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The Asymptotic Behavior of a Variation of Polarized Hodge Structure

By

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Introduction

0. 1. The purpose of this paper is to give the asymptotic behavior of variation of polarized Hodge structures in the several-dimensional case. We do not discuss here why and how the notion of variation of Hodge structures arises and it is developed by P. A. Griffiths, P. Deligne, W. Schmid and others. What motivates us is to generalize Zucker's result to the several-dimensional case. His result is as follows: the cohomology groups of a variation of Hodge structure on the compact curve with finite singular points have also a Hodge structure. He proceeds his proof as follows. As an analytic tool, he uses the harmonic analysis (Hodge-Kodaira theory) and as a geometric tool he uses W. Schmid's result that we discuss later. By using the Kähler metric on the curve which behaves with the special property at singular points and the Hermitian metric of the vector bundle which arises from the polarization of Hodge structure, he succeeds to express the cohomology groups of the variation as the L^2-cohomology groups. Since the L^2-cohomology group is isomorphic to the space of harmonic forms, by decomposing harmonic forms into (p, q)-forms, he obtains the Hodge decomposition of the cohomology group of Hodge structure. However, in order to prove the first step—to express the cohomology group by L^2-cohomology group—he is obliged to use the result of W. Schmid on the asymptotic behavior of variations of Hodge structures at singularity.

In this paper, we generalize W. Schmid's result to the several-

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dimensional case and in the forthcoming paper, we discuss the generalization of Zucker's result.

0.2. Now we are going to recall Schmid's result. For an integer n, a Hodge structure of weight n is a couple $(H_{\mathbf{Z}}, F)$ consisting of a finitely generated $\mathbf{Z}$-module $H_{\mathbf{Z}}$ and a finite decreasing filtration F of $H_{\mathbf{C}}=\mathbf{C}\otimes_{\mathbf{Z}} H_{\mathbf{Z}}$ such that $H_{\mathbf{C}}=\bigoplus_{n=p+q}(F^p\cap\bar{F}^q)$. Here, $\bar{F}$ is the complex conjugate of F. The Weil operator C is the automorphism given by $\mathrm{C}|_{H^{pq}}=i^{p-q}$ where $H^{p,q}=F^p\cap\bar{F}^q$. A polarization S is a non-degenerate bilinear form on $H_{\mathbf{Q}}=Q\otimes_{\mathbf{Z}} H_{\mathbf{Z}}$ such that $S(F^p, F^{n+1-p})=0$ and that $S(\mathrm{C}u, \bar{v})$ is a positive definite Hermitian form on $H_{\mathbf{C}}$.

Let X be a complex manifold. A *variation of Hodge structure* of weight n on X consists of data $(H_{\mathbf{Z}}, F, S)$: $H_{\mathbf{Z}}$ is a local system on X and S is a non-degenerate bilinear form S: $H_{\mathbf{Q}}\otimes H_{\mathbf{Q}}\to\mathbf{Q}_X$ and F is a finite filtration of holomorphic vector bundles such that at any point x the stalk $(H_{\mathbf{Z}}, F, S)$ gives a polarized Hodge structure and, for any holomorphic vector field v, $vF^p\subset F^{p-1}$.

Let D be the open unit disc of $\mathbf{C}$ and D^* the pictured disc. Let $(H_{\mathbf{Z}}, F, S)$ be a variation of Hodge structure and let M be the monodromy of $H_{\mathbf{Q}}$. Then M is quasi-unipotent (i.e. its eigenvalues are the root of unity).

Set $N=\frac{1}{m}\log M^m$ taking $m\geq 1$ so that M^m is unipotent. Then N is a nilpotent endomorphism of $H_{\mathbf{Q}}$. Let $W(N)$ be the monodromy weight filtration, i.e. the unique filtration such that $NW_k(N)\subset W_{k-2}(N)$ and N^k: $gr_k^{W(N)}\xrightarrow{\sim} gr_{-k}^{W(N)}$.

Then the theorem of W. Schmid says

Theorem. *For a flat section $u\in W_k$ with $u\notin W_{k-1}$, we have*

$$|u|_z^2\sim(-\log|z|)^k \text{ when } z\to 0.$$

Here $|\ |_z$ is the norm given by the polarization of the Hodge structure at $z\in D^*$.

In this paper, we give its generalization to several-dimensional case. To simplify the explanation, we consider the two-dimensional case. Let us consider a variation of Hodge structure on $(z_1, z_2)\in D^*\times D^*$. Let $M_j (j=1,2)$ be the monodromy at $z_j=0$ and define N_j from M_j

just as N and M in the one-dimensional case. Let W_0, W_1 and W_2 be the monodromy weight filtrations of N_1+N_2, N_1 and N_2, respectively. We divide $D^*\times D^*$ into two parts

$$A_1=\{z\in D^*\times D^*;\ \log|z_1|/\log|z_2|>\varepsilon\}$$

and

$$A_2=\{z\in D^*\times D^*;\ \log|z_2|/\log|z_1|>\varepsilon\}$$

Theorem. *Let us decompose* $H_C=\bigoplus U_{p,q}$ *with* $W_{0k}=\bigoplus_{p\leq k}U_{p,q}$ *and* $W_{1k}=\bigoplus_{q\leq k}U_{p,q}$ *and take a metric* $|\ |$ *on* H_C. *Then for a flat section* $u=\sum u_{p,q}$ *of* H_C *with* $u_{p,q}\in U_{p,q}$, *we have*

$$|u|_z^2\sim\sum_{p,q}(-\log|z_2|)^{p-q}(-\log|z_1|)^q|u_{pq}|^2 \text{ for } z\in A_1.$$

The estimate on A_2 is similar. For the more precise statement see Theorem 2.4.2, 3.4.1 and 3.4.2.

The author would like to express his gratitude to Professor T. Kawai for helpful discussions.

After he finished this paper, the author received a preprint by E. Cattani, A. Kaplan and W. Schmid which contains the same result. The proofs are different.

§ 1. Filtrations

1.1. Let A be an abelian category. A finite (decreasing) filtration F of an object M of A is by definition a decreasing sequence $\{F^p\}$ of subobjects of M such that $F^p=M$ for $p\ll 0$ and $F^p=0$ for $p\gg 0$. If there is no fear of confusion, we omit the phrase "finite". We write $gr_F^p=F^p/F^{p+1}$. As usual, by $F_p=F^{-p}$, we interchange freely increasing filtrations and decreasing filtrations.

1.2. Let T be an exact contravariant functor from A to another abelian category. For a filtration F of an object M of A, let us denote by $T(F)$ the filtration of $T(M)$ given by

(1.2.1) $$T(F)^p=T(M/F^{1-p}),$$

so that we have

(1.2.2) $$gr^p_{T(F)}=T(gr_F^{-p}).$$

1.3. Let F_1 and F_2 be two filtrations.

Lemma 1.3.1. *For any* k *we have*

$$\sum_{k=p+q} F_1^p \cap F_2^q = \bigcap_{p+q=k+1} (F_1^p + F_2^q)$$

Proof. Set V^k (resp. V'^k) the left-hand side (resp. right-hand side). We have $V^k \subset V'^k$. In fact, it is enough to show $F_1^p \cap F_2^q \subset F_1^{p'} + F_2^{q'}$ for $k=p+q$ and $k+1=p'+q'$. If $p \geqq p'$ or $q \geqq q'$, this is true. Hence we may assume $p \leqq p'-1$ and $q \leqq q'-1$. Then $k=p+q \leqq (p'-1)+(q'-1) = k-1$, which is a contradiction.

