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Decompositions of Linear Maps into Non-Separable C^*-Algebras

By

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Abstract

We study positive decompositions of bounded linear maps between C^*-algebras. A characterization of commutative injective C^*-algebras is given in terms of positive decompositions with certain norm condition of linear maps. We also provide under the Continuum Hypothesis a completely bounded map into the Calkin algebra which admits no positive decomposition.

§1. Introduction

In [17, Satz 4.5] Wittstock proved that if B is an injective C^*-algebra, then every self-adjoint completely bounded map from any C^*-algebra into B can be written as the difference of two completely positive maps (see [11] for another proof). Haagerup [4, Theorem 2.6] showed that for a von Neumman algebra B the converse of Wittsock's theorem holds. In the C^*-algebra case, if B is separable, the converse also is true [5]. But there exists a non-injective C^*-algebra B for which the converse fails to hold [6]. We recall that every positive (resp. bounded) linear map from a C^*-algebra into a commutative C^*-algebra is completely positive (resp. bounded) ([1, Proposition 1.2.2], [10, Lemma 1]).

In this paper we prove that for a commutative C^*-algebra B if every self-adjoint bounded linear map from any C^*-algebra into B can be decomposed into positive linear maps in the form $\phi=\phi^+-\phi^-$ with $\|\phi^++\phi^-\|=\|\phi\|$ then B is injective (Theorem 2). Characterization of general injective C^*-algebras seems to remain open. There exist several examples of bounded linear maps between commutative C^*-algebras which are not linear combinations of positive linear maps ([6, Theorem 2], [7, Proposition 9], [9], [13, Example 2.1], [15,

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1.3.4 Example II]). We show that such an indecomposable phenomenon occurs in the category of commutative C^*-algebras if there exists a sequence $\{T_i\}$ of disjoint open subsets of the compact Hausdorff space associated with a range algebra such that the intersection $\cap_{i=1}^{\infty} T_i^-$ of their closures is non-empty (Theorem 4). Applying this result, we give under the Continuum Hypothesis a completely bounded map from a commutative C^*-algebra into the Calkin algebra which is not a linear combination of positive linear maps (Theorem 8).

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§2. Preliminaries

Let A and B be C^*-algebras. A linear map ϕ: $A \to B$ is completely positive if every multiplicity map $\phi \otimes \mathrm{id}_n$: $A \otimes M_n \to B \otimes M_n$ is positive, and is completely bounded if $\sup_n \|\phi \otimes \mathrm{id}_n\| < \infty$. The supremum is called the completely bounded norm and is denoted by $\|\phi\|_{cb}$. A bounded linear map ϕ from a C^*-algebra into a commutative C^*-algebra satisfies $\|\phi\|_{cb} = \|\phi\|$ ([1, Proposition 1.2.2], [10, Lemma 1]). We use repeatedly these results without reference.

A linear map ϕ: $A \to B$ is said to admit a positive (resp. completely positive) decomposition if ϕ is a linear combination of positive linear (resp. completely positive) maps. It is known that ϕ: $A \to B$ admits a completely positive decomposition if and only if there exist completely positive maps ϕ_1, ϕ_2: $A \to B$ such that

$$(*) \qquad \phi_0(x) = \begin{pmatrix} \phi_1(x) & \phi^*(x) \\ \phi(x) & \phi_2(x) \end{pmatrix}$$

is a completely positive map from A into $B \otimes M_2$, where ϕ^* is the map given by $\phi^*(x) = \phi(x^*)^*$ for x in A [4, §1]. If ϕ: $A \to B$ admits a completely positive decomposition, we let $\|\phi\|_{\mathrm{dec}}$ denote the infimum of those $r \geqq 0$ for which there exist completely positive maps ϕ_1, ϕ_2: $A \to B$ such that $\|\phi_i\| \leqq r$, $i = 1$, 2 and $(*)$ holds [4, Definition 1.1]. In particular, if ϕ is a self-adjoint linear map ($\phi^* = \phi$) then $\|\phi\|_{\mathrm{dec}} = \inf\{\|\phi^+ + \phi^-\|\}$, where inf runs over all completely positive decompositions $\phi = \phi^+ - \phi^-$ [4, Proposition 1.3].

