

A System of Quasi Variational Inequalities and its Application to Reaction Diffusion Equations

By

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§ 1. Introduction

To control a flow and exchange of material through its boundary is the most important problem for the organism to maintain the order on the dynamic equilibrium. Mathematically this complicated control through the boundary or membrane of the organism is considered as various types of boundary conditions. For instance, by using the methods of variational inequalities (abr. V. I.) or quasi variational inequalities (abr. Q. V. I.), Duvaut and Lions studied the unilateral problem in [3] such that

$$(1.1) \quad -\Delta u + ku = 0 \quad \text{in } \Omega,$$

$$(1.2) \quad u \frac{\partial u}{\partial n} = 0, \quad \frac{\partial u}{\partial n} \geq 0, \quad u \geq 0 \quad \text{on } \partial\Omega$$

where u describes the concentration of a chemical substance in the domain Ω and $\frac{\partial}{\partial n}$ is the outward normal derivative. The boundary conditions (1.2) mean that the flow through the boundary into the domain is shut up on the place where $u|_{\partial\Omega} > 0$ and, on the other hand, the substance is allowed to enter into the domain on the place where $u|_{\partial\Omega} = 0$. In the last part of this paper we consider the boundary conditions called "a priori feedback control" model which was introduced by Lions [8];

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$$(1.3) \quad \begin{cases} \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega_+ = \{y \in \partial\Omega : u|_{\partial\Omega}(y) > Q(u|_{\partial\Omega})(y)\}, \\ \frac{\partial u}{\partial n} = M & \text{on } \partial\Omega_- = \{y \in \partial\Omega : u|_{\partial\Omega}(y) < Q(u|_{\partial\Omega})(y)\} \\ 0 \leq \frac{\partial u}{\partial n} \leq M \end{cases}$$

where $Q: L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$ is continuous and uniformly bounded. In the case of Lions [8] $Q(u|_{\partial\Omega})(y)$ is a constant function defined by

$$(1.4) \quad Q(u|_{\partial\Omega})(y) \equiv \text{mean}[u|_{\partial\Omega}] = \frac{1}{m(\partial\Omega)} \int_{\partial\Omega} u|_{\partial\Omega}(y) dy$$

which means the substrate is allowed to enter into the domain through the part of the boundary where the concentration is lower than the average and on the part where the concentration is higher than the average the flow is shut up. When $Q(u|_{\partial\Omega}) \equiv \varepsilon$ and $\varepsilon > 0$ is small enough, the unilateral problem is approximated by (1.3).

In this paper first we study a system of Q. V. I. General forms of Q. V. I. were introduced and studied by Mosco in [10] and Mossino also applied a general theory of Q. V. I. to the equations from Plasma Physics in [11]. She used the function type of Q. V. I. such that

$$(1.5) \quad F(u, u) \leq F(v, u) \quad \text{for every } v \in U,$$

that is,

$$(1.6) \quad 0 \in \partial F(u, u), \quad u \in U$$

where U, V are reflexive Banach spaces and $F: U \times V \rightarrow \mathbb{R}$ and the subdifferentials are taken with respect to the first variables.

To analyze reaction-diffusion systems we introduce a system of Q. V. I.

$$(1.7) \quad \begin{cases} 0 \in \partial F_1(s, p) \\ 0 \in \partial F_2(p, s) \end{cases}$$

and we prove the existence of a pair of solutions by applying the method of Mossino. Next we consider a system of reaction-diffusion equations which was mathematically formulated by Kernevez [7] or Banks [1] as an active transport model of double-layered membranes. We construct convex functionals. By applying the theory of Rockafellar about integrals of convex functions we calculate the subdifferentials of these functionals and show the existence of solutions in equilibrium

as minimal points of these convex functionals.

We study this reaction-diffusion system with the following two types of boundary conditions:

(1) The Dirichlet boundary conditions—We assume that the boundary is absorptive or chemical components are dissolved on the boundary.

(2) The Neumann boundary conditions—There exists the flux through the boundary which is controlled by the concentration of the other reacting chemical component. In this case we assume that the chemical components are dissolved at the rate proportional to their concentration in the domain.

It should be remarked that we can show the existence of the stationary solutions when the components are dissolved in the domain or on the boundary. These conditions of dissolution yield the saturated states chemically and also yield the coercive conditions mathematically. In our cases the coercivity is the most essential condition for the existence of the stationary solutions. When the coercive conditions are satisfied, we obtain the solutions as minimal points of convex functionals.

In § 2 we study a system of Q. V. I. and we show the existence of solutions. By applying this Q. V. I. system we show the existence of stationary solutions of reaction-diffusion equations with Dirichlet boundary conditions in § 3 and with Neumann boundary conditions in § 4 and, especially, we treat “a priori feedback control” problem in § 5.

§ 2. System of General Q. V. I.

Let U , V be real reflexive Banach spaces and assume that U is compactly imbedded in V , that is, (1) U is a vector subspace of V , and (2) the identity operator I defined on U into V is continuous and compact.

Let M_1 , M_2 be nonempty closed convex subsets of U and F_1 , F_2 be functions defined on $U \times V$ with values in $(-\infty, +\infty]$, satisfying the following conditions:

- (i. a) $F_1(\cdot, \cdot)$ is lower semi-continuous (abr. l. s. c.) on M_1 (relative weak topology of U) \times M_2 (relative strong topology of V);
- (i. b) $F_2(\cdot, \cdot)$ is l. s. c. on M_2 (relative weak top. of U) \times M_1 (relative strong top. of V);

- (i. c) for each $s \in M_1$, $F_1(s, \cdot)$ is continuous on M_2 (relative strong top. of V);
- (i. d) for each $q \in M_2$, $F_2(q, \cdot)$ is continuous on M_1 (relative strong top. of V);
- (ii. a) for each $p \in M_2$, $F_1(\cdot, p)$ is proper, strictly convex;
- (ii. b) for each $r \in M_1$, $F_2(\cdot, r)$ is proper, strictly convex;
- (iii. a) F_1 is coercive on M_1 in the following sense, there exists an element $s_0 \in M_1$ such that, if $s \in M_1$ and $\|s\|_U \rightarrow +\infty$, then $F_1(s, p) - F_1(s_0, p) \rightarrow +\infty$, uniformly w. r. t. p in M_2 ;
- (iii. b) F_2 is coercive on M_2 , that is, there exists an element q_0 in M_2 such that, if $q \in M_2$ and $\|q\|_U \rightarrow +\infty$, then $F_2(q, r) - F_2(q_0, r) \rightarrow +\infty$, uniformly w. r. t. r in M_1 .