Now we shall show $V'^k \subset V^k$. In order to see this, it is enough to prove

$$V'^k \cap F_2^q \subset V^k + F_2^{q+1}. \tag{1.3.1}$$

Now, we have

$$V'^k \cap F_2^q \subset (F_1^{k-q} + F_2^{q+1}) \cap F_2^q \subset F_1^{k-q} \cap F_2^q + F_2^{q+1} \subset V^k + F_2^{q+1}.$$

Q. E. D.

Definition 1.3.1. (Steenbrink-Zucker [S-Z]). *We define the* amalgum $F_1 * F_2$ *of* F_1 *and* F_2 *by*

$$(F_1 * F_2)^k = \sum_{k=p+q} F_1^p \cap F_2^q = \sum_{p+q=k+1} (F_1^p + F_2^q). \tag{1.3.2}$$

Remark 1.3.2. (i) This notion is self-dual, i.e. for an exact contravariant functor T, $T(F_1 * F_2) = T(F_1) * T(F_2)$.

(ii) For three filtrations F_1, F_2 and F_3, the relation, $(F_1 * F_2) * F_3 = F_1 * (F_2 * F_3)$ does not hold in general. Its sufficient condition is discussed in § 1.6.

1.4. Two filtrations F_1 and F_2 of M are called *n-opposed* (see Deligne [D]) if $M \simeq F_1^p \oplus F_2^q$ for $p+q=n+1$. This is equivalent to saying that $(F_1 * F_2)^n = M$ and $(F_1 * F_2)^{n+1} = 0$. The following is easy to prove (See [S-Z]).

1.5. Lemma 1.5.1. *Let* F_1, F_2 *be two filtrations of an object* M.

(i) ([S-Z]) $(F_1 * F_2)^p (gr_{F_1}^k) = F_2^{p-k} (gr_{F_1}^k)$

(ii) *Let* G *be another filtration of* M. *Then* $G = F_1 * F_2$ *if and only if* $F_1(gr_G^k)$ *and* $F_2(gr_G^k)$ *is k-opposed for any* k.

1.6. As a generalization of the amalgum of two filtrations, we make the following definition.

Definition 1.6.1. *For a family of filtrations $F_1, \dots, F_n$ of an object M, we define*

$$I(F_1, \dots, F_n)^k = \sum_{k=\Sigma p_j} \left(\bigcap_{j=1}^{n} F_j^{p_j}\right)$$

$$S(F_1, \dots, F_n)^k = \bigcap_{k+n-1=\Sigma p_j} \left(\sum_{j=1}^{n} F_j^{p_j}\right).$$

Remark 1.6.2. I and S are the dual notions to each other, i.e. for an exact contravariant functor T, we have

(1.6.1.1) $\qquad T(I(F_1, \dots, F_n)) = S(T(F_1), \dots, T(F_n))$

(1.6.1.2) $\qquad T(S(F_1, \dots, F_n)) = I(T(F_1), \dots, T(F_n))$.

Since the proof of the following lemma is simple and similar to that of Lemma 1.3.1, we omit its proof.

Lemma 1.6.2. *Let $F_1, \dots, F_n$ be a family of filtrations. Then we have*

(i) $I(F_1, \dots, F_n) \subset S(F_1, \dots, F_n)$.

(ii) $I(F_1, \dots, F_n) \subset I(I(F_1, \dots, F_l), F_{l+1}, \dots, F_n)$

(iii) $S(F_1, \dots, F_n) \supset S(S(F_1, \dots, F_l), F_{l+1}, \dots, F_n)$.

Definition 1.6.3. *A family $\{F_1, \dots, F_n\}$ of filtrations is called* distributive *if $I(F_1, \dots, F_n) = S(F_1, \dots, F_n)$.*

The naming comes from Remark 1.7.3. A single filtration and a couple of filtrations are distributive (Lemma 1.3.1).

1.7. We shall study the property of distributive families of filtrations. The following is a key lemma.

Proposition 1.7.1. *A family $\{F_1, \dots, F_n\}$ of filtrations of M is distributive if and only if the following two conditions are satisfied.*

(i) *For any q, $\{F_1 \cap F_n^q, \dots, F_{n-1} \cap F_n^q\}$ is a distributive family of filtrations of F_n^q and $I(F_1 \cap F_n^q, \dots, F_{n-1} \cap F_n^q) = I(F_1, \dots, F_{n-1}) \cap F_n^q$.*

(ii) *For any* q, $\{(F_1+F_n^q)/F_n^q, \dots, (F_{n-1}+F_n^q)/F_n^q\}$ *is a distributive family of filtrations of* M/F_n^q *and we have*

$$I((F_1+F_n^q)/F_n^q, \dots, (F_{n-1}+F_n^q)/F_n^q) = (I(F_1, \dots, F_{n-1})+F_n^q)/F_n^q.$$

Proof. Since the implication (i) + (ii) $\Rightarrow$ "$\{F_1, \dots, F_n\}$ is distributive" is easily proven, we shall show only the converse implication. Since (i) and (ii) are the dual statements, we shall show only (i). Let $\mathscr{F}$ be the family $\{F_1, \dots, F_{n-1}\}$. Then we have

$$\begin{array}{ccc} I(\mathscr{F}\cap F_n^q) & \subset & I(\mathscr{F})\cap F_n^q \\ \cap & & \cap \\ S(\mathscr{F}\cap F_n^q) & \subset & S(\mathscr{F})\cap F_n^q \end{array}$$

Hence it is enough to show

$$(A_{p,q}):\ S(\mathscr{F})^p\cap F_n^q\subset I(\mathscr{F}\cap F_n^q)^p$$

This is clear for $p\gg 0$ or $q\gg 0$. Therefore it is enough to prove $(A_{p,q+1})+(A_{p+1,q})\Rightarrow(A_{p,q})$. Now, we have, since $\{\mathscr{F}, F_n\}$ is distributive

$$\begin{aligned} S(\mathscr{F})^p\cap F_n^q &\subset (S(\mathscr{F})*F_n)^{p+q}\subset S(\mathscr{F}, F_n)^{p+q} \\ &= I(\mathscr{F}, F_n)^{p+q} \\ &= \sum_{\sum_{j=1}^{n-1} p_j+k=p+q} \left(\bigcap_{j=1}^{n-1} F_j^{p_j}\cap F_n^k\right) \end{aligned}$$

By dividing the summation into three parts

$$\{\textstyle\sum p_j=p, k=q\},\ \{\textstyle\sum p_j\le p-1,\ k\ge q+1\}$$

and

$$\{\textstyle\sum p_j\ge p+1,\ k\le q-1\},$$

we obtain

$$S(\mathscr{F})^p\cap F_n^q\subset I(\mathscr{F}\cap F_n^q)^p+F_n^{q+1}+I(\mathscr{F})^{p+1}.$$

This implies

$$S(\mathscr{F})^p\cap F_n^q\subset I(\mathscr{F}\cap F_n^q)^p+I(\mathscr{F})^p\cap F_n^{q+1}+I(\mathscr{F})^{p+1}\cap F_n^q.$$

Then $(A_{p,q+1})$ and $(A_{p+1,q})$ imply that the last two terms are contained in $I(\mathscr{F}\cap F_n^q)^p$, and hence we obtain $(A_{p,q})$. Q.E.D.