A unital C^*-algebra B is injective if for any C^*-algebra D such that $D \supseteq B$, there exists a projection of norm one from D onto B. For a compact Hausdorff space S, let $C(S)$ denote the C^*-algebra of all continuous functions on S. The space S is stonean (or extremally disconnected) if the closures of any disjoint open subsets are again disjoint. If $t \in S$, let $\rho(t)$ denote the supremum of 1 and those $n \geqq 2$ for which there exist n disjoint open subsets $S_1, \dots, S_n$ such that the intersection $\cap_{i=1}^{n} S_i^-$ of their closures contains t, and put $\rho(S) = \sup\{\rho(t): t \in S\}$. The C^*-algebra $C(S)$ is injective if and only if $\rho(S)=1$ (see [6] for example).

We refer to [14] for recent development of completely bounded maps.

§3. Norm $\|\cdot\|_{\mathrm{dec}}$ and Commutative Injective C^*-Algebras

In this section, we give a characterization of a commutative injective C^*-algebra in terms of the norm $\|\cdot\|_{\mathrm{dec}}$, and estimate $\|\cdot\|_{\mathrm{dec}}$ of linear maps which have been considered in [6].

The following is a basic lemma in this paper.

Lemma 1. *Let X be a compact Hausdorff space with n disjoint open subsets $\{X_i: i=1, \dots, n\}$ such that the intersection $\cap_{i=1}^{n} X_i^-$ of their closures is non-empty. Then there exist a commutative C^*-algebra A and a self-adjoint bounded linear map ϕ: $A \to C(X)$ such that $\|\phi\|_{\mathrm{dec}} = n\|\phi\|$ and $\|\phi^+\| \geqq n$ for any positive linear map $\phi^+ \geqq \phi$.*

Proof. For $i=1, \dots, n$, let Y_i denote the one-point compactification of X_i with the point ω_i at infinity. We put $A = \sum_{i=1}^{n} \oplus C(Y_i)$. Then each $C(Y_i)$ is canonically regarded as a C^*-subalgebra of A. Let ϕ: $A \to C(X)$ defined by, for f_i in $C(Y_i)$,

$$\phi(f_i)(s) = \begin{cases} f_i(s) - f_i(\omega_i) & \text{if } \ s \in X_i\,; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\phi^* = \phi$, $\phi(\sum_{i=1}^{n} f_i) \in C(X)$ and $\phi(f_i)\phi(f_j) = 0$ $(i \neq j)$ as $X_i \cap X_j$ is empty. Since $\|\phi | C(Y_i)\| = 2$ for $i=1, \dots, n$, we have

$$\|\phi\| = \max\{\|\phi | C(Y_i)\| : i=1, \dots, n\} = 2.$$

We put

$$\phi_0^+(f_i) = \phi(f_i) + f_i(\omega_i) I \quad \text{and} \quad \phi_0^-(f_i) = f_i(\omega_i) I$$

for f_i in $C(Y_i)$, where I denotes the unit of $C(X)$. It is easy to check that this induces positive linear maps ϕ_0^+, ϕ_0^-: $A \to C(X)$ with $\phi = \phi_0^+ - \phi_0^-$. If e_i denotes the unit of $C(Y_i)$, then

$$\|\phi_0^+ + \phi_0^-\| = \|\phi_0^+(\sum_{i=1}^n e_i) + \phi_0^-(\sum_{i=1}^n e_i)\| = \|2nI\| = 2n,$$

and hence $\|\phi\|_{\mathrm{dec}} \leqq 2n$.

If $s \in X_i$, we choose h_i in $C(Y_i)$ such that $\|h_i\| = 1$, $h_i(s) = 1$, $h_i \geqq 0$ and $h_i(\omega_i) = 0$. Put $g_i = e_i - h_i$. Let $\phi^+: A \to C(X)$ be a positive linear map such that $\phi \leqq \phi^+$. Put $\phi^- = \phi^+ - \phi$. Then

$$1 = e_i h_i(s) = \phi(e_i h_i)(s) \leqq \phi^+(e_i)(s),$$
$$1 = e_i g_i(\omega_i) = -\phi(e_i g_i)(s) \leqq \phi^-(e_i g_i)(s) \leqq \phi^-(e_i)(s).$$