Theorem 2.1. *Under the above assumptions, we have a pair of solutions (ξ, η) in $M_1 \times M_2$ of Q. V. I. given by*

$$(2.1) \quad \begin{cases} F_1(\xi, \eta) \leq F_1(s, \eta) & \text{for all } s \text{ in } M_1 \\ F_2(\eta, \xi) \leq F_2(q, \xi) & \text{for all } q \text{ in } M_2. \end{cases}$$

We use the following lemmas for the proof of Theorem.

Lemma 2.1. *For each $p \in M_2$, there exists a unique element s_p in M_1 ;*

$$F_1(s_p, p) \leq F_1(s, p) \quad \text{for all } s \text{ in } M_1,$$

and for each $r \in M_1$, there exists a unique element q_r in M_2 ;

$$F_2(q_r, r) \leq F_2(q, r) \quad \text{for all } q \text{ in } M_2.$$

Proof. It can be easily proved by the results of Ekeland-Temam [4] or Barbu [2], because $F_1(\cdot, p)$ is proper, strictly convex, coercive, and l. s. c. with the weak topology of U for each p in M_2 and also M_1 is closed convex in U . Furthermore, F_2, M_2 have same properties.

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Lemma 2.2. *We define mappings $T_1: M_2 \rightarrow M_1$, $T_2: M_1 \rightarrow M_2$ by $T_1 p = s_p$ for each p in M_2 and $T_2 r = q_r$ for each r in M_1 . Then T_1 and T_2 are continuous with the strong topology of V and*

$$\sup_{p \in M_2} \|T_1 p\|_U = K_1 < +\infty$$

and

$$\sup_{r \in M_1} \|T_2 r\|_U = K_2 < +\infty.$$

Proof. T_1, T_2 are well defined by Lemma 2.1. From the definition of T_1 , we have

$$F_1(T_1p, p) - F_1(s_0, p) \leq 0$$

where s_0 is the element defined in the coercive condition (iii). By the coerciveness we can easily see that

$$\sup_{p \in M_2} \|T_1p\|_U = K_1 < +\infty.$$

Now we shall show the continuity of T_1 . Consider a sequence $\{p_n\} \subset M_2$ convergent to p_1 in M_2 with the strong topology of V . Since $\|T_1p_n\|_U \leq K_1$ and M_1 is closed convex in U , there exist $s_1 \in M_1$ and a subsequence $\{p_{n_j}\}$ of $\{p_n\}$ such that

$$T_1p_{n_j} \rightarrow s_1 \text{ in } M_1 \text{ with the weak top. of } U$$

as $n_j \rightarrow +\infty$. Then we have

$$T_1p_{n_j} \rightarrow s_1 \text{ in } M_1 \text{ with the strong top. of } V,$$

because U is compactly imbedded in V . From the definition of T_1 we have

$$F_1(T_1p_{n_j}, p_{n_j}) \leq F_1(s, p_{n_j}) \quad \text{for all } s \text{ in } M_1.$$

Since $F_1(\cdot, \cdot)$ is l. s. c. and $F_1(s, \cdot)$ is continuous on M_2 for each s in M_1 in the sense of (i), we obtain

$$\begin{aligned} F_1(s_1, p_1) &\leq \liminf_{n_j \rightarrow +\infty} F_1(T_1p_{n_j}, p_{n_j}) \\ &\leq \liminf_{n_j \rightarrow +\infty} F_1(s, p_{n_j}) \\ &= F_1(s, p_1) \quad \text{for all } s \text{ in } M_1. \end{aligned}$$

Since the minimum point is unique, $s_1 = T_1p_1$. We can conclude that

$$T_1p_n \rightarrow T_1p_1 \text{ in } M_1 \text{ with the strong top. of } V.$$

Similar arguments imply the results about T_2 .

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Proof of Theorem 2.1. Let $B = M_2 \cap \{u \in U : \|u\|_U \leq K_2\}$. Then B is a nonempty weakly compact convex subset of U by Lemma 2.2 and the reflexivity of U . Furthermore, B is compact convex with the strong top. of V , because U is compactly imbedded in V .

Since $T_1M_2 \subset M_1, T_2M_1 \subset B$, and the composition $T_2 \circ T_1$ is continuous on M_2 with the strong top. of V by applying Lemma 2.2, there exists a fixed point η in $B \subset M_2$, i.e., $T_2 \circ T_1\eta = \eta$ by Schauder's fixed

point theorem. Let $\xi = T_1\eta$, then we have a pair of Q. V. I. solutions (ξ, η) in $M_1 \times M_2$:

$$F_1(\xi, \eta) \leq F_1(s, \eta) \quad \text{for all } s \text{ in } M_1$$

and

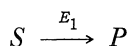
$$F_2(\eta, \xi) \leq F_2(q, \xi) \quad \text{for all } q \text{ in } M_2.$$

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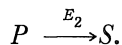
Remark. To show the uniqueness of solutions of this system is an interesting problem, but the assumptions for the uniqueness will be complicated, because we need the arguments about the uniqueness in the Schauder's fixed point theorem [6].

§ 3. Dirichlet Boundary Conditions

Let the cell Ω be a bounded open subset in R^n with the sufficiently smooth boundary $\Gamma = \partial\Omega$. Assume that the organs Ω_1, Ω_2 are measurable subsets of Ω and $\Omega_1 \cap \Omega_2 = \phi$. Enzymes E_1, E_2 are embedded in Ω_1, Ω_2 , respectively. We have coupled reactions between two chemical components S and P . In Ω_1 we have the irreversible reaction with competitive inhibition;



where S is the substrate of E_1 and the product P is an inhibitor of this reaction. In Ω_2 P is the substrate for the enzyme E_2 which catalyzes the irreversible reaction;



Here we consider a system of reaction-diffusion equations with the absorptive boundary Γ which can dissolve or absorb the substrate and the product. We assume that the diffusion coefficients of S and P are in the ratio of 1: λ . Then, in the stationary case, this model is described by the following system of the differential equations. (See [1] or [7] for further details.)

$$(3.1) \quad \begin{cases} -\Delta s + \chi_1 \frac{\sigma_1 s^+}{1 + \alpha p^+ + s^+} - \chi_2 \frac{\sigma_2 p^+}{1 + \beta p^+} - f_1 = 0 \\ -\lambda \Delta p + \chi_2 \frac{\sigma_2 p^+}{1 + \beta p^+} - \chi_1 \frac{\sigma_1 s^+}{1 + \alpha p^+ + s^+} - f_2 = 0, \end{cases}$$

$$(3.2) \quad \begin{cases} s|_{\Gamma} = 0 \\ p|_{\Gamma} = 0 \end{cases}$$

where

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$$

in the sense of distributions,

$$\chi_1(x) = \begin{cases} 1 & \text{if } x \in \Omega_1 \\ 0 & \text{if } x \notin \Omega_1, \end{cases}$$

$$\chi_2(x) = \begin{cases} 1 & \text{if } x \in \Omega_2 \\ 0 & \text{if } x \notin \Omega_2, \end{cases}$$

$s(x)$ is the concentration of S at $x \in \Omega$,

$p(x)$ is the concentration of P at $x \in \Omega$,

$p^+ = \max(p, 0)$, $s^+ = \max(s, 0)$,

$\sigma_1, \sigma_2, \alpha, \beta$, and λ are positive constants,

$f_1(x)$ is the supply of S at $x \in \Omega$,

$f_2(x)$ is the supply of P at $x \in \Omega$, $f_1, f_2 \in L^2(\Omega)$,

and we take a suitable unity of concentration.