Proposition 1.7.2. *Let* $\{F_1, \dots, F_n\}$ *be a distributive family of filtrations. Then we have*

(i) $\{F_1, F_1, \dots, F_n\}$ *is also distributive.*

(ii) $\{F_1*F_2, F_3, \dots, F_n\}$, $\{F_1+F_2, F_3, \dots, F_n\}$ *and* $\{F_1\cap F_2, F_3, \dots, F_n\}$ *are also distributive. Here* F_1+F_2 (*resp.* $F_1\cap F_2$) *is the filtration defined by* $(F_1+F_2)^p=F_1^p+F_2^p$ (*resp.* $(F_1\cap F_2)^p=F_1^p\cap F_2^p$).

(iii) *For* q *and* j *we denote by* $\tilde{F}_j^q$ *the filtration given by* $(\tilde{F}_j^q)^k=M, F_j^q, 0$ *for* $k=0, 1, 2$. *Then* $\{F_1, \dots, F_n, \tilde{F}_j^q\}$ *is distributive.*

(iv) *we have*

$$(F_i^p+F_j^q)\cap F_k^r=(F_i^p\cap F_k^r)+(F_j^q\cap F_k^r).$$

Since the proof is more or less direct, we omit it.

Remark 1.7.3. $\{F_1, \dots, F_n\}$ is a distributive family if and only if the lattice (by $+$ and $\cap$) generated by F_j^q's is distributive (i.e. $(a+b)\cap c=(a+c)\cap(b+c)$).

1.8. Lemma 1.8.1. *Let* $\{F_1, \dots, F_n\}$ *be a family of filtrations of* M. *We assume* $I(F_1, \dots, F_n)^l=M$ *and* $S(F_1, \dots, F_n)^{l+1}=0$. *Then we have*

(i) *Setting, for* $p_1, \dots, p_n$ *with* $l=\sum p_j$, $H^{p_1,\dots,p_n}=\bigcap_{j=1}^{n} F_j^{p_j}$, *we have*

$$M\simeq\bigoplus_{l=p_1+\dots+p_n} H^{p_1,\dots,p_n}$$

(ii) $$F_k^q\simeq\bigoplus_{q\leq p_k} H^{p_1,\dots,p_n}$$

where the indices run over $l=\sum p_j$ *and* $p_k\geqq q$.

Proof. Set $\tilde{H}^{p_1,\dots,p_n}=M/\sum_{j=1}^{n} F_i^{p_i+1}$. Then the image of φ: $\bigoplus H^p\to M$ is $I(F_1, \dots, F_n)^l$ and the kernel of ψ: $M\to\bigoplus\tilde{H}^p$ is $S(F_1, \dots, F_n)^{l+1}$. Hence we have $\bigoplus H^p\twoheadrightarrow M\rightarrowtail\bigoplus\tilde{H}^p$. On the other hand, $H^p\to\tilde{H}^{p'}$ is zero for $p\neq p'$. Therefore $H^p\to\tilde{H}^p$ is an isomorphism and we have $\bigoplus H^p\simeq M\simeq\bigoplus\tilde{H}^p$. Now we have $\psi(F_j^q)\subset\bigoplus_{p_j\geq q}\tilde{H}^p$ and hence $F_j^q\subset\sum_{p_j\geq q} H^p$. The other inclusion is evident and we have (ii). Q.E.D.

Definition 1.8.2. *If the assumption in Lemma* 1.8.1 *is verified we say* $\{F_1, \dots, F_n\}$ *is* l-opposed.

Proposition 1.8.3. *Let* $\{F_1, \dots, F_n\}$ *be a distributive family of filtrations of* M. *Set* $G=I(F_1, \dots, F_n)=S(F_1, \dots, F_n)$. *Then we have*

(i) $\{F_j(gr^k_G)\}_{j=1,\dots,n}$ *is* k*-opposed.*

(ii) *Setting* $H^{p_1,\dots,p_n}=\bigcap_j F_j^{p_j}(gr^k_G)$ *with* $k=\sum p_j$, *we have* $\bigcap F_j^{p_j}\twoheadrightarrow H^{p_1,\dots,p_n}$.

Corollary 1.8.3. *Let* $\{F_1,\dots,F_n\}$ *be a distributive family of filtrations of a semi-simple object* M. *Then there is a direct sum decomposition*

$$M=\bigoplus I^{p_1,\dots,p_n}$$

such that

$$F_j^q=\bigoplus_{p_j\geq q} I^{p_1,\dots,p_n}$$

Proof. We take $I^{p_1,\dots,p_n}$ such that $I^{p_1,\dots,p_n}\subset\bigcap F_j^{p_j}$ and $I^{p_1,\dots,p_n}\to H^{p_1,\dots,p_n}$ is an isomorphism. Then the last proposition implies the desired result.

Definition 1.8.4. *We call* $\{I^{p_1,\dots,p_n}\}$ *the splitting of* $\{F_1,\dots,F_n\}$.

1.9. Now let M be a finite-dimensional vector space over $\boldsymbol{R}$, and $\{W^1,\dots,W^n\}$ a distributive family of increasing filtrations of M. For a multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$, we set $W_\alpha=\bigcap_j W^j_{\alpha_j}$. Let T be a set and $f_\alpha(t)$ a non-negative valued function on T. We assume

(1.9.1) For α and β such that $\beta\leqq\alpha$ (i.e. $\beta_j\leqq\alpha_j$ for any j), there exists a positive constant C such that $f_\beta(t)\leqq Cf_\alpha(t)$.

Let $|\ |_t$ be a family of norms parametrized by $t\in T$. We fix a norm $|\ |$ on M.

Lemma 1.9.1. *Let* $M=\bigoplus I_\alpha$ *be a decomposition such that* $W_\alpha=\bigoplus_{\beta\leqq\alpha} I_\beta$. *We assume*

(1.9.2) *For* $u\in I_\alpha$, *there exists a constant* C *such that* $|u|_t\leqq Cf_\alpha(t)$ *for any* $t\in T$.

Then we have the following.

(1.9.3) *There exists a constant* C *such that for any* $u\in M$, *if we write* $u=\sum u_\alpha$ *with* $u_\alpha\in I_\alpha$, *then we have*

$$|u|_t\leqq C\sum_\alpha f_\alpha(t)\,|u_\alpha| \quad \textit{for any } t.$$

Proof. Let us take a base $\{u_j\}$ of I_α. Then there exists $C>0$ such that $|u_j|_t \leqq Cf_\alpha(t)$. Hence, for any $u=\sum\lambda_j u_j$ in I_α, $|u|_t \leqq C\sum|\lambda_j|f_\alpha(t) \leqq C'|u|f_\alpha(t)$. Hence there exists a constant $C>0$ such that for any α and any $u\in I_\alpha$, $|u|_t \leqq Cf_\alpha(t)|u|$. Thus if $u=\sum u_\alpha$, we have $|u|_t \leqq \sum|u_\alpha|_t \leqq C\sum_\alpha f_\alpha(t)|u_\alpha|$.

