscr{S}(\mathscr{A})^n\colon f(\omega_1,\dots,\omega_n)\geq c\},\ \text{for every real } c.$$

Then, the function f is called *quasi-concave* if $\mathfrak{M}(f;c)$ is convex for every choice of $c\in R^1$. Note that concavity of a realvalued function over $\mathscr{S}(\mathscr{A})^n$ (which is also often referred to as joint concavity) always implies quasi-concavity; the converse, however, is false in general. The reader should also remember the fact that a realvalued function f as above is *w^*-upper semicontinuous* (w^*-u.s.c. for short) iff $\mathfrak{M}(f;c)$ is w^*-closed for $\forall c\in R^1$, where the product w^*-topology is referred to over $\mathscr{S}(\mathscr{A})^n$. We call f *unitarily invariant* if $f(\omega_1^V,\dots,\omega_n^V)=f(\omega_1,\dots,\omega_n)$ for $\forall\omega_j\in\mathscr{S}(\mathscr{A})$ and all $V\in\mathscr{U}(\mathscr{A})$. With these notions in mind we have:

1.2 Lemma. *For states* $\nu_1,\dots,\nu_n$ *and* $\mu_1,\dots,\mu_n$ *over the unital* C^*-*algebra* $\mathscr{A}$ *the following conditions are equivalent*:

(i) *there is* $\phi\in\mathscr{C}_u(\mathscr{A})$ *with* $\nu_k=\phi(\mu_k)\ \forall k$;

(ii) $f(\nu_1,\ldots,\nu_n) \geq f(\mu_1,\ldots,\mu_n)$
for all quasi-concave, w-u.s.c., unitarily invariant functions f over* $\mathscr{S}(\mathscr{A})^n$.

Let $\mathscr{Q}_n(\mathscr{A})$ denote the set of quasi-concave, w^*-u.s.c., unitarily invariant functions over $\mathscr{S}(\mathscr{A})^n$. One consequence of the equivalence given by 1.1 is as follows: let $\mathscr{A}$ be an irreducible UHF-algebra over $\mathscr{H}$, and let g be a realvalued function over the n-fold product set $\mathscr{S}(\mathscr{H})_*^n$ of the *normal* states (identified with density operators, where a density operator over $\mathscr{H}$ is a positive operator of trace-class with trace one) over $\mathscr{B}(\mathscr{H})$ such that one finds $f \in \mathscr{Q}_n(\mathscr{A})$ with

$$g(\omega_1,\ldots,\omega_n) = f(\omega_{1/\mathscr{A}},\ldots,\omega_{n/\mathscr{A}}) \qquad \text{for all} \quad (\omega_1,\ldots,\omega_n) \in \mathscr{S}(\mathscr{H})_n^* ;$$

then, g is unitarily invariant with respect to the unitary group $\mathscr{U}(\mathscr{H})$ over $\mathscr{H}$: $g(\omega_1^V,\ldots,\omega_n^V) = g(\omega_1,\ldots,\omega_n) \quad \forall \mathscr{V} \in \mathscr{U}(\mathscr{H})$.

We collect all quasi-concave, unitarily invariant, w^*-u.s.c. functions over $\mathscr{S}(\mathscr{H})_*^n$ which allow for such a representation (with respect to an arbitrary irreducible UHF-algebra) into a set $\mathscr{Q}_n(\mathscr{H})$. Thus, $f \in \mathscr{Q}_n(\mathscr{H})$ iff there is an irreducible UHF-algebra $\mathscr{A}$ and $g \in \mathscr{Q}_n(\mathscr{A})$ such that

$$f(\omega_1,\ldots,\omega_n) = g(\omega_{1/\mathscr{A}},\ldots,\omega_{n/\mathscr{A}}) \quad \forall \omega_j \in \mathscr{S}(\mathscr{H})_* .$$

Then, due to 1.2 and 1.1 we may take for established:

1.3 Theorem. *For finite sets* $\omega_1,\ldots,\omega_n$ *and* $\sigma_1,\ldots,\sigma_n$ *of normal states over the bounded linear operators on a separable infinite-dimensional complex Hilbert-space* $\mathscr{H}$ *the following conditions are equivalent:*

(i) *there is a c.p., unital linear map T over* $\mathscr{B}(\mathscr{H})$ *such that*

$$\omega_k = \sigma_k \circ T \quad \forall k;$$

(ii) $f(\omega_1,\ldots,\omega_n) \geq f(\sigma_1,\ldots,\sigma_n)$ *for all* $f \in \mathscr{Q}_n(\mathscr{H})$.

Assume we are given a particular unitarily invariant, w^*-u.s.c., quasi-concave function f over $\mathscr{S}(\mathscr{H})_*^n$. Then, the proof on $f \in \mathscr{Q}_n(\mathscr{H})$ is facilitated considerably in numerous cases because only two general properties of the C^*-algebra $\mathscr{A}$ are really needed in these situations. This will be seen by the examples below. The essential points in this context are irreducibility and the fact that UHF-algebras intersect the compact operators $\mathscr{B}\mathscr{C}(\mathscr{H})$ over $\mathscr{H}$ only trivially. From functions of $\mathscr{Q}_n(\mathscr{H})$ one can easily construct further unitary invariants over $\mathscr{S}(\mathscr{H})_*^n$ which show monotonous behaviour under the simultaneous transformation of their arguments by means of c.p., unital linear maps. The case of tuples ($n=2$) deserves special interest. Among other things the following will be derived:

1.4 Theorem. *Let $\mathscr{P}(\mathscr{H})$ denote the set of realvalued functions over $\mathscr{S}(\mathscr{H})_* \times \mathscr{S}(\mathscr{H})_*$ characterized as follows: $p \in \mathscr{P}(\mathscr{H})$ iff*

(a) $p \in \mathscr{Q}_2(\mathscr{H})$;

(b) $p(\omega, \sigma) \in [0, 1] \quad \forall \omega, \sigma \in \mathscr{S}(\mathscr{H})_*$;

(c) $p(\omega, \sigma)=0$ *iff* $\omega\sigma=0$, *i.e.* ω, σ *are orthogonal states*;

(d) $p(\omega, \sigma)=1$ *iff* $\omega=\sigma$;

(e) *if* $\omega_x(\mathscr{A}):=(x, \mathscr{A}x)$, $\forall \mathscr{A} \in \mathscr{B}(\mathscr{H})$, *with* $x \in \mathscr{H}$, $\|x\|=1$, *then*

$$p(\omega_x, \omega_y)=|(x, y)|^2 .$$

The following conditions are equivalent: suppose $\omega, \sigma, \omega', \sigma' \in \mathscr{S}(\mathscr{H})_*$,

(i) $p(\omega', \sigma') \geq p(\omega, \sigma) \quad \forall p \in \mathscr{P}(\mathscr{H})$;

(ii) *there exists a unital, c.p. linear map T over $\mathscr{B}(\mathscr{H})$ such that*

$$\omega'=\omega \circ T \text{ and } \sigma'=\sigma \circ T.$$

1.5 *Remark.* Assume p is quasi-concave, unitarily invariant, (relatively) w^*-u.s.c. on $\mathscr{S}(\mathscr{H})_*^2$ and obeys conditions (b)–(e) of 1.4. If then, in addition, $p(\omega \circ T, \sigma \circ T) \geq p(\omega, \sigma)$ holds whenever T is unital and completely positive and $\omega, \sigma, \omega \circ T, \sigma \circ T \in \mathscr{S}(\mathscr{H})_*$, p will be referred to as a *generalized transition probability* (over the normal state space of $\mathscr{B}(\mathscr{H})$) throughout this paper. Let us collect all functions over $\mathscr{S}(\mathscr{H})_*^2$ being generalized transition probabilities in the sense explained into the set $\mathscr{T}\mathscr{P}(\mathscr{H})$. Then, the assertion of 1.4 is that the increase $p(\omega', \sigma') \geq p(\omega, \sigma)$ of all $p \in \mathscr{T}\mathscr{P}(\mathscr{H})$ is in fact equivalent to $\omega'=\omega \circ T$ and $\sigma'=\sigma \circ T$ for some c.p., unital linear map T.

Note that in the preceding definition of the term "generalized transition probability" symmetry, i.e. $p(\omega, \sigma)=p(\sigma, \omega)$, is *not* required, although all known explicite examples considered in literature (they all belong to $\mathscr{T}\mathscr{P}(\mathscr{H})$ as defined above) possess this additional property. We shall discuss this in a somewhat more general context throughout 5.5.

1.6 Example. Let $c_1, \ldots, c_n \in C$ be complex numbers and assume $v_1, \ldots, v_n$ be states over the unital C^*-algebra $\mathscr{A}$. Then, $f_{\mathscr{A}}(v_1, \ldots, v_n)=\|\sum_j c_j v_j\|_1$ ($\|\cdot\|_1$, the functional norm in $\mathscr{A}^*$) is convex and w^*-lower semicontinuous (w^*-l.s.c.) on $\mathscr{S}(\mathscr{A})^n$. Since $f_{\mathscr{A}}$ is also unitarily invariant we have $-f_{\mathscr{A}} \in \mathscr{Q}_n(\mathscr{A})$. Assume

$\mathscr{A}$ acts irreducibly over the Hilbert-space $\mathscr{H}$. Then, for *normal* states $\omega_1,\dots,\omega_n$ over $\mathscr{B}(\mathscr{H})f(\omega_1,\dots,\omega_n):=\|\sum_j c_j\omega_j\|_1=f_{\mathscr{A}}(\omega_{1/\mathscr{A}},\dots,\omega_{n/\mathscr{A}})$, due to the fact that the strong closure of the unit sphere of $\mathscr{A}$ gives the unit sphere of all bounded operators by irreducibility and the Kaplansky density theorem. Therefore, $-f\in\mathscr{Q}_n(\mathscr{H})$. This means $\|\sum_j c_j\omega_j\circ T\|_1\leq\|\sum_j c_j\omega_j\|_1$ for every c.p., unital linear map T, by 1.3. Thus, we have demonstrated a function on $\mathscr{S}(\mathscr{H})^n_*$ to be a member of $\mathscr{Q}_n(\mathscr{H})$. The decrease under the action of positive unital maps, however, in this case is not surprising at all since every unital positive linear map is a contraction, which fact is well-known since many years, see [14].

1.7 Example. Let $n=2$, and ω_1, ω_2 and σ_1, σ_2 be density operators such that both $\omega_1+\omega_2$ and $\sigma_1+\sigma_2$ are operators of rank 2. Then, by 1.6,

$$(*)\qquad \|c_1\omega_1+c_2\omega_2\|_1\leqq\|c_1\sigma_1+c_2\sigma_2\|_1\qquad\text{for all}\quad c_1,c_2\in C$$

whenever $\omega_j=\sigma_j\circ T$, $j=1, 2$, with a positive, unital linear map T. By [1] one knows that $(*)$ is not only necessary, it is also sufficient for the existence of a c.p., unital T such that $\omega_j=\sigma_j\circ T$, $j=1, 2$, so providing a special case for the validity of the implication (ii)→(i) of 1.3. Further examples where the implication mentioned is known to be true are given by [4], [20]. One can, of course, the n-tuple problem consider for state spaces of an arbitrary C^*-algebra. For the commutative case and all n the general solution of this problem has been given in [2], [3], see also the description of the problem in [9], [19]. Another illustration, now referring to 1.4 and 1.5, is given by the following:

1.8 Example. Let $\mathscr{A}$ be a unital $\mathscr{C}^*$-algebra, and $\omega,\sigma\in\mathscr{S}(\mathscr{A})$. One then associates to ω, σ a non-negative real $P_{\mathscr{A}}(\omega,\sigma)$ by the following setting:

$$P_{\mathscr{A}}(\omega,\sigma):=\sup_{\forall x_\omega,x_\sigma\forall\pi}|(x_\omega,x_\sigma)|^2,$$

where x_ω, $x_\sigma\in\mathscr{H}_\pi$ run over all vector representatives of ω, σ within all possible unital *-representations $\{\pi,\mathscr{H}_\pi\}$ of $\mathscr{A}$ on some Hilbert-space $\mathscr{H}_\pi$ where both ω and σ may be realized as vector states simultaneously. This definition is given in [18], and is in case of normal states over W^*-algebras equivalent to another definition given in [12] as it has been demonstrated in [10]. $P_{\mathscr{A}}$ fulfils conditions (b)–(d) of 1.4 if considered on $\mathscr{S}(\mathscr{A})^2$ and if "orthogonality" of two states ω, σ is replaced by the general notion of this term: $\omega\perp\sigma$ iff $\|\omega-\sigma\|_1=2$. For the special choice $\mathscr{A}=\mathscr{B}(\mathscr{H})$ also condition (e) of 1.4 holds. Let us show that $P_{\mathscr{A}}$ is quasi-concave. Assume ω_j, $\sigma_j\in\mathscr{S}(\mathscr{A})$, $j=1, 2$, and $r\in[0, 1]$.

Define $\nu_j = r\omega_j + (1-r)\sigma_j$. Let $c \in T$, and assume $P_{\mathscr{A}}(\omega_j, \sigma_j) \geq c$, $j=1, 2$. If $c \leqq 0$, $(\nu_1, \nu_2) \in \mathfrak{M}(P_{\mathscr{A}}; c)$ by triviality. For $c>0$, and $\varepsilon>0$ such that $c-\varepsilon>0$, we argue as follows. Let $\{\pi_j, \mathscr{H}_j\}$ be unital *-representations of $\mathscr{A}$ such that there are vector representatives $x_j, y_j \in \mathscr{H}_j$ of ω_j, σ_j with

$$(*) \qquad \begin{aligned} &(x_j, y_j) \geq 0, \\ &(x_j, y_j)^2 > c-\varepsilon, \quad j=1, 2. \end{aligned}$$

Such x_i, y_i exist by definition of $P_{\mathscr{A}}$.

