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On Grothendieck's Problem of Topologies

by

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Abstract

Some classes of locally convex spaces E and F are identified for which not every bounded set of the completed projective tensor product, $E\hat{\otimes}F$, is contained in the bipolar of some set $A\otimes B$, where A is bounded in E and B is bounded in F.

Statement and Proof of Results

One of the open problems in the theory of Topological Tensor Products is to characterize classes of locally convex spaces (l. c. s.) E and F, for which every bounded subset of the completed projective tensor product, $E\hat{\otimes}F$, is contained in the bipolar of some set $A\otimes B$, where A and B are bounded sets in E and F respectively (Problem of Topologies). For instance it is not known if the Problem of Topologies has a positive answer when E and F are general Fréchet spaces. But if one of the Fréchet spaces is nuclear, then it does ([4], Theorem 21. 5. 8).

It is the purpose of this note to investigate the Problem of Topologies for classes of l. c. s. that arise in the Theory of Distributions.

Before stating our main result, we make some remarks about the notation. In all that follows $\phi\equiv\bigoplus_{i=1}^{\infty}\boldsymbol{C}$ will be equipped with its largest locally convex topology and $\omega\equiv\prod_{i=1}^{\infty}\boldsymbol{C}$ with its product topology. An LF-space will be a strict inductive limit of a sequence of Fréchet spaces.

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Theorem. *The Problem of Topologies for* $E\hat{\otimes}F$ *has a negative solution if, either*

(i) E *and* F *are strong duals of LF-spaces, where one of these LF-spaces contains a non-normable Fréchet subspace, the other one is not a Fréchet space and either of them is nuclear, or*

(ii) E *is a Fréchet space that does not have a continuous norm and* F *a l. c. s. that contains* ϕ *as a complemented subspace.*

Three Lemmas will precede the proof of the Theorem.

Lemma 1. *Let* E *be a l. c. s. that has a non-normable Fréchet subspace and* F *a l. c. s. containing* ϕ *as a complemented subspace. Then* $E\hat{\otimes}F$ *and the completed inductive tensor product,* $E\bar{\otimes}F$, *have different duals.*

Proof. First let's assume that E is a non-normable Fréchet space. As E is non-normable and metrizable, there exists a family of continuous linear functionals $\{\phi_k\}_{k\geq 1}$ on E, such that for no sequence of strictly positive numbers $\{\alpha_k\}_{k\geq 1}$, is $\{\alpha_k\phi_k\}_{k\geq 1}$ equicontinuous ([1], Proposition 1.7. (b)). Let ψ_k be a continuous linear functional on ϕ whose kernel is $\bigoplus_{n=1}^{\infty} D_n^{(k)}$, where $D_k^{(k)}=\{0\}$ and $D_n^{(k)}=\boldsymbol{C}$ if $n\neq k$. Then $\sum_{k=1}^{\infty}\phi_k\psi_k$ is a separately continuous bilinear form on $E\times\phi$ which is not jointly continuous. As all tensor product topologies respect complemented subspaces, we get that $(E\bar{\otimes}F)'\neq(E\hat{\otimes}F)'$.

A Hahn-Banach extension argument applied to the sequence $\{\phi_k\}_{k\geq 1}$ gives the proof in the general case.

Lemma 2. *Every LF-space that is not a Fréchet space, has* ϕ *as a complemented subspace.*

Proof. Let $\{E_j\}$ be a defining sequence of Fréchet subspaces for E. There is a sequence $\{x_j\}_{j\geq 1}$ in E such that $x_j\in E_j$ and $x_{j+1}\notin E_j$ for all $j\geq 1$. If F_j is the smallest subspace containing $\{x_i\}_{1\leq i\leq j}$, an inductive argument shows that there are topological complements G_j for F_j in E_j such that $G_{j+1}\supset G_j$ for all $j\geq 1$. By construction $G=\bigcup_{j=1}^{\infty} G_j$ is a subspace of countable codimension of the barrelled space E and

therefore by the Saxon-Levin-Valdivia Theorem, G is barrelled ([6], Theorem 15-1-10). Since G is barrelled and is a union of a strictly increasing sequence of closed subspaces, it has the strict inductive limit topology ([6], Definition 13-3-14 and Theorem 13-3-15). So G is complete and therefore closed. As E is barrelled and G a closed subspace of countable codimension, $F=\bigcup_{j=1}^{\infty} F_j$, with its largest locally convex topology, is a complement of G ([6], Theorem 13-3-14). But F with its largest convex topology is isomorphic to ϕ (cf. [6], Problem 12-1-5).

