A Measure of Integrity for Local Analytic Algebras

By

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§ 0. Introduction

We will complete the study [I_1], [I_2] on orders of elements of a local analytic algebra. Let $(X, \mathcal{O}_X)$ be a complex space and $(\mathcal{O}, \mathfrak{m})$ $:=(\mathcal{O}_{X,\xi}, \mathfrak{m}_\xi)$ its local algebra at $\xi \in X$. $\mathcal{O}$ can be expressed as $\mathcal{O} = C\{x\}/I$ for some ideal I of the algebra $C\{x\}$ of convergent power series in $x=(x_1, \cdots, x_m)$. We define three kinds of orders for $f \in \mathcal{O}$.

algebraic order: $\nu(f) = \nu_\xi(f) := \sup\{p: f \in \mathfrak{m}^p\} = \sup\{$the degree of the lowest non-zero homogeneous term of $\hat{f}$: $\hat{f}$ is a representative of f in $C\{x\}\}$

reduced order ([Re_1], cf. [S]): $\bar{\nu}(f) := \lim_{k \to \infty} \nu(f^k)/k$

analytic order along $A \subset |X|$: $\mu_{A,\xi}(f) := \sup\{p:$ there exist $\alpha > 0$, neighbourhood U of ξ and representative $\tilde{f}$ of f over U such that $|\tilde{f}(x)| \leq \alpha |x-\xi|^p$ $(x \in U \cap A)\}$.

We can easily verify the following.

inequality for the reduced order:

$$(\mathrm{I}_0) \qquad \bar{\nu}(f) \geq \nu(f) \qquad (f \in \mathcal{O}).$$

inequality for products:

$$(\mathrm{I}_1) \qquad \nu(fg) \geq \nu(f) + \nu(g) \qquad (f, g \in \mathcal{O}).$$

inequality for pullbacks: If $\Phi: Y \to X$ is a morphism such that $\Phi(\eta) = \xi$,

$$(\mathrm{I}_2) \qquad \nu_\eta(f \circ \Phi) \geq \nu_\xi(f) \qquad (f \in \mathcal{O}).$$

inequality for the analytic order:

$$(\mathrm{I}_3) \qquad \mu_{A,\xi}(f) \geq \nu(f) \qquad (f \in \mathcal{O}).$$

We are interested in linear complementary inequalities (LCI) of

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these (I_i). By the strong valuation theorem of Rees [Re_3] on Noetherian rings, we see the following:

    If X is reduced at ξ, there exists $b_0 = b_0(\mathcal{O}) \geqq 0$ such that

$$(CI_0) \qquad \bar{\nu}(f) \leqq \nu(f) + b_0 \qquad (f \in \mathcal{O}).$$

We prove the following LCI as our main theorem, (3.4), assuming that X is reduced and irreducible at ξ i.e. $\mathcal{O}$ is an integral domain.

(1)   *There exist $a_1 = a_1(\mathcal{O}) \geqq 1$, $b_1 = b_1(\mathcal{O}) \geqq 0$ such that*

$$(CI_1) \qquad \nu(fg) \leqq a_1(\nu(f) + \nu(g)) + b_1 \qquad (f, g \in \mathcal{O}).$$

(2)   *For any morphism $\Phi: (Y, \eta) \to (X, \xi)$ with $\dim X_\xi = \mathrm{grnk}_\eta \Phi$ (see below), there exist $a_2 = a_2(\Phi) \geqq 1$, $b_2 = b_2(\Phi) \geqq 0$ such that*

$$(CI_2) \qquad \nu_\eta(f \circ \Phi) \leqq a_2 \nu_\xi(f) + b_2 \qquad (f \in \mathcal{O}).$$

(3)   *For any open subanalytic set $A \subset |X|$ adherent to ξ, there exist $a_3 = a_3(A) \geqq 1$, $b_3 = b_3(A) \geqq 0$ such that*

$$(CI_3) \qquad \mu_{A,\xi}(f) \leqq a_3 \nu(f) + b_3 \qquad (f \in \mathcal{O}).$$

In the above $\mathrm{grnk}_\eta \Phi$ denotes the *generic rank* defined for a morphism of germs of analytic spaces over $k = \boldsymbol{C}$ or $\boldsymbol{R}$:

    $\mathrm{grnk}_\eta \Phi = \varepsilon \cdot \inf \{$topological dimension of $\Phi(U): U$ is a neighbourhood of $\eta\}$ ($\varepsilon = 1/2$ if $k = \boldsymbol{C}$; $\varepsilon = 1$ if $k = \boldsymbol{R}$).

This has been used to study properties of a ring homomorphism induced by a morphism between spaces (see [Be]).

The constants a_i (or b_i) in the above LCI measure some intrinsic properties of $\mathcal{O}$, Φ and A as follows. If $\mathcal{O}$ is not an integral domain, a_1 does not exist ($a_1 = \infty$). So we may consider $\inf a_1 = \inf\{a_1:$ there exists b_1 such that (CI_1) holds$\}$ as a *measure of integrity* of the algebra $\mathcal{O}$ (or $1/\inf a_1$ as a distance of $\mathcal{O}$ from non-integral domains). $\inf a_2$ can be considered as a kind of order of the morphism Φ. Indeed, in case $(X, \xi) = (\boldsymbol{C}, 0)$, we see that $\inf a_2$ coincides with the reduced order, $\bar{\nu}(\Phi)$, of the function Φ. $\inf a_3$ seems to have a relation with the size of the germ A_ξ. Of course, if $A \subset B$, $\inf a_3(A) \geqq \inf a_3(B)$.

We have already proved that (1) and (2) of the main theorem are equivalent ([I_1], (1.2)). So we prove (1) by induction on $n := \dim X_\xi$. (The proof of (3) is postponed till the end of § 4.) In § 1 we recall the valuation theorem in a special form we need later. It implies the case $n = 1$ of (1) immediately. The case $n = 2$ can not be reduced to the case $n = 1$. We can manage to prove this case by

the use of the intersection matrix of Du Val–Mumford for a normal surface (§2). In §3 we reduce the case $n(\geq 3)$ to the case $n-1$ using a Bertini type theorem of Flenner, completing the proof of (1). In §4 we show a real analytic version of the main theorem. There we remark that Risler's criterion for real ideal (=real radical) gives an affirmative answer to Malgrange's conjecture ([Ma], p. 124) on the real analytic set associated to a complex analytic set. In §5, as an application of the main theorem, we treat "near solutions" of the algebraic equation $f^p u - g^p = 0$ in $C[[x]]$ for a given $u \in C\{x\}$ and a prime number p. This yields an intrinsic measure of nearness of u to p-th powers in $C\{x\}$ In Appendix we calculate the explicit values of inf a_1 and inf b_1 for some singularities.