Let $\pi = \pi_1 \oplus \pi_2$ be the representation of $\mathscr{A}$ acting on the orthogonal sum $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ in the usual manner. Putting $z_1 = \sqrt{r}\, x_1 + \sqrt{1-r}\, x_2$ and $z_2 = \sqrt{r}\, y_1 + \sqrt{1-r}\ y_2$, we see $\nu_j(A) = (z_j, \pi(A) z_j)$, $j=1, 2$. Therefore, by definition of $P_{\mathscr{A}}$ and $(*)$: $P_{\mathscr{A}}(\nu_1, \nu_2)^{1/2} \geq |(z_1, z_2)| = |r(x_1, y_1) + (1-r)(x_2, y_2)| = r(x_1, y_1) + (1-r)(x_2, y_2) > r\sqrt{c-\varepsilon} + (1-r)\sqrt{c-\varepsilon} = \sqrt{c-\varepsilon}$, so $P_{\mathscr{A}}(\nu_1, \nu_2) \geq c-\varepsilon$. This has to hold for any $\varepsilon > 0$ sufficiently small, so $P_{\mathscr{A}}(\nu_1, \nu_2) \geq c$, i.e. $\mathfrak{M}(P_{\mathscr{A}}; c)$ is convex. The latter means quasi-concavity of $P_{\mathscr{A}}$. Furthermore, in [6.] one shows $P_{\mathscr{A}}(\omega, \sigma) = \inf\limits_{A>0} \omega(A)\sigma(A^{-1})$, telling that $P_{\mathscr{A}}$ is the infimum of w^*-continuous functions. Hence, $P_{\mathscr{A}}$ is w^*-u.s.c. on $\mathscr{S}(\mathscr{A})^2$. Since unitary invariance is obvious we may take together all these facts and conclude that $P_{\mathscr{A}}$ is in $\mathscr{Q}_2(\mathscr{A})$. Let us assume now that $\mathscr{A}$ acts on a Hilbert-space $\mathscr{H}$. By a result of [6] one knows that for *normal* states ω, σ over $\mathscr{A}''$ (the double commutant of $\mathscr{A}$, i.e. the weak closure of $\mathscr{A}$) always $P_{\mathscr{A}''}(\omega, \sigma) = P_{\mathscr{A}}(\omega_{/\mathscr{A}}, \sigma_{/\mathscr{A}})$ holds true. Especially, for an irreducible C^*-subalgebra of $\mathscr{B}(\mathscr{H})$, i.e. $\mathscr{A}'' = \mathscr{B}(\mathscr{H})$, we find for normal states $\omega, \sigma \in \mathscr{S}(\mathscr{H})_*$: $P_{\mathscr{B}(\mathscr{H})}(\omega, \sigma) = P_{\mathscr{A}}(\omega_{/\mathscr{A}}, \sigma_{/\mathscr{A}})$. By our discussion above $P_{\mathscr{A}} \in \mathscr{Q}_2(\mathscr{A})$, hence we see that $P_{\mathscr{B}(\mathscr{H})}$ belongs to $\mathscr{Q}_2(\mathscr{H})$ since for $\mathscr{A}$ every irreducible UHF-algebra could have been chosen (once more again "UHF" is not important in the proof). Thus, $P_{\mathscr{B}(\mathscr{H})}$ fulfils 1.4 (a)–(e). Applying 1.3 we recognize that $P_{\mathscr{B}(\mathscr{H})}$ is a generalized transition probability, i.e. $P_{\mathscr{B}(\mathscr{H})} \in \mathscr{T}\mathscr{P}(\mathscr{H})$ in the sense of 1.5. In [18] one finds an explicit expression for $P_{\mathscr{B}(\mathscr{H})}$:

$$\text{if} \quad \omega, \sigma \in \mathscr{S}(\mathscr{H})_*, \text{ then } P_{\mathscr{B}(\mathscr{H})}(\omega, \sigma) = (\mathrm{Tr.}\ (\omega^{1/2}\sigma\omega^{1/2})^{1/2})^2,$$

where we identify normal states with density operators (Tr. means "trace of"). For later use, we give the following remarkable special case of 1.4 (see [5]):

1.9 Proposition. *Let $\omega, \sigma \in \mathscr{S}(\mathscr{H})_*$, and be ω_x, ω_y the vector states given by the unit vectors $x, y \in \mathscr{H}$. There exists a completely positive unital linear map T over $\mathscr{B}(\mathscr{H})$ with $\omega = \omega_x \circ T$, $\sigma = \omega_y \circ T$ iff*

$$P_{\mathscr{B}(\mathscr{H})}(\omega, \sigma) \geq P_{\mathscr{B}(\mathscr{H})}(\omega_x, \omega_y) = |(x, y)|^2.$$

If this condition holds true the T in question can be chosen to be a normal map.

1.10 *Remark.* For all questions relating operator algebras, positive and completely positive linear maps over operator algebras and their applications the reader is referred to textbooks, e.g. [17], [11] and their corresponding lists of references. Concerning positive maps see also the review [16]. Concerning "generalized transition probabilities" and references to that subject see [5]–[8], [10], [12], [13], [18], [19].

§2. Technical Preliminaries

Let $\mathscr{A}$ be a unital C^*-subalgebra of $\mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is a separable infinite-dimensional complex Hilbert-space. By $\mathscr{A}_h$, $\mathscr{B}(\mathscr{H})_h$, $\mathscr{A}_+$, $\mathscr{B}(\mathscr{H})_+$ the selfadjoint and positive portions of $\mathscr{A}$, $\mathscr{B}(\mathscr{H})$ will be denoted, respectively. Let n be an arbitrary but henceforth fixed natural number (to be non-trivial $n \geq 2$ is supposed).

2.1 Definition. *Let $\omega_1,\dots,\omega_n \in \mathscr{S}(\mathscr{H})_*$, $A_1,\dots,A_n \in \mathscr{B}(\mathscr{H})_h$, then*
$$K(\omega_1,\dots,\omega_n;A_1,\dots,A_n):=\sup_{U\in\mathscr{U}(\mathscr{H})}\sum_{j=1}^{n}\omega_j^U(A_j);$$
if $\omega_1,\dots,\omega_n \in \mathscr{S}(\mathscr{A})$, $A_1,\dots,A_n \in \mathscr{A}_h$, then
$$K_{\mathscr{A}}(\omega_1,\dots,\omega_n;A_1,\dots,A_n):=\sup_{U\in\mathscr{U}(\mathscr{A})}\sum_{j}\omega_j^U(A_j);$$
if $\omega_1,\dots,\omega_n \in \mathscr{S}(\mathscr{H})_*$, by convention we will agree to read
$$K_{\mathscr{A}}(\omega_1,\dots,\omega_n;A_1,\dots,A_n):=K_{\mathscr{A}}(\omega_{1/\mathscr{A}},\dots,\omega_{n/\mathscr{A}};A_1,\dots,A_n)$$
whenever $A_j \in \mathscr{A}_h$.

2.2 Lemma.

(i) *K and $K_{\mathscr{A}}$ are continuous functions with respect to the uniform product topology on $\mathscr{S}(\mathscr{H})_*^n \times \mathscr{B}(\mathscr{H})_h^n$ and $\mathscr{S}(\mathscr{A})^n \times \mathscr{A}_h^n(\mathscr{S}(\mathscr{H})_*^n \times \mathscr{A}_h^n)$, respectively;*

(ii) *$K(.,\dots,.;A_1,\dots,A_n)$ and $K_{\mathscr{A}}(.,\dots,.;A_1,\dots,A_n)$ are convex and w^*-l.s.c. on $\mathscr{S}(\mathscr{H})_*^n$, $\mathscr{S}(\mathscr{A})^n(\mathscr{S}(\mathscr{H})_*^n)$, respectively;*

(iii) *in case of an irreducible $\mathscr{A}$ i.e. $\mathscr{A}''=\mathscr{B}(\mathscr{H})$, one has $K(\omega_1,\dots,\omega_n;A_1,\dots,A_n)=K_{\mathscr{A}}(\omega_1,\dots,\omega_n;A_1,\dots,A_n)$ for all $\omega_j \in \mathscr{S}(\mathscr{H})_*$ and all $A_j \in \mathscr{A}_h$.*

Proof. (i) and (ii) are obvious from the definitions. (iii) follows from the fact that $\mathscr{U}(\mathscr{A})$ is strongly*-dense within $\mathscr{U}(\mathscr{A}'')$. In fact, the latter implies $\{(UA_1U^*,\dots,UA_nU^*)\colon U\in\mathscr{U}(\mathscr{A})\}$ to be strongly dense in $\{(UA_1U^*,\dots,UA_nU^*)\colon$

$U \in \mathscr{U}(\mathscr{A}'')\}$ for all choices $\{A_j\}$. For $\mathscr{A}''=\mathscr{B}(\mathscr{H})$ and normal states $\omega_1,\dots,\omega_n \in \mathscr{S}(\mathscr{H})_*$ then follows $K(\omega_1,\dots,\omega_n;\ A_1,\dots,A_n) \leq K_{\mathscr{A}}(\omega_1,\dots,\omega_n;\ A_1,\dots,A_n)$, from which fact together with $\mathscr{U}(\mathscr{H}) \supset \mathscr{U}(\mathscr{A})$ the validity of (iii) can be seen.

Let $\mathscr{M}_k$ denote the full matrix algebra of complex $k\times k$-matrices. Assume $\mathscr{M}$ is a unital C^*-algebra. We should remember that a linear map T acting from $\mathscr{M}$ into $\mathscr{M}$ is said to be *completely positive* (see [15]) if $T\otimes \mathrm{id}_k: \mathscr{M}\otimes\mathscr{M}_k \ni m=(m_{lj}) \mapsto T\otimes\mathrm{id}_k(m)=(T(m_{lj})) \in \mathscr{M}\otimes\mathscr{M}_k$ is a positive map for $k=1, 2, 3,\dots$. A linear map ϕ acting from $\mathscr{M}^*$ into $\mathscr{M}^*$ is said to be positive if $\phi(\mathscr{M}^*_+)\subset\mathscr{M}^*_+$, and will be referred to as a *stochastic map* if $\phi(\mathscr{S}(\mathscr{M}))\subset\mathscr{S}(\mathscr{M})$, and is said to be completely positive (c.p.) if the adjoint map ϕ^+ acting over the second dual $\mathscr{M}^{**}$ is c.p.. If ϕ is stochastic and c.p. we call ϕ *c.p.-stochastic*; ϕ is c.p.-stochastic iff ϕ^+ is c.p. and unital over $\mathscr{M}^{**}$. Each $f\in(\mathscr{M}\otimes\mathscr{M}_k)^*$ is uniquely determined by a family (f_{lj}), with $f_{lj}\in\mathscr{M}^*$, via the relation $f((m_{lj}))=\sum_{l,j} f_{lj}(m_{lj})$. A linear map over $\mathscr{M}^*$ is c.p. iff for any positive linear form $f=(f_{lj})$ over $\mathscr{M}\otimes\mathscr{M}_k$ always follows that $(\phi\otimes\mathrm{id}_k)(f)=(\phi(f_{lj}))$ defines a positive linear form over the same algebra for any natural k. In case of a W^*-algebra $\mathscr{M}$, with predual $\mathscr{M}_*$, a linear map over $\mathscr{M}_*$ is said to be: positive, if $\phi(\mathscr{M}_{*+})\subset\mathscr{M}_{*+}$; c.p. if ϕ^+ is c.p. over $\mathscr{M}$; stochastic if $\phi(\mathscr{M}_*\cap\mathscr{S}(\mathscr{M}))\subset\mathscr{M}_*\cap\mathscr{S}(\mathscr{M})$. If ϕ is c.p. and stochastic the term *c.p.-stochastic* on $\mathscr{M}_*$ will be in use. In case of $\mathscr{M}=\mathscr{B}(\mathscr{H})$, $\mathscr{B}(\mathscr{H})_*$ will be identified with $\mathscr{T}(\mathscr{H})$, the operators of trace-class, in the usual manner. It is not hard to see that a linear map ϕ over $\mathscr{B}(\mathscr{H})_*$ is c.p. iff $(\sigma_{lj})\geq 0$ within $\mathscr{T}(\mathscr{H})\otimes\mathscr{M}_k\subset\mathscr{B}(\mathscr{H})\otimes\mathscr{M}_k$ (inclusion as sets of bounded operators over the space $\mathscr{H}\otimes C^k$) always implies $(\phi(\sigma_{lj}))\geq 0$. One then easily sees that the following is true:

2.3 *Remark.*

(i) *Let $\mathscr{M}$ be a unital C^*-algebra. If T is c.p. on $\mathscr{M}$, then T^+ is c.p. on $\mathscr{M}^*$ in the above-discussed sense;*

(ii) *let $\mathscr{M}$ be a W^*-algebra. If ϕ is a* c.p. *linear map over $\mathscr{M}^*$ and $\phi(\mathscr{M}_*)\subset\mathscr{M}_*$, then the restriction $\phi_{\mathscr{M}_*}$ onto the predual is c.p. Especially, if T is a linear map over $\mathscr{M}$ which is c.p. and normal, then $T_+:=T^+_{/\mathscr{M}_*}$ is c.p. on $\mathscr{M}_*$.*

Let us take notice of the following very simple, but nevertheless very useful technical results:

2.4 Lemma. *Let P be an orthoprojection over $\mathscr{H}$, and $\nu_1,\dots,\nu_m$ states over*

$\mathscr{B}(\mathscr{H})$. Suppose a linear map ϕ is defined over $\mathscr{B}(\mathscr{H})^*$ by $\phi(\omega)(A) = \omega(\mathrm{PAP}) + \sum_i \omega(P_i)v_i(A)$, for all $\omega \in \mathscr{B}(\mathscr{H})^*$ and all $A \in \mathscr{B}(\mathscr{H})$, with a finite decomposition $\{P_i\}_{i=1,\dots,m}$ of $P^\perp = 1 - P$ into mutually orthogonal orthoprojections. Then

(i) *ϕ is c.p.-stochastic;*

(ii) *for $\dim P < \infty$, and $v_1,\dots, v_m \in \mathscr{S}(\mathscr{H})_*$ (normal states) $\phi(\mathscr{B}(\mathscr{H})^*) \subset \mathscr{B}(\mathscr{H})_*$;*

(iii) *under the assumptions of* (ii), *for any c.p.-stochastic Ψ over $\mathscr{B}(\mathscr{H})^*$ the composition map $\Omega = \phi \circ \Psi_{/\mathscr{B}(\mathscr{H})_*}$ is c.p.-stochastic on $\mathscr{B}(\mathscr{H})_*$, and hence Ω^+ is a normal unital c.p. map over $\mathscr{B}(\mathscr{H})$.*

Proof. The linear map T given by $T(A) = PAP + \sum_k v_k(A)P_k$, $A \in \mathscr{B}(\mathscr{H})$, is c.p. and unity preserving (since $P_k \geq 0$ and states are c.p.). Hence, $T^+ = \phi$ is c.p.-stochastic (see 2.3 (i)) and (i) is seen. To see (ii) one notes that in case of $\dim P < \infty$, $P\mathscr{B}(\mathscr{H})P$ is finite-dimensionally, so has only normal states. Therefore, $\omega(P(.)P) \in \mathscr{B}(\mathscr{H})_*$ for all $\omega \in \mathscr{B}(\mathscr{H})^*$. Moreover, the v_j's are supposed to be normal, so the assertion follows. (iii) is a consequence of 2.3 (ii).

2.5 **Lemma.** *Let $\omega_1,\dots,\omega_n$ be density operators over $\mathscr{H}$. There exist sequences $\{\omega_{1m}\},\dots,\{\omega_{nm}\}$ of density operators such that:*

(i) *all ω_{km} have finite rank, and $\|\cdot\|_1\text{-}\lim_m \omega_{km} = \omega_k$ $\forall k$;*

(ii) *there exist c.p.-stochastic maps ϕ_m over $\mathscr{B}(\mathscr{H})^*$ with $\phi_m(\omega_{km}) = \omega_m$ $\forall k, m$;*

(iii) *$\phi_m(\mathscr{B}(\mathscr{H})^*) \subset \mathscr{B}(\mathscr{H})_*$ for all m.*

Proof. Assume $\omega_k = \sum_{s=1}^{\infty} \beta_{ks}P_{ks}$, with $\{P_{ks}\}$ a decomposition of *1* into mutually orthogonal, onedimensional orthoprojections, and $\beta_{k1} \geq \beta_{k2} \geq \cdots$. Define $P_m = \bigvee_{k=1}^{n} \bigvee_{s=1}^{m} P_{ks}$. Let $Q_{1m},\dots, Q_{nm}$ be onedimensional, mutually orthogonal orthoprojections with $Q_{jm} < P_m^\perp$, $\forall j$. We define finite-rank density operators by setting $\omega_{km} = \sum_{s \leq m} \beta_{ks}P_{ks} + (\sum_{s>m} \beta_{ks})Q_{km}$. Then, (i) is easily followed. Let I_m be a set of indices defined as $I_m = \{j \in \{1,\dots, n\} : \sum_{s>m} \beta_{js} = 0\}$. Let us fix m. If $I_m = \{1,\dots, n\}$ we define $\phi_m(\omega) = \omega(P_m(.)P_m) + \omega(P_m^\perp)v$ with an arbitrarily chosen normal state v. Since in this case $P_m\omega_k = \omega_k$ $\forall k$, and $\omega_{km} = \omega_k$ $\forall k$, we see $\phi_m(\omega_k) = \phi_m(\omega_{km}) = \omega_k$ and the chosen ϕ_m satisfies (ii) and (iii), the latter following by applying 2.4 (ii). Assume $I_m \neq \{1,\dots, n\}$. We may suppose $n \notin I_m$. If $j \notin I_m$, let us define normal states v_{jm} by

$$v_{jm}(A)=(\sum_{s>m}\beta_{js})^{-1}\mathrm{Tr}.(\sum_{s>m}\beta_{js}P_{js})A, \qquad \text{for} \quad \forall A\in\mathscr{B}(\mathscr{H}).$$

Then, applying 2.4 (i), (ii) once more again we see that

$$\phi_m(\omega):=\omega(P_m(.)P_m)+\sum_{\substack{j\notin I_m\\ j\neq n}}\omega(Q_{jm})v_{jm}+\omega(P_m^{\perp}-\sum_{\substack{j\neq n\\ j\notin I_m}}Q_{jm})v_{nm}$$

is c.p.-stochastic and satisfies (iii). If $j\in I_m$, we have $P_m\omega_j=\omega_j$, i.e. $\omega_{jm}=\omega_j$, so $\phi_m(\omega_{jm})=\omega_j$. If $j\notin I_m$, $j\neq n$, then $P_m\omega_{jm}=\omega_{jm}P_m=\sum_{s\leq m}\beta_{js}P_{js}$, and

$$\omega_{jm}Q_{km}=\begin{cases}0 & \text{if} \quad k\neq j\\ (\sum_{s>m}\beta_{js})Q_{jm} & \text{if} \quad k=j\end{cases}, \quad \text{and} \quad \omega_{jm}(P_m^{\perp}-\sum_{\substack{l\neq n\\ l\notin Im}}Q_{lm})=0.$$

Hence, by definition of v_{jm}: $\phi_m(\omega_{jm})=\mathrm{Tr}.(\sum_{s\leq m}\beta_{js}P_{js})(\,.\,)+(\sum_{s>m}\beta_{js})v_{jm}=\mathrm{Tr}.\,\omega_j(.)$. Analogously $\phi_m(\omega_{nm})=\omega_n$ follows, so (ii) is seen.

2.6 Lemma. *Let $\omega_1,\dots,\omega_n$ be density operators over the separable Hilbert-space $\mathscr{H}$. There exist sequences $\{\omega_{1m}\},\dots,\{\omega_{nm}\}$ of density operators and linear maps ϕ_m over $\mathscr{B}(\mathscr{H})_*$ such that:*

(i) *all ω_{km} are of finite rank and $\|\cdot\|_1\text{-}\lim_m \omega_{km}=\omega_k$, $\forall k$;*

(ii) *the ϕ_m are c.p.-stochastic over $\mathscr{B}(\mathscr{H})_*$ with $\omega_{km}=\phi_m(\omega_k)$, $\forall k, m$;*

(iii) *$\phi_m^{++}(\mathscr{B}(\mathscr{H})^*)\subset\mathscr{B}(\mathscr{H})_*$ $\forall m$.*

Proof. Let $\omega_j=\sum_{s=1}^{\infty}\beta_{js}P_{js}$, with P_{js}, β_{js} and also P_m having the same meanings as introduced in the proof of 2.5. Take an orthogonal decomposition $\{Q_{sm}\}$ of $P_m^{\perp}$ into onedimensional orthoprojections: $P_m^{\perp}=\sum_{s=1}^{\infty}Q_{sm}$. Let us choose an onedimensional orthoprojection R_m, with $R_m<P_m^{\perp}$, and partial isometries $V_{ms}\in\mathscr{B}(\mathscr{H})$ with $V_{ms}^*V_{ms}=Q_{sm}$, $V_{ms}V_{ms}^*=R_m$, $\forall s$. We define linear maps ϕ_m over $\mathscr{B}(\mathscr{H})_*$ (in its identification with $\mathscr{T}(\mathscr{H})$) by $\phi_m(A)=P_mAP_m+\sum_{s=1}^{\infty}V_{ms}AV_{ms}^*$, $\forall A\in\mathscr{T}(\mathscr{H})\cong\mathscr{B}(\mathscr{H})_*$. Due to our assumptions ϕ_m is c.p.-stochastic over $\mathscr{B}(\mathscr{H})_*$. Let us define $\omega_{jm}:=\phi_m(\omega_j)$. By construction of ϕ_m, $\omega_{jm}=\sum_{s\leq m}\beta_{js}P_{js}+B_{jm}$, with some $B_{jm}\geq 0$. Since ϕ_m is stochastic, ω_{jm} is a density operator for all m. Now, $\|\cdot\|_1\text{-}\lim_m\sum_{s\leq m}\beta_{js}P_{js}=\omega_j$, therefore $\lim_m\mathrm{Tr}.\,B_{jm}=0$. From this, together with $B_{jm}\geq 0$, $B_{jm}\xrightarrow[m]{\|\cdot\|_1}0$ follows for $\forall j$. Hence ω_{jm} tends uniformly (functional norm) towards ω_j, for all j. By construction $\phi_m(A)$ has finite rank ($\leq nm+1$) for each $A\in\mathscr{T}(\mathscr{H})$, so ω_{jm} has finite rank. This proves (i) and (ii). Finally $\phi_m^{+}=P_m(.)P_m+\sum_{s=1}^{\infty}V_{ms}^*(.)V_{ms}$. Due to our assumptions $V_{ms}^*AV_{ms}=(\mathrm{Tr}\,.\,R_mA)Q_{sm}$ (see the choice of V_{ms}), so by 2.4 (ii) and since $\dim P_m<\infty$

we follow $\phi_m^{++}(\omega)=\omega(P_m(\,.\,)P_m)+\omega(P_m^\perp)\mathrm{Tr}\,.\,R_m(\,.\,)$ to be a c.p. map satisfying (iii).

2.7 Lemma. *Let $\omega_1,\ldots,\omega_n$ and $\sigma_1,\ldots,\sigma_n$ be normal states over $\mathscr{B}(\mathscr{H})$. Assume ϕ is a c.p.-stochastic linear map over $\mathscr{B}(\mathscr{H})^*$ such that $\omega_k=\phi(\sigma_k)$ for all k. Then, there exists a c.p., unital linear map T on $\mathscr{B}(\mathscr{H})$ such that $\omega_k=\sigma_k\circ T$, $\forall_k$.*

Proof. By 2.5 there is $\{\sigma_{km}\}\subset\mathscr{S}(\mathscr{H})_*$ with $\sigma_{km}\xrightarrow[m]{}\sigma_k$, $\forall k$, and c.p.-stochastic ϕ_m with $\phi_m(\mathscr{B}(\mathscr{H})^*)\subset\mathscr{B}(\mathscr{H})_*$ such that $\sigma_k=\phi_m(\sigma_{km})\forall k$, $\forall m$. By 2.6 there is $\{\omega_{km}\}\subset\mathscr{S}(\mathscr{H})_*$ with $\omega_{km}\xrightarrow[m]{}\omega_k$, $\forall k$, and c.p.-stochastic ϕ'_m over $\mathscr{B}(\mathscr{H})_*$ with $\omega_{km}=\phi'_m(\omega_k)$, $\forall m$, and $\phi'^{++}_m(\mathscr{B}(\mathscr{H})^*)\subset\mathscr{B}(\mathscr{H})_*$. Let us define $\Omega_m:=\phi'^{++}_m\circ\phi\circ\phi_m$. As a composition of c.p.-stochastic maps Ω_m is c.p.-stochastic, too. This is true for any m. Moreover, $\Omega_m(\mathscr{B}(\mathscr{H})_*)\subset\mathscr{B}(\mathscr{H})_*$, so $S_m:=\Omega_{m/\mathscr{B}(\mathscr{H})_!}$ are c.p.-stochastic maps on $\mathscr{B}(\mathscr{H})_*$, with $S_m(\sigma_{km})=\omega_{km}$, $\forall k$, m. Let T_m be the unital, c.p. linear map defined by $T_m:=S_m^+$. Then, $\omega_{km}=\sigma_{km}\circ T_m\forall m$, $\forall k$. Since the unit sphere of the bounded linear operators over $\mathscr{B}(\mathscr{H})$ is compact with respect to the weak operator topology over the duality $\langle\mathscr{B}(\mathscr{H}),\,\mathscr{B}(\mathscr{H})_*\rangle$ (see 1 for the definition of a w-topology) there exists a weakly converging subnet $\{T_{m_\beta}\}$: $w\text{-}\lim_\beta T_{m_\beta}=T$. Due to $T_{m_\beta}(I)=I$, $\forall\beta$, also $T(I)=I$ has to hold. It is clear that T is a completely positive map. For any $A\in\mathscr{B}(\mathscr{H})$ we have the following estimates: $|\sigma_k\circ T(A)-\sigma_{km_\beta}\circ T_{m_\beta}(A)|\le|\sigma_k\circ T(A)-\sigma_k\circ T_{m_\beta}(A)|+|(\sigma_k-\sigma_{km_\beta})(T_{m_\beta}(A))|$ $\le|\sigma_k(T(A)-T_{m_\beta}(A))|+\|A\|\|\sigma_k-\sigma_{km_\beta}\|$. Since σ_{km_β} tends uniformly towards σ_k and since T is the weak limit of $\{T_{m_\beta}\}$ the left hand side of our estimation tends to zero for any $A\in\mathscr{B}(\mathscr{H})$ (remember that σ_k is normal). Hence $\sigma_k\circ T=w^*\text{-}\lim_\beta\sigma_{km_\beta}\circ T_{m_\beta}$ in the w^*-topology on $\mathscr{B}(\mathscr{H})^*$ with respect to $\mathscr{B}(\mathscr{H})$. We know, however, $\sigma_{km_\beta}\circ T_{m_\beta}=\omega_{km_\beta}$ and $\omega_{km_\beta}\xrightarrow[\beta]{}\omega_k$ (even uniformly), so $\sigma_k\circ T=\omega_k$, $\forall k$, has to hold with a c.p., unital linear map T over $\mathscr{B}(\mathscr{H})$.