Lemma 3. *Let E and F be LF-spaces, one of them nuclear. Then $E\bar{\otimes}F$ is an LF-space and its strong dual is isomorphic to $E'_\beta\hat{\otimes}F'_\beta$.*

Proof. The first assertion follows from the fact that the inductive tensor product topology respects inductive limits ([5], p. 96 and p. 119, Exercise 22). Nuclearity guarantees that the inductive limit is strict (cf. [1], Theorem 1.4).

As $E\hat{\otimes}F$ and $E\bar{\otimes}F$ induce the same topology on the Fréchet subspaces of $E\bar{\otimes}F$, and every bounded subset of $E\bar{\otimes}F$ is contained in one of these subspaces, it follows from the Hahn-Banach theorem that $(E\hat{\otimes}F)'$ is strongly dense in $(E\bar{\otimes}F)'$. By the nuclear theorem and the positive solution of the problem of Topologies for $E\bar{\otimes}F$ ([4], Theorem 21.5.8), $E'\otimes F'$ is strongly dense in $(E\hat{\otimes}F)'$ and therefore also in $(E\bar{\otimes}F)'$. Since the strong dual of a nuclear LF-space is nuclear ([4], Proposition 21.5.1 and Corollary 21.5.5) and $(E\bar{\otimes}F)'_\beta$ is complete ($E\bar{\otimes}F$ is bornological), it will be enough to show that the strong topology on $(E\bar{\otimes}F)'$ induces the injective topology ε on its dense subspace $E'\otimes F'$. When A and B vary over all possible absolutely convex closed bounded sets in E and F, respectively, the seminorms

$$P_{A,B}(\xi)=\sup_{z\in\langle\overline{A\otimes B}\rangle}|\xi(z)|,\qquad \xi\in(E\bar{\otimes}F)',$$

($\langle\overline{A\otimes B}\rangle\equiv$absolutely closed convex hull of $A\otimes B$) define the strong topology on $(E\bar{\otimes}F)'$. But

$$P_{A,B}(\sum_{l=1}^{k}\phi_l\otimes\psi_l)=\sup_{x\in A,\,y\in B}|\sum_{l=1}^{k}\phi_l(x)\psi_l(y)|$$

$$= (P_A \otimes_\varepsilon q_B)\left(\sum_{l=1}^{k} \phi_l \otimes \psi_l\right),\ \phi_l \in E',\ \psi_l \in F' \quad (1 \leqq l \leqq k)$$

where $P_A(\phi) \equiv \sup_{x \in A} |\phi(x)|$ and $q_B(\psi) \equiv \sup_{y \in B} |\psi(y)|$.

This finishes the proof.

Proof of Theorem.

(i) By Lemmas 1 and 2 there is a continuous seminorm p on $E \bar{\otimes} F$ whose restriction to $E \otimes_\pi F$ ($E \otimes F$ equipped with the projective topology) is not continuous. The polar B_p^0 of the unit ball B_p of this seminorm is a bounded set in $(E \bar{\otimes} F)'_\beta \cong E'_\beta \hat{\otimes} F'_\beta$, such that for no absolutely convex closed bounded sets A and B in E'_β and F'_β, respectively, $B_p^0 \subset \langle \overline{A \otimes B} \rangle$. In effect this inclusion and the barrelledness of E and F would imply, as in the proof of Lemma 3, that the restriction of p to $E \otimes_\pi F$ would be continuous.

(ii) By Lemma 3 if $E = \phi$ and $F = \omega$ we have $(\phi \bar{\otimes} \omega)'_\beta \cong \omega \hat{\otimes} \phi$ as $\phi'_\beta \cong \omega$ and $\omega'_\beta \cong \phi$. By part (i), this proves part (ii) when $E = \omega$. The rest of the proof follows from the fact that every Fréchet space without a continuous norm has ω as a complemented subspace ([4], Theorem 7.2.7).

Remark (*a*). By Lemma 2, part (ii) of the Theorem applies to $E'_\beta \hat{\otimes} F$ if E and F are LF-spaces that are not Fréchet.

Remark (*b*). It can be shown that if E and F are arbitrary locally convex direct sums of Fréchet spaces equipped with continuous norms, then as sets $E \hat{\otimes} F = E \bar{\otimes} F$ and both completions have the same bounded sets. In particular this applies when E and F are LF-spaces with continuous norms and unconditional bases ([2], Lemma 5.43 and Exercise 5.59). Therefore a positive solution of the Problem of Topologies under these conditions for $E \bar{\otimes} F$ implies a positive solution of the corresponding problem for $E \hat{\otimes} F$. This applies, for example, when E and F are LF-spaces with unconditional bases, continuous norms, and one of them is nuclear.

Remark (*c*). A positive solution of the Problem of Topologies is

useful for deriving some general factorization Theorems in Quantum Field Theory (cf. [3], Proposition 3.1).

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