Corollary 1.9.2. *The following condition on* $|\ |_t$ *does not depend on the choice of a splitting* $M=\bigoplus I_\alpha$ *such that* $W_\alpha=\bigoplus_{\beta\leqq\alpha} I_\beta$.

(1.9.5) *there exists a constant* $C>0$ *such that for any* $u=\sum u_\alpha$ *with* $u_\alpha\in I_\alpha$, *we have*

(1.9.5.1) $|u|_t \leqq C\sum f_\alpha(t)|u_\alpha|$ *for any* $t\in T$.

(1.9.5.2) $\sum f_\alpha(t)|u_\alpha| \leqq C|u|_t$ *for any* $t\in T$.

Definition 1.9.3. *If the condition* (1.9.5) *is satisfied, we write*

$$|u|_t\sim f_\alpha(t) \text{ on } u\in W_\alpha \text{ and } t\in T.$$

§2. Polarized Hodge Structure

2.1. Let $H_{\boldsymbol{R}}$ be a finite-dimensional $\boldsymbol{R}$-vector space with a non-degenerate bilinear form $S(*,*)$ and let $H_{\boldsymbol{C}}$ be the complexification $\boldsymbol{C}\otimes_{\boldsymbol{R}}H_{\boldsymbol{R}}$ of $H_{\boldsymbol{R}}$. Let us fix a weight $n\in\boldsymbol{Z}$ and assume

(2.1.1) $$S(u,v)=(-)^n S(v,u).$$

Let $G=\mathrm{O}(S,H_{\boldsymbol{C}})$ and let $G_{\boldsymbol{R}}=\mathrm{O}(S,H_{\boldsymbol{R}})$ be its real form. Let us denote by $\mathfrak{g}$ and $\mathfrak{g}_{\boldsymbol{R}}$ the Lie algebras of G and $G_{\boldsymbol{R}}$, respectively.

Let $\check{D}$ be the classifying space of Hodge filtrations, that is,

(2.1.2) $$\check{D}=\{F; F \text{ is a finite filtration of } H_{\boldsymbol{C}},\ F^{p\perp}=F^{n+1-p} \text{ and } \dim F^p \text{ are the given one}\}.$$

Here $\perp$ denotes the orthogonal complement with respect to S. Then $\check{D}$ is a projective homogeneous space of $G_{\boldsymbol{C}}$. A Hodge filtration F is called a Hodge structure if F and its complex conjugate $\bar{F}$ is n-opposed. Then $H_{\boldsymbol{C}}=\bigoplus_{n=p+q} H^{p,q}$ with $H^{p,q}=F^p\cap\bar{F}^q$. We define the Weil operator $\mathrm{C}\in G_{\boldsymbol{R}}$ by $\mathrm{C}|_{H^{p,q}}=i^{p-q}$. We say that S is a polarization of the Hodge structure $(H_{\boldsymbol{R}},F)$ if the Hermitian form $S(\mathrm{C}u,\bar{u})$ is positive definite. In this case, we define the norm $|*|_F$ by

(2.1.3) $\quad |u|_F^2 = S(Cu, \bar{u})$.

We denote by D the set of $F \in \check{D}$ such that S gives a polarization of F. Then D is a homogeneous space of G_R and the isotropy subgroup is compact.

2.2. Let N be a nilpotent element in $\mathfrak{g}_R$.

Then there exists a unique finite filtration W of H_C such that

(2.2.1) $\quad NW_k \subset W_{k-2}$

(2.2.2) $\quad N^k\colon gr_k^W \overset{\sim}{\to} gr_{-k}^W$ for $k \geqq 1$.

This filtration W is denoted by $W(N)$ and called the *monodromy weight filtration* of N.

We have easily

(2.2.3) $\quad W(N)_k^{\perp} = W(N)_{-1-k}$.

For $k \geqq 0$, the kernel of $N^{k+1}\colon gr_k^W \to gr_{-k-2}^W$ is denoted by $P_k(N)$ and called the *primitive part*. Then we have

(2.2.4) $\quad gr_k^W \cong \bigoplus_{j \geqq 0} P_{|k|+2j}(N)$.

2.3. Let I be a *mutually commuting* set of nilpotent elements of $\mathfrak{g}_R$. We set

(2.3.1) $\quad C(I) = \{\sum_{N \in I} t_N N;\ t_N > 0\}$.

Definition 2.3.1. *For an $F \in \check{D}$ and I as above, we say that $\{I, F\}$ forms a* nilpotent orbit *if the following conditions are satisfied.*

(2.3.2) $\quad NF^p \subset F^{p-1}$ *for any* $N \in I$.

(2.3.3) *There exists $N_0 \in C(I)$ such that $e^{iN}F \in D$ for $N \in N_0 + C(I)$.*

The following theorem is due to W. Schmid.

Theorem 2.3.2 ([S]). *Let N be a nilpotent element in $\mathfrak{g}_R$, $F \in \check{D}$ and assume that $\{N, F\}$ forms a nilpotent orbit. Let W be the monodromy weight filtration of N. Then we have the following properties.*

(2.3.4) *(F, W) is a mixed Hodge structure of weight n, i.e. $(F(gr_k^W), \bar{F}(gr_k^W))$ is $(n+k)$-opposed.*

(2.3.5) *The bilinear form $S(u, N^k v)$ on gr_k^W gives a polarization of the Hodge structure on the primitive part $P_k(N)$.*

We have the partial converse of this theorem. Let $F\in\check{D}$, N a nilpotent element of $\mathfrak{g}_{\boldsymbol{R}}$ and let W be the monodromy weight filtration of N and P_k the primitive part.

Lemma 2.3.3([C-K]). *Assume the following conditions.*

(2.3.6) *(F, W) is a mixed Hodge structure of weight n.*

(2.3.7) *(F, W) is $\boldsymbol{R}$-split, that is, if we set $I^{p,q}=F^p\cap\bar{F}^q\cap W_{p+q-n}$, then $F^p=\bigoplus_q I^{p,q}$ and $W_k=\bigoplus_{p+q\leq k+n} I^{p,q}$.*

(2.3.8) *$S(u, N^k v)$ gives a polarization of the primitive part P_k.*

Then $e^{itN}F\in D$ for $t>0$.

The following theorem is due to Cattani-Kaplan.