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§1. The Valuation Theorem

Here we introduce the most basic result on orders on an analytic algebra, restricting ourselves to the case of the orders with respect to the maximal ideal. Let X be a positive dimensional complex space reduced at ξ and let $\mathcal{M}$ denote the class of all morphisms $\Pi: Y \to X$ such that Y are normal, Π induce proper morphisms onto neighbourhoods of ξ and the ideal sheaves defining $\Pi^{-1}(\xi)$ (with the canonical structure) are invertible. $\mathcal{M}$ is not empty; the composition of the normalization after the blowing-up with center ξ belongs to $\mathcal{M}$. If $\Pi \in \mathcal{M}$, the exceptional fiber E over ξ (=the reduction of $\Pi^{-1}(\xi)$) has a finite number of irreducible components $\{E_i\}_{i \in A(\Pi)}$. Let $\mathcal{I}$ and $\mathcal{I}_i$ denote the sheaves of ideals defining E and E_i respectively. By the normality and the Hauptidealsatz (cf. [G–R], p. 129, p. 223), there exists an complex analytic set $S \subset |Y|$ of codimension at least 2 such that $Y - S (= Y|_{|Y|-S})$ is a smooth manifold and $E - S$ and $E_i - S$ are submanifolds of $Y - S$ of codimension 1. For $f \in \Gamma(U, \mathcal{O}_X)$, we put

$$v_i(f) := \sup\{p: (f \circ \Pi)_y \in \mathcal{I}_i^p \text{ for all (some) } y \in |E_i| - S\}.$$

If $f \in \mathcal{O} := \mathcal{O}_{X,\xi}$, we define $v_i(f)$ using its representative. It is obvious that $v_i(fg) = v_i(f) + v_i(g)$ $(f, g \in \mathcal{O})$.

(1.1) Lemma. ([L-T],(4.1.8)). *If we put* $v_i(\mathfrak{m}) = \inf\{v_i(g) : g \in \mathfrak{m}\}$
$(\mathfrak{m} := m_\xi)$, *we have*

$$\inf\{v_i(f)/v_i(\mathfrak{m}) : i \in A(\varPi)\} = \sup\{p/q : (f \circ \varPi)^q \in \varGamma(U, \mathscr{I}^p)\}.$$

(1.2) Lemma. ([L-T], (4.1.6), (5.5), (7.2) cf. (1.12)). *Let X be a positive dimensional complex space reduced at ξ. Then, for any $f \in \varGamma(X, \mathcal{O}_X)$, $\bar{\nu}(f)$ is a rational number and the following conditions are equivalent for any p, $q \in N$.*

(r) $\bar{\nu}(f) \geqq p/q.$

(i)   *There exist $\sigma_i \in \mathcal{O}$ $(1 \leqq i \leqq k)$ such that $\nu(\sigma_i) \geqq ip/q$ and*

$$f^k - \sigma_1 f^{k-1} + \sigma_2 f^{k-2} - \cdots \pm \sigma_k = 0 \quad in \ \mathcal{O}.$$

(c)   *Let D denote the unit open disc in the complex plane $\mathbf{C}$. Then for any morphism $\varPhi : D \to X$ with $\varPhi(0) = \xi$, we have*

$$\nu_0(f \circ \varPhi) \geqq (p/q) \inf\{\nu_0(g \circ \varPhi) : g \in \mathfrak{m}\}.$$

(m)   *For any $\varPi : Y \to X$ in $\mathscr{M}$,*

$$\inf\{v_i(f)/v_i(\mathfrak{m}) : i \in A(\varPi)\} \geqq p/q.$$

(m′)   *There exists $\varPi : Y \to X$ in $\mathscr{M}$ such that*

$$\inf\{v_i(f)/v_i(\mathfrak{m}) : i \in A(\varPi)\} \geqq p/q.$$

(a) $\mu_{|X|,\xi}(f) \geqq p/q.$

(1.3) *Remark.* This theorem was proved in an algebraic form by Rees [Re$_1$], [Re$_2$] and Nagata [N$_1$] independently, developing Samuel's paper [S]. They treated more general rings. The conditions (c) and (a) were considered first by Lejeune and Teissier. They stated the result for general coherent sheaves on complex spaces.

(1.4) *Remark.* By (m) and (m′) we see that $\bar{\nu}(f) = \inf\{v_i(f)/v_i(\mathfrak{m}) : i \in A(\varPi)\}$ for any $\varPi \in \mathscr{M}$.

§2. LCI for Normal Surfaces

Here we use the notation in §1.

(2.1) Theorem. *If X is a 2-dimensional complex space normal at ξ, $\varPi : Y \to X$ a morphism in $\mathscr{M}$ and if Y is nonsingular, we have the following.*
  (i) *There exist $c_{ij} > 0$ $(c_{ii} = 1)$ such that $c_{ij}v_j(f) \geqq v_i(f)$ $(i, j \in A(\varPi))$.*

(ii) *If we put* $a_1 = (1 + \max\limits_{i,j} c_{ij} v_j(\mathfrak{m})/v_i(\mathfrak{m}))/2$, *we have* $\bar{\nu}(fg) \leqq$ $a_1(\bar{\nu}(f) + \bar{\nu}(g))$ *and hence* $\nu(fg) \leqq a_1(\nu(f) + \nu(g)) + 2a_1 b_0$ $(f, g \in \mathcal{O})$ *by* (CI_0).

Proof. (i) Let $A(\Pi) = \{E_1, \cdots, E_k\}$ and let S_f denote the strict transform of the divisor (f) of $f \in \mathcal{O}$ with respect to Π. Then no E_i appears in S_f and $(f \circ \Pi) = S_f + \sum m_i E_i$ $(m_i = v_i(f))$. Taking intersection number with E_j around E (cf. [Mu]), we have

$$\sum_i m_i e_{ij} = (f \circ \Pi) \cdot E_j - S_f \cdot E_j = -S_f \cdot E_j \leqq 0 \quad (j = 1, \cdots, k; \; e_{ij} = E_i \cdot E_j).$$

Hence the vector $(m_1, \cdots, m_k)$ is contained in the convex cone $K \subset \mathbf{R}^k$ defined by

(*) $$\sum_i x_i e_{ij} \leqq 0, \qquad x_j \geqq 0 \qquad (j = 1, \cdots, k).$$

We have only to prove that $K \backslash \{0\}$ is included in the strictly positive quadrant. Assume the contrary. By a rearrangement of the coordinates, we may assume that $(x_1, \cdots, x_p, 0, \cdots, 0) \in K$ $(x_1, \cdots, x_p > 0, 1 \leqq p < k)$. Since $e_{ij} \geqq 0$ $(i \neq j)$, (*) implies $e_{ij} = 0$ $(i \leqq p, j > p)$. This contradicts to the fact that E is connected.

