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A Condition in Constructing Chain Homotopies

Dedicated to Professor Nobuo Shimada on his 60th birthday

By

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§1. Introduction

The following question was asked by M. Morimoto:

(1. 1) Let G be a finite group and $R=\boldsymbol{Z}[G]$ be the group ring. Let C_* and D_* be chain complexes of free R-modules, $f.$ and $g.$ be chain equivalences from C_* to D_*. If

$$f_*=g_*: H_*(C_*)\longrightarrow H_*(D_*),$$

then is it true that $f.$ and $g.$ are chain homotopic to each other (and hence have the same torsion invariant)?

In this note we show by an example that the answer is negative, namely that $f.\simeq g.$ does not always hold. We also consider the case $R=K[G]$ where K is a field, and show that the answer is negative if and only if the characteristic of the field K divides the order of G when G is a finite group. We also obtain some condition for an infinite group G.

If 1. 1 were true, then the arguments of Morimoto in [3], which computes a homology class of the torsion invariant (cf. Theorem 8. 4 of Dovermann-Rothenberg [2]), would be considerably simplified. Thus the negativeness of 1. 1 suggests that we cannot do without delicate arguments as in [3] in computing torsion invariants.

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§2. The Example

Let G be a group, K be a ring and $R=K[G]$ be the group ring. We consider the negativity of the following:

Statement 2. 1. *Let C_*, D_* be chain complexes of (free) modules over $R=K[G]$ and $f.$, $g.$ be chain equivalences from C_* to D_*. If we assume that*

$$f_*=g_* : H_*(C_*)\longrightarrow H_*(D_*),$$

then we state that

$$f.\simeq g.: \text{ chain homotopic.}$$

Lemma 2. 2. *If there exist elements α, β of R which satisfy the following conditions:*

(2. 3)
i) $\alpha\beta=0$
ii) $\lambda\alpha+\beta\mu\neq 1$ *for any* $\lambda, \mu\in R$, *and*
iii) *if* $\alpha\gamma=0$ *then* $\gamma=\beta\delta$ *for some* $\delta\in R$,

then there is an example which shows that Statement 2. 1 does not hold.

Proof. Put

$$C_*=D_*=\{0\longleftarrow R^2\overset{\phi_0}{\longleftarrow}R^2\overset{\phi_1}{\longleftarrow}R^2\longleftarrow 0\},$$

with

$$\phi_0=\begin{pmatrix}0 & 0\\ 0 & \alpha\end{pmatrix},\quad \phi_1=\begin{pmatrix}0 & \beta\\ 0 & 0\end{pmatrix}.$$

Then

$$\text{Ker }\phi_0=\left\{\begin{pmatrix}x\\ y\end{pmatrix};\ \alpha y=0\right\},\quad \text{Im }\phi_1=\left\{\begin{pmatrix}\beta z\\ 0\end{pmatrix}\right\},$$

and C_* is a chain complex of free R-modules. Put

$$f.=\{f_0, f_1, f_2\}: C_*\longrightarrow D_*$$

to be

$$f_0=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ f_1=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\ f_2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Clearly $f_{\cdot}$ is a chain map and has an inverse. Further put

$$g_{\cdot}=\text{id.}:\ C_*\longrightarrow D_*.$$

Now we have

$$(f_1-g_1)\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}y\\ 0\end{pmatrix}=\begin{pmatrix}\beta\delta\\ 0\end{pmatrix}\in \text{Im}\ \phi_1$$

for $\begin{pmatrix}x\\ y\end{pmatrix}\in \text{Ker}\ \phi_0$, and so $f_*=g_*:\ H_*(C_*)\longrightarrow H_*(D_*)$.

Assume that $f_{\cdot}\simeq g_{\cdot}$. Then there are homomorphisms ψ_0, ψ_1 which satisfy

$$f_1-g_1=\psi_0\phi_0+\phi_1\psi_1.$$

If we put $\psi_0=\begin{pmatrix}p & q\\ r & s\end{pmatrix}$, $\psi_1=\begin{pmatrix}k & l\\ m & n\end{pmatrix}$, this means

$$\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}=\begin{pmatrix}0 & q\alpha\\ 0 & s\alpha\end{pmatrix}+\begin{pmatrix}\beta m & \beta n\\ 0 & 0\end{pmatrix},$$

which contradicts with condition ii). This completes the proof.