2.8 Lemma. *Let $\mathscr{H}_0$ be a finite-dimensional complex Hilbert-space, and $\mathscr{H}=\mathscr{H}_0\otimes\mathscr{H}_1$ with another complex Hilbert-space $\mathscr{H}_1$. Let ϕ be a completely positive, trace-preserving, linear map over $\mathscr{B}(\mathscr{H}_0)$. Assume $x\in\mathscr{H}_1$ is a unit vector, and E denotes the one-dimensional orthoprojection corresponding to x. There exists a unitary $U\in\mathscr{U}(\mathscr{H})$ such that*

$$\phi(A)=\mathrm{Tr}_1\,U(A\otimes E)U^*\qquad \textit{for}\quad\forall A\in\mathscr{B}(\mathscr{H}_0),$$

where Tr_1 means the relative trace over $\mathscr{H}_1$ with respect to $\mathscr{H}$.

Proof. By the Stinespring-theorem (see [15]), and since $\mathscr{H}_0$ is of finite dimen- sion, we find a representation π of $\mathscr{B}(\mathscr{H}_0)$ on some finite-dimensional Hilbertspace $\mathscr{H}'$, and a bounded linear operator V from $\mathscr{H}'$ into $\mathscr{H}_0$ with $VV^*=I$, such that $\phi^+(B)=V\pi(B)V^*$, $\forall B\in\mathscr{B}(\mathscr{H}_0)$. It is known that we may identify $\mathscr{H}'$ with a factorization $\mathscr{H}'=\mathscr{H}_0\otimes\mathscr{H}'_1$ such that $\pi(B)=B\otimes I$ for all $B\in\mathscr{B}(\mathscr{H}_0)$; hence $\phi^+(B)=W(B\otimes I)W^*$ for some bounded W, with $WW^*=I$. Let $\mathscr{H}''=\mathscr{H}_0\otimes\mathscr{H}''_1$, with $\mathscr{H}''_1=\mathscr{H}'_1\otimes\mathscr{H}'_2$ for some separable infinite-dimensional complex Hilbert-space $\mathscr{H}'_2$. Assume $z\in\mathscr{H}'_2$ is a unit vector. We may identify $\mathscr{H}'$ with $\mathscr{H}_0\otimes\mathscr{H}'_1\otimes[z]$ within $\mathscr{H}''$. Let P be the corresponding orthoprojection. Then, WP is a bounded operator acting from $\mathscr{H}''$ into $\mathscr{H}_0$, $WPW^*=I$, and $\phi^+(B)=WP(B\otimes I)PW^*$, $\forall B\in\mathscr{B}(\mathscr{H}_0)$, where $B\otimes I$ is understood to act on $\mathscr{H}''=\mathscr{H}_0\otimes\mathscr{H}''_1$. Since $\mathscr{H}_1$ is isomorphic to $\mathscr{H}''_1$, we might identify $\mathscr{H}$ with $\mathscr{H}''$ and have: there is R with $RR^*=I$ and $\phi^+(B)=R(B\otimes I)R^*$, $\forall B\in\mathscr{B}(\mathscr{H})$. Let $\mathfrak{S}$ be the isometry defined by S: $\mathscr{H}_0\ni y\mapsto y\otimes x\in\mathscr{H}$. Then, $S^*(A\otimes E)S=A$, and for $\forall A\in\mathscr{B}(\mathscr{H}_0)$:

$$(*)\qquad \mathrm{Tr}_0\, A\phi^+(B)=\mathrm{Tr}.\, S^*(A\otimes E)SR(B\otimes I)R^*=\mathrm{Tr}.\, R^*S^*(A\otimes E)SR(B\otimes I).$$

The orthoprojection Q onto $\mathscr{H}_0\otimes[x]$ is dimensionally finite, so $R^*S^*QSR=R^*R$ has finite dimension. Therefore, $Q^\perp$ is equivalent with $R^*R^\perp$, i.e. there is a unitary U with $UQ=R^*S^*Q$. From $(*)$ one follows:

$$\mathrm{Tr}_0\, A\phi^+(B)=\mathrm{Tr}.\, U(A\otimes E)U^*(B\otimes I),\quad \forall A,\, B\in\mathscr{B}(\mathscr{H}_0),$$

from which equality the assertion follows.

2.9 Lemma. *Let $\omega_1,\ldots,\omega_n$ and $\sigma_1,\ldots,\sigma_n$ be density operators over the finitedimensional Hilbert-space $\mathscr{H}_0$. Then, with $\mathscr{H}_1$, x, E having the same meanings as in the assumptions of 2.8, the following conditions are equivalent:*

(i) *there exists a completely positive stochastic ϕ over $\mathscr{B}(\mathscr{H}_0)^*$ (which may be identified with $\mathscr{B}(\mathscr{H}_0)$) with $\omega_k=\phi(\sigma_k)$, $\forall k$:*

(ii) *for every choice of $A_1,\ldots,A_n\in\mathscr{B}(\mathscr{H}_0)_+$ one has*

$$\sup_{U\in\mathscr{U}(\mathscr{H})}\sum_k \mathrm{Tr}.(\omega_k\otimes E)U(A_k\otimes I)U^*\le \sup_{U\in\mathscr{U}(\mathscr{H})}\sum_k \mathrm{Tr}.\,(\sigma_k\otimes E)U(A_k\otimes I)U^*.$$

Proof. Let $\mathfrak{M}(\sigma_1,\ldots,\sigma_n)=\{(\phi(\sigma_1),\ldots,\phi(\sigma_n))\colon \forall\phi \text{ c.p.-stochastic}\}$. This set is convex and closed in $\mathscr{B}(\mathscr{H}_0)^n_h$. Assume $(\omega'_1,\ldots,\omega'_n)\notin\mathfrak{M}(\sigma_1,\ldots,\sigma_n)$, and all ω'_k are hermitian. By a standard application of the separation theorem for convex sets in real topological vector spaces we provide us with a continuous real linear form f over $\mathscr{B}(\mathscr{H}_0)^n_h$ and real β such that $f(\omega'_1,\ldots,\omega'_n)>\beta\ge f(\phi(\sigma_1),\ldots,\phi(\sigma_n))$

$\forall\phi$ c.p.-stochastic. Now, f is given by some $A_1,\dots,A_n\in\mathscr{B}(\mathscr{H}_0)_h$ through the formula $f(\nu_1,\dots,\nu_n)=\sum_{i=1}^{n}\mathrm{Tr.}\,\nu_iA_i$, $\nu_j\in\mathscr{B}(\mathscr{H}_0)_h$, so $\sum_j\mathrm{Tr.}\,\omega'_jA_j>\beta\geq\sum_j\mathrm{Tr.}\,\phi(\sigma_j)A_j$, $\forall\phi$ c.p.-stochastic. Particularly

$$(*)\qquad \sup_{\phi}\sum_j\mathrm{Tr.}\,\phi(\omega'_j)A_j>\sup_{\phi}\sum_j\mathrm{Tr.}\,\phi(\sigma_j)A_j\,,$$

where the "sup" extends over all c.p.-stochastic linear maps. Since all ω'_j, σ_j have equal trace, we may suppose $A_j\geq 0$. By 2.8 we see that $(*)$ means that

$$\sup_{U\in\mathscr{U}(\mathscr{H})}\sum_j\mathrm{Tr.}\,(\omega'_j\otimes E)U(A_j\otimes I)U^*>\sup_{U\in\mathscr{U}(\mathscr{H})}\sum_j\mathrm{Tr.}\,(\sigma_j\otimes E)U(A_j\otimes I)U^*$$

for some $A_1,\dots,A_n\in\mathscr{B}(\mathscr{H}_0)_+$ if $(\omega'_1,\dots,\omega'_n)\notin\mathfrak{M}(\sigma_1,\dots,\sigma_n)$ has been supposed. Therefore, the implication (ii)→(i) is true. Since the set of c.p.-stochastic maps is a semigroup, with a view to 2.8 again, validity of the implication (i)→(ii) is also easily justified.

2.10 Proposition. *Let $\omega'_1,\dots,\omega'_n$ and $\sigma'_1,\dots,\sigma'_n$ be density operators over the separable infinite-dimensional Hilbert-space $\mathscr{H}$. Suppose all ω'_j and σ'_j have finite-dimensional range (i.e. are of finite rank). Then, the following conditions are equivalent:*

(i) *there exists a c.p.-stochastic linear map ϕ over $\mathscr{B}(\mathscr{H})_*$ with $\omega'_k=\phi(\sigma'_k)$ $\forall k$;*

(ii) *with $\mathfrak{K}$ defined in* 2.1

$$K(\omega'_1,\dots,\omega'_n;\,A_1,\dots,A_n)\leq K(\sigma'_1,\dots,\sigma'_n;\,A_1,\dots,A_n)$$

for every choice of $A_1,\dots,A_n\in\mathscr{A}_+$, where $\mathscr{A}$ is an arbitrary UHF-*algebra over $\mathscr{H}$.*

Proof. (i)→(ii). Let $\mathscr{A}_1\subset\mathscr{A}_2\subset\cdots$. $\mathscr{A}=\overline{\bigcup_{k=1}^{\omega}\mathscr{A}_k}$, with finite type-I-factors $\mathscr{A}_k$. Assume $\mathscr{A}_k$ is of type I_{n_k}. $\mathscr{A}$ is UHF if $n_k\to\infty$. Due to 2.2 (i) it is enough to show validity of (ii) for $A_1,\dots,A_n\in\bigcup_{k=1}^{\infty}\mathscr{A}_{k+}$. Let us suppose the latter. Let m be the dimension of the joint range of the family $\omega'_1,\dots,\omega'_n$, $\sigma'_1,\dots,\sigma'_n$. By assumption $m<\infty$. Now, there exists an index k such that $n_k\geq m$ and $A_1,\dots,A_n\in\mathscr{A}_k$. Since $\mathscr{A}_k$ is a type-I_{n_k}-factor over $\mathscr{H}$, we might identify $\mathscr{H}$ with $\mathscr{H}_0\otimes\mathscr{H}_1$, with $\dim\mathscr{H}_0=n_k$, and $\mathscr{A}_k$ is identified with $\mathscr{B}(\mathscr{H}_0)\otimes I$, i.e. $A_k=X_k\otimes I$ for some $X_k\in\mathscr{B}(\mathscr{H}_0)_+$. Let R be the orthoprojection onto the joint range of $\omega'_1,\dots,\sigma'_n$. With the notations of the assumptions of 2.9, we find $V\in\mathscr{U}(\mathscr{H})$ with $VR\mathscr{H}\subseteq\mathscr{H}_0\otimes[x]$. There exist density operators ω_j, $\sigma_j\in\mathscr{B}(\mathscr{H}_0)$ such that $V\omega'_jV^*=\omega_j\otimes E$, $V\sigma'_jV^*=\sigma_j\otimes E$. Therefore (see 2.1):

$$(*)\qquad \begin{aligned} K(\omega'_1,\dots,\omega'_n;\, A_1,\dots,A_n) &= \sup_U \sum_j \mathrm{Tr.}\,(\omega_j\otimes E)U(X_j\otimes I)U^*\,; \\ K(\sigma'_1,\dots,\sigma'_n;\, A_1,\dots,A_n) &= \sup_U \sum_j \mathrm{Tr.}\,(\sigma_j\otimes E)U(X_j\otimes I)U^*\,, \end{aligned}$$

where the "sup" runs over $\mathscr{U}(\mathscr{H})$. Assume (i) holds. Then $\phi' = V\phi(V^*(\cdot)V)V^*$ is c.p.-stochastic on $\mathscr{B}(\mathscr{H})_*$, and $\omega_j\otimes E = \phi'(\sigma_j\otimes E)\forall j$. But then, $T(A) := \mathrm{Tr}_1\,\phi'(A\otimes E)$ for all $A\in\mathscr{B}(\mathscr{H}_0)$ defines a c.p.-stochastic linear map on $\mathscr{B}(\mathscr{H}_0)$ with $\omega_j = T(\sigma_j)$, $\forall j$. 2.9 implies

$$\sup_{U\in\mathscr{U}(\mathscr{H})} \sum_j \mathrm{Tr.}\,(\omega_j\otimes E)U(X_j\otimes I)U^* \le \sup_{U\in\mathscr{U}(\mathscr{H})} \sum_j \mathrm{Tr.}\,(\sigma_j\otimes E)U(X_j\otimes I)U^*\,,$$

from which inequality by means of $(*)$

$$K(\omega'_1,\dots,\omega'_n;\, A_1,\dots,A_n) \le K(\sigma'_1,\dots,\sigma'_n;\, A_1,\dots,A_n)$$

follows. Therefore, (i)→(ii) is true.