Choose $s_\omega \in \bigcap_{i=1}^n X_i^-$ and a net $\{s_i(\lambda)\}$ in X_i such that $\lim_\lambda s_i(\lambda) = s_\omega$. Then

$$1 \leqq \lim_\lambda \phi^+(e_i)(s_i(\lambda)) = \phi^+(e_i)(s_\omega),$$
$$1 \leqq \lim_\lambda \phi^-(e_i)(s_i(\lambda)) = \phi^-(e_i)(s_\omega).$$

Hence we have

$$\|\phi^+ + \phi^-\| \geqq (\phi^+ + \phi^-)(\sum_{i=1}^n e_i)(s_\omega) \geqq 2n,$$
$$\|\phi^+\| \geqq \phi^+(\sum_{i=1}^n e_i)(s_\omega) \geqq n,$$

so that $\|\phi\|_{\mathrm{dec}} \geqq 2n$, $\|\phi^+\| \geqq n$ and this completes the proof.

Theorem 2. *Let B be a commutative C^*-algebra. If every self-adjoint bounded linear map ϕ from any commutative C^*-algebra into B admits a positive decomposition and $\|\phi\|_{\mathrm{dec}} = \|\phi\|$, then B is injective.*

Proof. Suppose that B is not unital. The self-adjoint linear map $\psi: B + \mathbf{C}I \to B$ defined by $\psi(a + \alpha I) = a$ admits no positive decomposition by the argument of the first paragraph of the proof of [7, Theorem 11]. Therefore we may assume that $B = C(X)$ for some compact Hausdorff space X.

Suppose that B is not injective, that is, X is not stonean. Then there exist disjoint open subsets X_1 and X_2 of X such that $X_1^- \cap X_2^-$ is non-empty. It follows from Lemma 1 that there exist a commutative C^*-algebra A and a self-adjoint bounded linear map $\phi: A \to C(X) = B$ such that $\|\phi\|_{\mathrm{dec}} = 2\|\phi\|$. This is a contradiction and completes the proof.

Remark. This result is related to a question of Tsui [15, pp. 97–98].

As stated in the introduction, we have the example of a non-injective C^*-algebra into which every completely bounded map ϕ from any C^*-algebra admits a positive decomposition [6]. Haagerup suggested at the GPOT-conference in Boulder Colorado, June 1983 that for such a map ϕ, $\|\phi\|_{\mathrm{dec}} \leqq 2\|\phi\|_{cb}$. We now include a proof of this estimate.

We recall the notation in [6]. Let S_1 and S_2 be stonean spaces with limit points s_1 and s_2 respectively. We put $T_i = S_i - \{s_i\}$. Let T denote the space obtained from S_1 and S_2 by indentifying s_1 and s_2. More precisely, T is the one-point compactification of the topological sum of locally compact spaces T_1 and T_2, with the point ω at infinity. Since S_i is homeomorphic to $T_i \cup \{\omega\}$, we identify S_i with $T_i \cup \{\omega\}$. Since T_1 and T_2 are open subsets of T and $T_1^- \cap T_2^- = \{\omega\}$, the space T is not stonean. Hence the C^*-algebra $C(T)$ is not injective. In [8], such a space $C(T)$ was studied as a Banach space.

Proposition 3. *With the above notation, if ϕ is a bounded linear map from a C^*-algebra A into $C(T)$, then*

$$\|\phi\|_{\mathrm{dec}} \leqq 2\|\phi\| .$$

Proof. For $i=1, 2$, let ϕ_i: $A \to C(S_i)$ be defined by $\phi_i(a) = \phi(a)|S_i$, the restriction to S_i of $\phi(a)$. Since $C(S_i)$ is injective, there exist completely positive maps $\phi_{i,1}$, $\phi_{i,2}$: $A \to C(S_i)$ such that $\|\phi_{i,1}\|$, $\|\phi_{i,2}\| \leqq \|\phi_i\|$ and

$$\Phi_i(a) = \begin{pmatrix} \phi_{i,1}(a) & \phi_i^*(a) \\ \phi_i(a) & \phi_{i,2}(a) \end{pmatrix}$$

defines a completely positive map from A into $C(S_i) \otimes M_2$ by [4, Theorem 1.6].