(ii) We may assume that $v_\alpha(f)/v_\alpha(\mathfrak{m}) = \bar{\nu}(f) \leqq \bar{\nu}(g) = v_\beta(g)/v_\beta(\mathfrak{m})$ (see (1.4)). Then we have

$$\bar{\nu}(fg) = \min (v_i(f) + v_i(g))/v_i(\mathfrak{m}) \leqq (v_\beta(f) + v_\beta(g))/v_\beta(\mathfrak{m})$$
$$\leqq c_{\beta\alpha}(v_\alpha(\mathfrak{m})/v_\beta(\mathfrak{m}))v_\alpha(f)/v_\alpha(\mathfrak{m}) + v_\beta(g)/v_\beta(\mathfrak{m})$$
$$\leqq (2a_1 - 1)\bar{\nu}(f) + \bar{\nu}(g) \leqq a_1(\bar{\nu}(f) + \bar{\nu}(g)). \qquad \square$$

(2.2) *Remark.* c_{ij} are obviously determined by (e_{ij}). We can show that a_1 is also majorized by a constant determined by (e_{ij}). Indeed there exists $f \in \mathcal{O}_{X,\xi}$ such that $v_i(f) = v_i(\mathfrak{m})$ $(i = 1, \cdots, k)$. Then $v_j(\mathfrak{m}) \leqq$ $c_{ji} v_i(\mathfrak{m})$ and $a_1 \leqq (1 + \max\limits_{i,j} c_{ij} c_{ji})/2$.

§3. LCI in General Case

We borrow an important result, a Bertini type theorem of Flenner:

(3.1) Lemma ([F], (4.10)). *Let* $(\mathcal{O}, \mathfrak{m})$ *be a local k-algebra over a field k of characteristic 0 and let* $\varphi_1, \cdots, \varphi_m \in m$. *Suppose that the completion $\widehat{\mathcal{O}}$ is an integral domain and* $\dim \mathcal{O}_p \geqq 3$, *Prof* $\mathcal{O}_p \geqq 2$ *for any prime ideal p that contains* $\varphi_1, \cdots, \varphi_m$ $(\mathcal{O}_p$: *the localization). Then for any* $e =$

$(e_1, \cdots, e_m)$ in a nonempty Zariski open subset of k^m, $\sum e_i \varphi_i$ is a prime element of $\mathcal{O}$ (even of $\hat{\mathcal{O}}$).

(3.2) Lemma. *Let B be an infinite set of complex hyperplanes (linear subspaces of codimension 1) through 0 in C^m. Then $\nu_0(f) = \min_{H \in B} \nu_0(f|H)$ ($f \in C\{x\}$).*

Proof. The inequality $\nu_0(f) \leqq \nu_0(f|H)$ is a special case of the inequality for pullbacks (I_2). The case $f = 0$ is trivial. If $f \neq 0$ and if H is not contained in the tangent cone of the hypersurface defined by f, then the initial form (the non-zero homogeneous term of the lowest degree) survives after restriction to H and $\nu_0(f) = \nu_0(f|H)$. Since B is an infinite set, such an H really exists in B. $\square$

(3.3) Lemma. *([I₁], Added in proof).* *Let $\Phi: (Y, \eta) \to (X, \xi)$ be the normalization of a reduced and irreducible germ of a complex space. If (Y, η) satisfies (CI_1) in (3.4), (X, ξ) also satisfies it.*

Proof. Since Φ is finite (CI_2) holds $([\mathrm{I}_1], (2.3)$ or $(4.1))$: $\nu_\eta(f \circ \Phi) \leqq a\nu_\xi(f) + b$ ($f \in \mathcal{O}_{X,\xi}$) for some $a \geqq 1$ and $b \geqq 0$. Then we have

$$\nu_\xi(fg) \leqq \nu_\eta((f \circ \Phi)(g \circ \Phi)) \leqq a_1(\nu_\eta(f \circ \Phi) + \nu_\eta(g \circ \Phi)) + b_1$$
$$\leqq a_1 a(\nu_\xi(f) + \nu_\xi(g)) + 2a_1 b + b_1 \qquad (f, g \in \mathcal{O}_{X,\xi}). \qquad \square$$

(3.4) Main Theorem. *Let X be a complex space reduced and irreducible at ξ (i.e. $\mathcal{O} := \mathcal{O}_{X,\xi}$ is an integral domain) such that $n := \dim X_\xi \geqq 1$. Then it has the following three properties.*

(1) *There exist $a_1 = a_1(\mathcal{O}) \geqq 1$, $b_1 = b_1(\mathcal{O}) \geqq 0$ such that*

$$(\mathrm{CI}_1) \qquad \nu(fg) \leqq a_1(\nu(f) + \nu(g)) + b_1 \qquad (f, g \in \mathcal{O}).$$

(2) *For any morphism $\Phi: (Y, \eta) \to (X, \xi)$ with $\mathrm{grnk}_\eta \Phi = \dim X_\xi$ (see §0 for the symbol grnk), there exist $a_1 = a_1(\mathcal{O}) \geqq 1$, $b_1 = b_1(\mathcal{O}) \geqq 0$ such that*

$$(\mathrm{CI}_2) \qquad \nu_\eta(f \circ \Phi) \leqq a_2 \nu_\xi(f) + b_2 \qquad (f \in \mathcal{O}).$$

(3) *For any open subanalytic set $A \subset X$ adherent to ξ, there exist $a_3 = a_3(A) \geqq 1$, $b_3 = b_3(A) \geqq 0$ such that*

$$(\mathrm{CI}_3) \qquad \mu_{A,\xi}(f) \leqq a_3 \nu(f) + b_3 \qquad (f \in \mathcal{O}).$$

Proof. The proof of (3) is postponed to (4. 8). Since (1) and (2) are equivalent ([I$_1$], (1. 2)), we have only to prove one of them. The case $n=1$ of (1) follows from (1. 4), since $v_i(fg) = v_i(f) + v_i(g)$ and $\sharp A(\pi) = 1$. The case $n=2$ follows from (2. 1) and (3. 3). As to the case $n \geqq 3$, we. prove (1) by induction on n.