Now we deal with the problem (3.1)-(3.2) by using the results of § 2. Consider the functions ϕ_1, ϕ_2, ψ_1 , and $\psi_2: R \times R \rightarrow R$ defined by

$$\phi_1(a, b) = \sigma_1 \{a^+ - (1 + \alpha b^+) \log(1 + a^+ + \alpha b^+)\},$$

$$\phi_2(a, b) = -\sigma_2 \frac{b^+}{1 + \beta b^+ a},$$

$$\psi_1(a, b) = -\sigma_1 \left\{ \frac{b^+}{1 + b^+} (a - a^+) + \frac{b^+}{\alpha} \log(1 + \alpha a^+ + b^+) \right\},$$

$$\psi_2(a, b) = \psi_2(a) = \frac{\sigma_2}{\beta} \left\{ a^+ - \frac{1}{\beta} \log(1 + \beta a^+) \right\}.$$

Let G_1, G_2 be real valued functions defined on $H_0^1(\Omega) \times L^2(\Omega)$:

$$G_1(s, p) = \frac{1}{2} \int_{\Omega} |\nabla s|^2 dx + \int_{\Omega} \chi_1(x) \phi_1(s(x), p(x)) dx \\ + \int_{\Omega} \chi_2(x) \phi_2(s(x), p(x)) dx - \int_{\Omega} f_1 s dx,$$

$$G_2(q, r) = \frac{1}{2} \lambda \int_{\Omega} |\nabla q|^2 dx + \int_{\Omega} \chi_2(x) \psi_2(q(x), r(x)) dx$$

$$+ \int_{\Omega} \chi_1(x) \phi_1(q(x), r(x)) dx - \int_{\Omega} f_2 q dx.$$

The integrals are well defined, because the composition of a Borel measurable function and a Lebesgue measurable function is Lebesgue measurable and also $\phi_i(s(x), p(x)), \phi_i(q(x), r(x)), i=1, 2$, are summable by using the inequalities

$$(3.3) \quad \frac{t}{1+t} \leq \log(1+t) \leq t \quad \text{for all } t \geq 0.$$

Theorem 3. 1. *There exists a pair of solutions (s, p) in $H_0^1(\Omega) \times H_0^1(\Omega)$, which satisfies the system of Q, V. I. given by*

$$(3.4) \quad \begin{cases} G_1(s, p) \leq G_1(u, p) & \text{for all } u \in H_0^1(\Omega) \\ G_2(p, s) \leq G_2(v, s) & \text{for all } v \in H_0^1(\Omega). \end{cases}$$

Proof. In view of Theorem 2. 1, let $U = M_1 = M_2 = H_0^1(\Omega)$, $V = L^2(\Omega)$, $F_1 = G_1$, and $F_2 = G_2$, then it is sufficient to show that G_1 and G_2 satisfy the conditions (i), (ii) and (iii) in § 2.

(i) : Consider a sequence $\{s_n\} \subset H_0^1(\Omega)$ convergent to an element s_1 in $H_0^1(\Omega)$ with the weak topology of $H_0^1(\Omega)$ and a sequence $\{p_n\} \subset L^2(\Omega)$ convergent to p_1 with the strong topology of $L^2(\Omega)$. Since $H_0^1(\Omega)$ is compactly imbedded in $L^2(\Omega)$, $s_n \rightarrow s_1$ in $L^2(\Omega)$ strongly. For an element v in $L^2(\Omega)$, we define v^+, v^- by $v^+ = \max(v, 0), v^- = -\min(v, 0)$. It can be easily shown that

$$s_n^+ \rightarrow s_1^+, \quad s_n^- \rightarrow s_1^- \quad \text{in } L^2(\Omega), \text{ strongly,}$$

and

$$p_n^+ \rightarrow p_1^+, \quad p_n^- \rightarrow p_1^- \quad \text{in } L^2(\Omega), \text{ strongly.}$$

We claim that, as $n \rightarrow +\infty$,

$$\int_{\Omega} \chi_1(x) \phi_1(s_n, p_n) dx \rightarrow \int_{\Omega} \chi_1(x) \phi_1(s_1, p_1) dx$$

and

$$\int_{\Omega} \chi_2(x) \phi_2(s_n, p_n) dx \rightarrow \int_{\Omega} \chi_2(x) \phi_2(s_1, p_1) dx,$$

which is the direct consequence of the following calculations.

$$\begin{aligned} \left| \int_{\Omega_1} \phi_1(s_n, p_n) - \int_{\Omega_1} \phi_1(s_1, p_1) \right| &= \sigma_1 \left| \int_{\Omega_1} \{s_n^+ - (1 + \alpha p_n^+) \log(1 + s_n^+ + \alpha p_n^+)\} \right. \\ &\quad \left. - \int_{\Omega_1} \{s_1^+ - (1 + \alpha p_1^+) \log(1 + s_1^+ + \alpha p_1^+)\} \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \sigma_1 \int |s_n^+ - s_1^+| + \sigma_1 \int | (1 + \alpha p_n^+) \log(1 + s_n^+ + \alpha p_n^+) \\
 &\quad - (1 + \alpha p_1^+) \log(1 + s_1^+ + \alpha p_1^+) | \\
 &\leq \sigma_1 \int |s_n^+ - s_1^+| + \sigma_1 \int | \alpha (p_n^+ - p_1^+) \log(1 + s_n^+ + \alpha p_n^+) | \\
 &\quad + \sigma_1 \int (1 + \alpha p_1^+) | \log(1 + s_n^+ + \alpha p_n^+) - \log(1 + s_1^+ + \alpha p_1^+) | \\
 &\leq \sigma_1 \int |s_n^+ - s_1^+| + \sigma_1 \alpha \int |p_n^+ - p_1^+| |s_n^+ + \alpha p_n^+| \\
 &\quad + \sigma_1 \int (1 + \alpha p_1^+) \frac{|s_n^+ - s_1^+ + \alpha p_n^+ - \alpha p_1^+|}{1 + s_1^+ + \alpha p_1^+} \\
 &\leq c \|s_n^+ - s_1^+\|_{L^2} + c \|p_n^+ - p_1^+\|_{L^2} \{ \|s_n^+\|_{L^2} + \alpha \|p_n^+\|_{L^2} \} \\
 &\quad + c \{ \|s_n^+ - s_1^+\|_{L^2} + \alpha \|p_n^+ - p_1^+\|_{L^2} \},
 \end{aligned}$$