For $j=1, 2$, let ψ_j: $A \to C(T)$ be defined by

$$\psi_j(a)(t) = \phi_{1,j}(a)(t) + \phi_{2,j}(a)(\omega) \quad \text{if} \quad t \in S_1 ;$$
$$\psi_j(a)(t) = \phi_{1,j}(a)(\omega) + \phi_{2,j}(a)(t) \quad \text{if} \quad t \in S_2 .$$

Then each ψ_j is completely positive and

$$\|\psi_j\| \leqq \|\phi_{1,j}\| + \|\phi_{2,j}\| \leqq \|\phi_1\| + \|\phi_2\| \leqq 2\|\phi\| .$$

Let Φ: $A \to C(T) \otimes M_2$ be defined by

$$\Phi(a) = \begin{pmatrix} \phi_1(a) & \phi^*(a) \\ \phi(a) & \phi_2(a) \end{pmatrix}.$$

We now show that Φ is completely positive. For convenience, we define completely positive maps Ψ_1, Ψ_2: $A \to M_2$ by

$$\Psi_i(a) = \begin{pmatrix} \phi_{i,1}(a)(\omega) & 0 \\ 0 & \phi_{i,2}(a)(\omega) \end{pmatrix}$$

since $\phi_{i,1}$ and $\phi_{i,2}$ are positive. For a compact Hausdorff space T_0, the C^*-algebra $C(T_0) \otimes M_n$ can be identified with the C^*-algebra of M_n-valued continuous functions on T_0 and a is a positive element of $C(T_0)\otimes M_n$ if and only if $a(t)$ for each t in T_0 is a positive matrix in the $n\times n$ matrix algebra M_n. We have, for a in $A\otimes M_n$,

$$(\Phi\otimes\mathrm{id}_n)(a)(t)=\Phi_1\otimes\mathrm{id}_n(a)(t)+\Psi_2\otimes\mathrm{id}_n(a) \quad \text{if} \quad t\in S_1\,;$$
$$(\Phi\otimes\mathrm{id}_n)(a)(t)=\Psi_1\otimes\mathrm{id}_n(a)+\Phi_2\otimes\mathrm{id}_n(a)(t) \quad \text{if} \quad t\in S_2\,.$$

Hence Φ is completely positive and the proof is complete.

We remark that the number "2" in Proposition 3 is the best posibility by Lemma 1.

§4. Commutative Non-injective C^*-Algebras

We recall the notation in the introduction. For t in a compact Hausdorff space S let $\rho(t)$ denote the supremum of 1 and those n for which there exist n disjoint open subsets $S_1,\dots, S_n$ such that $t\in\bigcap_{i=1}^{n} S_i^-$, and $\rho(S)=\sup\{\rho(t): t\in S\}$.

In this section, we show that if X is a compact Hausdorff space with $\rho(X)=\infty$, then there exist a commutative C^*-algebra A and a bounded linear map from A into $C(X)$ which admits no positive decomposition. Therefore, if the C^*-algebra $C(T)$ for a compact Hausdorff space T satisfies the condition that every bounded linear map from any commutative C^*-algebra into $C(T)$ admits a positive decomposition, then $\rho(T)<\infty$.

The following result is an improvement of [6, Theorem 2].

Theorem 4. *Let X be a compact Hausdorff space with $\rho(X)=\infty$. Then there exist a commutative C^*-algebra A and a bounded linear map from A into $C(X)$ which admits no positive decomposition.*

Proof. For each integer $m\geqq 2$, we have m^3 disjoint open subsets $X(m, 1),\dots, X(m, m^3)$ of X such that $\bigcap_{j=1}^{m^3} X(m, j)^-$ is non-empty. For each $j\leqq m^3$, let $Y(m, j)$ be the one-point compactification of $X(m, j)$ with the point $\omega(m, j)$ at infinity and let A_m be the direct sum $\sum_{j=1}^{m^3}\oplus C(Y(m, j))$. Let A be the $C(\infty)$-direct sum $\sum_{i=2}^{\infty}\oplus A_i$ of $\{A_i\}$. Then each $C(Y(m, j))$ is canonically regarded as a $*$-subalgebra of A. We define $\phi\colon A\to C(X)$ by, for $f_{(m,j)}$ in $C(Y(m, j))$,

$$\phi(f_{(m,j)})(s)=\begin{cases}(1/m)^2(f_{(m,j)}(s)-f_{(m,j)}(\omega(m,j))) & \text{if} \quad s\in X(m,j)\,;\\ 0 & \text{otherwise.}\end{cases}$$

By a similar argument of the first paragraph of the proof of Lemma 1, it is easy to chech that ϕ is bounded.