Suppose that the case $n-1$ $(n \geqq 3)$ has been proved already. We may assume that X is normal by (3. 3). We may also assume that $\xi = 0 \in \mathcal{C}^m$ and X is a closed complex subspace of a product open neighbourhood $U = V \times W (V \subset \mathcal{C}^n, W \subset \mathcal{C}^{m-n})$ of 0 such that the projection $\Pi: X \to V$ is finite proper open and locally biholomorphic on $X - \Pi^{-1}(D)$ for some proper analytic subset $D \subset V$ (local parametrization theorem). If we choose a sufficiently small U and such a coordinate space $\mathcal{C}^{m-n}$ that intersects the tangent cone of X_0 only at 0, then $|\Pi(x)| \leqq |x| \leqq c|\Pi(x)|$ $(x \in X)$ for some $c > 0$. Let P denote the complex projective space of hyperplanes of $\mathcal{C}^n$ through 0 and, if $H \in P$, let $X(H)$ denote the subspace $\Pi^{-1}(H)$ of X with the canonical complex structure. Let us put

$B := \{H \in P: X(H)$ is reduced and irreducible at 0$\}$. Since Prof $\mathcal{O} \geqq 2$ by normality (cf. [G–R], III, 1, Satz 11) and since the completion $\hat{\mathcal{O}}$ is also an integral domain by Nagata's theorem (cf. [Ma], 1), we may apply (3. 1) to $\mathcal{O}$ and its parameter system induced from the affine coordinates of $\mathcal{C}^n$: B includes a nonempty Zariski open subset of P. If $H \in B$, there exist $a(H) \geqq 1$, $b(H) \geqq 0$ such that

$$a(H)(\nu_{X(H)}(u) + \nu_{X(H)}(v)) + b(H) \geqq \nu_{X(H)}(uv) \quad (u, v \in \mathcal{O}_{X(H),0})$$

by the inductive hypothesis. ($\nu(\ )$, $\nu_H(\ )$ and $\nu_{X(H)}(\ )$ denote the algebraic orders respectively on X, H and $X(H)$ at 0.) If we put $B_h = \{H: a(H) \leqq h, b(H) \leqq h\}$, we have $B = \bigcup_{1 \leqq h < \infty} B_h$. Therefore there exists $k \in N$ such that B_k is an infinite set.

Now suppose that $f, g \in \mathcal{O}$ and $\nu(fg) = p$. Since $\nu_{X(H)}(fg|X(H)) \geqq \nu(fg)$ by the pullback inequality (I$_2$), we have

$$\mathrm{k}(\nu_{X(H)}(f|X(H)) + \nu_{X(H)}(g|X(H))) + k \geqq p \quad (H \in B_k).$$

Then, we may assume that $k\nu_{X(H)}(f|X(H)) + k/2 \geqq p/2$ $(H \in B')$ for some infinite subset $B' \subset B_k$ without loss of generality. Let s denote the order of the covering $X - \Pi^{-1}(D) \to V - D$ and let $\sigma_i(y)$ $(1 \leqq i \leqq s)$ denote the i-th elementary symmetric polynomial of values of f on the fiber $\Pi^{-1}(y) = \{x_1, \cdots, x_s\}$ for $y \in V - D$. $\sigma_i(y)$ has a holomorphic

extension over V. If $H \in B'$,

$$\mu_{X(H)|,0}(f|X(H)) \geq \nu_{X(H)}(f|X(H)) \geq (p-k)/2k$$

by (I_3). Hence there exist $\varepsilon_H > 0$ and $c_H > 0$ such that

$$|f(x)| \leq c_H |x|^{(p-k)/2k} \qquad (x \in |X(H)|, \ |x| < \varepsilon_H; \ H \in B').$$

Then there exists $\varepsilon'_H > 0$ such that

$$\frac{|\sigma_i(y)|}{(c|y|)^{i(p-k)/2k}} \leq \sum_{\alpha_1 \leq \cdots \leq \alpha_i} \frac{|f(x_{\alpha_1})|}{|x_{\alpha_1}|^{(p-k)/2k}} \cdots \frac{|f(x_{\alpha_i})|}{|x_{\alpha_i}|^{(p-k)/2k}} = \binom{s}{i} c_H^i$$

$$(y \in |H| \backslash D, \ |y| < \varepsilon'_H; \ H \in B').$$

Since $\{H: |H| \subset D\}$ is a finite set, by a choice of B', we may assume that $D \cap |H|$ is thin in $|H|$ for $H \in B'$. Then, if $H \in B'$, we have $|\sigma_i(y)| \leq d_{H,i} |y|^{i(p-k)/2k} (y \in |H|, |y| < \varepsilon'_H)$ for some $d_{H,i} > 0$. This proves that

$$\nu_H(\sigma_i|H) = \mu_{|H|,0}(\sigma_i|H) \geq i(p-k)/2k \qquad (H \in B').$$

(The left equality follows from the regularity of H.) Thus we have $\nu(\sigma_i \circ \Pi) \geq \nu_{\mathfrak{c}^n}(\sigma_i) \geq i(p-k)/2k$ by (I_2) and (3.2). Since $f^s - (\sigma_1 \circ \Pi) f^{s-1} + \cdots \pm \sigma_s \circ \Pi = 0$ on X_0, $\bar{\nu}(f) \geq (p-k)/2k$ by (1.2), $(i) \Rightarrow (r)$. Then we obtain (CI_1) for the case n, using (CI_0): $\bar{\nu}(f) \leq \nu(f) + b_0$. □

(3.5) Corollary. *Let X be a reduced complex space and let $E \subset |X|$ be an exceptional set such that the set of regular points of X is connected in an arbitrarily small neighbourhood of E. Then for any $\xi, \xi' \in E$ there exist $a \geq 1$, $b \geq 0$ such that $\nu_\xi(f) \leq a\nu_{\xi'}(f) + b$ for any $f \in \Gamma(U, \mathcal{O}_X)$ defined in a neighbourhood U of E.*

Proof. By the definition of exceptional set ([G], p. 339), there exists a proper surjective morphism $\Phi: X \to Y$ such that $\Phi(E) = \{\eta\}$, Φ is biholomorphic on $|X| - E$ and, if V is a neighbourhood of η, any $f \in \Gamma(\Phi^{-1}(V), \mathcal{O}_X)$ can be expressed as $f = g \circ \Phi$ for some $g \in \Gamma(V, \mathcal{O}_Y)$. Since U can be expressed as $\Phi^{-1}(V)$, we may apply this to given f. Obviously Y_η is reduced and irreducible and $\mathrm{grnk}_x \Phi = \dim Y_\eta (x \in E)$. Then $\nu_\xi(f)$ and $\nu_{\xi'}(f)$ are linearly comparable to $\nu_\eta(g)$ by (I_2) and (CI_2) of the theorem. Hence $\nu_\xi(f)$ and $\nu_{\eta'}(f)$ are also linearly comparable.

