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Non-Uniqueness in the Cauchy Problem for Partial Differential Operators with Multiple Characteristics, II

By

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§ 0. Introduction

In this paper, we shall consider non-uniqueness of C^∞-solutions of the non-characteristic Cauchy problem for a class of operators with C^∞-coefficients containing some degenerate elliptic operators. Then we shall extend the non-uniqueness results of our preceding papers [6] and [7].

A typical example of the operators treated here is the following operator in $\boldsymbol{R}^2$:

$$P=(\partial_t-\sqrt{-1}\,t^lC(t,x)\partial_x)^p+t^kA(t,x)(\sqrt{-1}\,\partial_x)^q-t^mB(t,x)(\sqrt{-1}\,\partial_x)^{q-r},$$

where $p\geqq q>r\geqq 1$. Plis [10] treated the case $l=m=0$, $A=B=C=1$ (i.e. elliptic case) and Nakane [6] treated the case $A=B=C=1$ (i.e. degenerate elliptic case). These results show that under some conditions on k, l, m, there exist C^∞-functions u and f satisfying

$$Pu-fu=0, \qquad (0,\ 0)\in \operatorname{supp} u\subset\{t\geqq 0\}.$$

An important property of P is that the imaginary parts of its characteristic roots have finite order zeros on the initial surface $t=0$. For these operators, [6], Ōkaji [9], Roberts [11] and Uryu [13] showed uniqueness under Levi type conditions on the lower order terms. But they considered mainly the case of variable multiple characteristics. We are much interested in the case of constant multiplicity (i.e. $p>q$). As for this case, there are few results. In [6] and [9], they treated the case $p=q=2$, $r=1$ and $A=0$. For first order operators,

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see Strauss-Treves [12] or Zuily [14].

The main purpose of this paper is to give a necessary condition for uniqueness on the lower order terms for operators of the above type. Then we conclude that uniqueness does not hold unless the lower order terms degenerate according to the order of degeneracy of the imaginary parts of characteristic roots. Another purpose of this paper is to investigate the effect of the behaviour of imaginary parts (or of real parts) of symbols of operators.

The method of proofs of our results is a modification of those of Alinhac-Zuily [1], Lascar-Zuily [5], Nakane [7] and Zuily [14]. That is, we shall construct the functions u and f by using the method of geometrical optics. Our results will show that this method is very powerful to get necessary conditions for uniqueness.

§ 1. Statement of Results

Let $P=P(t,x;\partial_t,D_x)$ be the following operator of order p in $\mathbb{R}^{d+1}$:

$$P=(\partial_t-t^l C(t,x;D_x))^p+t^k A(t,x;D_x)-t^m B(t,x;D_X)+\sum_{j=1}^{p}\sum_{i\leqq j} t^{m(j,i)} B_{j,i}(t,x;D_x)\partial_t^{p-j}. \tag{1.1}$$

Here $\partial_t=\partial/\partial t$, $D_x=\frac{1}{\sqrt{-1}}(\partial/\partial x_1,\cdots,\partial/\partial x_d)$, $k,l,m,m(j,i)\in\mathbb{N}=\{0,1,2,\cdots\}$, $A,B,C,B_{j,i}$ are partial differential operators, homogeneous order q, $q-r$, 1, i with respect to D_x respectively, whose coefficients are C^∞ in U, an open neighborhood of the origin in $\mathbb{R}^{d+1}$ and $p\geqq q>r\geqq 1$. Let ξ be the dual variable of x.

Then the following theorem is a corollary of Theorem 1.1 of [7].

Theorem 1.1. *Suppose*

$$\frac{pr+qm}{q-r}<k<\frac{prl+(p-q)m}{p-q+r}, \tag{1.2}$$

$$m(j,i)>\frac{jk}{p}+\frac{(ip-jq)(k-m)}{pr}. \tag{1.3}$$

We also assume that there exist $\xi^0\in\mathbb{R}^d\setminus\{0\}$ *and a branch* $D(\xi^0)$ *of* $\{B(0,0;\xi^0)-A(0,0;\xi^0)\}^{1/p}$ *satisfying*

$$\operatorname{Re} D(\xi^0)>0, \tag{1.4}$$

(1.5) $$\operatorname{Re}\left\{\left(\frac{A(0,0;\xi^0)}{B(0,0;\xi^0)-A(0,0;\xi^0)}+1-\frac{q}{r}\right)D(\xi^0)\right\}>0.$$

Then there exist an open neighborhood U' of the origin and C^∞-functions u and f in U' such that

(1.6) $$Pu-fu=0, \qquad (0,0)\in \operatorname{supp} u\subset\{t\geqq 0\}.$$

Remark 1.1. Assumption (1.2) is equivalent to assumption (1.9) or (1.13) of Theorem 2 of [6]. Hence this theorem is a generalization of Theorem 2 of [6].

Remark 1.2. As in Remark 1.4 of [7], we introduce Newton polygons. In the (X, Y)-plane, we plot the following points:

$$R_1=(q/p,-1),\quad R_2=(0,k/p),\quad R_3=(r/p,m/p),$$
$$R_4=(q/p-1,l),\quad P_{j,i}=(q/p-i/j,m(j,i)/j).$$

The first inequality of (1.2) implies that R_3 is located below the line passing through the points R_1 and R_2. The second inequality of (1.2) implies that R_4 is located above the line passing through the points R_2 and R_3. Assumption (1.3) implies that all the points $P_{j,i}$ are located above the line passing through the points R_2 and R_3. Hence above theorem is a corollary of Theorem 1.1 of [7].

Now we consider the case $k>\dfrac{prl+(p-q)m}{p-q+r}$.

Theorem 1.2. *Suppose*

(1.7) $$k>\frac{prl+(p-q)m}{p-q+r},$$

(1.8) $$m<(l+1)(q-r)-p,$$

(1.9) $$m(j,i)>lj+\frac{m-pl}{p-q+r}(j-i).$$

We also assume that there exist $\xi^0\in \boldsymbol{R}^d\setminus\{0\}$ and a branch $B(0,0;\xi^0)^{1/p}$ satisfying

(1.10) $$\operatorname{Re} C(0,0;\xi^0)+\operatorname{Re} B(0,0;\xi^0)^{1/p}>0,$$

(1.11) $$p\operatorname{Re} C(0,0;\xi^0)+(q-r)\operatorname{Re} B(0,0;\xi^0)^{1/p}<0.$$

Then the same conclusion as in Theorem 1.1 holds.