Suppose that there exists a positive linear map ϕ^+: $A \to C(X)$ such that $\phi \leqq \phi^+$. Since the restriction $m^2\phi|A_m$ is the map ϕ for $n=m^3$ obtained in Lemma 1, we have $m^3 \leqq \|m^2\phi^+|A_m\|$. Hence

$$m \leqq \|\phi^+|A_m\| \leqq \|\phi^+\| .$$

This implies the unboundedness of ϕ^+.

A map h: $X \to Y$ between two topological spaces is called minimal if it is continuous, and no closed proper subset of X is carried onto $h(X)$ by h. If S is a compact Hausdorff space, then there exist a stonean space G_S and a minimal map g_S from G_S onto S [3]. We call G_S the Gleason space of S and g_S the Gleason map of S.

Proposition 5. *Let X be a compact Hausdorff space. If $\rho(X) < \infty$ and $\{t \in X: \rho(t) \geqq 2\}$ is a finite set, then every bounded linear map ϕ from any C^*-algebra A into $C(X)$ admits a positive decomposition.*

Proof. Since $\rho(X) < \infty$, it follows from [8, Lemma 7] that $g_X^{-1}(t)$ for each t in X is finite, so that we put $\{t_1, \ldots, t_n\} = g_X^{-1}(\{t \in X: \rho(t) \geqq 2\})$. Let $G_1, \ldots, G_n$ be disjoint open and closed subsets of the Gleason space G_X such that $\cup_{i=1}^n G_i = G_X$ and $t_i \in G_i$ for each $i \leqq n$. The restriction $g_X|G_i$ is a homeomorphism from G_i onto $X_i = g_X(G_i)$ and $X_i \cap X_j \subseteq \{t \in X: \rho(t) \geqq 2\}$ if $i \neq j$. Since G_i is stonean, so is X_i. The intersection $X_i \cap X_j$ for any pair (i, j) is finite and $\cup_{i=1}^n X_i = X$. Hence ϕ admits a positive decomposition [6, Remark (ii) of Theorem 1].

The following proposition gives an example of a compact Hausdorff space T such that $\rho(T)=2$, $F=\{t \in T: \rho(t)=2\}$ is an infinite closed subset and every bounded linear map from any C^*-algebra into $C(T)$ admits a positive decomposition. If the proposition is compered with Proposition 3, the range algebra of the proposition is in a restricted form because we have in general no simultaneous extension from $C(F)$ into $C(T)$ (cf. [12, Proposition 5.3]).

Let S be a stonean space with a closed subset F and put $S_1 = S$ and $S_2 = S$. Let S_0 be the topological sum of S_1 and S_2. For a homeomorphism ψ: $S_1 \to S_2$, let $C_F(S) = \{f \in C(S_0): f(t) = f(\psi(t))$ for all t in $F\}$. The algebra $C_F(S)$ is $*$-isomorphic to the C^*-algebra of all continuous functions on the space S_F obtained from S_0 by the identification of the naturally corresponding points of F and

$\psi(F)$. It is easy to see that $\rho(S_F) \leqq 2$. If F is a nowhere dense closed set then $\rho(S_F) = 2$ by [8, Lemma 2].