where we use the inequality (3.3),

$$\begin{aligned}
 &\left| \int_{\Omega_2} \phi_2(s_n, p_n) - \int_{\Omega_2} \phi_2(s_1, p_1) \right| = \sigma_2 \left| \int_{\Omega_2} \left\{ \frac{p_n^+}{1 + \beta p_n^+} s_n - \frac{p_1^+}{1 + \beta p_1^+} s_1 \right\} \right| \\
 &\leq \sigma_2 \int \left| \frac{p_n^+}{1 + \beta p_n^+} (s_n - s_1) \right| + \sigma_2 \int \left| \frac{p_n^+}{1 + \beta p_n^+} - \frac{p_1^+}{1 + \beta p_1^+} \right| |s_1| \\
 &\leq \sigma_2 \int \frac{1}{\beta} |s_n - s_1| + \sigma_2 \int \frac{|p_n^+ - p_1^+| |s_1|}{(1 + \beta p_n^+) (1 + \beta p_1^+)} \\
 &\leq c \|s_n - s_1\|_{L^2} + c \|p_n^+ - p_1^+\|_{L^2},
 \end{aligned}$$

where c is a positive suitable constant which is independent of n and we use the abbreviation

$$\int = \int_{\Omega} dx.$$

Furthermore, we have

$$\int f_1 s_n \rightarrow \int f_1 s_1 \quad \text{as } n \rightarrow +\infty$$

and also $\int |\mathcal{F} \cdot|^2$ is weakly l. s. c. on $H_0^1(\Omega)$. Similarly, we can show the lower semi-continuity of $G_2(\cdot, \cdot)$ and the continuity of $G_2(q, \cdot)$ for each q in M_2 .

(ii) : The strictly convex property is easily shown by using the fact that $\phi_1(\cdot, b)$, $\phi_2(\cdot, b)$, $\psi_1(\cdot, b)$ and $\psi_2(\cdot, b)$ are convex for each b in R and $\int |\cdot|^2$ is strictly convex.

(iii) : Let $s_0=0$, $q_0=0$. The following calculations show the coerciveness of G_1 , G_2 .

$$\begin{aligned}
G_1(s, p) - G_1(0, p) &= \frac{1}{2} \int |\nabla s|^2 + \int \sigma_1 \chi_1 \{s^+ - (1 + \alpha p^+) \log(1 + \alpha p^+ + s^+)\} \\
&\quad - \int \sigma_2 \chi_2 \frac{p^+}{1 + \beta p^+} s - \int \sigma_1 \chi_1 \{- (1 + \alpha p^+) \log(1 + \alpha p^+)\} - \int f_1 s \\
&= \frac{1}{2} \int |\nabla s|^2 + \int \sigma_1 \chi_1 \left\{ s^+ - (1 + \alpha p^+) \log \frac{1 + s^+ + \alpha p^+}{1 + \alpha p^+} \right\} \\
&\quad - \int \sigma_2 \chi_2 \frac{p^+}{1 + \beta p^+} s - \int f_1 s \\
&\geq \frac{1}{2} \int |\nabla s|^2 + \int \sigma_1 \chi_1 \left\{ s^+ - (1 + \alpha p^+) \frac{s^+}{1 + \alpha p^+} \right\} \\
&\quad - \int \sigma_2 \chi_2 \cdot \frac{p^+}{1 + \beta p^+} \cdot |s| - \int |f_1| |s| \\
&\geq C_1 \|s\|_{H_0^1}^2 - C_2 \|s\|_{H_0^1}, \quad C_1 > 0, C_2 > 0
\end{aligned}$$

where we use Poincaré's inequality and the fact that the injection $I: H_0^1(\Omega) \rightarrow L^2(\Omega)$ is continuous. Similarly, we have

$$\begin{aligned}
G_2(q, r) - G_2(0, r) &= \frac{\lambda}{2} \int |\nabla q|^2 + \int \frac{\sigma_2}{\beta} \chi_2 \left\{ q^+ - \frac{1}{\beta} \log(1 + \beta q^+) \right\} \\
&\quad - \int \sigma_1 \chi_1 \left\{ \frac{r^+}{1 + r^+} (q - q^+) + \frac{r^+}{\alpha} \log(1 + \alpha q^+ + r^+) \right\} \\
&\quad - \int f_2 q + \int \sigma_1 \chi_1 \frac{r^+}{\alpha} \log(1 + r^+) \\
&\geq \frac{\lambda}{2} \int |\nabla q|^2 + \int \frac{\sigma_2}{\beta} \chi_2 \left\{ q^+ - \frac{1}{\beta} \cdot \beta q^+ \right\} \\
&\quad - \int \sigma_1 \chi_1 \left\{ \frac{r^+}{1 + r^+} (q - q^+) + \frac{r^+}{\alpha} \log \left(1 + \frac{\alpha q^+}{1 + r^+} \right) \right\} - \int f_2 q \\
&\geq \frac{\lambda}{2} \int |\nabla q|^2 - \int \sigma_1 \chi_1 \left\{ \frac{r^+}{1 + r^+} (q - q^+) + \frac{r^+ \cdot \alpha q^+}{\alpha(1 + r^+)} \right\} \\
&\quad - \int |f_2| |q| \\
&\geq C_3 \|q\|_{H_0^1}^2 - C_4 \|q\|_{H_0^1}, \quad C_3 > 0, C_4 > 0.
\end{aligned}$$

Q. E. D.

Definition 3.1. Let X be a real Banach space with its dual X^* and (\cdot, \cdot) denote the dual pairing between X^* and X . Given the proper convex function $f: X \rightarrow (-\infty, +\infty]$, the subdifferential of such a function is the generally multivalued mapping $\partial f: X \rightarrow X^*$ defined by

$$\partial f(x) = \{x^* \in X^* : f(u) - f(x) \geq (x^*, u - x) \text{ for } \forall u \in X\}.$$

The elements $x^* \in \partial f(x)$ are called subgradients of f at x .

Definition 3.2. Consider any function $f: X \rightarrow (-\infty, +\infty]$. The function $f^*: X^* \rightarrow [-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup \{(x^*, x) - f(x) : x \in X\}$$

is called the conjugate function of f .

We obtain the following Theorem from the definition of subdifferentials and Theorem 3.1.