Theorem 2.3.4 ([C-K]). *Let I be a mutually commuting set of nilpotent elements of $\mathfrak{g}_{\boldsymbol{R}}$, $F\in\check{D}$ and assume that $\{I, F\}$ forms a nilpotent orbit. Then we have*

(2.3.9) *For any $J\subset I$, there exists a filtration $W(J)$ such that $W(J)$ is the monodromy filtration of any $N\in C(J)$.*

(2.3.10) *There exists $g\in G$ with the following properties*

(2.3.10.1) *g commutes with I.*

(2.3.10.2) *$g|_{gr_k^{W(I)}}=\mathrm{id}$ for any k.*

(2.3.10.3) *If we set $F_0=gF$, then $\{F_0, W(I)\}$ is $\boldsymbol{R}$-split and $e^{iN}F_0\in D$ for any $N\in C(I)$.*

2.4. Admitting these results in § 2.3, we shall start our arguments by the following lemma.

Lemma 2.4.1. *Let $F\in\check{D}$ and let I be a commuting finite set of nilpotent elements in $\mathfrak{g}_{\boldsymbol{R}}$. Assume that $\{I, F\}$ forms a nilpotent orbit. Then, for any decreasing sequence $I=I_0\supset I_1\supset\ldots\supset I_m$ of subsets I, $\{F, W(I_0), W(I_1), \ldots, W(I_m)\}$ is a distributive family.*

Proof. We shall prove this by the induction on m. Therefore, we may assume that

(2.4.1) $\{W(I_j)\}_{j\geq 1}$ is a distributive family.

By (2.3.10), there exists $g\in G$ commuting with I and satisfying the conditions (2.3.10.2)$\sim$(2.3.10.3). Since $gW(I_k)=W(I_k)$, we may replace F with gF, which is $\boldsymbol{R}$-split. Hence we may assume from the beginning that $(F, W(I))$ is $\boldsymbol{R}$-split. Set $I^{p,q}=F^p\cap\bar{F}^q\cap W_{p+q-n}(I)$. Then we have $H_{\boldsymbol{C}}=\bigoplus I^{p,q}$ and $F^p=\bigoplus_q I^{p,q}$.

Let us define $Y_1, Y_2\in\mathrm{End}(H_{\boldsymbol{C}})$ by

$$Y_1|_{I^{p,q}}=p \text{ and } Y_2|_{I^{p,q}}=q. \tag{2.4.2}$$

Then since $NI^{p,q}\subset I^{p-1,q-1}$ for $N\in I$, we have $[Y_1, N]=-N$, $[Y_2, N]=-N$ for $N\in I$. Hence Y_1 and Y_2 preserves the filtrations F and $W(I_k)$. By setting $I_k^{p,q}=W(I_k)\cap I^{p,q}$, we have $W(I_k)=\bigoplus_{p,q} I_k^{p,q}$.

Now by the assumption $\{W(I_j)\}_{j\geqq 1}$ is a distributive family, and hence $\{I_j^{p,q}\}_{j\geqq 1}$ is a distributive family of filtrations on $I^{p,q}$. This immediately implies the desired result. Q.E.D.

Theorem 2.4.2. *Let K be a compact subset of $\check{D}$ and let $I=\{N_1,\dots, N_l\}$ be a commuting set of nilpotent elements in $\mathfrak{g}_{\boldsymbol{R}}$. We set $I_j=\{N_i; i\geqq j\}$.*

We assume the following conditions.

(2.4.3.1) *If $F\in K$ and $N\in I$ then $NF^p\subset F^{p-1}$*

(2.4.3.2) *For $F\in K$ and $N\in \mathrm{C}(I)$ we have $e^{iN}F\in D$*

(2.4.3.3) *For any p, k and $J\subset I$, $\dim F^p\cap W_k(J)$ does not depend on $F\in K$.*

Then for any $\varepsilon>0$ we have

$$|u|^2_{e^{iN}F}\sim t_1^{p_1}(t_2/t_1)^{p_2}\dots(t_l/t_{l-1})^{p_l} \tag{2.4.4}$$

$$\text{on } u\in\bigcap_{j=1}^{l} W_{p_j}(I_j) \text{ and } N=\sum_{j=1}^{l} t_jN_j \text{ with } t_1>\varepsilon,$$

$$t_j/t_{j-1}>\varepsilon \text{ for } 2\leqq j\leqq l.$$

For this notation, see Definition 1.9.3.

Corollary 2.4.3. *Under the same notation, we have*

$$|e^{iN}u|^2_{e^{iN}F}\sim t_1^{p_1}(t_2/t_1)^{p_2}\dots(t_l/t_{l-1})^{p_l} \tag{2.4.5}$$

on $u\in\bigcap_{j=1}^{l} W_{p_j}(I_j)$ and $N=\sum_{j=1}^{l} t_jN_j$ with $t_1>\varepsilon$, $t_j/t_{j-1}>\varepsilon$ for $2\leqq j\leqq l$.

Proof. We shall prove this by the induction on l. Set $W=W(I)$. Let us take a semi-simple element Y in $\mathfrak{g}_{\mathbf{R}}$ such that

(2.4.5) $$[Y, N]=-2N \text{ for } N\in I$$

(2.4.6) $$Y|_{gr_k^W}=k\cdot\mathrm{id}$$

For example, take Y_1+Y_2-n in the proof of preceding lemma. If V_k denotes the eigen-space of Y with eigenvalue k, then $W_k=\bigoplus_{j\leq k} V_j$. Now, (2.4.5) implies

(2.4.7) $$e^{isN}=s^{-Y/2}e^{iN}s^{Y/2}$$
$$\text{for } N\in C(I) \text{ and } s>0.$$

Now, let $N=\sum_{j=1}^{l} t_jN_j$ with $t_j>\varepsilon$ be an element of $C(I)$. We set $N'=\sum_{j\geqq 2} t_jN_j/t_1$. Then we have $N=t_1(N_1+N')$. Therefore (2.4.7) implies

(2.4.8) $$\begin{aligned} e^{iN}F &= t_1^{-Y/2}e^{i(N_1+N')}t_1^{Y/2}F \\ &= t_1^{-Y/2}e^{iN'}(e^{iN_1}t_1^{Y/2}F). \end{aligned}$$

Lemma 2.4.4. *For any* $\varepsilon>0$, *let* K' *be the closure of* $\{e^{iN_1}s^{Y/2}F; F\in K, s>\varepsilon\}$. *Then* (2.4.3.1) $\sim$ (2.4.3.3) *for* (K', I_2) *are satisfied.*

Admitting this lemma for a while, we shall continue the proof of the theorem. By the hypothesis of the induction, we can apply the theorem for I_2 and K'. Therefore we have for $N'=\sum_{j=2}^{l}(t_j/t_1)N_j$ with $t_j/t_{j-1}>\varepsilon$ $(2\leqq j\leqq l)$.