Proposition 6. *With the above notation, if ϕ is a bounded linear map from any C^*-algebra A into $C_F(S)$ then ϕ admits a positive decomposition and $\|\phi\|_{\mathrm{dec}} \leqq 2\|\phi\|$.*

Proof. Since $C(S_0)$ is an injective C^*-algebra, there exist, by Haagerup [4, Theorem 1.6], completely positive maps ϕ_1, ϕ_2: $A \to C(S_0)$ such that $\|\phi_1\|$, $\|\phi_2\| \leqq \|\phi\|$ and the map

$$\Phi(a) = \begin{pmatrix} \phi_1(a) & \phi^*(a) \\ \phi(a) & \phi_2(a) \end{pmatrix}$$

defines a completely positive map from A into $C(S_0) \otimes M_2$. For $i = 1, 2$, we define a completely positive map ϕ_i': $A \to C(S_0)$ by

$$\phi_i'(a)(t) = \begin{cases} \phi_i(a)(\psi(t)) & \text{if } t \text{ in } S_1 ; \\ \phi_i(a)(\psi^{-1}(t)) & \text{if } t \text{ in } S_2 . \end{cases}$$

We put $\phi_i'' = \phi_i + \phi_i'$. Then for all t in F and a in A,

$$\phi_i''(a)(t) = \phi_i(a)(t) + \phi_i'(a)(\psi(t)) = \phi_i''(a)(\psi(t)) .$$

We then can define a completely positive map Φ' from A into $C_F(S) \otimes M_n$ by

$$\Phi'(a) = \begin{pmatrix} \phi_1''(a) & \phi^*(a) \\ \phi(a) & \phi_2''(a) \end{pmatrix} .$$

Hence ϕ admits a positive decomposition. It is easy to check that $\|\phi_i''\| \leqq 2\|\phi\|$ and $\|\phi\|_{\mathrm{dec}} \leqq 2\|\phi\|$. This completes the proof.

Remark. Let βN be the Stone–Čech compactification of the discrete space N of all positive integers. Isbell and Semadeni [8, Proposition 1] proved that if $S = \beta N$ and $F = \beta N - N$ then $C(S_F)$ is not injective as a Banach space and $\rho(S_F) = 2$.

§ 5. Linear Maps into the Calkin Algebra

In this section, assuming the Continuum Hypothesis, we give a bounded linear map from a commutative C^*-algebra A into l^∞ / c_0 which admits no positive decomposition, where l^∞ and c_0 denote the C^*-algebra of bounded sequences and the C^*-algebra of sequences convergent to 0. This map also induces a completely bounded map from the C^*-algebra A into the Calkin algebra which admits no positive decomposition.

Let S be a compact Hausdorff space. A subset S_0 of S is called a zero-set if there exists g_0 in $C(S)$ with $S_0=\{x\in S: g_0(x)=0\}$. A subset S_1 of S is called a cozero-set if there exists g_1 in $C(S)$ with $S_1=\{x\in S: g_1(x)\neq 0\}$. Hence S_0 is a zero-set if and only if $S-S_0$ is a cozero-set.

The following lemma is based on an idea of Gillman [16, Proposition 3.30].

Lemma 7. *Let βN be the Stone–Čech compactification of the discrete space N of all positive integers. Assume the Continuum Hypothesis. Let $p\in\beta N-N$. Then $\rho(p)=\infty$.*

Proof. We choose a base of cardinality of the continuum zero-set neighbourhoods of p since $\beta N-N$ has a base consisting of cardinality of the continuum open and closed subsets [16, Corollary 3.17] and every neighbourhood of p contains a zero-set neighbourhood of p. By the Continuum Hypothesis, the basis is indexed by the first uncountable cardinal ω_1 and written $\{Z_\alpha: \alpha<\omega_1\}$. Proceeding by transfinite induction, we assume for a given $\alpha<\omega_1$ that cozero-sets $\{A_{i,\sigma}: i\in N, \sigma<\alpha\}$ such that

$A_{i,\lambda}\cap A_{j,\tau}$ is empty, the union $\cup_{i=1}^{\infty} A_{i,\lambda}\subseteq Z_\lambda$ and $p\notin A_{i,\lambda}$ for all i, j in N $i\neq j$ and all λ, $\tau<\alpha$. A countable union of cozero-sets is again a cozero-set [2, 1.14]. We put

$$A_\alpha=Z_\alpha\cap(\cap_{\sigma<\alpha}\{(\beta N-N)-(\cup_{i=1}^{\infty} A_{i,\sigma})\}).$$

Then A_α contains p and thus is a non-empty zero-set of $\beta N-N$. Hence A_α has a non-empty interior by [16, Corollary 3.28]. Since $\beta N-N$ contains no isolated points [16, Proposition 3.12], there exists a family $\{A_{i,\alpha}: i\in N\}$ of disjoint countable cozero-sets in $A_\alpha-\{p\}$. Then the induction hypothesis is satisfied for all λ, $\tau\leqq\alpha$.