(3.6) Corollary. *Let A be a ring finitely generated over C and $\mathfrak{m}$ one*

of its maximal ideal. Let us put $\nu_{\mathfrak{m}}(f) = \sup\{p: f \in \mathfrak{m}^p\}$ for $f \in A$. Then the following conditions are equivalent.

(i) There exist $a_1 \geq 1$, $b_1 \geq 0$ such that $\nu_{\mathfrak{m}}(fg) \leq a_1(\nu_{\mathfrak{m}}(f) + \nu_{\mathfrak{m}}(g)) + b_1$ $(f, g \in A)$.

(ii) The completion $(A_{\mathfrak{m}})^\wedge$ of the localization $A_{\mathfrak{m}}$ is an integral domain.

Proof. We may assume that $A = C[x]/I$ $(x = (x_1, \cdots, x_m))$. Since C is algebraically closed we may assume that $I \subset \mathfrak{m}_0 := (x_1, \cdots, x_m) C[x]$, $\mathfrak{m} = \mathfrak{m}_0 A$ without loss of generality. Then $\tilde{A} = C\{x\}/IC\{x\}$ is an analytic local algebra with the maximal ideal $\mathfrak{m}\tilde{A}$ such that $A_{\mathfrak{m}} \subset \tilde{A}$. $(A_{\mathfrak{m}})^\wedge$ is the completion of both $\tilde{A}$ and $A_{\mathfrak{m}}$ and hence faithfully flat over them. Then $\tilde{A}$ is faithfully flat over $A_{\mathfrak{m}}$ and $\mathfrak{m}^p A_{\mathfrak{m}} = \mathfrak{m}^p \tilde{A} \cap A_{\mathfrak{m}}$. On the other hand symbolic powers $\mathfrak{m}^{(p)} := \mathfrak{m}^p A_{\mathfrak{m}} \cap A$ coincide with $\mathfrak{m}^p$ by the maximality of $\mathfrak{m}$. These prove that $\nu_{\mathfrak{m}}(f) = \nu_{\mathfrak{m}A_{\mathfrak{m}}}(f) = \nu_{\mathfrak{m}\tilde{A}}(f)$ $(f \in A)$. Then (i) follows from (ii) by (CI_1) for the integral domain $\tilde{A}$.

Conversely suppose that (ii) is false. Then there exist $\hat{f}, \hat{g} \in (A_{\mathfrak{m}})^\wedge$ such that $\hat{f}\hat{g} = 0$, $\hat{f} \neq 0$, $\hat{g} \neq 0$. We can identify $(A_{\mathfrak{m}})^\wedge$ with $C[[x]]/IC[[x]]$. Take representatives $f, g \in C[[x]]$ of $\hat{f}, \hat{g}$ respectively such that $\nu_{\mathfrak{m}'}(f) = \nu_{\hat{\mathfrak{m}}}(\hat{f})$, $\nu_{\mathfrak{m}'}(g) = \nu_{\hat{\mathfrak{m}}}(\hat{g})$ $(\mathfrak{m}' = \mathfrak{m}_0 C[[x]], \hat{\mathfrak{m}} = \mathfrak{m}(A_{\mathfrak{m}})^\wedge)$. Let f_p (resp. g_p) be the polynomial which consists of the homogeneous terms of f (resp. g) of degree less than $p+1$. If $p \geq \max\{\nu_{\hat{\mathfrak{m}}}(\hat{f}), \nu_{\hat{\mathfrak{m}}}(\hat{g})\}$, then $\nu_{\mathfrak{m}}(f_p \mod I) = \nu_{\hat{\mathfrak{m}}}(\hat{f}) < \infty$, $\nu_{\mathfrak{m}}(g_p \mod I) = \nu_{\hat{\mathfrak{m}}}(\hat{g}) < \infty$. On the other hand $\nu_{\mathfrak{m}}(f_p g_p \mod I) \geq p+1$. Hence (i) does not hold.

§4. Real Analytic Algebras

Let $\mathcal{O}$ be a local real analytic algebra. It can be considered as the local ring $\mathcal{O}_{X,\xi}$ of a germ X_ξ of a real analytic space. We define the dimension of X_ξ by the Krull dimension of $\mathcal{O}$: $\dim X_\xi := \dim \mathcal{O}$. In general $\dim X_\xi \geq \dim |X|_\xi$. We call $C \otimes_R \mathcal{O}$ the complexification of $\mathcal{O}$. It determines a germ X_ξ^c of a complex space called the complexification of X_ξ. Since $C \otimes_R \mathcal{O}$ is integral over $\mathcal{O}$,

$$\dim X_\xi^c = \dim C \otimes_R \mathcal{O} = \dim \mathcal{O} = \dim X_\xi$$

by a theorem of Krull-Cohen-Seidenberg.

(4.1) Theorem. *Let $\mathcal{O}$ be a local real analytic algebra of positive*

dimension whose complexification is an integral domain. Then $\mathcal{O}$ satisfies the real versions of (1), (2), (3) *of the main theorem* (3.4).

Proof. (1) Obvious by (3.4), (1).

(2) Let Φ^c, f^c be the complexifications of Φ and f respectively. Since $\nu_\xi(f^c) = \nu_\xi(f)$, $\nu_\eta(f^c \circ \Phi^c) = \nu_\eta(f \circ \Phi)$, $\mathrm{grnk}_\eta \Phi^c \geqq \mathrm{grnk}_\eta \Phi$ (cf. [I$_1$], p. 461, p. 465 (*correction*: paracompact$\to\sigma$-compact)), our assertion follows from (3.4), (2).

(3) This follows from (3.4), (1) and [I$_1$], (3.1).   □

If S is a germ of a real analytic set at $0 \in \mathbf{R}^n$, $I(S)$ denotes the ideal of elements of $\mathbf{R}\{x\}$ vanishing on S. If J is an ideal of $\mathbf{R}\{x\}$, $V(J)$ denotes the maximal germ of a real analytic set at 0 on which all elements of J vanish.

(4.2) Lemma. ([Ri], (4.1)). *The following conditions are equivalent for an ideal J of $R\{x\}$.*

(C) $f_1^2 + \cdots + f_p^2 \in J$ *implies* $f_1, \cdots, f_p \in J$.