Proposition 2.4. *Let R be one of the following:*

a) *G is any group and $R=\mathbf{Z}[G]$,*

b) *K is a ring, G is a group which contains a finite subgroup of order multiple of the characteristic of K, and $R=K[G]$,*

c) *K is a ring, G is a group which contains an element of infinite order, and $R=K[G]$.*

Then Statement 2.1 *does not hold.*

Proof. When R contains an element α which is neither a divisor of zero nor a unit, Condition 2.3 is satisfied if we put $\beta=0$. In case a), $\alpha=2\in\mathbf{Z}$ satisfies this.

In case b), let $x\in G$ be of order k with $k\equiv 0$ in K, and put

$$\alpha=\sum_{i=0}^{k-1} x^i \text{ and } \beta=1-x.$$

Clearly Condition 2.3 i) holds. Define a K-linear map

$$\varepsilon:\ R=K[G]\longrightarrow K \quad \text{by}$$

$$\varepsilon\Big(\sum_{g\in G} c(g)g\Big)=\sum_{g\in G} c(g).$$

ε is a ring homomorphism and $\varepsilon(\alpha)=\varepsilon(\beta)=0$, and hence Condition 2. 3 ii) holds.

Assume that $\alpha\gamma=0$ with $\gamma=\sum_{g\in G}c(g)g$. Then

$$0=\sum_{i=0}^{k-1}\sum_{g}c(g)x^{i}g=\sum_{g}\sum_{i=0}^{k-1}c(x^{-i}g)g,$$

and

$$\sum_{i=0}^{k-1}c(x^{i}g)=0 \qquad \text{for any } g\in G.$$

Let $\{g_{\omega}\}_{\omega\in\Lambda}$ be a representative of the set of right cosets $\langle x\rangle\backslash G$. Then $G=\{x^{i}g_{\omega};\ i=0,\cdots,k-1,\ \omega\in\Lambda\}$. Put

$$\delta=\sum_{g\in G}d(g)g,\quad d(g)=\sum_{j=1}^{i}c(x^{j}g_{\omega}) \qquad \text{for } g=x^{i}g_{\omega}.$$

Then

$$\beta\delta=(1-x)\sum d(g)g=\sum(d(g)-d(x^{-1}g))g.$$

If $g=x^{i}g_{\omega}$, $1\leqq i\leqq k-1$, then

$$d(g)-d(x^{-1}g)=c(x^{i}g_{\omega})=c(g).$$

If $g=x^{0}g_{\omega}$, then

$$d(g)-d(x^{-1}g)=0-\sum_{j=1}^{k-1}c(x^{j}g_{\omega})=c(x^{0}g_{\omega})=c(g).$$

Hence $\beta\delta=\gamma$, and Condition 2. 3 iii) holds.

In case c), let $x\in G$ be of infinite order and put

$$\alpha=1-x.$$

Then $\varepsilon(\alpha)=0$ and α is not a unit in $R=K[G]$. On the other hand, if we assume that $\alpha\gamma=0$ with $\gamma=\sum_{g\in G}c(g)g$, then we have

$$0=\sum_{g\in G}(c(g)-c(x^{-1}g))g,$$

namely

$$c(g)=c(x^{i}g) \qquad \text{for any } i.$$

Because x is of infinite order, this means that $\gamma=0$, and that α is not a divisor of zero. Hence $\alpha=1-x$ and $\beta=0$ satisfy Condition 2. 3. The proof is complete.

§ 3. The Semisimple Case

In this section let G be a finite group and K be a field whose characteristic does not divide the order of G (including the case $\mathrm{ch}\, K = 0$). In this case any $K[G]$-module is completely reducible by Maschke's theorem ([1], §10), namely any $K[G]$-submodule is a direct summand. Now we prove:

Proposition 3.1. *Let $R=K[G]$ be as above. Then Statement* 2.1 *holds.*

Proof. Let $C_*=\{C_n, \phi_n\}$ and $D_*=\{D_n, \psi_n\}$ be R-chain complexes, and let $h.=f.-g.: C_* \longrightarrow D_*$ be an R-chain map which satisfy

$$h_*=0: H_*(C_*) \longrightarrow H_*(D_*).$$

We shall construct an R-chain homotopy $\{\lambda_n\}$ between $h.$ and 0, as in the diagram:

$$\begin{array}{ccccccc}
\cdots \xleftarrow{\phi_{n-1}} & C_{n-1} & \xleftarrow{\phi_n} & C_n & \xleftarrow{\phi_{n+1}} \cdots \\
 & \downarrow{\scriptstyle h_{n-1}} & \searrow^{\lambda_n} & \downarrow{\scriptstyle h_n} & \\
\cdots \xleftarrow{\psi_{n-1}} & D_{n-1} & \xleftarrow{\psi_n} & D_n & \xleftarrow{\psi_{n+1}} \cdots.
\end{array}$$

Let $C_n^0=\mathrm{Ker}\, \phi_n$. Then $C_n=C_n^0 \oplus C_n^1$ (direct sum of R-modules) and $\phi_n|_{C_n^1}: C_n^1 \longrightarrow C_{n-1}$ is a monomorphism.