Theorem 3.2. *We have a solution (s, p) in $H_0^1(\Omega) \times H_0^1(\Omega)$ of the following system;*

$$(3.5) \quad \begin{cases} 0 \in \partial G_1(s, p) \\ 0 \in \partial G_2(p, s) \end{cases}$$

where we consider the subdifferentials of G_1, G_2 with respect to the first variables.

Now we can show the main theorem of this section.

Theorem 3.3. *The system (3.1)–(3.2) admits a solution (s, p) in $[H_0^1(\Omega) \cap H^2(\Omega)]^2$.*

Proof. By applying Theorem 3.2 we have a solution (s, p) of (3.5) in $H_0^1(\Omega) \times H_0^1(\Omega)$. Now we calculate the subdifferentials of G_1 and G_2 actually and show that the solution of (3.5) satisfies System (3.1)–(3.2).

We consider the functions h_1, h_2, l_1 , and $l_2: \Omega \times R \rightarrow R$ defined by

$$\begin{aligned} h_1(x, t) &= \chi_1(x) \phi_1(t, p(x)), \\ h_2(x, t) &= \chi_2(x) \phi_2(t, p(x)), \\ l_1(x, t) &= \chi_1(x) \phi_1(t, s(x)), \end{aligned}$$

and

$$l_2(x, t) = \chi_2(x) \phi_2(t, s(x)).$$

We need the following Lemma for the proof of Theorem.

Lemma. $h_i, l_i, i=1, 2$, satisfy the following conditions.

- (a) $h_i(x, \cdot), l_i(x, \cdot)$ are proper, convex and continuous for each $x \in \Omega$.
- (b) $h_i(\cdot, t), l_i(\cdot, t)$ are measurable for each $t \in R$.
- (c) $\text{int } D_{h_i}(x) \equiv \text{int } \{t \in R: h_i(x, t) < +\infty\}$ and $\text{int } D_{l_i}(x) \equiv \text{int } \{t \in R: l_i(x, t) < +\infty\}$ are nonempty for each x in Ω .
- (d) $h_i(x, u_i(x))$ [resp. $l_i(x, v_i(x))$] is majorized by a summable function of x for at least one choice of u_i [resp. v_i] in $L^2(\Omega)$, $i=1, 2$.
- (e) The conjugate function $h_i^*(x, u_i^*(x))$ [resp. $l_i^*(x, v_i^*(x))$] of h_i [resp. l_i] is majorized by a summable function of x for at least one u_i^* [resp. v_i^*] in $L^2(\Omega)$, $i=1, 2$, where the conjugate functions are given by

$$h_i^*(x, s^*) = \sup \{s^*t - h_i(x, t) : t \in R\}, \quad s^* \in R,$$

$$l_i^*(x, r^*) = \sup \{r^*t - l_i(x, t) : t \in R\}, \quad r^* \in R.$$

Proof. (a), (b) and (c) are obvious. We shall show (d) and (e). Taking $u_i(x) = v_i(x) = 0$, we obtain

$$h_1(x, 0) = -\sigma_1 \chi_1(x) (1 + \alpha p(x)^+) \log(1 + \alpha p(x)^+),$$

$$h_2(x, 0) = 0,$$

$$l_1(x, 0) = -\sigma_1 \chi_1(x) \frac{s(x)^+}{\alpha} \log(1 + s(x)^+),$$

and

$$l_2(x, 0) = 0.$$

Thus (d) is satisfied. Take

$$u_1^*(x) = 0, \quad u_2^*(x) = -\sigma_2 \chi_2(x) \frac{p(x)^+}{1 + \beta p(x)^+}, \quad v_1^*(x) = -\sigma_1 \chi_1(x) \frac{s(x)^+}{1 + s(x)^+}$$

and

$$v_2^*(x) = 0,$$

then (e) is obtained by the following calculations.

$$h_1^*(x, 0) = \sup_{t \in R} \{-h_1(x, t)\}$$

$$= \sup \sigma_1 \chi_1(x) \{(1 + \alpha p(x)^+) \log(1 + t^+ + \alpha p(x)^+) - t^+\}$$

$$\leq \sigma_1 \chi_1(x) (1 + \alpha p(x)^+) \log(1 + \alpha p(x)^+),$$

$$h_2^*\left(x, -\sigma_2 \chi_2(x) \frac{p(x)^+}{1 + \beta p(x)^+}\right)$$

$$\begin{aligned}
 &= \sup_{t \in \mathbb{R}} \left\{ -t\sigma_2\chi_2(x) \frac{p(x)^+}{1+\beta p(x)^+} + \sigma_2\chi_2(x) \frac{p(x)^+}{1+\beta p(x)^+} t \right\} \\
 &= 0, \\
 l_1^*(x, -\sigma_1\chi_1(x) \frac{s(x)^+}{1+s(x)^+}) &= \sup_{t \in \mathbb{R}} \left[t \left\{ -\sigma_1\chi_1(x) \frac{s(x)^+}{1+s(x)^+} \right\} + \sigma_1\chi_1(x) \left\{ \frac{s(x)^+}{1+s(x)^+} (t-t^+) \right. \right. \\
 &\quad \left. \left. + \frac{s(x)^+}{\alpha} \log(1+\alpha t^+ + s(x)^+) \right\} \right] \\
 &= \sup_{t \in \mathbb{R}} \sigma_1\chi_1(x) \left\{ \frac{s(x)^+}{\alpha} \log(1+\alpha t^+ + s(x)^+) - t^+ \frac{s(x)^+}{1+s(x)^+} \right\} \\
 &\leq \sigma_1\chi_1(x) \frac{s(x)^+}{\alpha} \log(1+s(x)^+), \\
 l_2^*(x, 0) &= \sup_{t \in \mathbb{R}} \{-l_2(x, t)\} \\
 &= \sup_{t \in \mathbb{R}} \frac{\sigma_2}{\beta} \chi_2(x) \left\{ \frac{1}{\beta} \log(1+\beta t^+) - t^+ \right\} \\
 &\leq 0.
 \end{aligned}$$