(2.4.9) $$|u|^2_{e^{iN'}F'}\sim(t_2/t_1)^{p_2}\dots(t_l/t_{l-1})^{p_l}$$
$$\text{on } u\in\bigcap_{j=2}^{l} W_{p_j}(I_j) \text{ and } F'\in K'.$$

If $F'=e^{iN_1}t_1^{Y/2}F$ with $F\in K$, $t_1>\varepsilon$, then (2.4.8) implies $e^{iN}F=t_1^{-Y/2}e^{iN'}F'$. Hence, $t_1^{Y/2}\in G_{\mathbf{R}}$ gives

(2.4.10) $$|u|_{e^{iN}F}=|t_1^{Y/2}u|_{e^{iN'}F'}.$$

Now, $W(I_j)$ is invariant by Y. Hence we have $W(I_j)=\bigoplus_k(W(I_j)\cap V_k)$. Since $\{W(I_j)\}_{2\leqq j}$ is distributive, $\{W(I_j)\cap V_k\}_{j\geqq 2}$ is also a distributive family of filtrations of V_k. Hence we can write

(2.4.11) $$V_k=\bigoplus_p U_{k,p}$$

where $p=(p_2, \dots, p_l)\in \boldsymbol{Z}^{l-1}$ and

$$W_q(I_j)=\bigoplus_{p_j\leqq q} U_{k,p} \qquad \text{for } j\geqq 2. \tag{2.4.12}$$

Then (2.4.9) means that, fixing a norm $|\ |$ of H_C,

$$|u|^2_{e^{iN'}F'}\sim\sum_{p,k}(t_2/t_1)^{p_2}\dots(t_l/t_{l-1})^{p_l}|u_{k,p}|^2 \tag{2.4.13}$$

$$\text{where } u=\sum u_{k,p} \text{ with } u_{k,p}\in U_{k,p}.$$

Since $t_1^{Y/2}u=\sum_k t_1^{k/2}u_{k,p}$, (2.4.10) implies

$$|u|^2_{e^{iN}F}\sim\sum(t_2/t_1)^{p_2}\dots(t_l/t_{l-1})^{p_l}|t_1^{k/2}u_{k,p}|^2.$$

This is nothing but the meaning of (2.4.4). Q.E.D.

2.5. *Proof of Lemma* 2.4.4. We shall prove first

Sublemma 2.5.1. (i) $t^{-Y}F(t>0,\ F\in K)$ *can be continued to a continuous function from* $\{(t, F);\ t\geqq 0,\ F\in K\}$ *into* $\check{D}$.

(ii) *If we set* $F_0=t^{-Y}F|_{t=0}$, *then* $F_0(gr^W)=F(gr^W)$ *and* (F_0, W) *is* $\boldsymbol{R}$-*split.*

(iii) $e^{iN}F_0\in D$ *for any* $N\in C(I)$.

Proof. Since $\dim(F^p\cap W_k)$ is a constant function in $F\in K$, $F^p(gr_k^W)$ forms a vector sub-bundle of gr_k^W on K. Let $F_k^p\subset V_k$ be the inverse image of $F^p(gr_k^W)$ by the isomorphism $V_k\xrightarrow{\sim}gr_k^W$. Then F_k^p depends continuously on F. Hence, locally in F, there exists an isomorphism depending continuously on F

$$\varphi^p:\ \bigoplus_k F_k^p\xrightarrow{\sim}F^p \tag{2.5.1}$$

such that

$$\varphi^p(F_k^p)\subset F^p\cap W_k$$

and

$$\psi_k^p(u)=u-\varphi^p(u)\in W_{k-1} \text{ for } u\in F_k^p.$$

Therefore

$$t^{-Y}F^p=\bigoplus_k\{t^{-Y}\varphi^p(u);\ u\in F_k^p\}$$
$$=\bigoplus_k\{t^{k-Y}(u+\psi_k^p(u));\ u\in F_k^p\}.$$

Since $\psi_k^p(u)\in W_{k-1}$, $t^{k-1-Y}\psi_k^p(u)$ is a polynomial in t, and hence $t^{k-Y}\cdot$

$(u+\psi_k^p(u))$ is a polynomial in t, whose value at $t=0$ is u. Therefore we have (i) and $F_0^p=\bigoplus_k F_k^p$. Since $(F(gr_k^W),\ \bar{F}(gr_k^W))$ is a Hodge structure of weight $n+k$, we have $V_k=\bigoplus_{p+q=k+n} F_k^p\cap\bar{F}_k^q$. Therefore (F_0, W) is $\boldsymbol{R}$-split. Then (iii) is an immediate consequence of Theorem 2.3.2 and Lemma 2.3.3. Q.E.D.

Now we resume the proof of Lemma 2.4.4. The condition (2.4.3.1) for (K', I_2) is evident. We shall prove (2.4.3.2) and (2.4.3.3) for (K', I_2).

If $F'\in K'$, then there are two cases by the preceding sublemma.

(2.5.2) $$F'=e^{iN_1}t_1^{Y/2}F \qquad \text{for } F\in K,\ t_1>\varepsilon.$$

(2.5.3) $$F'=e^{iN_1}F_0 \text{ with } F_0=s^{-Y}F\,|_{s=0} \text{ for an } F\in K.$$

In the first case, for $N'\in C(I_2)$, $e^{iN'}F'=t_1^{Y/2}e^{t_1(N_1+N')}F$ belongs to D by the assumption (2.3.3.2) for (K, I). In the second case $e^{iN'}F'=e^{i(N'+N_1)}F_0$ belongs to D by Sublemma 2.5.1 (iii). For $J\subset I_2$, we shall calculate $\dim F'^p\cap W_k(J)$. In the first case, $\dim F'^p\cap W_k(J)=\dim F^p\cap W_k(J)$. In the second case, we shall show $\dim(F'^p\cap W_k(J))=\dim F^p\cap W_k(J)$, which completes the proof of Lemma 2.4.4. Now, we shall use the fact that $\{F, W(J), W\}$ and $\{F_0, W(J), W\}$ are distributive (Lemma 2.4.1). We have

$$\begin{aligned}\dim(F'^p\cap W_k(J)) &= \dim(F_0^p\cap W_k(J))\\ &= \sum_j \dim(F_0^p(gr_j^W)\cap W_k(J)(gr_j^W))\\ &= \sum_j \dim(F^p(gr_j^W)\cap W_k(J)(gr_j^W))\\ &= \dim(F^p\cap W_k(J)).\end{aligned}$$

2.6. *Proof of Corollary* 2.4.3.

For $u\in\bigcap_{j=1}^{l} W_{p_j}(I_j)$, we have

$$\begin{aligned}|e^{iN}u|^2_{e^{iN}F} &\leqq \sum\Big(\frac{1}{\alpha!}t_1^{\alpha_1}\cdots t_l^{\alpha_l}\,|N_1^{\alpha_1}\cdots N_l^{\alpha_l}u\,|_{e^{iN}F}\Big)^2\\ &\leqq \text{const. } t_1^{p_1}(t_2/t_1)^{p_2}\cdots(t_l/t_{l-1})^{p_l},\end{aligned}$$

because $N_iW_k(I_j)\subset W_k(I_j)$ and $N_iW_k(I_j)\subset W_{k-2}(I_j)$ for $i\in I_j$. Then

Lemma 1.9.1 implies that there exists $C>0$ such that $|e^{iN}u|_{e^{iN}F} \leqq C|u|_{e^{iN}F}$. Similarly we have $|e^{-iN}u|_{e^{iN}F} \leqq C|u|_{e^{iN}F}$. These two imply $|e^{iN}u|_{e^{iN}F} \sim |u|_{e^{iN}F}$.