Remark 1.3. In terms of Remark 1.2, assumption (1.7) implies that R_4 is located below the line passing through the points R_2 and R_3. Assumption (1.8) implies that R_3 is located below the line passing through the points R_1 and R_4. Assumption (1.9) implies that all the points $P_{j,i}$ are located above the line passing through the points R_3 and R_4.

Example. We consider the following operator in $\boldsymbol{R}^2$:

$$P=(\partial_t-t^l D_x)^p-t^m B(t,x) D_x^{q-r},$$

where $B\in C^\infty(U)$ and $p\geqq q>r\geqq 1$. Since it corresponds to the case $A=B_{j,i}=0$, assumptions (1.7) and (1.9) are automatically satisfied. We assume (1.8) and we consider assumption (1.10) and (1.11). By considering the effect of the similarity transformation: $x\mapsto hx$ for some $h\in\boldsymbol{R}$, we have the following:

Case 1. *When* $p\geqq 3$, *assumptions* (1.10) *and* (1.11) *are satisfied if* $B(0,0)\neq 0$.

Case 2. *When* $p=q=2$ *and* $r=1$, *assumptions* (1.10) *and* (1.11) *are satisfied if* $B(0,0)\notin \boldsymbol{C}\setminus[0,\infty)$.

Remark 1.4. Consider the following operator in $\boldsymbol{R}^2$:

$$P=(\partial_t-t^l D_x)^2-t^m B(t,x) D_x+C(t,x),$$

where B, $C\in C^\infty(U)$. In [6], we showed that uniqueness holds for P if $m>l-1$. Recently Ōkaji [9] showed that uniqueness holds for P if $m\geqq l-1$. Furthermore, Professor K. Watanabe pointed us that uniqueness holds for P with $m<l-1$ if $B(t,x)>0$. Hence assumptions (1.8), (1.10) and (1.11) are indispensable.

Finally we consider the case $k=\dfrac{prl+(p-q)m}{p-q+r}$.

Theorem 1.3. *Suppose* (1.8), (1.9) *and*

$$k=\frac{prl+(p-q)m}{p-q+r}. \tag{1.12}$$

We also assume that there exist $\xi^0\in\boldsymbol{R}^d\setminus\{0\}$ *and a branch* $D(\xi^0)$ *of* $\{B(0,0;\xi^0)-A(0,0;\xi^0)\}^{1/p}$ *satisfying*

(1. 13) $$\mathrm{Re}\, C(0, 0;\, \xi^0) + \mathrm{Re}\, D(\xi^0) > 0,$$

(1. 14) $$p\, \mathrm{Re}\, C(0, 0;\, \xi^0) + \mathrm{Re}\left\{\left(\frac{(p-q+r)\,(m-k)\,A(0, 0;\, \xi^0)}{B(0, 0;\, \xi^0) - A(0, 0;\, \xi^0)} + q - r\right) D(\xi^0)\right\} < 0.$$

Then the same conclusion as in Theorem 1. 1 *holds.*

Remark 1. 5. Assumption (1. 12) implies that R_4 is located on the line passing through the points R_2 and R_3.

Since the proof of Theorem 1. 3 is similar to that of Theorem 1. 2, we omit its proof.

§ 2. Proof of Theorem 1. 2

Let ω be a sufficiently small open neighborhood of the origin in $\boldsymbol{R}^d$. We construct the function $u(t, x)$ as a superposition of functions $u_n(t, x)$, defined in $U_n = (b_{n+1}, b_{n-1}) \times \omega$, of the form:

$$u_n(t, x) = \exp\left\{\sqrt{-1}\, \tau_n\left(\xi^0 x - \frac{\sqrt{-1}}{l+1}(t^{l+1} - b_n^{l+1})\right)\right\} \exp\left\{\phi\left(\frac{t}{b_n}, x, b_n\right)\right\} \times \exp\{-\gamma_n(x)\}\, w_n\left(\frac{t}{b_n}, x\right),$$

so that the function $f = Pu/u$ becomes C^∞ near the origin. The above form of u_n is a modification of the one in [7]. Considering the degenerate elliptic part $t^l C(t, x;\, D_x)$ of P, we introduce a complex phase function $\xi^0 x - \frac{\sqrt{-1}}{l+1}(t^{l+1} - b_n^{l+1})$. Then a similar argument as in [7] works for this case.

This method was originally introduced by Cohen [3] and [10], etc. for a special type of operators. In order to treat more general class of operators, Hörmander [4] constructed u_n by using the method of geometrical optics. In [1] and [14], they have developed his idea and have obtained a more systematic way to construct u_n. As in [7], we treat the case of higher multiplicity. Then we come across a new difficulty when we solve the transport equations.

Put $t = \delta s$, where δ is a small positive parameter. Then P is transformed into

$$\tilde{P}=(\delta^{-1}\partial_s-\delta^l s^l C(\delta s, x;\ D_x))^p+\delta^k s^k A(\delta s, x;\ D_x)$$
$$-\delta^m s^m B(\delta s, x;\ D_x)+\sum_{j=1}^{p}\sum_{i\leqq j}\delta^{m(j,i)+j-p}s^{m(j,i)}B_{j,i}(\delta s, x;\ D_x)\,\partial_s^{p-j}.$$

Let T be a small open neighborhood of 1 in $\boldsymbol{R}$ and let B^∞ be the set of functions $f=f(s, x, \delta)$ which are C^∞ with respect to (s, x) and satisfy

$$|\partial_s^j D_x^\alpha f|\leqq C_{j,\alpha} \qquad \text{in } T\times\omega$$

for any $(j, \alpha)\in\boldsymbol{N}^{d+1}$ as δ tends to 0. We construct the asymptotic solution $u=u(s, x, \delta)$ of the equation $\tilde{P}u=0$ in $T\times\omega$ in the form:

$$u(s, x, \delta)=\exp\left\{\sqrt{-1}\,\tau\left(\xi^0 x-\frac{\sqrt{-1}}{l+1}(t^{l+1}-\delta^{l+1})\right)\right\}\exp\{\phi(s, x, \delta)\}$$
$$\times\exp\{-\gamma(x, \delta)\}\,w(s, x, \delta),$$

where

$$\phi(s, x, \delta)=\sum_{j=1}^{N}\nu_j\phi_j(s, x, \delta), \qquad \phi_j\in B^\infty,$$
$$\gamma(x, \delta)=\sum_{j=1}^{N}\tilde{\gamma}_j(x, \delta)=\sum_{j=1}^{N}\nu_j\gamma_j(x, \delta), \qquad \gamma_j\in B^\infty,$$
$$\tau=\delta^{-d_0},$$
$$\nu_j=\delta^{-d_j}, \qquad 1\leqq j\leqq N,$$
$$d_0>d_1>\cdots>d_N>0,$$
$$N\in\boldsymbol{N},$$

all these are determined later.