Now we return to the proof of Theorem. By using the results of Rockafellar [13], [14] and the above Lemma we obtain

$$w_i^* \in \partial I_{h_i}(s) \quad [\text{resp. } z_i^* \in \partial I_{l_i}(p)]$$

if and only if

$$w_i^*(x) \in \partial h_{i,x}(s(x)) \quad [\text{resp. } z_i^*(x) \in \partial l_{i,x}(p(x))]$$

for a. e. $x \in \Omega$, $i = 1, 2$, where

$$I_{h_i}(s) = \int_{\Omega} h_i(x, s(x)) \, dx,$$

$$I_{l_i}(p) = \int_{\Omega} l_i(x, p(x)) \, dx,$$

$$h_{i,x}(\cdot) = h_i(x, \cdot)$$

and

$$l_{i,x}(\cdot) = l_i(x, \cdot).$$

By calculating the ordinary differentials, we have

$$\frac{\partial h_1}{\partial t}(x, t) = \sigma_1\chi_1(x) \frac{t^+}{1+\alpha p(x)^+ + t^+},$$

$$\frac{\partial h_2}{\partial t}(x, t) = -\sigma_2 \chi_2(x) \frac{p(x)^+}{1 + \beta p(x)^+},$$

$$\frac{\partial l_1}{\partial t}(x, t) = -\sigma_1 \chi_1(x) \frac{s(x)^+}{1 + \alpha t^+ + s(x)^+}$$

and

$$\frac{\partial l_2}{\partial t}(x, t) = \sigma_2 \chi_2(x) \frac{t^+}{1 + \beta t^+} \quad \text{for a. e. } x \in \Omega.$$

So we have

$$\partial h_{1,x}(s(x)) = \sigma_1 \chi_1(x) \frac{s(x)^+}{1 + \alpha p(x)^+ + s(x)^+},$$

$$\partial h_{2,x}(s(x)) = -\sigma_2 \chi_2(x) \frac{p(x)^+}{1 + \beta p(x)^+},$$

$$\partial l_{1,x}(p(x)) = -\sigma_1 \chi_1(x) \frac{s(x)^+}{1 + \alpha p(x)^+ + s(x)^+}$$

and

$$\partial l_{2,x}(p(x)) = \sigma_2 \chi_2(x) \frac{p(x)^+}{1 + \beta p(x)^+}.$$

By calculating the subdifferentials of the other terms, $\int |\nabla s|^2, \int f_1 s, \int |\nabla p|^2, \int f_2 p$ (see proposition 2.7, 2.8 of chapter 2 in Barbu [2]), we can show that the subdifferentials of G_1, G_2 are single valued and correspond to the terms in (3.1). Thus we have a solution (s, p) in $[H_0^1(\Omega)]^2$ which satisfy (3.1)-(3.2). By using the arguments about the smoothness of solutions of regular elliptic problems in [9] (p.212) we have

$$s, p \in H_0^1(\Omega) \cap H^2(\Omega).$$

Q. E. D.

Remark. When the system depends to time-variables, it is important to analyze the time-dependent behavior of solutions in relation to the solutions in equilibrium. Under suitable boundary conditions, such as convergency and periodicity, we show the convergence to the equilibrium states ([5]) and the periodicity ([12]) of solutions of reaction-diffusion systems.

§ 4. Neumann Boundary Conditions

We consider the constant flow of the substrate S and the product P through the boundary Γ and assume that S and P are dissolved at the rate proportional to their concentration, $k_1s(x)$, $k_2p(x)$ at x in Ω , where k_1, k_2 are positive constants. Then the system is given by

$$(4.1) \quad \begin{cases} -\Delta s + k_1s + \chi_1 \frac{\sigma_1 s^+}{1 + \alpha p^+ + s^+} - \chi_2 \frac{\sigma_2 p^+}{1 + \beta p^+} - f_1 = 0 \\ -\lambda \Delta p + k_2p + \chi_2 \frac{\sigma_2 p^+}{1 + \beta p^+} - \chi_1 \frac{\sigma_1 s^+}{1 + \alpha p^+ + s^+} - f_2 = 0 \end{cases}$$

in Ω ,

$$(4.2) \quad \begin{cases} \frac{\partial s}{\partial n} = g_1 \\ \frac{\partial p}{\partial n} = g_2 \end{cases}$$

on Γ ,

where $g_1, g_2 \in H^{\frac{1}{2}}(\Gamma)$, $\frac{\partial}{\partial n}$ is the normal derivative taken toward the exterior of Ω , and we use the same notations as those in § 3.

In order to use the Q. V. I. method, we consider the functions $J_1, J_2: H^1(\Omega) \times L^2(\Omega) \rightarrow R$ defined by

$$\begin{aligned} J_1(s, p) &= \frac{1}{2} \int_{\Omega} |\nabla s|^2 \, dx + \frac{k_1}{2} \int_{\Omega} |s|^2 \, dx + \int_{\Omega} \chi_1 \phi_1(s, p) \, dx \\ &\quad + \int_{\Omega} \chi_2 \phi_2(s, p) \, dx - \int_{\Omega} f_1 s \, dx - \int_{\Gamma} g_1 s \, dy, \\ J_2(q, r) &= \frac{\lambda}{2} \int_{\Omega} |\nabla q|^2 \, dx + \frac{k_2}{2} \int_{\Omega} |q|^2 \, dx + \int_{\Omega} \chi_1 \phi_1(q, r) \, dx \\ &\quad + \int_{\Omega} \chi_2 \phi_2(q, r) \, dx - \int_{\Omega} f_2 q \, dx - \int_{\Gamma} g_2 q \, dy \end{aligned}$$

where ϕ_1, ϕ_2, ψ_1 , and ψ_2 are defined in § 3.

Theorem 4.1. *The system of the following Q. V. I. admits a solution (s, p) in $H^1(\Omega) \times H^1(\Omega)$.*

$$(4.3) \quad \begin{cases} J_1(s, p) \leq J_1(u, p) & \text{for all } u \text{ in } H^1(\Omega) \\ J_2(p, s) \leq J_2(v, s) & \text{for all } v \text{ in } H^1(\Omega). \end{cases}$$

Proof. Let $U=M_1=M_2=H^1(\Omega)$, $V=L^2(\Omega)$, $F_1=J_1$, and $F_2=J_2$ in § 2. By using the argument similar to § 3 and applying Theorem 2.1, it may be sufficient to show that J_1, J_2 satisfy the coercive condition (iii), but it is immediately clear from the calculations in § 3 and the inequalities

$$\begin{aligned} \frac{1}{2} \int |\nabla s|^2 + \frac{k_1}{2} \int |s|^2 &\geq \frac{1}{2} \min(1, k_1) \|s\|_{H^1(\Omega)}^2, \\ \frac{\lambda}{2} \int |\nabla q|^2 + \frac{k_2}{2} \int |q|^2 &\geq \frac{1}{2} \min(\lambda, k_2) \|q\|_{H^1(\Omega)}^2. \end{aligned}$$

Q. E. D.

From the definition of subdifferentials we have the following Theorem.

Theorem 4.2. *We have a solution (s, p) in $H^1(\Omega) \times H^1(\Omega)$ which satisfies*

$$(4.4) \quad \begin{cases} 0 \in \partial J_1(s, p) \\ 0 \in \partial J_2(p, s) \end{cases}$$

where the subdifferentials with respect to the first variables are considered.