To see the implication (ii)→(i), let us suppose (ii) holds. Especially, (ii) holds for all $A_1,\dots,A_n\in\mathscr{A}_k$, with k such that $n_k\ge m$. With the notations of the first part of this proof (see $(*)$) we may conclude that 2.9 (ii) is valid. By 2.9 there exists a c.p.-stochastic ϕ'' over $\mathscr{B}(\mathscr{H}_0)$, with $\omega_k = \phi''(\sigma_k)$, $\forall k$. Hence, by well-known facts, there are $V_j\in\mathscr{B}(\mathscr{H}_0)$, with $\sum_j V_j^* V_j = I$, such that $\phi''(\cdot) = \sum_j V_j(\cdot)V_j^*$. Let $W_j\in\mathscr{B}(\mathscr{H}_0\otimes\mathscr{H}_1)$ be defined by $W_j = V_j\otimes I$. Then, $\phi'(\cdot) = \sum_j W_j(\cdot)W_j^*$ is c.p.-stochastic over $\mathscr{B}(\mathscr{H}_0\otimes\mathscr{H}_1)_*$, due to $\sum_j W_j^* W_j = I$. Finally, $\phi := V^*\phi'(V(\cdot)V^*)V$ is a c.p.-stochastic map on $\mathscr{B}(\mathscr{H})_*$ with $\omega'_j = \phi(\sigma'_j)$, $\forall j$.

§3. Proofs of 1.1–1.3

3.1 Proposition. *Let $\omega_1,\dots,\omega_n$ and $\sigma_1,\dots,\sigma_n$ be normal states over the bounded linear operators on some infinite-dimensional separable complex Hilbertspace $\mathscr{H}$. The following conditions are equivalent*:

- (i) *there is a completely positive stochastic linear map ϕ over $\mathscr{B}(\mathscr{H})^*$ with $\omega_k = \phi(\sigma_k)$, $\forall k$;*
- (ii) *$K(\omega_1,\dots,\omega_n;\; A_1,\dots,A_n) \le K(\sigma_1,\dots,\sigma_n;\; A_1,\dots,A_n)$ for all $A_1,\dots,A_n\in\mathscr{A}_+$, where $\mathscr{A}$ is an arbitrary* UHF-*algebra over $\mathscr{H}$;*
- (iii) *there exists a c.p., unital linear map T acting in $\mathscr{B}(\mathscr{H})$ such that $\omega_k = \sigma_k\circ T$, $\forall k$.*

Proof. The equivalence (i)↔(iii) is clear by 2.7. We show the implication (iii)→(ii). Again we identify normal states with density operators. By 2.5 there are finite-rank density operators $\{\sigma_{km}\}$, with $\|\cdot\|_1-\lim_m \sigma_{km}=\sigma_k$, $\forall k$, and c.p.-stochastic maps ϕ_m on $\mathscr{B}(\mathscr{H})^*$ with

$$\phi_m(\sigma_{km})=\sigma_k,\ \phi_m(\mathscr{B}(\mathscr{H})^*)\subset\mathscr{B}(\mathscr{H})_*\quad \forall m,\ \forall k.$$

By 2.6 there are another sequences $\{\omega_{km}\}$ of density operators, all of finite rank, with $\|\cdot\|_1$-$\lim_m \omega_{km}=\omega_k$, and c.p.-stochastic maps ϕ'_m over $\mathscr{B}(\mathscr{H})_*$ such that $\phi'^{++}_m(\mathscr{B}(\mathscr{H})^*)\subset\mathscr{B}(\mathscr{H})_*$, $\forall m$, and $\phi'^{++}_m(\omega_k)=\phi'_m(\omega_k)=\omega_{km}$ for all m, k. Let us define a c.p.-stochastic map Ω_m on $\mathscr{B}(\mathscr{H})^*$ by

$$\Omega_m=\phi'^{++}_m\circ T^+\circ\phi_m \qquad \text{(see 2.3).}$$

Then, $\Omega_m(\sigma_{km})=\omega_{km}$, $\forall m$, k. Moreover, by the above-listed properties of ϕ'_m and ϕ_m, we have $\Omega_m(\mathscr{B}(\mathscr{H})_*)\subset\mathscr{B}(\mathscr{H})_*$, $\forall m$, therefore $\Omega_{m/\mathscr{B}(\mathscr{H})_*}$ is a c.p.-stochastic linear map on $\mathscr{B}(\mathscr{H})_*$ (see 2.3 (ii)). Hence, all the assumptions for an application of 2.10 are given and we may apply 2.10 with the result that

$$K(\omega_{1m},\dots,\omega_{nm};A_1,\dots,A_n)\le K(\sigma_{1m},\dots,\sigma_{nm};A_1,\dots,A_n)$$

for any choice of $A_1,\dots,A_n\in\mathscr{A}_+$. With a view to 2.1(i) and since $\{\sigma_{km}\}$ and $\{\omega_{km}\}$ are sequences which approximate σ_k and ω_k in a uniform sense, respectively, the validity of (ii) becomes evident. We show the implication (ii)→(iii). Assume (ii) holds for any choice of $A_1,\dots,A_n\in\mathscr{A}_+$, and ϕ_m, ϕ'_m, (σ_{km}), (ω_{km}) be the maps and sequences, respectively, introduced in the first part of the proof where we have already seen that (iii), which is equivalent to (i), implies (ii). Particularly, this last-mentioned implication applies to the situation $\omega_{km}=\phi'^{++}_m(\omega_k)$, $\forall m$, with the c.p.-stochastic ϕ'^{++}_m on $\mathscr{B}(\mathscr{H})^*$, and we get

$$(*)\qquad K(\omega_{1m},\dots,\omega_{nm};A_1,\dots,A_n)\le K(\omega_1,\dots,\omega_n;A_1,\dots,A_n)$$

for all choices $A_1,\dots,A_n\in\mathscr{A}_+$. Since ϕ_m is c.p.-stochastic on $\mathscr{B}(\mathscr{H})^*$, too, by the same argument and since $\phi_m(\sigma_{km})=\sigma_k$, $\forall k$,

$$(**)\qquad K(\sigma_1,\dots,\sigma_n;A_1,\dots,A_n)\le K(\sigma_{1m},\dots,\sigma_{nm};A_1,\dots,A_n)$$

for all $A_1,\dots,A_n\in\mathscr{A}_+$ has to hold. By assumption (ii), (*) and (**) may be taken together and result in

$$K(\omega_{1m},\dots,\omega_{nm};A_1,\dots,A_n)\le K(\sigma_{1m},\dots,\sigma_{nm};A_1,\dots,A_n)$$

$\forall A_j\in\mathscr{A}_+$. By 2.10 we get a sequence $\{\Omega_m\}$ of c.p.-stochastic linear maps over $\mathscr{B}(\mathscr{H})_*$ with $\omega_{jm}=\Omega_m(\sigma_{jm})$, $\forall m$, $\forall j$. Defining the c.p., unital linear map T_m over $\mathscr{B}(\mathscr{H})$ by $T_m=\Omega_m^+$, we have $\omega_{km}=\sigma_{km}\circ T_m$, $\forall k$, m. Let T be a weak accumulation point of the sequence $\{T_m\}$ with respect to the weak operator topology with respect to the duality $\langle \mathscr{B}(\mathscr{H}), \mathscr{B}(\mathscr{H})_*\rangle$. Arguing as in the proof of 2.7 we see $\omega_k=\sigma_k\circ T$, $\forall k$, i.e. (iii) holds with a c.p., unital linear map T over $\mathscr{B}(\mathscr{H})$.

3.2 Proposition. *Let $\mathscr{A}$ be a unital C^*-algebra, and $\nu_1,\dots,\nu_n$ and $\mu_1,\dots,\mu_n$ be states over $\mathscr{A}$. Then, the following conditions are equivalent* (*see* 1. 1.2, 2.1):

(i) *there is $T\in\mathscr{C}_u(\mathscr{A})$ with $\nu_k=T(\mu_k)$ $\forall k$;*

(ii) $K_{\mathscr{A}}(\nu_1,\dots,\nu_n;A_1,\dots,A_n)\le K_{\mathscr{A}}(\mu_1,\dots,\mu_n;A_1,\dots,A_n)$ $\forall A_j\in\mathscr{A}_+$;

(iii) $f(\nu_1,\dots,\nu_n)\ge f(\mu_1,\dots,\mu_n)$ $\forall f\in\mathscr{Q}_n(\mathscr{A})$.

Proof. Let $\mathfrak{M}(\mu)$ be defined as $\mathfrak{M}(\mu)=\{(T(\mu_1),\dots,T(\mu_n)):T\in\mathscr{C}_u(\mathscr{A})\}$, which is a subset of $\mathscr{A}_h^{*n}$. $\mathfrak{M}(\mu)$ is convex and w^*-closed. Therefore, an application of the standard separation theorem for convex sets to the situation $\nu'_j\in\mathscr{S}(\mathscr{A})$, $(\nu'_1,\dots,\nu'_n)\notin\mathfrak{M}(\mu)$ guaranties the existence of $B_j\in\mathscr{A}_+$ such that $K_{\mathscr{A}}(\nu'_1,\dots,\nu'_n;B_1,\dots,B_n)>K_{\mathscr{A}}(\mu_1,\dots,\mu_n;B_1,\dots,B_n)$ (the argumentation running formally as in the proof of 2.9 with obvious modifications, so we omit the details). Therefore, (ii) implies (i). The $K_{\mathscr{A}}$-functions are w^*-l.s.c., unitarily invariant and convex, so the implication (i)→(ii) is easily derived (we omit the details). $(-K_{\mathscr{A}})$ is w^*-u.s.c., unitarily invariant and concave on $\mathscr{S}(\mathscr{A})^n$. Since concavity implies quasiconcavity, we have $-K_{\mathscr{A}}(\cdot,\dots,\cdot;A_1,\dots,A_n)\in\mathscr{Q}_n(\mathscr{A})$. Hence, the implication (iii)→(i) is true. On the other hand,

$$\mathfrak{M}'(\mu)=\bigcap_{f\in\mathscr{Q}_n(\mathscr{A})}\{(\nu_1,\dots,\nu_n)\in\mathscr{S}(\mathscr{A})^n: f(\nu_1,\dots,\nu_n)\ge f(\mu_1,\dots,\mu_n)\}$$

is an intersection of convex, w^*-closed, unitarily invariant sets (by quasi-concavity, unitary invariance and w^*-u.s. continuity for $f\in\mathscr{Q}_n(\mathscr{A})$). Hence $\mathfrak{M}'(\mu)$ is w^*-closed, convex, unitarily invariant and contains $(\mu_1,\dots,\mu_n)$. By the already established equivalence (i)↔(ii), $\mathfrak{M}(\mu)$ is the smallest set of this specification, so $\mathfrak{M}(\mu)=\mathfrak{M}'(\mu)$. This shows (ii)→(iii).

Now, we are ready to prove even a sharpening of 1.1:

3.3 Theorem. *Let $\mathscr{H}$ be an infinite-dimensional separable complex Hilbertspace. Assume $\omega_1,\dots,\omega_n$ and $\sigma_1,\dots,\sigma_n$ are normal states over $\mathscr{B}(\mathscr{H})$. The following conditions are equivalent to each other:*

(i) *there is a completely positive, stochastic linear map ϕ with $\omega_k=\phi(\sigma_k)$, $\forall k$;*

(ii) *there is a c.p., unital linear map T over $\mathscr{B}(\mathscr{H})$ such that $\omega_k=\sigma_k\circ T$, $\forall k$;*

(iii) *there exists an irreducible* UHF-*algebra* $\mathscr{A}$ *of bounded linear operators over* $\mathscr{H}$ *such that there is* $T \in \mathscr{C}_u(\mathscr{A})$ *with* $\omega_{k/\mathscr{A}} = T(\sigma_{k/\mathscr{A}})$, $\forall k$;

(iv) *for all* $f \in \mathscr{Q}_n(\mathscr{H})$ *(see* 1.2*)*

$$f(\omega_1,\dots,\omega_n) \geq f(\sigma_1,\dots,\sigma_n).$$

If one of the conditions (and so all of them) happens to be true, then (iii) is true for every irreducible UHF-*algebra over* $\mathscr{H}$.

Proof. Let $\mathscr{A}$ be as in (iii). By 2.1, 2.2(iii) we have

$$(*) \qquad K(\rho_1,\dots,\rho_n;\, A_1,\dots,A_n) = K_{\mathscr{A}}(\rho_1,\dots,\rho_n;\, A_1,\dots,A_n)$$

for any choice of $\rho_1,\dots,\rho_n \in \mathscr{S}(\mathscr{H})_*$ and all $A_j \in \mathscr{A}_+$. If (iii) is true, 3.2 and $(*)$ show equivalence of (iii) with

$$K(\omega_1,\dots,\omega_n;\, A_1,\dots,A_n) \leq K(\sigma_1,\dots,\sigma_n;\, A_1,\dots,A_n)$$

for all $A_j \in \mathscr{A}_+$. This, however, is equivalent with (i) and also equivalent with (ii), which can be seen by means of 3.1. Therefore, (i)$\leftrightarrow$(ii)$\leftrightarrow$(iii) has been established. From 3.1 follows that (iii) is true for any irreducible UHF-algebra $\mathscr{A}$ if one of the conditions (i), (ii) is full-filled. Finally, from 3.2 and the definition of $\mathscr{Q}_n(\mathscr{H})$ (see 1.2) follows equivalence of (iv) with (iii).