Let us consider

$$I=\exp\left\{-\sqrt{-1}\,\tau\left(\xi^0 x-\frac{\sqrt{-1}}{l+1}(t^{l+1}-\delta^{l+1})\right)-\phi+\gamma\right\}\times\tilde{P}u.$$

We determine ϕ and γ so that I shall be written in the form:

(2.1) $$I=\delta^{-p}\nu_1^{p-1}(L_0 w+\delta^\varepsilon L_1 w),$$

where

(2.2) $$L_0=C_0(s, x, \delta)\,\partial_s+C_1(s, x, \delta), \qquad C_0,\ C_1\in B^\infty,$$

(2.3) $$C_0(s, x, \delta)\neq 0,$$

and $L_1=L_1(s, x, \delta;\ \partial_s, D_x)$ is a partial differential operator of order p

with coefficients in B^∞ and $\varepsilon>0$ is an appropriate constant. We note that $\xi^0 x-\frac{\sqrt{-1}}{l+1}(t^{l+1}-\delta^{l+1})$ and ϕ are phase functions and that γ is the normalization term.

We put

$$\begin{aligned} I_1= & \{\delta^l\tau s^l(1-C(\delta s, x;\ \xi^0)+\delta^{-1}\nu_1\phi_{1,s}\}^p \\ & +\delta^k\tau^q s^k A(\delta s, x;\ \xi^0)-\delta^m\tau^{q-r}s^m B(\delta s, x;\ \xi^0) \\ & +\sum_{j=1}^{p}\sum_{i\leqq j}\delta^{m(j,i)}\tau^i s^{m(j,i)}B_{j,i}(\delta s, x;\ \xi^0)(\delta^l\tau s^l+\delta^{-1}\nu_1\phi_{1,s})^{p-j}, \end{aligned}$$

$$\begin{aligned} I_g= & \{\delta^l\tau s^l(1-C(\delta s, x; \xi^0))+\delta^{-1}\sum_{h=1}^{g}\nu_h\phi_{h,s}\}^p \\ & +\delta^k s^k A(\delta s, x; \tau\xi^0+\sqrt{-1}\sum_{h=1}^{g-1}\nu_h(\gamma_{h,x}-\phi_{h,x})) \\ & -\delta^m s^m B(\delta s, x; \tau\xi^0+\sqrt{-1}\sum_{h=1}^{g-1}\nu_h(\gamma_{h,x}-\phi_{h,x})) \\ & +\sum_{j=1}^{p}\sum_{i\leqq j}(\delta s)^{m(j,i)}B_{j,i}(\delta s, x; \tau\xi^0+\sqrt{-1}\sum_{h=1}^{g-1}\nu_h(\gamma_{h,x}-\phi_{h,x})) \\ & \qquad \times(\delta^l\tau s^l+\delta^{-1}\sum_{h=1}^{g}\nu_h\phi_{h,s})^{p-j}, \qquad (2\leqq g\leqq N). \end{aligned}$$

First we determine ϕ_1 so that $I_1=0$. Next we determine γ_1 from ϕ_1. Then we find ϕ_2 so that $I_2=0(\delta^{\varepsilon-p}\nu_1^{p-1})$ and find γ_2 from ϕ_2. In the same way, once we have determined ϕ_j and γ_j $(1\leqq j\leqq g)$, we find ϕ_{g+1} so that $I_{g+1}=O(\delta^{\varepsilon-p}\nu_1^{p-1})$ and then find γ_{g+1} from ϕ_{g+1}. We shall see that there exists $N\in\boldsymbol{N}$ such that $I_{N+1}=O(\delta^{\varepsilon-p}\nu_1^{p-1})$ even if we take $\phi_{N+1}=\gamma_{N+1}=0$. Then, by the choice of τ, ν_j $(1\leqq j\leqq N)$, it is easy to see that I is of the form (2. 1).

(2. a) *Determination of* ϕ_1.

Now we consider I_1. We set $\delta^l\tau=\delta^{-1}\nu_1$ and $\delta^{pl}\tau^p=\delta^m\tau^{q-r}$. That is, we have

$$\tau=\delta^{\frac{m-pl}{p-q+r}}, \qquad d_0=\frac{pl-m}{p-q+r},$$

$$\nu_1=\tau\delta^{l+1}, \qquad d_1=d_0-(l+1).$$

From (1. 8), we can easily see that $d_0>d_1>0$. Here we have, from (1. 7) and (1. 9),

$$\delta^k\tau^q=O(\delta^m\tau^{q-r}),$$

$$\delta^{m(j,i)}\tau^i(\delta^l\tau)^{p-j}=O(\delta^{pl}\tau^p).$$

Then there exist a small constant $\varepsilon>0$ and $D_{1,j}(s,x,\delta)\in B^\infty(1\leqq j\leqq p)$ such that

$$I_1=\delta^{-p}\nu_1^p[\{\phi_{1,s}+s^l(1-C(0,x;\xi^0))\}^p-s^mB(0,x;\xi^0)+\delta^\varepsilon\sum_{j=1}^{p}D_{1,j}(s,x,\delta)\phi_{1,s}^{p-j}].$$

Let ϕ_1 be the solution of the Cauchy problem:

$$\begin{cases}\{\phi_{1,s}+s^l(1-C(0,x;\xi^0))\}^p-s^mB(0,x;\xi^0)+\delta^\varepsilon\sum_{j=1}^{p}D_{1,j}\phi_{1,s}^{p-j}=0,\\ \phi_1|_{s=1}=0.\end{cases}\tag{2.4}$$

We note that

$$\phi_{1,s}(1,x,0)=-1+C(0,x;\xi^0)+B(0,x;\xi^0)^{1/p},\tag{2.5}$$

$$\phi_{1,ss}(1,x,0)=-l+lC(0,x;\xi^0)+\frac{m}{p}B(0,x;\xi^0)^{1/p}.\tag{2.6}$$

We put $\beta_{j,1}(x,\delta)=\mathrm{Re}\phi_{j,s}(1,x,\delta)$ and $2\beta_{j,2}(x,\delta)=\mathrm{Re}\,\phi_{j,ss}(1,x,\delta)$, $(1\leqq j\leqq N)$. Then, from (1.10), it follows that

$$1+\beta_{1,1}(x,\delta)>0\tag{2.7}$$

by taking δ and ω small if necessary. Furthermore, since $B(0,0;\xi^0)$, $C(0,0;\xi^0)\neq 0$ from (1.10) and (1.11), we have

$$\phi_{1,s}+s^l(1-C(0,x,\xi^0))\neq 0\tag{2.8}$$

for $(s,x)\in T\times\omega$ and for small δ.