Definitions. $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $H^1(\Omega)$ and $[H^1(\Omega)]'$ and $\langle \cdot, \cdot \rangle_\gamma$ denotes the dual pairing between $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$. The inner products on $L^2(\Omega)$, $L^2(\Gamma)$ are denoted by (\cdot, \cdot) , $(\cdot, \cdot)_\gamma$, respectively.

I, I_0 are the continuous injections from $H^1(\Omega)$ into $L^2(\Omega)$, from $H^{\frac{1}{2}}(\Gamma)$ into $L^2(\Gamma)$, respectively, and γ is the trace operator. Their dual operators are denoted by I^*, I_0^* and γ^* .

Theorem 4.3. *The system (4.1)-(4.2) admits a solution (s, p) in $H^2(\Omega) \times H^2(\Omega)$.*

Proof. By calculating the subdifferentials of J_1, J_2 and applying Theorem 4.2, we have

$$\begin{aligned} \langle -\Delta s + I^* \left\{ k_1 s + \chi_1 \frac{\sigma_1 s^+}{1 + \alpha p^+ + s^+} - \chi_2 \frac{\sigma_2 p^+}{1 + \beta p^+} - f_1 \right\} \\ - \gamma^* I_0^* g_1, v \rangle = 0 \quad \text{for all } v \text{ in } H^1(\Omega). \end{aligned}$$

By Green's formula we obtain

$$\begin{aligned} & \left(-\Delta s + k_1 s + \chi_1 \frac{\sigma_1 s^+}{1 + \alpha p^+ + s^+} - \chi_2 \frac{\sigma_2 p^+}{1 + \beta p^+} - f_1, v \right) \\ & + \left\langle \frac{\partial s}{\partial n} - g_1, v \right\rangle_\Gamma = 0 \quad \text{for all } v \in H^1(\Omega). \end{aligned}$$

Taking $v = \omega \in \mathcal{D}(\Omega)$ in the above equation, we have

$$\begin{aligned} & \left(-\Delta s + k_1 s + \chi_1 \frac{\sigma_1 s^+}{1 + \alpha p^+ + s^+} - \chi_2 \frac{\sigma_2 p^+}{1 + \beta p^+} - f_1, \omega \right) = 0 \\ & \text{for all } \omega \in \mathcal{D}(\Omega). \end{aligned}$$

Since $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$, we have

$$-\Delta s + k_1 s + \chi_1 \frac{\sigma_1 s^+}{1 + \alpha p^+ + s^+} - \chi_2 \frac{\sigma_2 p^+}{1 + \beta p^+} - f_1 = 0 \quad \text{a. e. in } \Omega.$$

Thus we have

$$\left\langle \frac{\partial s}{\partial n} - g_1, \gamma v \right\rangle_\Gamma = 0 \quad \text{for all } v \text{ in } H^1(\Omega).$$

It implies that

$$\frac{\partial s}{\partial n} = g_1 \quad \text{a. e. on } \Gamma.$$

By the same argument about J_2 , we obtain

$$-\lambda \Delta p + k_2 p + \chi_2 \frac{\sigma_2 p^+}{1 + \beta p^+} - \chi_1 \frac{\sigma_1 s^+}{1 + \alpha p^+ + s^+} - f_2 = 0 \quad \text{a. e. in } \Omega$$

and

$$\frac{\partial p}{\partial n} = g_2 \quad \text{a. e. on } \Gamma.$$

By using the argument about the regularity of regular elliptic problems in [9] (p. 212), we have s, p in $H^2(\Omega)$.

Q. E. D.

§ 5. A Priori Feedback Control

Now we consider ‘‘a priori feedback control’’ model. We start from the theorem of Mossino about Q. V. I. Let U, V be real reflexive Banach spaces and assume that U is compactly imbedded in V . Let K be a function defined on $U \times V$ with values in $(-\infty, +\infty]$ and assume that K satisfies the following conditions (I), (II) and (III).

(I) $K(\cdot, \cdot)$ is lower semi-continuous on $U(\text{weak}) \times V(\text{strong})$.
 For each s in U , $K(s, \cdot)$ is continuous on $V(\text{strong})$.

(II) For each p in V , $K(\cdot, p)$ is proper, strictly convex.

(III) K is coercive in the following sense; there exists an element s_0 in U and a constant $m_0 > 0$ such that $K(s_0, p) < +\infty$ for each p in V and, if $\|s\|_U > m_0$,

$$K(s, p) - K(s_0, p) \geq 0 \quad \text{for all } p \text{ in } V.$$

The above coercive condition (III) is weaker than (iii) in § 2. Furthermore, this condition can be weakened to the type which appeared in [15]. The following theorem is proved by using the same method as the proof Theorem 2.1.

Theorem 5.1. *Under the above assumptions there exists u in U such that*

$$(5.1) \quad K(u, u) \leq K(v, u) \quad \text{for all } v \text{ in } V,$$

that is,

$$(5.2) \quad 0 \in \partial K(u, u)$$

where the subdifferential is taken with respect to the first variable.

First we consider the following system.

$$(5.3) \quad -\Delta u + ku = f \quad \text{in } \Omega$$

with the boundary condition

$$(5.4) \quad \begin{cases} \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_+ = \{y \in \Gamma : u|_{\Gamma}(y) > Q(u|_{\Gamma})(y)\} \\ \frac{\partial u}{\partial n} = m & \text{on } \Gamma_- = \{y \in \Gamma : u|_{\Gamma}(y) < Q(u|_{\Gamma})(y)\} \\ 0 \leq \frac{\partial u}{\partial n} \leq m & \text{on } \Gamma_0 = \Gamma - \Gamma_+ \cup \Gamma_- \end{cases}$$

where k, m are positive constants and $f \in L^2(\Omega)$ and $Q: L^2(\Gamma) \rightarrow L^2(\Gamma)$ is continuous and $\sup_{w \in L^2(\Gamma)} |Q(w)| < +\infty$.