We define

$$A_i=\cup_{\alpha<\omega_1} A_{i,\alpha}.$$

Then $\{A_i\}$ consists of disjoint open sets. Since each basic neighbourhood Z_α of p contains $\cup_{i=1}^{\infty} A_{i,\alpha}$, every neighbourhood of p meets all A_i so that $\cap_{i=1}^{\infty} A_i^-$ contains p. This completes the proof.

Let H be an infinite dimensional Hilbert space and let $L(H)$ and $K(H)$ be the C^*-algebra of bounded linear operators on H and the ideal of compact linear operators on H, respectively. We put $Q(H)=L(H)/K(H)$ and denote by π the quotient map $L(H) \to Q(H)$. If H is a separable infinite dimensional space, $Q(H)$ is called the Calkin algebra. Let M be the σ-weak closure of the algebra generated by a family $\{p_i\}$ of countable mutually orthogonal minimal projections in $L(H)$. Then

$$M = \{x \in L(H): x = \sum_{n=1}^{\infty} x(n)p_n,\ x(n) \in \mathbb{C} \text{ and } \sup_n |x(n)| < \infty\}$$

and

$$M \cap K(H) = \{x \in M: p_n x = x p_n = x(n)p_n,\ \lim_n x(n) = 0\}.$$

Hence $M \simeq l^{\infty}$ and $M \cap K(H) \simeq c_0$, so that $C(\beta N - N) \simeq l^{\infty}/c_0$. Since $\pi(M) \simeq C(\beta N - N)$, the algebra $C(\beta N - N)$ is regarded as a C^*-subalgebra of $Q(H)$.

Theorem 8. *Assume the Continuum Hypothesis. With the above notation, there exist a commutative C^*-algebra A and a completely bounded map $\phi: A \to Q(H)$ with $\phi(A) \subseteq C(\beta N - N)$ which admits no positive decomposition.*

Proof. Let $\Phi: L(H) \to M$ defined by

$$\Phi(x) = \sum_{i=1}^{\infty} p_i x p_i.$$

Then Φ is a projection of norm one from $L(H)$ onto M. If $x \in K(H)$, then $\pi \circ \Phi(x) = 0$. Hence we define the projection Ψ of norm one from $Q(H)$ onto $C(\beta N - N)$ by

$$\Psi(x + K(H)) = \pi \circ \Phi(x)$$

for x in $L(H)$.

By Theorem 4 and Lemma 7, there exist a commutative C^*-algebra A and a self-adjoint bounded linear map $\phi: A \to C(\beta N - N)$ which is not a linear combination of positive linear maps. Since $\phi(A) \subseteq C(\beta N - N)$, ϕ is a completely bounded map from A into $Q(H)$. Suppose that there exist positive linear maps $\phi^+, \phi^-: A \to Q(H)$ such that $\phi = \phi^+ - \phi^-$. Then

$$\phi = \Psi \circ \phi = \Psi \circ \phi^+ - \Psi \circ \phi^-.$$

Both $\Psi \circ \phi^+$ and $\Psi \circ \phi^-$ are positive linear maps from A into $C(\beta N - N)$. This is a contradiction and completes the proof.

Remark. Answering a question of Paulsen, Haagerup [4, Corollary 2.9] showed that there exist a non-commutative von Neumann algebra B and an ideal J such that for every infinite dimensional C^*-algebra D, there exists a completely bounded map $\psi: D \to B/J$ which has no completely bounded lifting $\psi^{\sim}: D \to B$. The map ϕ in Theorem 8 also is regarded as a map $\phi_0: A \to l^{\infty}/c_0$. Then

ϕ_0 admits no positive decomposition. Since $L(H)$ and l^∞ are injective, neither ϕ nor ϕ_0 has a completely bounded lifting.

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