(2.b) *Determination of* γ_1.

Put $\delta=b_n=n^{-\rho}$, $\tau=\tau_n=b_n^{-d_0}$ and $\nu_1=\nu_{1,n}=b_n^{-d_1}$, where $n\in N$, $n\geqq n_0$, n_0 is sufficiently large and $\rho>0$ is a constant determined later. For $t\in(b_{n+1},b_{n-1})$, we define

$$G_{1,n}(t,x)=\frac{1}{l+1}\{\tau_n(t^{l+1}-b_n^{l+1})-\tau_{n+1}(t^{l+1}-b_{n+1}^{l+1})\}+\nu_{1,n}\,\mathrm{Re}\,\phi_1\Big(\frac{t}{b_n},x,b_n\Big)-\nu_{1,n+1}\,\mathrm{Re}\,\phi_1\Big(\frac{t}{b_{n+1}},x,b_{n+1}\Big).$$

Lemma 2.1. *We put* $m_n=(b_n+b_{n+1})/2$, $l_n=b_n-b_{n+1}$ *and* $I_{1,n}(x)=G_{1,n}(m_n,x)$. *Then we have*

$$I_{1,n}(x)\sim-(1+\beta_{1,1}(x,0))\rho n^{\rho d_1-1}.\tag{2.9}$$

Proof. Since $\left|\frac{t}{b_n}-1\right|$, $\left|\frac{t}{b_{n+1}}-1\right|\leqq\text{const.}\cdot n^{-1}$ for $t\in(b_{n+1}, b_{n-1})$ and for $n\geqq n_0$, we may take

$$\operatorname{Re}\phi_1(s, x, \delta)=\beta_{1,1}(x, 0)(s-1)+\beta_{1,2}(x, 0)(s-1)^2.$$

Then we have

$$\begin{aligned}I_{1,n}(x)&=\frac{1}{l+1}\left\{\tau_n(m_n-b_n)\sum_{j=0}^{l}m_n^j b_n^{l-j}-\tau_{n+1}(m_n-b_{n+1})\sum_{j=0}^{l}m_n^j b_{n+1}^{l-j}\right\}\\&\quad+\beta_{1,1}(x, 0)\left\{\frac{\nu_{1,n}}{b_n}(m_n-b_n)-\frac{\nu_{1,n+1}}{b_{n+1}}(m_n-b_{n+1})\right\}\\&\quad+\beta_{1,2}(x, 0)\left\{\frac{\nu_{1,n}}{b_n^2}(m_n-b_n)^2-\frac{\nu_{1,n-1}}{b_{n+1}^2}(m_n-b_{n+1})^2\right\}\\&=①+②+③.\end{aligned}$$

Since

$$①\sim-\rho n^{\rho d_1-1},$$

$$②=-\frac{1}{2}\beta_{1,1}(x, 0)\,l_n\left(\frac{\nu_{1,n}}{b_n}+\frac{\nu_{1,n+1}}{b_{n+1}}\right)-\beta_{1,1}(x, 0)\,\rho n^{\rho d_1-1},$$

$$|③|\leqq\frac{1}{4}|\beta_{1,2}(x, 0)|\,l_n^2\left(\frac{\nu_{1,n+1}}{b_{n+1}^2}-\frac{\nu_{1,n}}{b_n^2}\right)\leqq\text{const.}\cdot n^{\rho d_1-3},$$

we have (2.9). This completes the proof.

We define $\gamma_{1,n}(x)$ by

$$\gamma_{1,n}(x)=-\sum_{j=n_0}^{n-1}I_{1,j}(x)\sim d_1^{-1}(1+\beta_{1,1}(x, 0))\,n^{\rho d_1}.$$

It is easy to see that there exists $\gamma_1(x, \delta)\in B^\infty$ satisfying

$$\gamma_{1,n}(x)=\nu_{1,n}\gamma_1(x, b_n)=\tilde{\gamma}_1(x, b_n).$$

Note that, from the definition,

$$\gamma_{1,n+1}(x)-\gamma_{1,n}(x)=-I_{1,n}(x).$$

(2. c) *Determination of* ϕ_j, γ_j $(j\geqq 2)$.

In order to determine ϕ_2, we consider I_2. Since $I_1=0$, we have

$$\begin{aligned}I_2&=\delta^{-p}\sum_{h=1}^{p}\binom{p}{h}\nu_1^{p-h}\nu_2^h\{s^l(1-C)+\phi_{1,s}\}^{p-h}\phi_{2,s}^h\\&\quad+\sum_{\beta\neq0}\frac{1}{\beta!}\{\delta^k s^k A^{(\beta)}(\delta s, x; \tau\xi^0)-\delta^m s^m B^{(\beta)}(\delta s, x; \tau\xi^0)\}\\&\qquad\times\{\sqrt{-1}\,\nu_1(\gamma_{1,x}-\phi_{1,x})\}^\beta\end{aligned}$$

$$+\sum_{j=1}^{p}\sum_{i\leqq j}(\delta s)^{m(j,i)}\sum_{\beta\neq 0}\frac{1}{\beta!}B_{j,i}^{(\beta)}(\delta s, x;\tau\xi^0)$$
$$\times\{\sqrt{-1}\,\nu_1(\delta_{1,x}-\phi_{1,x})\}^{\beta}(\tau\delta^l s^l+\delta^{-1}\sum_{h=1}^{2}\nu_h\Lambda_{h,s})^{p-j}$$
$$+\sum_{j=1}^{p}\sum_{i\leqq j}(\delta s)^{m(j,i)}B_{j,i}(\delta s, x;\xi^0)$$
$$\times\sum_{h=1}^{p-j}\binom{p-j}{h}(\delta^{-1}\nu_1)^{p-j-h}(\delta^{-1}\nu_2)^h(s^l+\phi_{1,s})^{p-j-h}\phi_{2,s}^h$$
$$=①+②+③+④,$$

where

$$C=C(0, x, \xi^0),$$
$$A^{(\beta)}(t, x;\xi)=\partial_\xi^\beta A(t, x;\xi),$$
$$\{\sqrt{-1}\,\nu_1(\gamma_{1,x}-\phi_{1,x})\}^\beta=\prod_{h=1}^{d}\{\sqrt{-1}\,\nu_1(\gamma_{1,x_h}-\phi_{1,x_h})\}^{\beta_h}$$

and so on.

