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Construction of a Continuous $SL(3, \mathbb{R})$ Action on 4-Sphere

Dedicated to Professor Nobuo Shimada on his 60th birthday

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§0. Introduction

Let Φ_0: $SO(3)\times M_3(R)\to M_3(R)$ denote the $SO(3)$ action on the vector space $M_3(R)$ of all real matrices of degree 3, defined by $\Phi_0(A, X)=AXA^{-1}$ for $A\in SO(3)$ and $X\in M_3(R)$. Put $(X, Y)=\text{trace}\,{}^tXY$ for $X, Y\in M_3(R)$. Then (X, Y) is an $SO(3)$ invariant inner product of $M_3(R)$. Denote by V and $S(V)$ the linear subspace of $M_3(R)$ consisting of symmetric matrices of trace 0 and its unit sphere, respectively. Then V and $S(V)$ are $SO(3)$ invariant.

Let Φ: $SO(3)\times S(V)\to S(V)$ denote the restricted action of Φ_0. This is an orthogonal $SO(3)$ action on the 4-sphere $S(V)$. In this note, we shall show that the $SO(3)$ action Φ on $S(V)$ is extended to a continuous $SL(3, R)$ action Ψ on $S(V)$, but the action Ψ is not C^1-differentiable. It is still open whether the $SO(3)$ action Φ can be extended to a C^1-differentiable $SL(3, R)$ action or not.

The problem is motivated by the following (cf. [1]). We studied real analytic $SL(n, R)$ actions on spheres, and it was important to consider the restricted $SO(n)$ actions. Moreover, we gave an orthogonal $SO(4)$ action on 6-sphere which was not extendable to any continuous $SL(4, R)$ action.

§1. An Action of $GL(2, R)$ on 2-Disk

1.1. Denote by D the set of complex numbers with modulus $\leqq 1$. We regard D as a closed unit 2-disk. Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $GL(2, R)$, and put

$$\alpha=(a+d+(b-c)i)/2, \quad \beta=(a-d-(b+c)i)/2.$$

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Then $\det A=|\alpha|^2-|\beta|^2$ and

$$TAT^{-1}=\begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{pmatrix} \quad \text{for} \quad T=\begin{pmatrix} i & 1\\ i & -1\end{pmatrix}.$$

Define a map $\psi_1\colon \boldsymbol{GL}(2, \boldsymbol{R})\times \boldsymbol{D}\to\boldsymbol{D}$ by

$$\psi_1(A, w)=\begin{cases}(\alpha w+\beta)/(\bar{\beta}w+\bar{\alpha}) & \text{if} \quad \det A>0\\ (\beta\bar{w}+\alpha)/(\bar{\alpha}\bar{w}+\bar{\beta}) & \text{if} \quad \det A<0.\end{cases}$$

The map ψ_1 is well-defined, because

$$|\bar{\beta}w+\bar{\alpha}|\geqq|\bar{\alpha}|-|\bar{\beta}w|\geqq|\alpha|-|\beta|>0 \qquad \text{for} \quad |w|\leqq 1,\ \det A>0,$$

$$|\bar{\alpha}\bar{w}+\bar{\beta}|\geqq|\bar{\beta}|-|\bar{\alpha}\bar{w}|\geqq|\beta|-|\alpha|>0 \qquad \text{for} \quad |w|\leqq 1,\ \det A<0$$

and

$$|\alpha+\beta\bar{w}|^2-|\alpha w+\beta|^2=(|\alpha|^2-|\beta|^2)(1-|w|^2)$$

for any complex numbers α, β, w. Moreover, we see that the map ψ_1 is a continuous action of $\boldsymbol{GL}(2, \boldsymbol{R})$ on $\boldsymbol{D}$ and $\psi_1(A, 1)=1$ if and only if A is of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, by a routine work.

Here we describe a distinct property of the action ψ_1. Define $M_1(x-iy)=\begin{pmatrix} x & y\\ y & -x\end{pmatrix}$ for real numbers x, y. Then

$$(*) \qquad M_1(\psi_1(A, w))=AM_1(w)A^{-1} \qquad \text{for} \quad w\in\boldsymbol{D},\ A\in\boldsymbol{O}(2).$$

1.2. Finally we notice the following fact. Consider a correspondence $w\to z$ of complex numbers defined by

$$z=i(1+w)/(1-w), \quad w=(z-i)/(z+i).$$

The correspondence induces a homeomorphism of the interior $\mathring{\boldsymbol{D}}$ onto the upper half plane $\boldsymbol{H}$, and the action ψ_1 corresponds to an action ψ_2 of $\boldsymbol{GL}(2, \boldsymbol{R})$ on $\boldsymbol{H}$. We see that the action ψ_2 is well-known, in fact,

$$\psi_2(A, z)=\begin{cases}(az+b)/(cz+d) & \text{if} \quad \det A>0,\\ (a\bar{z}+b)/(c\bar{z}+d) & \text{if} \quad \det A<0,\end{cases}$$

where $z\in\boldsymbol{H}$ and $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$.