(Z) $J = I(S)$

for some germ S of a real analytic set.

J *is called* real *if it satisfies the conditions in* (4.2).

(4.3) Lemma. *Let J be a prime ideal of $\mathbf{R}\{x\}$.*

(i) J *is real if and only if* $\dim \mathbf{R}\{x\}/J = \dim V(J)$ ([R$_i$], (4.3)).

(ii) *If J is real, $\mathbf{C} \otimes_\mathbf{R} J$ is prime in $\mathbf{C}\{x\}$* ([G-T], p. 371; [R$_i$], (6.1)).

(4.4) Remark. Let S be an irreducible germ of a real analytic set at $0 \in \mathbf{R}^n$. Then (4.1) is applicable to the integral domain $\mathbf{R}\{x\}/I(S)$ by (4.3), (ii). There exists, however, a non-real ideal J with a prime complexification $\mathbf{C} \otimes_\mathbf{R} J$ (e.g. $J = (x^2+y^2+z^2)\mathbf{R}\{x,y,z\}$). In such a case $\dim |X|_\xi < \dim X_\xi$, $\mathrm{grnk}_\eta \Phi < \dim X_\xi$ and hence (2) and (3) of (4.1) is meaningless. Note also that in such a case the complexifications of X_ξ and $|X|_\xi$ are not compatible.

For $f(z) = \sum \alpha_k z^k \in \mathbf{C}\{z\}$ ($z = (z_1, \cdots, z_m)$), we put

$$f^*(z) := \sum \bar{\alpha}_k z^k \in \mathbf{C}\{z\},$$
$$\mathrm{Re}\, f(x+iy) := (f(x+iy) + f^*(x-iy))/2 \in \mathbf{R}\{x,y\},$$

$$\mathrm{Im}\, f(x+iy):=(f(x+iy)-f^*(x-iy))/2i\in R\{x,y\}.$$

If I is an ideal of $C\{z\}$, I^r denote the ideal of $R\{x,y\}$ generated by $\{\mathrm{Re}\ f(x+iy)\colon\ f\in I\}$ (or, equivalently, by $\{\mathrm{Im}\ f(x+iy)\colon\ f\in I\}$). We put

$$\mathcal{O}:=C\{z\}/I,\quad \mathcal{O}^r:=R\{x,y\}/I^r,\quad \tilde{I}:=C\otimes_R I^r,$$

$$\tilde{\mathcal{O}}:=C\{x,y\}/\tilde{I}=C\otimes_R\mathcal{O}^r.$$

Further we put

$$J:=\{f(u)\colon\ f(z)\in I\},\quad J^*:=\{f^*(v)\colon\ f(z)\in I\},$$

$$K:=(J\otimes 1)\,C\{u,v\}+(1\otimes J^*)\,C\{u,v\}$$

$$\subset C\{u\}\hat{\otimes}C\{v\}=C\{u,v\}$$

$(\otimes, \hat{\otimes}$: analytic tensor product, see [G–R]). Let us define an isomorphism $\theta\colon\ C\{u,v\}\to C\{x,y\}$ by $\theta(f(u,v)):=f(x+iy,x-iy)$.

(4.5) Lemma. (*cf.* [Ma], § 3, *which treats analytic sets*).

$$\tilde{I}=\theta(K)\qquad i.e.\quad \tilde{\mathcal{O}}\cong\mathcal{O}\hat{\otimes}\mathcal{O}^*\qquad (\mathcal{O}^*=C\{v\}/J^*).$$

Proof. Obvious by the following implication.

$$f\in\theta(K)\Leftrightarrow \exists g\in K \text{ s. t. } f(x,y)=g(x+iy,x-iy)$$

$$\Leftrightarrow \exists\alpha_\mu,\ \exists\beta_\nu\in C\{u,v\};\ \exists h_\mu,\ \exists k_\mu\in J\ (\mu=1,\cdots,p)$$

$$\text{s. t. } f(x,y)=\sum_\mu\{\alpha_\mu(x+iy,x-iy)\,h_\mu(x+iy)$$

$$+\beta_\nu(x+iy,x-iy)\,k_\mu^*(x-iy)\}$$

$$\Leftrightarrow f\in\tilde{I}.\qquad\qquad\qquad\square$$

(4.6) Proposition. *If $\mathcal{O}$ is an integral domain in the above, we have the following.*

(i) $\tilde{\mathcal{O}}$ *is an integral domain.*

(ii) I^r *is a real ideal.*

(ii) is just the affirmative answer to Malgrange's conjecture [Ma], p. 124.

Proof. (i) By (4.5) we have only to prove that $\mathcal{O}\hat{\otimes}\mathcal{O}^*$ is an integral domain. Since $\mathcal{O}^*$ is isomorphic to $\mathcal{O}$ as a ring, it is an integral domain. Then $\mathcal{O}\hat{\otimes}\mathcal{O}^*$ is also so by Nagata's theorem [N$_2$], (47.5).

(ii)   Since $\tilde{\mathcal{O}}$ is an integral domain and since
$$\dim \mathcal{O}^r = \dim \tilde{\mathcal{O}} = \dim \mathcal{O} \hat{\otimes} \mathcal{O}^* = 2 \dim \mathcal{O} = \dim V(I^r),$$
we have the assertion by (4.3), (i).   □

(4.7) *Remark.* If $\mathcal{O}$ is an integral domain, we may well call $\mathcal{O}^r$ the real analytic algebra associated to $\mathcal{O}$. But, if not, $\mathcal{O}^r$ is not fit to be called so. For example, consider the case $I = I_1 \cap I_2, I_i = z_i C \{z_1, z_2\}$ $(i=1, 2)$. Since $K = (u_1 u_2, v_1 v_2) C \{u, v\}$, $V(\tilde{I}) \cong V(K)$ has four components at $0 \in C^4$. Therefore $V(\tilde{I})$ is bigger than the join of $V(\tilde{I}_1)$ and $V(\tilde{I}_2)$. Returning to the algebras of the real parts, we see that $\mathcal{O}^r$ is too big to be called the real analytic space associated to $\mathcal{O}$.

(4.8) *Remark.* If $f_\mu (\mu = 1, \cdots, p)$ generate I, $\mathrm{Re} f_\mu (x+iy)$, $\mathrm{Im} f_\mu$ $(x+iy)$ generate I^r. Then, for any complex space X, we can construct the real analytic space X_r (defined by a coherent sheaf of ideals) and its complexification $\tilde{X}$ in the sense of [H].