Here, by a direct calculation, we have

$$①=\delta^{-p}\nu_1^{p-1}\nu_2\sum_{h=1}^{p}\binom{p}{h}(\nu_2/\nu_1)^{h-1}\{\phi_{1,s}+s^l(1-C)\}^{p-h}\phi_{2,s}^h,$$
$$②=\delta^m\tau^{q-r-1}\nu_1 D_2(s, x, \delta),$$
$$③=\delta^{\varepsilon-p}\nu_1^{p+1}\tau^{-1}\sum_{h=0}^{p-1}(\nu_2/\nu_1)^h D_{2,h}(s, x, \delta)\,\phi_{2,s}^h,$$
$$④=\delta^{\varepsilon-p}\nu_1^p\sum_{h=1}^{p-1}(\nu_2/\nu_1)^h D'_{2,h}(s, x, \delta)\,\phi_{2,s}^h,$$

where D_2, $D_{2,h}$, $D'_{2,h}\in B^\infty$.

If $\delta^m\tau^{q-r-1}\nu_1=0(\delta^{-p}\nu_1^{p-1})$, we take $\phi_j=0$ for $j\geqq 2$. Then the remainder term ② is absorbed in C_1 of (2.2). If not so, we set $\delta^{-p}\nu_1^{p-1}\nu_2=\delta^m\tau^{q-r-1}\nu_2$, which implies $\nu_2=\tau^{-1}\nu_1^2=\delta^{d_0-2d_1}$ and $d_2=2d_1-d_0=d_1+(d_1-d_0)<d_1$. Then we have

$$I_2=\delta^{-p}\nu_1^{p-1}\nu_2[p\{\phi_{1,s}+s^l(1-C)\}^{p-1}\phi_{2,s}$$
$$+\sum_{h=2}^{p}\binom{p}{h}\delta^{(d_0-d_1)(h-1)}\{\phi_{1,s}+s^l(1-C)\}^{p-h}\phi_{2,s}^h$$
$$+D_2(s, x, \delta)$$
$$+\delta^\varepsilon\sum_{h=0}^{p-1}\delta^{(d_0-d_1)h}D_{2,h}(s, x, \delta)\,\phi_{2,s}^h$$

$$+\delta^{\varepsilon}\sum_{h=1}^{p-1}\delta^{(d_0-d_1)(h-1)}D'_{2,h}(s, x, \delta)\phi_{2,s}^{h}].$$

We construct ϕ_2 so that $I_2=O(\delta^{\varepsilon-p}\nu_1^{p-1})$ and $\phi_2|_{s=1}=0$. Such ϕ_2 really exists. In fact, if we take $\varepsilon>0$ sufficiently small and take ϕ_2 in the form:

$$\begin{cases}\phi_2(s, x, \delta)=\sum_{j=0}^{N_2}\delta^{\varepsilon j}\phi_{2,j}(s, x, \delta), & \text{for some } N_2\in N,\\ \phi_2|_{s=1}=0,\end{cases}$$

then $\phi_{2,j}$ are determined successively. Here we use (2.9).

Now we construct γ_2 from ϕ_2 in the same way as we have constructed γ_1 from ϕ_1. We define for $t\in(b_{n+1}, b_{n-1})$

$$G_{2,n}(t, x)=\nu_{2,n}\operatorname{Re}\phi_2\Big(\frac{t}{b_n}, x, b_n\Big)-\nu_{2,n+1}\operatorname{Re}\phi_2\Big(\frac{t}{b_{n+1}}, x, b_{n+1}\Big),$$

$$I_{2,n}(x)=G_{2,n}(m_n, x).$$

Then we have

Lemma 2.2. $I_{2,n}(x)\sim-\beta_{2,1}(x, 0)\rho n^{\rho d_2-1}$.

Since this lemma can be proved in the same way as Lemma 2.1, we omit the proof.

We define $\gamma_{2,n}(x)$ by

$$\gamma_{2,n}(x)=-\sum_{j=n_0}^{n-1}I_{2,n}(x)\sim d_2^{-1}\beta_{2,1}(x, 0)n^{\rho d_2}.$$

Then it is easy to see that there exists $\gamma_2(x, \delta)\in B^{\infty}$ satisfying

$$\gamma_{2,n}(x)=\nu_{2,n}\gamma_2(x, b_n)=\tilde{\gamma}_2(x, b_n).$$

Suppose we have constructed ϕ_j and γ_j $(1\leqq j\leqq g)$. Then $I_g=O(\delta^{\varepsilon-p}\nu_1^{p-1})$ and we have

$$I_{g+1}=\delta^{-p}\sum_{j=1}^{p}\binom{p}{j}\{\nu_1 s^l(1-C)+\sum_{h=1}^{g}\nu_h\phi_{h,s}\}^{p-j}\nu_{g+1}^{j}\phi_{g+1,s}^{j}$$

$$+\sum_{\beta\neq0}\frac{1}{\beta!}\{\delta^k s^k A^{(\beta)}(\delta s, x; \tau\xi^0+\sqrt{-1}\sum_{h=1}^{g-1}\nu_h(\gamma_{h,x}-\phi_{h,x}))$$

$$-\delta^m s^m B^{(\beta)}(\delta s, x; \tau\xi^0+\sqrt{-1}\sum_{h=1}^{g-1}\nu_h(\gamma_{h,x}-\phi_{h,x}))\}$$
$$\times\{\sqrt{-1}\,\nu_g(\gamma_{g,x}-\phi_{g,x})\}^\beta$$
$$+\sum_{j=1}^{p}\sum_{i\leqq j}(\delta s)^{m(j,i)}\sum_{\beta\neq 0}\frac{1}{\beta!}B_{j,i}^{(\beta)}(\delta s, x; \tau\xi^0+\sqrt{-1}\sum_{h=1}^{g-1}\nu_h(\gamma_{h,x}-\phi_{h,x}))$$
$$\times\{\sqrt{-1}\,\nu_g(\gamma_{g,x}-\phi_{g,x})\}^\beta\delta^{j-p}(\nu_1 s^l+\sum_{h=1}^{g+1}\nu_h\phi_{h,s})^{p-j}$$
$$+\sum_{j=1}^{p}\sum_{i\leqq j}(\delta s)^{m(j,i)}\delta^{j-p}B_{j,i}(\delta s, x; \tau\xi^0+\sqrt{-1}\sum_{h=1}^{g-1}\nu_h(\gamma_{h,x}-\phi_{h,x}))$$
$$\times\sum_{h=1}^{p-j}\binom{p-j}{h}(\nu_1 s^l+\sum_{h=1}^{g-1}\nu_h\phi_{h,s})^{p-j-h}\nu_{g+1}^h\phi_{g+1,s}^h$$
$$+O(\delta^{\varepsilon-p}\nu_1^{p-1})$$
$$=①+②+③+④+⑤.$$