§ 3. Variations of Polarized Hodge Structure

3.1. Let X be a complex manifold. A *variation of Hodge structure* of weight n on X is a couple $(H_{\boldsymbol{Z}}, F)$ of a locally constant $\boldsymbol{Z}_X$-module $H_{\boldsymbol{Z}}$ of finite rank and a finite filtration $\{F^k\}_{k\in\boldsymbol{Z}}$ of $\mathcal{O}_X \underset{\boldsymbol{Z}}{\otimes} H_{\boldsymbol{Z}}$ by vector subbundles, satisfying

(3.1.1) At each point $x\in X$, $(H_{\boldsymbol{Z},x}, F(x))$ is a Hodge structure of weight n.

(3.1.2) $vF^p \subset F^{p-1}$ for any holomorphic vector field v.

A variation of Hodge structure is called polarized when a bilinear homomorphism S: $H_{\boldsymbol{Z}}\otimes H_{\boldsymbol{Z}} \to \boldsymbol{Q}_X$ is given in such a way that

(3.1.3) $(H_{\boldsymbol{Z},x}, F(x), S_x)$ is a polarized Hodge structure at any $x\in X$.

3.2. Now, let X be a complex manifold and Y a closed analytic subset. Let (H_Z, F, S) be a variation of Hodge structure of weight n on $X\backslash Y$. For $x\in X\backslash Y$, let us denote by $\mathrm{C}(x)$ the Weil operator of the Hodge structure $(H_{Z,x}, F(x), S_x)$. Then

(3.2.1)
$$\langle u|v\rangle_x = S_x(\mathrm{C}(x)u, \bar{v}), \quad u, v\in H_{C,x}$$
$$||u||_x = (\langle u|u\rangle_x)^{1/2}$$

defines the Hermitian metric on the vector bundle H_C, which depends really analytically on $x\in X\backslash Y$.

We shall discuss the behavior $||u||_x$ when x goes to Y.

3.3. Before studying the asymptotic behavior of variation of Hodge structure, we shall discuss the canonical extension of integrable connections. Let X be a complex manifold and Y a closed analytic subset and $\mathscr{F}$ a $\mathscr{D}_{X\backslash Y}$-module coherent over $\mathcal{O}_{X\backslash Y}$. Let j: $X\backslash Y \hookrightarrow X$ be the open embedding. We shall define the coherent $\mathcal{O}_X$-submodule $E^{\ddagger}_{X\backslash Y}(\mathscr{F})$ of $j_*(F)$ as follows.

3. 3. 1 Assume first Y to be a normally crossing hypersurface. Then $E^{+}_{X\setminus Y}(\mathscr{F})$ is the unique locally free $\mathcal{O}_X$-submodule of $j_*\mathcal{O}_X$ which has the following property.

(3. 3. 1. 1) For any holomorphic vector field v tangent to Y, we have

$$vE^{\pm}_{X\setminus Y}(\mathscr{F})\subset E^{\pm}_{X\setminus Y}(\mathscr{F}).$$

(3. 3. 1. 2) At any non singular point y of Y and any vector field v tangent to Y such that $v\,|I_Y/I_Y^2=\mathrm{id}$, any eigenvalue of v: $E^{\pm}_{X\setminus Y}(\mathscr{F})/I_Y E^{\pm}_{X\setminus Y}(\mathscr{F})$ is contained in $\{\lambda;\ 0\leqq\pm\mathrm{Re}\lambda<1\}$.

(3. 3. 1. 3) $E^{\pm}_{X\setminus Y}(\mathscr{F})\,|_{X\setminus Y}=\mathscr{F}$

Then, $\mathscr{F}\mapsto E^{\pm}_{X\setminus Y}(\mathscr{F})$ is an exact functor. Moreover we have

(3. 3. 2) $E^{+}_{X\setminus Y}(\mathscr{F})\subset E^{-}_{X\setminus Y}(\mathscr{F})$

(3. 3. 3) $E^{\pm}_{X\setminus Y}(\mathscr{F})=(E^{\mp}_{X\setminus Y}(\mathscr{F}^{\vee}))^{\vee}$. Here ${}^{\vee}=\mathscr{H}om_{\mathcal{O}}(*,\mathcal{O})$.

3. 3. 2 In general, let f: $X'\to X$ be a proper morphism such that $Y'=f^{-1}Y$ is a normally crossing hypersurface and that $X'\setminus Y'\to X\setminus Y$ is an isomorphism. We define $E^{\pm}_{X\setminus Y}(\mathscr{F})$ as

(3. 3. 2. 1) $E^{\pm}_{X\setminus Y}(\mathscr{F})=f_*(E^{\pm}_{X'\setminus Y'}(f^{-1}\mathscr{F}))$.

This does not depend on the choice of f.

3. 4. Now, let X be a complex manifold, Y a closed analytic subset of X and $(H_{\mathbf{Z}}, F, S)$ a variation of polarized Hodge structure.

We assume

(3. 4. 1) Y is a normal crossing hypersurface.

Let $Y=\bigcup_{j\in J}Y_j$ be the decomposition to irreducible components. For a finite subset α of J set $Y_\alpha=\bigcap_{j\in\alpha}Y_j$.

Let M_j be the monodromy of $H_{\mathbf{Z}}$ around Y_j. Hence M_j defines the automorphism of $H_{\mathbf{Z}}$ on a neighborhood of Y_j. Then by [S], M_j is quasi-unipotent. Taking a positive integer m_j such that $M_j^{m_j}$ is unipotent, set $N_j=\dfrac{1}{m_j}\log M_j^{m_j}$. For a finite subset α let $W(\alpha)$ be the monodromy weight filtration of $\sum_{j\in\alpha}N_j$.

Then there exists an open neighborhood U_α of Y_α such that $W(\alpha)$ is the filtration of $H_{\mathbf{Q}}|_{U_\alpha\setminus Y}$.

Let us take $x_0 \in X$ and $\alpha = \{j;\ x_0 \in Y_j\} = \{j_1, \dots, j_l\}$. Let f_j be the defining function of Y_j, and $y_\nu = -\log|f_{j_\nu}|$. Set $\alpha_\nu = \{j_\mu;\ \mu \geq \nu\}$.

Theorem 3.4.1. *There exists $M>0$ such that for any relatively compact subanalytic set U in $X \setminus Y$ and for any $\varepsilon > 0$, we have, for a flat section u on $H_C|_U$,*

$$|u|_x^2 \sim y_1^{p_1}(y_2/y_1)^{p_2} \cdots (y_l/y_{l-1})^{p_l}$$

on $u \in \bigcap_{\nu=1}^{l} W_{p_j}(\alpha_j)$, $x \in U$, $y_\nu > M$ $(v=1, \dots, l)$ and $y_\nu/y_{\nu-1} > \varepsilon$ for $\nu = 2, \dots, l$.