§4. The K-Functions

4.1 Proposition. *Let* $\mathscr{A}$ *be a* UHF-*algebra over the infinite-dimensional separable complex Hilbert-space* $\mathscr{H}$, *and* $A_1,\dots,A_n \in \mathscr{A}_+$. *With the K-functions defined in* 2.1 *we have*:

(i) $K(\omega,\dots,\omega;\, A_1,\dots,A_n) = \|\sum_{j=1}^{n} A_j\|$ *for* $\forall \omega \in \mathscr{S}(\mathscr{H})_*$;

(ii) *if* $\omega_1,\dots,\omega_n \in \mathscr{S}(\mathscr{H})_*$ *are mutually orthogonal density operators, i.e.* $\omega_j\omega_k = 0$ $\forall j \neq k$, *then*

$$K(\omega_1,\dots,\omega_n;\, A_1,\dots,A_n) = \sum_{j=1}^{n} \|A_j\|.$$

Proof. To see (i), we consider the linear map ϕ over $\mathscr{B}(\mathscr{H})_*$ defined by $\phi(\rho) = (\mathrm{Tr}.\,\rho)\omega$, $\forall \rho \in \mathscr{B}(\mathscr{H})_*$. By 2.4, ϕ is c.p.-stochastic on $\mathscr{B}(\mathscr{H})_*$, and $\omega = \phi(\sigma)$ for each density operator σ. Application of 3.1 yields $K(\omega,\dots,\omega;\, A_1,\dots,A_n) = \inf_{\forall \sigma_j \in \mathscr{S}(\mathscr{H})_*} K(\sigma_1,\dots,\sigma_n;\, A_1,\dots,A_n)$. Especially, the value of $K(\omega,\dots,\omega; A_1,\dots,A_n)$ does not depend on the particular $\omega \in \mathscr{S}(\mathscr{H})_*$ chosen. Taking an one-dimensional orthoprojection E, with $Ex = x$ for the unit vector $x \in \mathscr{H}$, as a special case of a density operator, we see $K(\omega,\dots,\omega;A_1,\dots,A_n)=K(E,\dots,E;A_1,\dots,A_n)=\sup_{U\in\mathscr{U}(\mathscr{H})}\sum_j \mathrm{Tr}.EUA_jU^*=\sup_{U\in\mathscr{U}(\mathscr{H})}(Ux,\sum_j A_jUx)=\|\sum_j A_j\|$.

To derive (ii), assume $\omega_1,\dots,\omega_n$ are mutually orthogonal density operators, and $\{P_j\}$ be an orthogonal decomposition of 1 into orthoprojections with $P_j\omega_j=\omega_j$, $\forall j$. Let $\{\sigma_1,\dots,\sigma_n\}$ be an arbitrarily chosen n-tuple of density operators. We introduce a linear map over $\mathscr{B}(\mathscr{H})_*$ by $\phi(\cdot)=\sum_{j=1}^n \sigma_j\mathrm{Tr}.P_j(\cdot)$. ϕ preserves the trace and is c.p. by 2.4, hence, it is c.p.-stochastic over $\mathscr{B}(\mathscr{H})_*$. By construction $\phi(\omega_j)=\sigma_j$, $\forall_j$. Applying 3.1 once more again, and having in mind that the n-tuple $\{\sigma_1,\dots,\sigma_n\}$ could have been chosen at will from the set of all n-tuples of density operators, we see $K(\omega_1,\dots,\omega_n;A_1,\dots,A_n)=\sup_{\forall\sigma_j\in\mathscr{S}(\mathscr{H})}K(\sigma_1,\dots,\sigma_n;A_1,\dots,A_n)$. Especially, the value $K(\omega_1,\dots,\omega_n;A_1,\dots,A_n)$ has to be independent of the specific orthogonal family $\omega_1,\dots,\omega_n$ chosen. Let us take n mutually orthogonal one-dimensional orthoprojections $E_1,\dots,E_n$ as our special choice. Then,

$$(*)\qquad K(E_1,\dots,E_n;A_1,\dots,A_n)=K(\omega_1,\dots,\omega_n;A_1,\dots,A_n)\leqq\sum_j\|A_j\|,$$

the inequality being an obvious consequence of the definition of K. To see (ii) we need some auxiliary construction.

4.2 Lemma. *Let Q, P be orthoprojections, with* $\dim P=\infty$, $\dim Q<\infty$. *Then,* $\dim P\wedge Q^\perp=\infty$.

Proof of 4.2. Let $\{u_k\}$ be a complete orthonormal system in $P\mathscr{H}$. Then, $u_k=Qu_k+Q^\perp u_k$. Let $J=\{k_j\colon Qu_{k_j}\neq 0\}$. If card $\mathscr{N}\backslash J=\infty$ ($\mathscr{N}$ means the natural numbers), $u_k=Q^\perp u_k \forall k\in\mathscr{N}\backslash J$, and the assertion follows. Assume card $\mathscr{N}\backslash J<\infty$. Then, $\overline{[\{Qu_{k_j}\}_{j\in\mathscr{N}}]}\subset Q\mathscr{H}$. Let $\dim Q=m$, and let us consider the systems Γ_j of vectors given by $\Gamma_j=\{Qu_{k_{j(m+1)+1}},\dots,Qu_{k_{(j+1)(m+1)}}\}$, $j=0,1,2,\dots$. Each of these systems contains $m+1$ non-vanishing vectors. Since $\dim Q=m$, there exist nontrivial systems $\{\beta_r^{(j)}\}_{r=1}^{m+1}$ of complex numbers such that $Q(\sum_{r=1}^{m+1}\beta_r^{(j)}u_{k_{j(m+1)+r}})=0$, from which by means of the orthogonality of the u_k's follows that $\sum_{r=1}^{m+1}\beta_r^{(j)}u_{k_{j(m+1)+r}}=:w_j\neq 0$, with $w_j=Q^\perp w_j$, $\forall j$. By construction, the w_j are mutually orthogonal: $(w_j,w_k)=0$, $\forall j\neq k$, and $\overline{[w_1,\dots,w_k,\dots]}\subset Q^\perp\mathscr{H}$. On the other hand, since $w_j\in\overline{[u_1,u_2,\dots]}$, $Pw_j=w_j$, so the infinite-dimensional subspace $\overline{[w_1,\dots,w_k,\dots]}$ of $\mathscr{H}$ is contained in both $Q^\perp\mathscr{H}$ and $P\mathscr{H}$, and $\dim Q^\perp\wedge P=\infty$ follows.

4.3 Lemma. *Let $P_1,\dots P_n$ be orthoprojections over $\mathscr{H}$, with* $\dim P_j=\infty$ *for* *all j. There are mutually orthogonal unit vectors* $x_1,\dots, x_n$ *such that* $P_jx_j = x_j| \ \forall j$.

Proof of 4.3. For x_1 let us take a vector $\|x_1\|=1$, $P_1x_1=x_1$. Assume, we have yet built a system $\{x_1,\dots, x_k\}$ of vectors, being mutually orthogonal, with $P_jx_j=x_j$, $j=1,\dots, k$. We construct x_{k+1} as follows: let Q_k be the orthoprojection onto $[x_1,\dots, x_k]$, then $\dim Q_k=k<\infty$, and 4.2 can be applied showing $\dim Q_k^\perp \wedge P_{k+1}=\infty$. Take $\|x_{k+1}\|=1$, $x_{k+1}\in\mathscr{H}$, with $Q_k^\perp\wedge P_{k+1}x_{k+1}=x_{k+1}$. Then, $\{x_1,\dots, x_{k+1}\}$ is an orthogonal system of unit vectors fulfilling $P_jx_j=x_j$ $\forall j\leq k+1$. Successive application of this procedure gives the result.

We continue in the proof of 4.1: due to 2.2(i) it is sufficient to prove (ii) for $A_1,\dots, A_n\in\bigcup_{k=1}^{\infty}\mathscr{A}_{k+}$, $A_j\neq 0$. Hence, assume $A_j\in\mathscr{A}_{k+}$, $j=1,\dots, n$, non-trivial. Since $\mathscr{A}_+$ is a finite type-I-factor we may assume a spectral representation of A_j, $A_j=\sum_k \alpha_{jk}P_{jk}$, with $\{P_{jk}\}$ the corresponding orthogonal decomposition of I, and $\alpha_{j1}>\alpha_{j2}>\cdots\geq 0$ the points in the spectrum of A_j. Especially, $\|A_j\|=\alpha_{j1}\neq 0$. Since $\mathscr{A}_k\cap\mathscr{B}\mathscr{C}(\mathscr{H})=\{0\}$, we must have $\dim P_{j1}=\infty$, $\forall j$. Then, by 4.3 we provide us with an orthonormal system $\{x_1,\dots, x_n\}$ such that $P_{j1}x_j=x_j$, $j=1,\dots, n$. Let us make a special choice for $E_1,\dots, E_n$ in $(*)$: take E_j as the orthoprojection onto $[x_j]$. Then, by the choice of x_j we obtain

$$\sum_j \|A_j\|=\sum_j \mathrm{Tr.}\, E_jA_j\leq \sup_{U\in\mathscr{U}(\mathscr{H})}\sum_j \mathrm{Tr.}\, E_jUA_jU^*=K(E_1,\dots, E_n;\, A_1,\dots, A_n),$$

from which inequality together with $(*)$ the assertion (ii) follows.

4.4 Lemma. *Let* $\mathscr{A}$ *be a* UHF-*algebra over* $\mathscr{H}$. *Let* $\omega_1,\dots,\omega_n\in\mathscr{S}(\mathscr{H})_*$. *Assume* $K(\omega_1,\dots,\omega_n;\, A_1,\dots, A_n)=\sum_{j=1}^{n}\|A_j\|$ *holds for every choice of* $A_1,\dots, A_n\in\mathscr{A}_+$. *Then, the* ω_j *are mutually orthogonal.*

Proof. Let $E_1,\dots, E_n$ be mutually orthogonal one-dimensional orthoprojections. By assumption and since $K(\nu_1,\dots,\nu_n;\, A_1,\dots, A_n)\leq\sum_j\|A_j\|$ for any choice $\nu_1,\dots,\nu_n$ of density operators and every subset $\{A_1,\dots, A_n\}\subset\mathscr{A}_+$ we may apply 3.1 to see the existence of a c.p., unital linear map T over $\mathscr{B}(\mathscr{H})$ such that $E_k=\omega_k\circ T$, $\forall k$. By 1.9 we then have $P_{\mathscr{B}(\mathscr{H})}(E_i, E_k)\geq P_{\mathscr{B}(\mathscr{H})}(\omega_i,\omega_k)$ $\forall i, k$, and having in mind the explicite formula for $P_{\mathscr{B}(\mathscr{H})}$ (see 1.8) we might conclude as follows: let $i\neq k$, then $0=(\mathrm{Tr.}\, E_iE_k)^2=(\mathrm{Tr.}\,(E_i^{1/2}E_kE_i^{1/2})^{1/2})^2=P_{\mathscr{B}(\mathscr{H})}(E_i, E_k)\geq P_{\mathscr{B}(\mathscr{H})}(\omega_i,\omega_k)=(\mathrm{Tr.}\,(\omega_i^{1/2}\omega_k\omega_i^{1/2})^{1/2})^2\geq 0$, so $\omega_i\omega_k=0$ for $i\neq k$ has to be followed.

4.5 Lemma. *Let* $\mathscr{A}$ *be a* UHF-*algebra over* $\mathscr{H}$, *and* $\omega_1,\dots,\omega_n\in\mathscr{S}(\mathscr{H})_*$

given. Assume $K(\omega_1,\dots,\omega_n;\ A_1,\dots,A_n)=\|\sum_j A_j\|$ *for every choice of* $A_1,\dots,A_n\in\mathscr{A}_+$. *Then* $\omega_1=\cdots=\omega_n$.

Proof. The argumentations of the proof of 4.1(i) show that $\|\sum_j A_j\|$ is the minimal value of $K(\nu_1,\dots,\nu_n;\ A_1,\dots,A_n)$ for $\nu_1,\dots,\nu_n\in\mathscr{S}(\mathscr{H})_*$. Therefore, our assumption tells us that

$$K(\nu_1,\dots,\nu_n;\ A_1,\dots,A_n)\geq K(\omega_1,\dots,\omega_n;\ A_1,\dots,A_n)\text{for all } A_j\in\mathscr{A}_+$$

and an arbitrary choice of $\nu_j\in\mathscr{S}(\mathscr{H})_*$, $j=1,\dots,n$. Applying 3.1 once more again gives a c.p., unital linear map T_ν over $\mathscr{B}(\mathscr{H})$ with $\omega_k=\nu_k\circ T_\nu$ $\forall k$. Making the special choice $\nu_1=\cdots=\nu_n$ we see $\omega_1=\cdots=\omega_n$.