By a direct calculation, we have

$$①=\delta^{-p}\,\nu_1^{p-1}\nu_{g+1}\sum_{j=1}^{p}\binom{p}{j}\left\{s^l(1-C)+\sum_{h=1}^{g}\frac{\nu_h}{\nu_1}\phi_{h,s}\right\}^{p-j}\left(\frac{\nu_{g+1}}{\nu_1}\right)^{j-1}\phi_{g+1,s}^j,$$
$$②=\delta^m\tau^{q-r-1}\nu_g D_{g+1}(s, x, \delta),$$
$$③=\delta^{\varepsilon-p}\tau^{-1}\nu_1^p\nu_g\sum_{h=0}^{p-1}\left(\frac{\nu_{g+1}}{\nu_1}\right)^h D_{g+1,h}(s, x, \delta)\,\phi_{g+1,s}^h,$$
$$④=\delta^{\varepsilon-p}\nu_1^p\sum_{h=1}^{p-1}\left(\frac{\nu_{g+1}}{\nu_1}\right)^h D'_{g+1,h}(s, x, \delta)\,\phi_{g+1,s}^h,$$

where $D_{g+1}, D_{g+1,h}, D'_{g+1,h}\in B^\infty$.

If $\delta^m\tau^{q-r-1}\nu_g=O(\delta^{-p}\nu_1^{p-1})$, we take $\phi_j=\gamma_j=0$ for $j\geqq g+1$. Then the remainder term ② is absorbed in C_1 of (2.2). If not so, we set $\delta^{-p}\nu_1^{p-1}\nu_{g+1}=\delta^m\tau^{q-r-1}\nu_g$, which means $\nu_{g+1}=\tau^{-1}\nu_1\nu_g$ and $d_{g+1}=d_g-(d_0-d_1)$ $<d_g$. Consequently we have

$$I_{g+1}=\delta^{-p}\nu_1^{p-1}\nu_{g+1}\Big[p\left\{s^l(1-C)+\sum_{h=1}^{g}\frac{\nu_h}{\nu_1}\phi_{h,s}\right\}^{p-1}\phi_{g+1,s}$$
$$+\sum_{j=2}^{p}\binom{p}{j}\left\{s^l(1-C)+\sum_{h=1}^{g}\frac{\nu_h}{\nu_1}\phi_{h,s}\right\}^{p-j}\left(\frac{\nu_{g+1}}{\nu_1}\right)^{j-1}\phi_{g+1,s}^j$$
$$+D_{g+1}(s, x, \delta)$$
$$+\delta^\varepsilon\sum_{h=0}^{p-1}\left(\frac{\nu_{g+1}}{\nu_1}\right)^h D_{g+1,h}(s, x, \delta)\,\phi_{g+1,s}^h$$
$$+\delta^\varepsilon\sum_{h=1}^{p-1}\left(\frac{\nu_{g+1}}{\nu_1}\right)^{h-1}D_{g+1,h}(s, x, \delta)\,\phi_{g+1}^h\Big]$$
$$+O(\delta^{\varepsilon-p}\nu_1^{p-1}).$$

We take ϕ_{g+1} so that $I_{g+1}=O(\delta^{\varepsilon-p}\nu_1^{p-1})$ and $\phi_{g+1}|_{s=1}=0$. By the same way as before, we can construct such ϕ_{g+1}. We can also construct γ_{g+1} from ϕ_{g+1} as before.

Since $d_g=d_{g-1}-(d_0-d_1)=\cdots=d_0-g(d_0-d_1)=d_0-g(l+1)$, there exists $N\in \boldsymbol{N}$ such that $I_{N+1}=O(\delta^{\varepsilon-p}\nu_1^{p-1})$ even if we take $\phi_{N+1}=\gamma_{N+1}=0$.

It is easy to see that the remainder terms of I are absorbed in the term $\delta^\varepsilon L_1 w$ of (2.1) by taking ε small if necessary. We remark that, from (2.8),

$$C_0(s, x, \delta)=p\left\{s^l(1-C)+\sum_{h=1}^{N}\frac{\nu_h}{\nu_1}\phi_{h,s}\right\}^{p-1}\neq 0$$

for sufficiently small δ, which implies (2.3).

(2.d) *Transport equations.*

Now we consider the formal solution of the equation $L_0 w+\delta^\varepsilon L_1 w=0$. Let w_j $(j\geqq 0)$ be the solutions of the following equations :

$$(2.10)\qquad \begin{cases} L_0 w_0=0, \\ w_0|_{s=1}=1, \end{cases}\qquad \begin{cases} L_0 w_j=-L_1 w_{j-1}, \\ w_j|_{s=1}=0. \end{cases}\quad (j\geqq 1)$$

It is easy to see that there exists a function $g=g(s, x, \delta, \eta)$ such that for any $(j, \alpha)\in \boldsymbol{N}^{d+1}$, $K\in \boldsymbol{N}$, there exists $C=C_{j,\alpha,K}>0$ satisfying

$$(2.11)\qquad |\partial_s^j D_x^\alpha(g-\sum_{i=0}^{K-1}\eta^i w_i)|\leqq C|\eta|^K.$$

Then $w=w(s, x, \delta)=g(s, x, \delta, \delta^\varepsilon)$ is the desired solution.