Proof of (3.4), (3). Let X_ξ be a reduced and irreducible germ of a complex space. Then X_ξ^r has a reduced and irreducible complexification $\tilde{X}_\xi$ by (4.6). Hence (4.1) holds for X_ξ^r. If $f \in \mathcal{O}_{X,\xi}$, we have only to apply (4.1), (3) for X_ξ^r to the real and the imaginary part of f. (If we consider the proof of (4.1), (3), we see that (3.4), (3) for n-dimensional X_ξ results from (3.4), (2) for $2n$-dimensional $\tilde{X}_\xi$.)
                                                                      □

§5. Nearness to p-th Power in $C\{x_1, \cdots, x_m\}$

In this section we put $A = \mathcal{O}_{m,0} = C\{x_1, \cdots, x_m\}$, $\hat{A} = C[[x_1, \cdots, x_m]]$, $\nu( ) = \nu_0( )$. Let p be a prime number and u not a p-th power in A. The functional equation $f^p u - g^p = 0$ has unique solution $(f, g) = (0, 0)$. If $\nu(f^p u - g^p)$ is large, (f, g) is a "near solution". We claim that a near solution is near to the unique solution:

(5.1) **Proposition.** *If $u \in A$ is not a p-th power in A, there exist $a \geq p, b \geq \nu(u), b' \geq 0$ depending onlyon n, p, u such that $a\nu(f) + b \geq \nu(f^p u - g^p)$, $a\nu(g) + b' \geq \nu(f^p u - g^p)$ for any $f, g \in \hat{A}$.*

(5.2) *Remark.* By Artin's strong approximation theorem [A₂],

(6.1), we see that, if u is a polynomial, there exists a function e: $N \to N$ such that $\nu(f)$, $\nu(g) \geq e[\nu(f^p u - g^p)]$. (5.1) asserts that we can choose a linear e.

Proof. Let us put $F = \nu(f)$, $G = \nu(g)$, $K = \nu(f^p u - g^p)$, $U = \nu(u)$ and $T = \max\{F, G, K\} + 1$. We have to prove that

(*) $\qquad aF + b \geq K, \qquad aG + b' \geq K.$

Considering the cases $g = 0$ and $f = 0$ we have necessary conditions

(i) $\qquad a \geq p,\ b \geq \nu(u),\ b' \geq 0.$

This assures (*) if $T = \infty$. Thus we have only to treat the case $T < \infty$. If $f - f'$, $g - g' \in \mathfrak{m}^T$ then $\nu(f') = F$, $\nu(g') = G$, $\nu(f'^p u - g'^p) = K$. Hence we may assume that $f, g \in A$. If the initial form (the homogeneous term of the lowest degree) In u is not a p-th power, $K = \min (pF + U, pG)$ and (*) is obvious ($a = p$). So we may assume that $U = pV (V \in N)$.

Let us introduce a new variable z and consider

$$\mathcal{O}_{m+1,0} := C\{x_1, \cdots, x_m, z\},$$
$$\mathcal{O} := \mathcal{O}_{m+1,0}/(u - (-z)^p)\, \mathcal{O}_{m+1,0}.$$

Let ν and $\nu_\mathcal{O}$ denote the algebraic order of $\mathcal{O}_{m+1,0}$ and $\mathcal{O}$ with respect to the respective maximal ideals (these are extensions of the original ν). $h \in \mathcal{O}$ has a unique representative of form

$$\sum_{i=0}^{p-1} h_i z^i \in \mathcal{O}_{m+1,0} \quad (h_i \in A).$$

$\nu_\mathcal{O}(h) = \min (\nu(h_1) + i)$ by the assumption $U = pV \geq p$. If we put

$$h := (fg^{p-2} z^{p-1} - g^{p-1} z^{p-2}) + (f^{p-1} uz^{p-3} - f^{p-2} guz^{p-4}$$
$$+ \cdots + f^3(-g)^{p-4} uz + f^2(-g)^{p-3} u) \qquad (\text{if } p = 2,\ h := fz - g),$$

we have

$$(fz + g)\,h \equiv (f^p u - g^p) z^{p-2} \mod u - (-z)^p \quad (f, g \in A).$$

Then, applying (3.4), (1) to the integral domain $\mathcal{O}$, we have

(ii) $\qquad a_1(\nu(fz + g) + \nu(h)) + b_1 \geq \nu((f^p u - g^p) z^{p-2}) = K + p - 2$
$$(f, g \in A).$$

If $pG \neq pF + U$, we can directly prove (*) using (i). So we have only to treat the case $pG = pF + U$. Then, since

$$\nu(g^{p-1}z^{p-2}) - \nu(fg^{p-2}z^{p-1}) = V-1,$$
$$\nu(f^{p-1}uz^{p-3}) - \nu(fg^{p-2}z^{p-1}) = 2(V-1),$$
$$\nu(f^{p-2}guz^{p-4}) - \nu(fg^{p-2}z^{p-1}) = 3(V-1),$$
$$\cdots\cdots$$
$$\nu(f^2g^{p-3}u) - \nu(fg^{p-2}z^{p-1}) = (p-1)(V-1),$$

we have

(iii) $\qquad \nu(h) = \nu(fg^{p-2}z^{p-1}) = (p-1)F + (p-2)V + p - 1$
$$= (p-1)G - V + p - 1.$$

Since $G - (F+1) = V - 1 \geqq 0$,

(iv) $\qquad \nu(fz+g) = F+1 = G - V + 1.$

Combining (ii), (iii) and (iv), we have

$$a_1(pF + (p-2)V + p) + b_1 - p + 2 \geqq K,$$
$$a_1(pG - 2V + p) + b_1 - p + 2 \geqq K.$$

Hence (*) holds by a suitable choice of a and b $(a = a_1 p)$. $\qquad\square$

(5.3) Definition. $\theta(p, u) := \inf a/p$ $(a:$ in $(5.1))$.

This can be regarded as a *measure of nearness to p-th power*. In view of the proof above, we have an estimate $1 \leqq \theta(p, u) \leqq a_1$.

(5.4) Example. If $\ln u$ is not a p-th power, $\theta(p, u) = 1$. This means that u is very far from p-th powers.