Let $\mathscr{A}$ be a UHF-algebra over $\mathscr{H}$. Then, $\mathscr{A}$ is *separable*. Especially, there is a normdense countable subset $\{A_k\}_{k=1}^\infty\subset\mathscr{A}_+$ for $\mathscr{A}_+$: $\overline{\{A_k\}_{k=1}^\infty}=\mathscr{A}_+$. It is not hard to get convinced that we may choose $\{A_k\}$ in such a way that

$$\sum_{i=1}^n\|A_{k_i}\|>\|\sum_{i=1}^n A_{k_i}\|\quad\text{for any choice } (k_1,\dots,k_n)\in\mathscr{N}_0^n,\text{ with}$$

$\mathscr{N}_0^n:=\mathscr{N}^n\setminus\{(m,m,\dots,m):\ m\in\mathscr{N}\}$. Let us define

$$R_k(\omega_1,\dots,\omega_n)=\Big(\sum_{i=1}^n\|A_{k_i}\|-\|\sum_{i=1}^n A_{k_i}\|\Big)^{-1}\Big(\sum_{i=1}^n\|A_{k_i}\|-K(\omega_1,\dots,\omega_n;\ A_{k_1},\dots,A_{k_n})\Big)$$

for any $k=(k_1,\dots,k_n)\in\mathscr{N}_0^n$. By 4.1 we know that $R_k(\omega_1,\dots,\omega_n)\in[0,1]$ for all $\omega_j\in\mathscr{S}(\mathscr{H})_*$, and R_k is concave and w^*-u.s.c. over $\mathscr{S}(\mathscr{H})_*^n$. Let $\mathscr{L}^n$ be the set of sequences

$$\mathscr{L}^n=\{=\lambda(\lambda_k)_{k\in\mathscr{N}_0^n}:\ \lambda_k>0\quad\forall k,\ \sum_{k\in\mathscr{N}_0^n}\lambda_k=1\},$$

and define

$$P_\lambda(\mathscr{A};\ \omega_1,\dots,\omega_n)=\sum_{k\in\mathscr{N}_0^n}\lambda_k R_k(\omega_1,\dots,\omega_n)\qquad\text{for}\quad\lambda\in\mathscr{L}^n.$$

We then have:

4.6 Proposition. *Let* $\mathscr{A}$ *be an* UHF-*algebra over* $\mathscr{H}$; $\{P_\lambda(\mathscr{A};\cdots),\ \lambda\in\mathscr{L}^n\}$ *is a system of* w^**-u.s.c., quasi-concave and unitarily invariant functions on* $\mathscr{S}(\mathscr{H})_*^n$, *with the following additional properties*:

(i) $P_\lambda(\mathscr{A};\ \omega_1,\dots,\omega_n)\in[0,1]$ $\forall\omega_j\in\mathscr{S}(\mathscr{H})_*$;

(ii) $P_\lambda(\mathscr{A};\ \omega_1,\dots,\omega_n)=0$ *iff* $\omega_j\omega_k=0$ $\forall j\neq k$;

(iii) $P_\lambda(\mathscr{A};\ \omega_1,\dots,\omega_n)=1$ *iff* $\omega_1=\cdots=\omega_n$;

(iv) *if $\mathscr{A}$ is irreducible, then $P_\lambda \in \mathscr{Q}_n(\mathscr{H})$ and $P_\lambda(\mathscr{A}; \omega_1,\dots, \omega_n) \geq P_\lambda(\mathscr{A}; \sigma_1,\dots, \sigma_n)$ for all $\lambda \in \mathscr{L}^n$ if and only if there is a c.p.-stochastic linear ϕ over $\mathscr{B}(\mathscr{H})^*$ such that $\omega_k = \phi(\sigma_k)$, $\forall k$.*

Proof. Quasi-concavity, unitary invariance and (i) are obvious properties. By definition, there exist $\beta_\lambda \in R^1_+$ and $\alpha_k \in R^1_+$ such that $P_\lambda(\mathscr{A}; \omega_1,\dots, \omega_n) = \beta_\lambda - \sum_{k \in \mathscr{N}^n_0} \alpha_k K(\omega_1,\dots, \omega_n; A_{k_1},\dots, A_{k_n}) = \inf_{\mathscr{M}} \{\beta_\lambda - \sum_{k \in \mathscr{M} \subset \mathscr{N}^n_0} \alpha_k K(\omega_1,\dots, \omega_n; A_{k_1},\dots, A_{k_n})\}$, where the infimum runs over all finite subsets $\mathscr{M}$ of $\mathscr{N}^n_0$. The infimum of u.s.c.-functions is u.s.c., so $P_\lambda(\mathscr{A}; .\,,\dots, .)$ is w^*-u.s.c. By construction of P_λ, we have

$$P_\lambda(\mathscr{A}; \omega_1,\dots, \omega_n) = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \text{iff } K(\omega_1,\dots, \omega_n; A_{k_1},\dots, A_{k_n}) = \begin{Bmatrix} \sum_{i=1} \|A_{k_i}\| \\ \|\sum_{i=1} A_{k_i}\| \end{Bmatrix} \text{for all } k \in \mathscr{N}^n_0.$$

This, together with density of $\{A_k\}$ within the positive cone of $\mathscr{A}$ implies (ii), (iii) by 4.1, 4.4 and 4.5. Let $k \in \mathscr{N}^n_0$. Then, there is a sequence $\{\lambda^{(m)}\} \subset \mathscr{L}^n$ such that $\lim_m P_{\lambda^{(m)}}(\mathscr{A}; \nu_1,\dots, \nu_n) = R_k(\nu_1,\dots, \nu_n)$ $\forall \nu_j \in \mathscr{S}(\mathscr{H})_*$. Hence, $P_\lambda(\mathscr{A}; \omega_1,\dots, \omega_n) \geq P_\lambda(\mathscr{A}; \sigma_1,\dots, \sigma_n)$ $\forall \lambda \in \mathscr{L}^n$ implies $R_k(\omega_1,\dots, \omega_n) \geq R_k(\sigma_1,\dots, \sigma_n)$ which is equivalent with

$$K(\omega_1,\dots, \omega_n; A_{k_1},\dots, A_{k_n}) \leq K(\sigma_1,\dots, \sigma_n; A_{k_1},\dots, A_{k_n}),\ k \in \mathscr{N}^n_0.$$

This implication holds for an arbitrarily chosen $k \in \mathscr{N}^n_0$. By 2.1(i), and since $\{A_k\}$ is dense in $\mathscr{A}_+$ we may use 3.1 to see the existence of c.p.-stochastic ϕ over $\mathscr{B}(\mathscr{H})^*$ with $\omega_k = \phi(\sigma_k)$, $\forall k$, and the one direction of (iv) is proved. The other way around follows by 3.3 and the fact that $P_\lambda(\mathscr{A}; \cdots) \in \mathscr{Q}_n(\mathscr{H})$ in case of an irreducible $\mathscr{A}$.

§5. The Case of Tuples

Throughout this part we specialize to $n=2$. Let $\mathscr{A}$ be an arbitrary (but throughout fixed) irreducible UHF-algebra over $\mathscr{H}$. Fix $A, B \in \mathscr{A}_+$. If $x, y, x', y' \in \mathscr{H}$ are unit vectors, with $|(x, y)| = |(x', y')|$, then there is $U \in \mathscr{U}(\mathscr{H})$ such that $x' = Ux$, $y' = Uy$ ($\mathscr{H}$ is a complex Hilbert-space). Therefore, if ω_x denotes the state $\omega_x = (x, (\cdot)x)$, etc., we have $K(\omega_x, \omega_y; A, B) = K(\omega_{x'}, \omega_{y'}; A, B)$. This shows that $f_{A,B}(t) := K(\omega_x, \omega_y; A, B)$ with $t = |(x, y)|^2$ is a weldefined realvalued function.

5.1 Lemma. *For fixed $A, B \in \mathscr{A}_+, f := f_{A,B}$ is a continuous and monoto-*

neously decreasing function over the unit interval.

Proof. Having in mind that $P_{\mathscr{B}(\mathscr{H})}(\omega_x, \omega_y)=|(x, y)|^2$, an application of a combination of 1.8 with 3.1 gives the decreasing behaviour. Let $t\in[0, 1]$, and define $x_t=\sqrt{t}\,x+\sqrt{1-t}\,y$, with x, y unit vectors being orthogonal to each other: $(x, y)=0$. Assume $\{U_m\}\subset\mathscr{U}(\mathscr{H})$ have been chosen in such a way that, with $A_m=U_mAU_m^*$, $B_m=U_mBU_m^*$, the following limits exist (use a compactness argument repeatedly):

$$f(t)=K(\omega_x, \omega_{x_t};\, A, B)=\lim_m\ \{\omega_x(A_m)+\omega_{x_t}(B_m)\},\ \alpha(t)=\lim_m\,(x, A_mx),$$

$$\gamma(t)=\lim_m\,(y, B_my),\ \delta(t)=\lim_m\,(x, B_my),\ \beta(t)=\lim_m\ \{(x, B_mx)-(y, B_my)\}.$$

A simple calculation shows that

$$(*)\qquad f(t)=\alpha(t)+\gamma(t)+t\beta(t)+2\sqrt{t(1-t)}\,\mathrm{Re}\,\delta(t)\quad (\mathrm{Re}-\text{the real part}).$$

If $\gamma(t)=|\delta(t)|\exp i\varDelta$, put $z=\sqrt{t}\,x+(\exp\text{-}i\varDelta)\sqrt{1-t}\,y$. Then, $|(x, z)|^2=t$, so $K(\omega_x, \omega_z;\, A, B)\geq\lim_m\ \{\omega_x(A_m)+\omega_z(B_m)\}=\alpha(t)+\gamma(t)+t\beta(t)+2\sqrt{t(1-t)}\,|\delta(t)|\geq f(t)$, by $(*)$. Therefore, $\delta(t)\geq 0$ has to hold for $t\neq 0, 1$, and can be supposed for $t=0, 1$. Now, the sequence $\{U_m\}\subset\mathscr{U}(\mathscr{H})$ depends on the particular choice of $t\in[0, 1]$. Hence, in general for $s\neq t$: $f(s)\geq\lim_m\ \{\omega_x(A_m)+\omega_{x_s}(B_m)\}=\alpha(t)+\gamma(t)+s\beta(t)+2\sqrt{s(1-s)}\,\delta(t)$, $\forall s$. We define a function $F_t(s):=\alpha(t)+\gamma(t)+s\beta(t)+2\sqrt{s(1-s)}\,\delta(t)$ over the unit interval. For the second derivative $F_t(s)''$ we see $F_t(s)''=-\sqrt{s(1-s)}^{-1}(2+(1-2s)^2(2s(1-s))^{-1})\delta(t)\leq 0$, due to $\delta(t)\geq 0$, for s taken from the unit interval. Consequently, $F_t(s)$ is concave for $s\in[0, 1]$. We take together all these informations and know: to every $t\in[0, 1]$ there is a concave continuous function $F_t(s)$ over $[0, 1]$ such that

(i) $f(s)\geq F_t(s)\quad\forall s$;

(ii) $f(t)=F_t(t)$;

(iii) $F_t(s)\geq 0\quad\forall s$.

The function f is l.s.c. (the supremum of continuous functions by definition) and decreasing so that it is continuous from the right. Let $\{t_n\}$ be an increasing sequence within $[0, 1)$ and $t_n\nearrow t$. From concavity of $F_t(s)$ together with (i), (ii) and (iii) we close as follows: $f(s)\geq F_{t_n}(s)\geq(1-s)(1-t_n)^{-1}F_{t_n}(t_n)+(s-t_n)(1-t_n)^{-1}F_{t_n}(1)\geq(1-s)(1-t_n)^{-1}f(t_n)$ for all $s\in[t_n, 1)$. If $t\in[t_n, 1)$, for $s=t$ this yields

$$f(t)\geq(1-t)(1-t_n)^{-1}f(t_n)\geq(1-t)(1-t_n)^{-1}f(t),$$

where in the last step we also used monotonicity of f. From the last written inequality follows that $\lim_n f(t_n)=f(t)$. For $t=1$ this follows from $|F_{t_n}(1)-f(t_n)| \leq (1-t_n)|\beta(t_n)| + 2\sqrt{t_n(1+t_n)}\,\delta(t_n) \underset{n}{\rightarrow} 0$, since $|\beta(t_n)|$, $\delta(t_n)$ are uniformly bounded. In fact, since $\lim f(t_n)$ exists (monotonicity), we have $\lim f(t_n)= \lim F_{t_n}(1)$. From $F_{t_n}(1)\leq f(1)\leq f(t_m)$ (see (i) and monotonicity) comes $\lim f(t_n)\leq f(1)\leq f(t_m)$, so $\lim f(t_n)=f(1)$ follows. Thus, f is also continuous from the left.

We will have need for the following elementary, but very useful result:

5.2 Lemma. *Let g be a monotonously increasing continuous function over an interval $[\alpha, \beta]\subset R^1$. Let h be a quasi-concave and u.s.c. function over a convex subset $\mathscr{X}$ in a topological vector space. Assume $h(\mathscr{X})\subset[\alpha, \beta]$. Then, $g\circ h$ is quasi-concave and u.s.c. over $\mathscr{X}$.*

Proof. Let $c\in R^1$, and define $\mathfrak{M}_c=\{x\in\mathscr{X}: g\circ h(x)\geq c\}$. Assume h is quasi-concave and u.s.c. Let $\mathfrak{N}_c=\{r\in[\alpha, \beta]: g(r)\geq c\}$. $\mathfrak{N}_c$ is closed. If $\mathfrak{N}_c=\phi$, then $\mathfrak{M}_c=\phi$. If $\mathfrak{N}_c\neq\phi$, by monotonicity and continuity of g, there is $c'\in R^1$ with $\mathfrak{N}_c=[c', \beta]$. Since $x\in\mathfrak{M}_c$ iff $h(x)\in\mathfrak{N}_c$, we have $\mathfrak{M}_c=\{x\in X: h(x)\geq c'\}$. By assumptions, $\mathfrak{M}_c$ is convex and closed. Hence, $g\circ h$ is quasi-concave and u.s.c.