We set $w_n(s, x)=w(s, x, b_n)$ and $\gamma_n(x)=\gamma(x, b_n)$ and we define in U_n,

$$v_n(t, x)=\exp\left\{\sqrt{-1}\,\tau_n\left(\xi^0 x-\frac{\sqrt{-1}}{l+1}(t^{l+1}-b_n^{l+1})\right)\right\}\exp\left\{\phi\left(\frac{t}{b_n}, x, b_n\right)\right\}$$
$$\times\exp\{-\gamma_n(x)\}\,w_n\left(\frac{t}{b_n}, x\right).$$

By the argument above, we have

Proposition 2.3. *In U_n, we define $f_n=Pv_n/v_n$. Then, for any $(j, \alpha)\in \boldsymbol{N}^{d+1}$, there exists $a>0$ such that for any $K\in \boldsymbol{N}$ there exists $C=C_{j,\alpha,K}>0$ satisfying*

$$(2.12)\qquad |\partial_t^j D_x^\alpha f_n|\leqq Cn^{-aK}$$

for $(t, x)\in U_n$ and for $n\geqq n_0$.

(2. e) *The set where* $|v_n| = |v_{n+1}|$.

In U_n, we put $F_n(t, x) = \mathrm{Log}\left|\frac{v_n}{v_{n+1}}\right|$. Then we have

Proposition 2. 4. *Suppose* $\rho > 2/d_1$. *Then there exist* $C > 0$ *and* $\sigma > \rho + 1$ *such that*

$$\frac{\partial F_n}{\partial t} \geqq C n^{\sigma} \tag{2. 13}$$

for $(t, x) \in U_n$ *and for* $n \geqq n_0$.

Proof. As before, we may take

$$\mathrm{Re}\phi(s, x, \delta) = \sum_{j=1}^{N} \nu_j \{\beta_{j,1}(x, 0)(s-1) + \beta_{j,2}(x, 0)(s-1)^2\}.$$

Then it follows that

$$\begin{aligned}\frac{\partial F_n}{\partial t} &= t^l(\tau_n - \tau_{n+1}) + \sum_{j=1}^{N} \beta_{j,1}(x, 0)\left(\frac{\nu_{j,n}}{b_n} - \frac{\nu_{j,n+1}}{b_{n+1}}\right) \\ &\quad + \sum_{j=1}^{N} 2\beta_{j,2}(x, 0)\left\{\frac{\nu_{j,n}}{b_n^2}(t - b_n) - \frac{\nu_{j,n+1}}{b_{n+1}^2}(t - b_{n+1})\right\} \\ &\quad + \left(\frac{w_{n,s}}{w_n} b_n^{-1} - \frac{w_{n+1,s}}{w_{n+1}} b_{n+1}^{-1}\right) \\ &= ① + ② + ③ + ④.\end{aligned}$$

Here

$$\begin{aligned}① &\sim -\rho d_0 n^{\rho(d_1+1)-1}, \\ ② &\sim -\sum_{j=1}^{N} \beta_{j,1}(x, 0)\rho(d_j + 1) n^{(\rho d_j+1)-1} \\ &\sim -\beta_{1,1}(x, 0)\rho(d_1 + 1) n^{\rho(d_1+1)-1}, \\ ③ &= -\sum_{j=1}^{N} 2\beta_{j,2}(x, 0)\left\{\frac{\nu_{j,n+1}}{b_{n+1}^2}(b_n - b_{n+1}) + (t - b_n)\left(\frac{\nu_{j,n+1}}{b_{n+1}^2} - \frac{\nu_{j,n}}{b_n^2}\right)\right\} \\ &\sim -\sum_{j=1}^{N} 2\beta_{j,2}(x, 0)\{\rho n^{\rho(d_j+1)-1} + O(n^{\rho(d_j+1)-2})\} \\ &\sim -2\beta_{1,2}(x, 0) n^{\rho(d_1+1)-1}, \\ |④| &\leqq \text{const.} \cdot n^{\rho}.\end{aligned}$$

If we take $\rho > 2/d_1$, we have $\rho(d_1 + 1) - 1 > \rho + 1$. Hence, if we put $\sigma = \rho(d_1 + 1) - 1 > \rho + 1$, we have

$$\frac{\partial F_n}{\partial t} \sim -\rho\{d_0 + \beta_{1,1}(x, 0)(d_1 + 1) + 2\beta_{1,2}(x, 0)\} n^{\sigma}.$$

From assumption (1. 11) and (2. 5), (2. 6), it follows that

$$\begin{aligned} &-\{d_0+\beta_{1,1}(0,0)(d_1+1)+2\beta_{1,2}(0,0)\} \\ &=-\frac{pl-m}{p-q+r}\{p \operatorname{Re} C(0,0;\xi^0)+(q-r)\operatorname{Re} B(0,0;\xi^0)\} \\ &>0. \end{aligned}$$

Hence, if we take ω sufficiently small, there exists $C>0$ such that

$$\frac{\partial F_n}{\partial t}\geqq Cn^\alpha \qquad \text{in } U_n.$$

This completes the proof.

By virtue of (2. 13) and the implicit function theorem, there exists $m_n(x)\in C^\infty(\omega)$ satisfying

$$F_n(m_n(x),\ x)=0. \tag{2.14}$$

Lemma 2. 5. *For sufficiently large* n, $m_n(x)\in(b_{n+1}, b_{n-1})$.

Proof. There exists $m_n^*(x)$ for each $x\in\omega$, satisfying

$$m_n(x)-m_n=\frac{F_n(m_n(x),x)-F_n(m_n,x)}{\dfrac{\partial F_n}{\partial t}(m_n^*(x),x)}.$$

From the choice of γ_n, it follows that $F_n(m_n, x)=O(1)$. Hence, from (2. 13) and (2. 14), we have

$$|m_n(x)-m_n|\leqq \text{const.}\cdot n^{-\sigma}. \tag{2.15}$$

Since $\sigma>\rho+1$, $m_n(x)\in(b_{n+1}, b_{n-1})$ for sufficiently large n. This completes the proof.

(2. f) *Modification of* v_n.

We set

$$\begin{aligned} u_n(t,x)=\exp\left\{\sqrt{-1}\,\tau_n\left(\xi^0x-\frac{\sqrt{-1}}{l+1}(t^{l+1}-b_n^{l+1})\right)\right\}\exp\left\{\phi\left(\frac{t}{b_n}, x, b_n\right)\right\} \\ \times\exp\{-\gamma_n(x)\}\left\{w_n\left(\frac{t}{b_n}, x\right)+z_n\left(\frac{t}{b_n}, x\right)\right\}. \end{aligned}$$

As in the proof of Proposition 3. 5 of [7] or in the proof of fundamental lemma of [1], we have the following proposition.