Take $U = H^1(\Omega)$ and $V = H^\epsilon(\Omega)$ where ϵ is a positive constant such that $\frac{1}{2} < \epsilon < 1$, then U is compactly imbedded in V . We shall deal

with this system by considering the function $K: H^1(\Omega) \times H^e(\Omega) \rightarrow R$ defined by

$$(5.5) \quad K(u, v) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx + \frac{k}{2} \int_{\Omega} |u|^2 \, dx + m \int_{\Gamma} \{u|_{\Gamma} - Q(v|_{\Gamma})\}^- \, dy - \int_{\Omega} f u \, dx.$$

When $v \in H^e(\Omega)$, $\frac{1}{2} < \varepsilon < 1$, the trace of v , $v|_{\Gamma}$ is in $H^{\varepsilon-\frac{1}{2}}(\Gamma)$. Since $Q: L^2(\Gamma) \rightarrow L^2(\Gamma)$ is continuous, if $v_n \rightarrow v$ in $H^e(\Omega)$ as $n \rightarrow +\infty$, we have $Q(v_n|_{\Gamma}) \rightarrow Q(v|_{\Gamma})$ in $L^2(\Gamma)$. Thus, by using the same argument as the proof of Theorem 3.1 (i), it follows that K satisfies the condition (I). As for the condition (II), since the operator on $R; t \rightarrow \{t\}^-$ is convex, we can use the argument of the proof of Theorem 3.1 (ii). For the condition (III), take $s_0 = 0$, then by using the uniform boundedness of Q we have

$$\begin{aligned} K(s, p) - K(0, p) &= \frac{1}{2} \int_{\Omega} |\nabla s|^2 \, dx + \frac{k}{2} \int_{\Omega} |s|^2 \, dx + m \int_{\Gamma} [\{s|_{\Gamma} - Q(p|_{\Gamma})\}^- - \{-Q(p|_{\Gamma})\}^-] \, dy - \int_{\Omega} f s \, dx \\ &\geq c_1 \|s\|_{H^1(\Omega)}^2 - m \int_{\Gamma} \{-Q(p|_{\Gamma})\}^- \, dy - c_2 \|s\|_{H^1(\Omega)} \\ &\geq c_1 \|s\|_{H^1(\Omega)}^2 - c_2 \|s\|_{H^1(\Omega)} - c_3 \end{aligned}$$

where c_1, c_2 and c_3 are positive constants.

Thus there exist a constant $m_0 > 0$ which satisfies the condition (III).

Theorem 5.2. *The system (5.3) - (5.4) admits a solution u in $H^1(\Omega)$.*

Proof. It is sufficient to calculate the subdifferential of K . By using the argument in the proof of Theorem 3.3 and 4.3 we obtain

$$0 \in \langle -\Delta u + ku - f + m\gamma^* I_0^* \partial \int_{\Gamma} \{\gamma u - Q(\gamma u)\}^-, w \rangle$$

for all w in $H^1(\Omega)$. Taking $w = \omega \in \mathcal{D}(\Omega)$ in the above equation, we obtain

$$-\Delta u + ku = f$$

in the sense of distribution. By Green's formula, we have

$$0 \in (-\Delta u + ku - f, w) + \left\langle \frac{\partial u}{\partial n} + mI_0^* \partial \int_{\Gamma} \{\gamma u - Q(\gamma u)\}^-, \gamma w \right\rangle_{\Gamma}$$

for all w in $H^1(\Omega)$. And hence

$$0 \in \left\langle \frac{\partial u}{\partial n} + mI_0^* \partial \int_{\Gamma} \{\gamma u - Q(\gamma u)\}^-, \gamma w \right\rangle_{\Gamma}.$$

Let ϕ be a convex function from $L^2(\Gamma)$ to R defined by

$$\phi(v) = \int_{\Gamma} \{v\}^- dy, \quad v \text{ in } L^2(\Gamma),$$

then it is easily known that

$$\partial \phi(u) = \begin{cases} 0 & \text{on } \{y \in \Gamma : u|_{\Gamma} > 0\} \\ -1 & \text{on } \{y \in \Gamma : u|_{\Gamma} < 0\} \\ [-1, 0] & \text{otherwise.} \end{cases}$$

Thus we conclude that

$$\begin{cases} \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_+ \\ \frac{\partial u}{\partial n} = m & \text{on } \Gamma_- \\ 0 \leq \frac{\partial u}{\partial n} \leq m & \text{on } \Gamma_0. \end{cases}$$

Q.E.D.

Now we can study the reaction diffusion system with a priori feedback control by applying the previous results. We consider the system (4.1) with the following boundary conditions

$$(5.6) \quad \begin{cases} \frac{\partial s}{\partial n} = 0 & \text{on } \{y \in \Gamma : \gamma s > Q_1(\gamma p)\} \\ \frac{\partial s}{\partial n} = m_1 & \text{on } \{y \in \Gamma : \gamma s < Q_2(\gamma p)\} \\ 0 \leq \frac{\partial s}{\partial n} \leq m_1, \end{cases}$$

$$(5.7) \quad \begin{cases} \frac{\partial p}{\partial n} = 0 & \text{on } \{y \in \Gamma : \gamma p > Q_2(\gamma s)\} \\ \frac{\partial p}{\partial n} = m_2 & \text{on } \{y \in \Gamma : \gamma p < Q_2(\gamma s)\} \\ 0 \leq \frac{\partial p}{\partial n} \leq m_2. \end{cases}$$

In (5.6) and (5.7) m_1, m_2 are positive constants, $Q_i: L^2(\Gamma) \rightarrow L^2(\Gamma)$ are continuous and $\sup_{w \in L^2(\Gamma)} |Q_i(w)| < +\infty, i=1, 2$. By these functions Q_1, Q_2 , the flow of each substance through the boundary is controlled according to the boundary value of the other chemical component.

Define $F_1, F_2: H^1(\Omega) \times H^1(\Omega) \rightarrow R$ as follows:

$$(5.8) \quad F_1(s, p) = \frac{1}{2} \int_{\Omega} |\nabla s|^2 dx + \frac{k_1}{2} \int_{\Omega} |s|^2 dx + \int_{\Omega} \chi_1 \phi_1(s, p) dx \\ + \int_{\Omega} \chi_2 \phi_2(s, p) dx + m_1 \int_{\Gamma} \{\gamma s - Q_1(\gamma p)\}^- dy - \int_{\Omega} f_1 s dx,$$

$$(5.9) \quad F_2(q, r) = \frac{1}{2} \int_{\Omega} |\nabla q|^2 dx + \frac{k_2}{2} \int_{\Omega} |q|^2 dx + \int_{\Omega} \chi_1 \phi_1(q, r) dx \\ + \int_{\Omega} \chi_2 \phi_2(q, r) dx + m_2 \int_{\Gamma} \{\gamma q - Q_2(\gamma r)\}^- dy - \int_{\Omega} f q dx$$

where $\phi_1, \phi_2, \psi_1, \psi_2$ are defined in § 3. Then, by Theorem 2.1 and the previous arguments, we obtain the following theorem.

Theorem 5.3. *System (4.1)-(5.6)-(5.7) admits a solution (s, p) in $H^1(\Omega) \times H^1(\Omega)$.*

Remark. Enzymatically catalyzed membranes are of primary importance in various biological process, because they control the flow of chemical substances ([16]). Mechanism of this control is very complicated, but the recent progress in biochemistry shows that proteins in membrane play the most important role by catalyzing the reaction which occurs active transport. In our models the functions Q_1 and Q_2 correspond to this control. To give an actual form of Q_i in each biochemical model will be a complicated, but an interesting problem.

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