§2. An Action of $SL(3, R)$ on 4-Sphere

2.1. Let $N(3)$ and $T(3)$ denote the closed subgroups of $SL(3, R)$ consisting of matrices of the forms

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \quad \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

respectively. Let $\pi: N(3) \to GL(2, R)$ be a projection defined by

$$\pi\begin{pmatrix} * & * & * \\ 0 & a & b \\ 0 & c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Put

$$M(x-iy) = \frac{1}{\sqrt{8}}\begin{pmatrix} 2m & 0 & 0 \\ 0 & x-m & y \\ 0 & y & -x-m \end{pmatrix}, \quad m = \sqrt{(4-x^2-y^2)/3}$$

for real numbers x, y such that $x^2+y^2 \leqq 1$. Then we have an injection $M: D \to S(V)$. Define a map $\psi: N(3) \times M(D) \to M(D)$ by

$$\psi(A, M(w)) = M(\psi_1(\pi(A), w)) \qquad \text{for} \quad w \in D, \ A \in N(3).$$

We see that the map ψ is a continuous action of $N(3)$ on $M(D)$ and,

(a) $$\psi(A, M(1)) = M(1) \quad \textit{if and only if} \quad A \in T(3).$$

By the property $(*)$ for ψ_1, we see that

(b) $$\psi(A, M(w)) = AM(w)A^{-1} \qquad \text{for} \quad w \in D, \ A \in SO(3) \cap N(3).$$

In addition, for each $w \in D$, there is an element $A \in SO(3) \cap N(3)$ such that

(c) $$M(w) = AM(|w|)A^{-1}.$$

2.2. Denote by $S_+(V)$ (resp. $S_-(V)$) the set of $X \in S(V)$ such that $\det X \geqq 0$ (resp. $\det X \leqq 0$). If $X \in S_+(V)$ (resp. $X \in S_-(V)$), then $X = AM(x)A^{-1}$ (resp. $X = -AM(x)A^{-1}$) for some $A \in SO(3)$ and a unique real number x such that $0 \leqq x \leqq 1$. Notice that $\det X = 0$ if and only if $x = 1$; in addition, $AM(x)A^{-1} = M(x)$ if and only if

(d) $$\begin{aligned} &A \in SO(3) \cap T(3) && \text{for} \quad 0 < x \leqq 1, \\ &A \in SO(3) \cap N(3) && \text{for} \quad x = 0. \end{aligned}$$

Let $P \in \boldsymbol{SL}(3, \boldsymbol{R})$ and $A \in \boldsymbol{SO}(3)$. We can express

(i) $\quad PA = A_1 N_1 \quad \text{and} \quad {}^t(PA)^{-1} = A_2 N_2$

for some $A_p \in \boldsymbol{SO}(3)$ and $N_p \in N(3)$. Put

$$\text{(ii)} \quad \begin{aligned} &P\Delta AM(x)A^{-1} = A_1 \psi(N_1, M(x)) A_1^{-1}, \\ &P\nabla(-AM(x)A^{-1}) = -A_2 \psi(N_2, M(x)) A_2^{-1}. \end{aligned}$$

If $AM(x)A^{-1} = A'M(x)A'^{-1}$, then $A' = AK$ for some $K \in \boldsymbol{SO}(3) \cap N(3)$ by (d); hence $PA' = A_1(N_1 K)$ and ${}^t(PA')^{-1} = A_2(N_2 K)$ where $N_p K \in N(3)$. Therefore we see that the definition (ii) does not depend on the choice of A, by the condition (b).