Proposition 2.6. *Put* $S_n=\{t=m_n(x)\}$. *Then there exists* $z_n\in B^\infty$ *such that*

(i) $\tilde{F}_n=\mathrm{Log}\left|\dfrac{u_n}{u_{n+1}}\right|$ *satisfies* (2.13) *and* (2.14),

(ii) $g_n=Pu_n/u_n$ *satisfies* (2.12),

(iii) g_n *is flat on* S_n *and on* S_{n-1}.

Remark 2.1. In order to construct z_n, we use *Whitney's extension theorem with estimates*, which can be easily proved (see [1] or [7]).

(2.g) *Smoothness of* u *and* f.

Let $\chi\in C_0^\infty(\boldsymbol{R})$ be a function satisfying

$$\chi(s)=1 \qquad \text{for } |s|\leqq 3/4, \qquad \mathrm{supp}\ \chi\subset[-1,1],$$

and put $\chi_n(t)=\chi\left(\dfrac{t-b_n}{l_n}\right)$.

We define the desired functions u and f by

$$u(t,x)=\begin{cases}\sum_{n\geqq n_0}\chi_n(t)u_n(t,x) & t>0,\\ 0 & t\leqq 0,\end{cases} \tag{2.16}$$

$$f(t,x)=\begin{cases}Pu/u & t>0,\\ 0 & t\leqq 0.\end{cases} \tag{2.17}$$

First we show the smoothness of f. Note that, from the above definition, f may fail to be smooth only on $t=0$ or on S_n.

(2.g.1) For $t\in\left[b_n-\frac{3}{4}l_n,\ b_{n+1}+\frac{3}{4}l_{n+1}\right]$, we have

$$f=\frac{P(u_n+u_{n+1})}{u_n+u_{n+1}}=\frac{g_nu_n+g_{n+1}u_{n+1}}{u_n+u_{n+1}}.$$

From (2.15), we can easily see that $m_n(x)\in\left[b_n-\frac{3}{4}l_n, b_{n+1}+\frac{3}{4}l_{n+1}\right]$. If $t\geqq m_n(x)$, $|u_n|\geqq|u_{n+1}|$ and

$$|u_n+u_{n+1}|\geqq|u_n|\left(1-\left|\frac{u_{n+1}}{u_n}\right|\right)=|u_n|(1-e^{-\tilde{F}_n}).$$

Using the inequality:

$$1-e^{-x}\geqq\begin{cases}x/4 & \text{for } x\in[0,2],\\ 1/2 & \text{for } x\geqq 1,\end{cases}$$

and (2. 13), we have

$$|u_n+u_{n+1}|\geqq\begin{cases}Cn^{\sigma}(t-m_n(x))\,|u_n| & \text{if } \tilde{F}_n\in[0,2],\\ |u_n|/2 & \text{if } \tilde{F}_n\geqq 1.\end{cases}$$

Since g_n and g_{n+1} are flat on $t=m_n(x)$, this implies that f is C^∞ on S_n and that

$$|f|\leqq\max\left\{\frac{|g_n|+|g_{n+1}|}{Cn^{\sigma}|t-m_n(x)|},\ 2(|g_n|+|g_{n+1}|)\right\}. \tag{2. 18}$$

The same holds for $t\leqq m_n(x)$ and for the derivatives of f.

(2. g. 2) For $t\in\left[b_{n+1},\ b_n-\frac{3}{4}l_n\right]$, we have

$$f=\frac{P(u_{n+1}+\chi_n u_n)}{u_{n+1}+\chi_n u_n}.$$

From (2. 15), it follows that

$$t-m_n(x)=t-m_n+m_n-m_n(x)\leqq l_n/4+O(n^{-\sigma}).$$

Then, from (2. 13), we have

$$\begin{aligned}\left|\frac{u_n}{u_{n+1}}\right|&=\exp(\tilde{F}_n)\\ &=\exp\{(t-m_n(x))\,\partial_t\tilde{F}_n(m_n^*(x),x)\}\\ &\leqq\exp(-Cn^{\sigma-\rho-1}).\end{aligned} \tag{2. 19}$$

This implies, for large n,

$$|u_{n+1}+\chi_n u_n|\geqq|u_{n+1}|/2\,.$$

Hence we conclude that

$$\begin{aligned}|f|&\leqq\frac{2(|g_{n+1}u_{n+1}|+|[P,\chi_n]u_n|+\chi_n|g_n u_n|}{|u_{n+1}|}\\ &\leqq 2\{|g_n|+|g_{n+1}|+O(n^M)\exp(-Cn^{\sigma-\rho-1})\}.\end{aligned} \tag{2. 20}$$

The same holds for the derivatives of f.

(2. g. 3) For $t\in\left[b_{n+1}+\frac{3}{4}l_{n+1},\ b_n\right]$, we get (2. 20) by the same argument as in (2. g. 2).

From the above argument, it follows that f is C^∞ near the origin. Now we show the smoothness of u. From the definition of $\gamma_n(x)$ and (2.7), we have, for some

$$\gamma_n(x) \geqq \text{const.} \cdot n^{\rho d_1}.$$

On the other hand, we have, for $t \in [b_{n+1}, b_{n-1}]$,

$$\tau_n |t^{l+1} - b_n^{l+1}| \leqq \text{const.} \cdot n^{\rho d_1 - 1},$$

$$\left|\phi\left(\frac{t}{b_n}, x, b_n\right)\right| \leqq \text{const.} \cdot \nu_{1,n} \left|\frac{t}{b_n} - 1\right| \leqq \text{const.} \cdot n^{\rho d_1 - 1}.$$

From these, we can easily see that u is C^∞ at $t=0$. This completes the proof of Theorem 1.2.

Remark 2.2. If $p=2$ or if the coefficients of P are independent of x, we can take $\phi_j = \gamma_j = 0$ for $j \geqq 2$. Especially in the latter case, we can take u_n in the form:

$$u_n(t, x) = \exp\left\{\sqrt{-1}\, \tau_n\left(\xi^0 x - \frac{\sqrt{-1}}{l+1}(t^{l+1} - b_n^{l+1})\right)\right\} \exp\left\{\phi\left(\frac{t}{b_n}, b_n\right)\right\}$$
$$\times \exp(-\gamma_n) \left\{w_n\left(\frac{t}{b_n}\right) + z_n\left(\frac{t}{b_n}\right)\right\}.$$

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