Let us now specialize to the case of tuples within all constructions and definition of 4, especially 4.6. By 5.1 we know that $f_{A_{k_1}A_{k_2}}(t)=K(\omega_x, \omega_y; A_{k_1}, A_{k_2})$, with $t=|(x, y)|^2$, is a decreasing and continuous function over $[0, 1]$. Then, the function $G_\lambda(t):=P_\lambda(\mathscr{A}; \omega_x, \omega_y)$ as a uniform limit of increasing, continuous functions over $[0, 1]$ has to be so, too. Combining 1.9 with 3.1 we see for $\lambda\in\mathscr{L}^2$ and $t, s\in[0, 1]$

$$G_\lambda(t)>G_\lambda(s) \text{ iff } \quad t>s,$$

so G_λ a strictly increasing continuous function which maps $[0, 1]$ into itself (see 4.6 (i)–(iii)). Then, G_λ^{-1} exists and is an increasing continuous function on $[0, 1]$ with range $[0, 1]$. We apply 5.2 with $\mathscr{X}=\mathscr{S}(\mathscr{H})_*\times\mathscr{S}(\mathscr{H})_*$, $\alpha=0$, $\beta=1$, $g=G_\lambda^{-1}$, $h=P_\lambda(\mathscr{A}; .,\ldots, .)$ and see that $P_\lambda(\omega, \sigma):=G_\lambda^{-1}\circ P_\lambda(\mathscr{A}; \omega, \sigma)$ for $\forall\lambda\in\mathscr{L}^2$ is a quasi-concave, unitarily invariant, w^*-u.s.c. function over $\mathscr{S}(\mathscr{H})_*\times\mathscr{S}(\mathscr{H})_*$ with the additional property $P_\lambda(\omega_x, \omega_y)=|(x, y)|^2$ for unit vectors $x, y\in\mathscr{H}$. Since G_λ is strictly monotonous and continuous, $P_\lambda(\omega, \sigma)\geq P_\lambda(\omega', \sigma')$, $\forall\lambda\in\mathscr{L}^2$ iff $P_\lambda(\mathscr{A}; \omega, \sigma)\geq P_\lambda(\mathscr{A}; \omega', \sigma')$ $\forall\lambda\in\mathscr{L}^2$. By construction, $P_\lambda\in\mathscr{Q}_2(\mathscr{H})$, $\forall\lambda\in\mathscr{L}^2$. With a view to 3.3, 4.6 we may take together the results of this section in the following form:

5.3 Proposition. *Let $\mathcal{TP}(\mathcal{H})$ be the set of realvalued functions over $\mathcal{S}(\mathcal{H})_* \times \mathcal{S}(\mathcal{H})_*$ characterized by: $p \in \mathcal{TP}(\mathcal{H})$ if and only if*

(i) $p(\omega, \sigma) \in [0, 1]$ $\forall \omega, \sigma \in \mathcal{S}(\mathcal{H})_*$;

(ii) *p is w^*-u.s.c., quasi-concave and unitarily invariant;*

(iii) *$p(\omega, \sigma)=0$ iff ω, σ are mutually orthogonal states;*

(iv) *$p(\omega, \sigma)=1$ iff $\omega=\sigma$;*

(v) *$p(\omega_x, \omega_y)=|(x, y)|^2$ for all unit vectors $x, y \in \mathcal{H}$;*

(vi) *$p(\omega\circ T, \sigma\circ T) \geq p(\omega, \sigma)$ for any unital, completely positive linear map T over $\mathcal{B}(\mathcal{H})$ and all $\omega, \sigma \in \mathcal{S}(\mathcal{H})_*$ such that $\omega\circ T, \sigma\circ T \in \mathcal{S}(\mathcal{H})_*$.*

Then, for normal states $\omega, \sigma, \omega', \sigma' \in \mathcal{S}(\mathcal{H})_*$ we have:

$$p(\omega, \sigma) \geq p(\omega', \sigma') \qquad \text{for any} \quad p \in \mathcal{TP}(\mathcal{H})$$

is equivalent with the existence of a completely positive, unital linear map T over $\mathcal{B}(\mathcal{H})$ such that $\omega=\omega'\circ T$, $\sigma=\sigma'\circ T$.

Proof. The P_λ's constructed above belong to $\mathcal{Q}_2(\mathcal{H})$ and fulfil (i)–(v). By 4.6 (iv) and our discussions above all P_λ obey (vi), so the assertion is seen.

Note that in proving 5.3 we also proved 1.4. Finally, as the discussion of 1.8 shows, it is useful to take notice of the following fact:

5.4 Corollary. *Let $p(.,.)$ be a realvalued function over $\mathcal{S}(\mathcal{H})_* \times \mathcal{S}(\mathcal{H})_*$ with properties* 5.3 (i)–(v). *Then, $p \in \mathcal{Q}_2(\mathcal{H})$ (see* 1.2*) implies $p \in \mathcal{TP}(\mathcal{H})$.*

Proof. From $p \in \mathcal{Q}_2(\mathcal{H})$ and 3.3 follows 5.3 (vi).

5.5 *Remark.* As we know from 5.3 each of the functions of $\mathcal{TP}(\mathcal{H})$ reduces to $|(x, y)|^2$ if it is considered in restriction to the set of pure states $\{\omega_x : x \in \mathcal{H}, \|x\|=1\}$, i.e. $p(\omega_x, \omega_y)=|(x, y)|^2$ for all $p \in \mathcal{TP}(\mathcal{H})$. Hence, in context of quantum statistical mechanics, one might think of each of the functions p with properties 5.3 (i)–(vi) as a "reasonable" extension of the notion of the quantum mechanical transition probability (together with some of its properties) from pure to mixed states in the sense of quantum statistics. Whether or not such an extension is sound or useful cannot be decided or even discussed within this mathematical treatment. This task together with a discussion of the physical backgrounds of the search for such extensions is a matter of foundation of quantum physics within the algebraic approach. This is to be done elsewhere. Nevertheless, some heuristic and methodological aspects should be touched on. The reader who is not willing to follow such speculations beyond mathematics might skip the rest of this point. In the literature on mathematical physics several definitions for "generalized transition probabilities" exist. All of them which are known to the author fall into the class of functions described by 5.3 (i)–(vi) if they are considered in restriction to normal states over $\mathscr{B}(\mathscr{H})$. As an additional property these examples are symmetic functions, i.e. $p(\omega, \sigma)=p(\sigma, \omega)$, $\forall \omega, \sigma \in \mathscr{S}(\mathscr{H})_*$. A really physical reasoning why such a symmetry should be imposed is not known to the author (besides the simple fact that $|(x, y)|^2$ is symmetric). For a general element of $\mathscr{T}\mathscr{P}(\mathscr{H})$ such symmetry is not required. Only in certain special cases the restriction onto the symmetric members of $\mathscr{T}\mathscr{P}(\mathscr{H})$ exclusively will be sufficient to maintain the validity of both directions of the assertion of 5.3 (see 1.9 for such an example). Even the fact that for the description of irreversible behaviour in quantum statistical mechanics one has to deal mainly with dissipative (directed, non-reversible) motions (instead of a description by means of groups and semigroups of automorphisms) seems to indicate that also non-symmetric extensions of $|(x, y)|^2$ should be meaningful. Granting an interpretation for p in sense of a transition probability, one should better refer to the number $p(\omega, \sigma)$ by convention as giving a transition probability from ω into the one-point-set $\{\sigma\}$ of states than speaking merely of the transition probability *between* ω and σ (which suggests symmetry). The result of 5.3 might be used to give mathematical support to the arguments in favour of considering also non-symmetric extensions of $|(x, y)|^2$. This gives, in a natural way, an imbedding of the case of two states into the case of n-tuples (see 4.6) and suggests to refer to the values of certain "canonical" functions $p(\omega_1,\dots, \omega_n)$ with properties as described in 4.6 as giving a "generalized transition probability from ω_1 into the convex set conv $\{\omega_j\}_{j=2}^n$". In this case it is not clear, however, how to impose "boundary conditions" on the pure states (compare 5.3 (v) in case of tuples) without loosing the equivalence of $p(\omega_1,\dots, \omega_n)\geq p(\sigma_1,\dots, \sigma_n)$ with the existence of a transformation T such that $\omega_j=\sigma_j\circ T$, $\forall j$. Also, it seems that there is no natural pendant of $|(x, y)|^2$ in quantum mechanics for more than two vectors, where generalizations to the general quantum statistical situation could be based on in sense of the n-tuple-problem. Thus, these general considerations don't seem to be as significant as in the case $n=2$. These very incomplete remarks are thought to serve as an impulse for further discussions about generalized transition probabilities between mathematicians and physicists.

§6. The Case of a Non-Separable Hilbert Space

Assume the Hilbert-space $\mathscr{H}$ is *not* separable. Let $\omega_1,\dots,\omega_n$ and $\sigma_1,\dots,\sigma_n$ be density operators over $\mathscr{H}$. Let E, P be the orthoprojections onto the joint range of the family $\{\omega_1,\dots,\omega_n\}$, $\{\sigma_1,\dots,\sigma_n\}$, respectively. Since all operators considered are of trace-class, $E\mathscr{H}$ and $P\mathscr{H}$ are separable. $U\in\mathscr{U}(\mathscr{H})$ might be chosen such that $UEU^*=E^u\geq P$, or $E^u\leq P$ holds. Assume T is a c.p., unital linear map over $\mathscr{B}(\mathscr{H})$ such that $\omega_j=\sigma_j\circ T$, $\forall j$. Then, $\omega_j^U=\sigma_j\circ T''$, $\forall j$, with the unital c.p.-map $T''=T(U\cdot U^*)$. Moreover, for $\forall A\in\mathscr{B}(\mathscr{H})$:

$$\mathrm{Tr}.\,\sigma_j T''(A)=\mathrm{Tr}.\,\omega_j^U A=\mathrm{Tr}.\,\omega_j^U E^U A E^U=\mathrm{Tr}.\,\sigma_j\mathfrak{T}''(E^U A E^U),\ \text{i.e. } \forall j:$$
$$\mathrm{Tr}.\,\sigma_j(PT''(A)P-PT''(E^U A E^U)P)=0\,.$$

Especially, the latter holds for $A=\underline{1}$. Since T'' is unital, we see $B=PT''1()P-PT''(E^U)P\geq P-P=0$, so the equality above amounts to $\mathrm{Tr}.\,\sigma_j B=0$ $\forall j$, which implies $B=0$, since $P^\perp B=BP^\perp=0$. With other words, $P=PT''(E^U)P$. We define $Q=E^U\vee P$. Then, $Q\mathscr{H}$ is separable. Over $\mathscr{B}(Q\mathscr{H})\cong Q\mathscr{B}(\mathscr{H})Q$ let us define a completely positive linear map T' by $T'=PT''(E^U(\cdot)E^U)P+(Q-P)(\cdot)(Q-P)$. We have $T'(Q)=PT''(E^U)P+(Q-P)=P+(Q-P)=Q$, so T' is unital on $\mathscr{B}(Q\mathscr{H})$. Now, over $Q\mathscr{H}$, we may write: $\mathrm{Tr}.\,\sigma_j T'(A)=\mathrm{Tr}.\,\sigma_j PT''(E^U A E^U)P=\mathrm{Tr}.\,\sigma_j^{T''}(E^U A E^U)=\mathrm{Tr}.\,\omega_j^U E^U A E^U=\mathrm{Tr}.\,\omega_j^U A$, $\forall A\in\mathscr{B}(Q\mathscr{H})$, so $\omega_j^U=\sigma_j\circ T'$, $\forall j$, if ω_j^U, σ_j are thought of as density operators over $Q\mathscr{H}$.

What we have shown is the following: if $\omega_k=\sigma_k\circ T$ for a c.p., unital linear map T, then there exists a unitary $U\in\mathscr{U}(\mathscr{H})$, an orthoprojection Q with $Q\mathscr{H}$ separable, and a c.p., unital linear mapping T' over $\mathscr{B}(Q\mathscr{H})$ such that $\{\omega_k^U,\sigma_k\}\subset Q\mathscr{B}(\mathscr{H})_*Q\cong\mathscr{B}(Q\mathscr{H})_*$ and $\omega_k^U=\sigma_k\circ T'$, $\forall k$. On the other hand, let $\omega_1,\dots,\omega_n$, $\sigma_1,\dots,\sigma_n$ be given density operators and an orthoprojection Q, with $Q\mathscr{H}$ separable, $U\in\mathscr{U}(\mathscr{H})$ such that $Q\omega_j^U=\omega_j^U$, $Q\sigma_j=\sigma_j$, $\forall j$. Assume, there is a completely positive, unital map T over $\mathscr{B}(Q\mathscr{H})$ such that $\omega_j^U=\sigma_j\circ T$, $\forall j$, if ω_j^U, σ_j are thought of as density operators on $Q\mathscr{H}$. Let V be an unitary in $\mathscr{U}(\mathscr{H})$ such that $VQ^\perp V^*=Q^\perp$, and define a linear map T' by:

$$T'=QT(QU(\cdot)U^*Q)Q+Q^\perp V^*U(\cdot)U^*VQ^\perp,\quad\text{over}\quad\mathscr{B}(\mathscr{H})\,.$$

Then, T' is completely positive, and $T'(1)=QT(Q)Q+Q^\perp=Q+Q^\perp=1$ shows that $\mathfrak{T}'$ is unital. Moreover, for $\forall A\in\mathscr{B}(\mathscr{H})$: $\mathrm{Tr}.\,\sigma_j\ T'(A)=\mathrm{Tr}.\,\sigma_j QT'(A)Q$ $=\mathrm{Tr}.\,\sigma_j QT(QUAU^*Q)Q=\mathrm{Tr}.\,\sigma_j T(QUAU^*Q)=\mathrm{Tr}.\,\omega_j^U QUAU^*Q=\mathrm{Tr}.\,\omega_j^U UAU^*$

$=\mathrm{Tr}.\,\omega_j A$. Hence, $\omega_j=\sigma_j \circ T'$, with a c.p., unital linear map T' on $\mathscr{B}(\mathscr{H})$. Taking together all this we see that our restriction of the n-tuple-problem to considerations over separable Hilbert-spaces throughtout §§1–5 is not a severe one. All results might be adapted to the non-separable situation by obvious modifications as indicated above.

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