Next we show that the definition (ii) does not depend on the expression (i) by the condition (b). Suppose

$$PA = A_1 N_1 = A_1' N_1' \quad \text{and} \quad {}^t(PA)^{-1} = A_2 N_2 = A_2' N_2'$$

for $A_p' \in \boldsymbol{SO}(3)$, $N_p' \in N(3)$. Then $A_p' = A_p B_p$ and $N_p' = B_p^{-1} N_p$ for some $B_p \in \boldsymbol{SO}(3) \cap N(3)$. Hence

$$\begin{aligned} A_p' \psi(N_p', M(x)) A_p'^{-1} &= A_p B_p \psi(N_p', M(x)) B_p^{-1} A_p^{-1} \\ &= A_p \psi(B_p N_p', M(x)) A_p^{-1} = A_p \psi(N_p, M(x)) A_p^{-1}. \end{aligned}$$

Consequently we can define continuous mappings

$$\Psi_+ : \boldsymbol{SL}(3, \boldsymbol{R}) \times S_+(V) \longrightarrow S_+(V), \quad \Psi_- : \boldsymbol{SL}(3, \boldsymbol{R}) \times S_-(V) \longrightarrow S_-(V)$$

by $\Psi_+(P, X) = P\Delta X$ (resp. $\Psi_-(P, X) = P\nabla X$) for $P \in \boldsymbol{SL}(3, \boldsymbol{R})$ and $X \in S_+(V)$ (resp. $X \in S_-(V)$).

2.3. Next we show that Ψ_+ (resp. Ψ_-) is an action of $\boldsymbol{SL}(3, \boldsymbol{R})$ on $S_+(V)$ (resp. $S_-(V)$). Let $P, Q \in \boldsymbol{SL}(3, \boldsymbol{R})$ and $A \in \boldsymbol{SO}(3)$. Express

$$PA = A_1 N_1, \; QA_1 = A_1' N_1'; \quad {}^t(PA)^{-1} = A_2 N_2, \; {}^t(QA_2)^{-1} = A_2' N_2'$$

for some $A_p, A_p' \in \boldsymbol{SO}(3)$ and $N_p, N_p' \in N(3)$. Then

$$QPA = A_1' N_1' N_1 \quad \text{and} \quad {}^t(QPA)^{-1} = A_2' N_2' N_2 .$$

By the conditions (b) and (c), we see that

$$\psi(N_p, M(x)) = B_p M(x_p) B_p^{-1} = \psi(B_p, M(x_p))$$

for some $B_p \in \boldsymbol{SO}(3) \cap N(3)$ and a real number x_p such that $0 \leqq x_p \leqq 1$. Since

$$QA_1 B_1 = A_1'(N_1' B_1) \quad \text{and} \quad {}^t(QA_2 B_2)^{-1} = A_2'(N_2' B_2),$$

we see that

$$
\begin{aligned}
&Q\Delta(P\Delta AM(x)A^{-1})=Q\Delta(A_1\psi(N_1, M(x))A_1^{-1})=Q\Delta(A_1B_1M(x_1)B_1^{-1}A_1^{-1})\\
&\quad =A_1'\psi(N_1'B_1, M(x_1))A_1'^{-1}=A_1'\psi(N_1'N_1, M(x))A_1'^{-1}=QP\Delta AM(x)A^{-1},\\
&Q\nabla(P\nabla(-AM(x)A^{-1}))=Q\nabla(-A_2\psi(N_2, M(x))A_2^{-1})=Q\nabla(-A_2B_2M(x_2)B_2^{-1}A_2^{-1})\\
&\quad =-A_2'\psi(N_2'B_2, M(x_2))A_2'^{-1}=-A_2'\psi(N_2'N_2, M(x))A_2'^{-1}=QP\nabla(-AM(x)A^{-1}).
\end{aligned}
$$

Thus we obtain $Q\Delta(P\Delta X)=QP\Delta X$ for $X\in S_+(V)$ and $Q\nabla(P\nabla X)=QP\nabla X$ for $X\in S_-(V)$, respectively; hence Ψ_+ and Ψ_- are actions.

2.4. Here we show that the actions Ψ_+ and Ψ_- coincide on the intersection $S_+(V)\cap S_-(V)$. Let $X\in S_+(V)\cap S_-(V)$. Then

$$X=AM(1)A^{-1}=-ASM(1)S^{-1}A^{-1}$$

for some $A\in SO(3)$, where $S=\begin{pmatrix} & & 1\\ & -1 & \\ 1 & & \end{pmatrix}\in SO(3)$. Let $P\in SL(3, R)$. We can express

$${}^t(PAS)^{-1}=A_1N_1$$

for some $A_1\in SO(3)$ and $N_1\in T(3)$. Then

$$PA=A_1{}^tN_1^{-1}S^{-1}=(A_1S^{-1})(S\,{}^tN_1^{-1}S^{-1}),$$

where $A_1S^{-1}\in SO(3)$ and $S\,{}^tN_1^{-1}S^{-1}\in T(3)$. Therefore, we see that by the condition (a),

$$
\begin{aligned}
&P\Delta AM(1)A^{-1}=A_1S^{-1}\psi(S\,{}^tN_1^{-1}S^{-1}, M(1))SA_1^{-1}=A_1S^{-1}M(1)SA_1^{-1},\\
&P\nabla(-ASM(1)S^{-1}A^{-1})=-A_1\psi(N_1, M(1))A_1^{-1}=-A_1M(1)A_1^{-1}.
\end{aligned}
$$