Since $h_*=0$ on homology, $h_n(C_n^0) \subset \psi_{n+1}(D_{n+1}) \subset D_n$. Thus we also have a direct sum decomposition

$$D_{n+1}=\mathrm{Ker}\, \psi_{n+1} \oplus D_{n+1}^1 \oplus D_{n+1}^2,$$

where

$$\psi_{n+1}|_{D_{n+1}^1}: D_{n+1}^1 \xrightarrow{\cong} h_n(C_n^0)$$

is an isomorphism and

$$\psi_{n+1}|_{D_{n+1}^2}: D_{n+1}^2 \longrightarrow D_n$$

is a monomorphism.

Now define $\lambda'_{n+1}: C_n^0 \longrightarrow D_{n+1}$ to be the composite

$$\lambda'_{n+1} = (\psi_{n+1}|_{D^1_{n+1}})^{-1} \circ h_n\colon\ C^0_n \longrightarrow h_n(C^0_n) \xrightarrow{\cong} D^1_{n+1} \subset D_{n+1}.$$

We shall define $\lambda''_{n+1}\colon C^0_n \longrightarrow \operatorname{Ker}\psi_{n+1} \subset D_{n+1}$ later. Put

$$\lambda_{n+1} = (\lambda'_{n+1} + \lambda''_{n+1}) \oplus 0\colon\ C_n = C^0_n \oplus C^1_n \longrightarrow D_{n+1}.$$

Then on $C^0_n = \operatorname{Ker}\phi_n$, we have

$$\lambda_n \circ \phi_n + \psi_{n+1} \circ \lambda_{n+1} = \psi_{n+1} \circ \lambda_{n+1} = \psi_{n+1} \circ \lambda'_{n+1} = h_n$$

as is needed, for any λ''_{n+1}.

On the other hand, on C^1_n,

$$\lambda_n \circ \phi_n + \psi_{n+1} \circ \lambda_{n+1} = \lambda_n \circ \phi_n = \lambda'_n \circ \phi_n + \lambda''_n \circ \phi_n$$

since $\phi_n(C^1_n) \subset C^0_{n-1}$. Now we have

$$\psi_n \circ \lambda'_n \circ \phi_n = h_{n-1} \circ \phi_n = \psi_n \circ h_n,$$

namely

$$(h_n - \lambda'_n \circ \phi_n)(C^1_n) \subset \operatorname{Ker}\psi_n.$$

Using the decomposition

$$C^0_{n-1} = \phi_n(C^1_n) \oplus C^2_{n-1},$$

where

$$\phi_n|_{C^1_n}\colon\ C^1_n \xrightarrow{\cong} \phi_n(C^1_n)$$

is an isomorphism, we define $\lambda''_n\colon C^0_{n-1} \longrightarrow \operatorname{Ker}\psi_n$ by

$$\lambda''_n|_{\phi_n(C^1_n)} = (h_n - \lambda'_n \circ \phi_n) \circ (\phi_n|_{C^1_n})^{-1}$$

and

$$\lambda''_n|_{C^2_{n-1}} = 0.$$

Then on C^1_n, we have

$$\lambda_n \circ \phi_n + \psi_{n+1} \circ \lambda_{n+1} = \lambda'_n \circ \phi_n + \lambda''_n \circ \phi_n = \lambda'_n \circ \phi_n + (h_n - \lambda'_n \circ \phi_n) = h_n,$$

as is needed. This completes the proof.

Combining the results of Sections 2 and 3, we have:

Main Theorem. a) *For any group G and $R = \boldsymbol{Z}[G]$, Statement* 2. 1 *does not hold.*

b) *When K is a field, G is a finite group and $R = K[G]$, Statement*

2.1 *holds if and only if* (ch K, $|G|$) $=1$ (*including the case* ch $K=0$).

c) *When K is a ring, G is a group which contains an element of infinite order and $R=K[G]$, Statement* 2.1 *does not hold.*

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