(5.5) Example. Suppose that $u := v^p - w$ is not a p-th power in A and $k := \nu(w) \geqq p\nu(v) + 1$. Using Tailor expansion

$$\sqrt[p]{1 + t + t^2 + \cdots} = 1 + c_1 t + c_2 t^2 + \cdots,$$

we have

$$(1 + c_1 t + c_2 t^2 + \cdots + c_n t^n)^p (1-t) - 1$$
$$= d_{n+1} t^{n+1} + d_{n+2} t^{n+2} + \cdots + d_{np+1} t^{np+1}$$

for suitable $d_{n+1}, \cdots, d_{np+1} \in C$. Hence

$$(v^{np} + c_1 v^{(n-1)p}w + c_2 v^{(n-2)p}w^2 + \cdots + c_n w^n)^p (v^p - w) - v^{(np+1)p}$$
$$= d_{n+1} v^{(np-n)p} w^{n+1} + d_{n+2} v^{(np-n-1)p} w^{n+2} + \cdots + d_{np+1} w^{np+1}.$$

If we put $f_n = v^{np} + c_1 v^{(n-1)p}w + \cdots + c_n w^n$, $g_n = v^{np+1}$, we have

$$\nu(f_n^p u - g_n^p) = \nu(v^{(np-n)p} w^{n+1}) = n(p-1)ph + (n+1)k \qquad (h := \nu(v)).$$

Hence $\theta(p, u) \geqq (p^2 h - ph + k)/p^2 h = 1 + (k - hp)/hp^2 (>1).$

(5.6) Example. If $u = x_1^p - x_2^k$ ($k > p$, $k \neq 0 \mod p$), a_1 for $\mathcal{O}$ in the proof of (5.1) is estimated as $a_1 \leqq k/p$ by the proof of $[I_1]$, (1.5). Hence we have

$$1 - 1/p + k/p^2 \leqq \theta(p, u) \leqq k/p.$$

Appendix : inf a_1 and inf b_1 for Some Elementary Singularities

Let $\underline{a}_1$ denote the infimum of a_1 in (3.4) for a given local complex analytic integral domain $\mathcal{O} = \mathcal{O}_{X,\xi}$. In case $\underline{a}_1$ is attained, $\underline{b}_1$ denotes the infimum of b_1 that satisfies (3.4), (1) with $a_1 = \underline{a}_1$. The author does not know whether $\underline{a}_1$ is always attained and whether $\underline{a}_1$ is a rational number. Here we exhibit the explicit values of $\underline{a}_1$ and $\underline{b}_1$ for some elementary singularities. $\underline{a}_1$ in (i), (iii), (iv) and $\underline{b}_1$ in (i) were first taught by Tomari.

(i) $\nu(fg) = \nu(f) + \nu(g)$ ($f, g \in \mathcal{O}$) (i.e. $\underline{a}_1$ and $\underline{b}_1$ are attained and $\underline{a}_1 = 1$, $\underline{b}_1 = 0$) iff $\mathrm{Gr}_m \mathcal{O} := \sum_{p=0}^{\infty} \mathfrak{m}^p / \mathfrak{m}^{p+1}$ is an integral domain. The proof is standard.

(ii) If $\dim \mathcal{O} = 1$, the exceptional fiber of the normalization of the blowing-up of X with center $\mathfrak{m}_\xi$ is irreducible. Then $\underline{a}_1$ is attained and $\underline{a}_1 = 1$ by (1.4).

(iii) Let $(X, 0)$ be a hypersurface singularity in $\mathbb{C}^3$ defined by $z^2 - \varphi(x, y) = 0$ ($\varphi(0, 0) = 0$). Let ν and $\nu_\mathcal{O}$ denote the algebraic orders on $C\{x, y, z\}$ and on $\mathcal{O} := C\{x, y, z\}/(z^2 - \varphi)C\{x, y, z\}$ respectively. Further we assume that $r := \nu(\varphi)$ is an odd number greater than 2. Here we claim about $\mathcal{O}$ that $\underline{a}_1 = 1$ and $\underline{b}_1 = r - 2$ and that they are attained. Obviously every $\bar{f} \in \mathcal{O}$ has a unique representative of the form $f = f_1(x, y) z + f_2(x, y) \in C\{x, y, z\}$ such that $\nu_\mathcal{O}(\bar{f}) = \nu(f) = \min(\nu(f_1) + 1, \nu(f_2))$. Suppose that

(*) $\nu_\mathcal{O}(\bar{f}\bar{g}) > \nu_\mathcal{O}(\bar{f}) + \nu_\mathcal{O}(\bar{g}) + r - 2$

for some $\bar{f}, \bar{g} \in \mathcal{O}$. This implies that

$$\nu(f_1 g_2 + f_2 g_1) > \nu(f) + \nu(g) + r - 3,$$
$$\nu(f_1 g_1 \varphi + f_2 g_2) > \nu(f) + \nu(g) + r - 2.$$

By a standard calculation we have

$$\nu(f_1^2 \varphi - f_2^2) > 2\nu(f) + r - 2.$$

On the other hand, since r is odd,

$$\nu(f_1^2 \varphi - f_2^2) = \min(2\nu(f_1) + r, \ 2\nu(f_2)) \leqq 2\nu(f) + r - 2,$$

a contradction. Thus (*) never holds and we may put $a_1 = 1$, $b_1 = r - 2$. Since

$$\nu_{\mathcal{O}}(z^2) = r = (\nu_{\mathcal{O}}(z) + \nu_{\mathcal{O}}(z)) + r - 2,$$

b_1 can never be smaller than $r - 2$. Hence we have proved the assertion.

In the case 0 is an isolated singularity of X (i. e. X is normal at 0), we can prove that the exceptional fiber of the normalization of the blowing-up of X with center $\mathfrak{m}_0$ is irreducible, which again implies that $\underline{a}_1$ is attained and $\underline{a}_1 = 1$. This case includes the rational singularities of types D_k $(k \geqq 4)$, E_6, E_7, E_8 (cf. [Br]).

(iv)   The remaining two dimensional rational singularity of multiplicity 2 is A_k $(k \geqq 1)$ defined by the equation $x^2 + y^2 + z^{k+1} = 0$ in $\mathbb{C}^3$. By the proof of [I$_1$], (1.5), we see that (3.4), (1) holds for $a_1 = (k+1)/2$, $b_1 = 0$. On the other hand

$$\begin{aligned} a_1 &\geqq \{\nu((x+iy)^p(x-iy)^p) - b_1\} / \{\nu((x+iy)^p) + \nu((x-iy)^p)\} \\ &= \{(k+1)p - b_1\}/2p \qquad (p \in \mathbb{N}). \end{aligned}$$

Hence $a_1 \geqq (k+1)/2$. Thus we have proved $\underline{a}_1 = (k+1)/2$, $\underline{b}_1 = 0$ and they are attained.

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