Hence we see that the actions Ψ_+ and Ψ_- coincide on $S_+(V)\cap S_-(V)$. Thus we obtain a continuous action Ψ of $SL(3, R)$ on $S(V)$ whose restriction on $S_+(V)$ (resp. $S_-(V)$) is the action Ψ_+ (resp. Ψ_-).

By the definition of Ψ, we see that

$$\Psi(P, X)=PXP^{-1}=\Phi(P, X)$$

for each $P\in SO(3)$ and $X\in S(V)$. Hence the action Ψ is a desired continuous action of $SL(3, R)$ on $S(V)$.

§3. Non-Differentiability of Ψ

Denote by $S_d(V)$ the set consisting of the diagonal matrices of $S(V)$. Then $S_d(V)$ is a one-dimensional C^∞-submanifold of $S(V)$. Put $G_t=\mathrm{diag}(e^{-2t}, e^t, e^t)$

for each real number t. The correspondence $X \to \Psi(G_t, X)$ defines a homeomorphism h_t of $S_d(V)$ onto itself. We shall show that the homeomorphism h_t is not C^1-differentiable for each $t \neq 0$. Put

$$D(\theta) = (1/\sqrt{6})\,\mathrm{diag}\,(\cos\theta + \sqrt{3}\sin\theta,\ \cos\theta - \sqrt{3}\sin\theta,\ -2\cos\theta)$$

for each real number θ. The correspondence $\theta \to D(\theta)$ defines a C^∞-differentiable submersion of $\boldsymbol{R}$ onto $S_d(V)$. The point $D(\pi/6) = M(1)$ is a fixed point of the homeomorphism h_t for each real number t. Define a function $f(t, \theta)$ by

$$h_t(D(\theta)) = \mathrm{diag}\,(-,\ -,\ f(t, \theta))$$

for each real numbers t, θ. We show that $f(t, \theta)$ is not C^1-differentiable at $\theta = \pi/6$ for each $t \neq 0$. Suppose first $\pi/6 \leqq \theta \leqq \pi/3$. Then

$$D(\theta) = M(\sqrt{3}\cos\theta - \sin\theta)$$

and hence

$$h_t(D(\theta)) = M(\psi_1(\mathrm{diag}\,(e^t, e^t), \sqrt{3}\cos\theta - \sin\theta)) = D(\theta).$$

Therefore $f(t, \theta) = (-2/\sqrt{6})\cos\theta$; hence

$$\lim_{\theta \to \pi/6+} \frac{\partial}{\partial\theta} f(t, \theta) = 1/\sqrt{6}.$$

Suppose next $0 \leqq \theta \leqq \pi/6$. Then

$$D(\theta) = -SM(2\sin\theta)S^{-1} \qquad \text{for} \quad S = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix},$$

and hence

$$h_t(D(\theta)) = -SM(\psi_1(\mathrm{diag}\,(e^t, e^{-2t}), 2\sin\theta)S^{-1} = -SM(x(t, \theta))S^{-1},$$

where

$$x(t, \theta) = \frac{2(e^t + e^{-2t})\sin\theta + (e^t - e^{-2t})}{2(e^t - e^{-2t})\sin\theta + (e^t + e^{-2t})}$$

and $f(t, \theta) = -\sqrt{(4 - x(t, \theta)^2)/6}$. Therefore we obtain

$$\lim_{\theta \to \pi/6-} \frac{\partial}{\partial\theta} f(t, \theta) = e^{-3t}/\sqrt{6}.$$

Consequently, we see that $f(t, \theta)$ is not C^1-differentiable at $\theta = \pi/6$ for each $t \neq 0$, and hence the action Ψ of $\boldsymbol{SL}(3, \boldsymbol{R})$ on $S(V)$ is not C^1-differentiable.

Reference

[1] Uchida, F., Real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ actions on spheres, *Tôhoku Math. J.* **33** (1981), 145–175.