Theorem 3.4.2. *$\{E^+(\mathcal{O} \otimes W(\alpha_1)), \dots, E^+(\mathcal{O} \otimes W(\alpha_l))\}$ is a distributive family of filtrations of $E^+(\mathcal{O} \otimes H_C)$. Let us take a splitting*

$$E^+(H_C) = \bigoplus U_{p_1, \dots, p_l}$$

with $E^+(\mathcal{O} \otimes W_p(\alpha_\nu)) = \bigoplus_{p_\nu \leq p} U_{p_1, \dots, p_l}$ and a C^∞ Hermitian norm $|*|$ on $E^+(\mathcal{O}_X \otimes H_C)$. Then on a neighborhood of x_0 we have

$$||u||_x^2 \sim \sum y_1^{p_1} (y_2/y_1)^{p_2} \cdots (y_l/y_{l-1})^{p_l}\ |u_{p_1 \dots p_l}|^2$$

$$\text{on } x \in \{x; y_j/y_{j-1} > \varepsilon \text{ for } j = 2, \dots, l\},$$

where $u = \sum u_{p_1 \dots p_l}$ is a section of the vector bundle $E^+(\mathcal{O} \otimes H)$ and

$$u_{p_1, \dots, p_l} \in U_{p_1, \dots, p_l}.$$

§ 4. Proof of Theorem 3.4.1 and 3.4.2

4.1. As the question is local, we may assume

(4.1.1) $\qquad X = \Delta^l$.

Here Δ is the unit disc $\{z \in \boldsymbol{C};\ |z| < 1\}$.

(4.1.2) $\qquad Y = \{z \in X; z_1 \cdots z_l = 0\}$.

Considering a branched covering by z_j^{1/m_j} if necessary, we may assume from the beginning

(4.1.3) $\qquad$ The monodromy M_j is unipotent.

Let p: $\widetilde{X \setminus Y} = \boldsymbol{C}_+^l \to X \setminus Y$ be a universal covering given by $(\tau_1, \dots, \tau_l) \mapsto$

$(z_1, \ldots, z_l)$ with $z_j = e^{2\pi i \tau_j}$. We shall trivialize $p^{-1}H_{\boldsymbol{Z}}$. Then at each point $\tilde{x} \in \widetilde{X \backslash Y}$, the Hodge filtration $F(p(\tilde{x}))$ on a fixed vector space $H_{\boldsymbol{C}}$ is given so that this defines a holomorphic map

(4.1.4) $\quad \tilde{\Phi}\colon \widetilde{X \backslash Y} \to D$

to the appropriate classification space D of polarized Hodge structure (see §2). Let $\check{D} \supset D$ be the classification space of Hodge filtration. We use the notations G, $G_{\boldsymbol{R}}$, and $\mathfrak{g}_{\boldsymbol{R}}$ as in §2. Then $N_j \in \mathfrak{g}_{\boldsymbol{R}}$ and $\tilde{\Psi}(\tau) = e^{-2\pi i \Sigma \tau_j N_j} \tilde{\Phi}(\tau)$ gives the holomorphic map $\tilde{\Psi}\colon \widetilde{X \backslash Y} \to \check{D}$, which are invariant by $\tau \mapsto \tau + m$ $(m \in \boldsymbol{Z}^l)$. Hence it decomposes

(4.1.5)
$$\begin{array}{ccc} \widetilde{X \backslash Y} & \xrightarrow{\tilde{\Psi}} & \check{D} \\ \downarrow & \nearrow_{\Psi} & \\ X \backslash Y & & \end{array}$$

By the nilpotent orbit theorem ([S]), we have

(4.1.6) $\quad \Psi$ is continued to a holomorphic map $\Psi\colon X \to \check{D}$.

Set $F_0 = \Psi(0)$ and define

$$\tilde{\Phi}_0(\tau) = e^{2\pi i \Sigma \tau_j N_j} \circ F_0.$$

Then Lemmas 8.25 and 8.27 in [S] say that there exists $g(\tau) \in G_{\boldsymbol{R}}$ such that

(4.1.7) $\quad g(\tau)\tilde{\Phi}(\tau) = a$ is a fixed point of D,

and there exist $\beta, C, M > 0$ such that

(4.1.8) $\quad \mathrm{d}_M(g(\tau)\tilde{\Phi}_0(\tau), a) \leqq C(\sum \operatorname{Im} \tau_j)^{\beta} (\sum_j e^{-2\pi \operatorname{Im} \tau_j})$ for $\operatorname{Im} \tau_j > M$.

Here d_M is a metric invariant by the action of a compact form M of G.

In particular if δ is small enough we have

(4.1.9) $\quad \tilde{\Phi}_0(\tau) \in D$

for τ with $(\sum \operatorname{Im} \tau_j)^{\beta} (\sum_j e^{-2\pi \operatorname{Im} \tau_j}) < \delta$ and $\operatorname{Im} \tau_j > M$. Since D is pseudo-convex to the horizontal direction (See [G-W], Lemma 4.2.1), $\{\tau \in \boldsymbol{C}^m_+;\ \tilde{\Phi}_0(\tau) \in D\}$ is also pseudo-convex. Moreover this contains

$$E = \{\tau;\ (\sum \operatorname{Im} \tau_j)^{\beta} (\sum_j e^{-2\pi \operatorname{Im} \tau_j}) < \delta,\ \operatorname{Im} \tau_j > M\}.$$

Remark that

(4.1.10) A connected tube domain which is pseudo-convex is convex.

Since the convex hull of a connected component of E contains $\{\tau;\ \mathrm{Im}\ \tau_j > M'\}$ for $M' \gg 0$, we finally obtain

(4.1.11) $\qquad \tilde{\Phi}_0(\tau) \in D$ for $\mathrm{Im}\ \tau_j > M'$.

Proposition 4.1.1. $|u|_{\tilde{\Phi}_0(\tau)} \sim |u|_{\tilde{\Phi}(\tau)}$ for $\mathrm{Im}\ \tau_j > M'$.

If this proposition is proven, then Theorem 3.4.1 and Theorem 3.4.2 are immediate consequence of Theorem 2.4.2 and Corollary 2.4.3.

4.2. In order to prove Proposition 4.1.1, we shall introduce $\delta(x, x')$, the function on $D \times D$.

For $x \in D$, let $\mathrm{C}(x)$ denotes the Weil operator of the Hodge structure at x. For $x, x' \in D$ set

(4.2.1) $$\delta(x, x') = \mathrm{tr}\mathrm{C}(x)^{-1}\mathrm{C}(x')$$

we have

(4.2.2) $$\langle \mathrm{C}(x)^{-1}\mathrm{C}(x')u | v\rangle_x = \langle u | v\rangle_{x'}.$$

Hence for a suitable base, $\mathrm{C}(x)^{-1}\mathrm{C}(x')$ is a positive-definite symmetric matrix so that its eigenvalues are positive. Since $\mathrm{C}(x)^{-1}\mathrm{C}(x')$ is conjugate to its inverse, if λ is an eigenvalue of $\mathrm{C}(x)^{-1}\mathrm{C}(x')$, then λ^{-1} is also its eigenvalue with the same multiplicity. Hence we have

(4.2.3) $$\delta(x, x') \geq \dim H_C.$$

If the equality holds, then $\mathrm{C}(x)^{-1}\mathrm{C}(x')$ is unipotent. Hence we have

(4.2.4) $$\delta(x, x') = \dim H_C \text{ implies } \mathrm{C}(x) = \mathrm{C}(x').$$

The relation (4.2.2) implies

(4.2.5) $$|u|_x \leqq \delta(x, x')\, |u|_{x'}.$$

Lemma 4.2.1. (i) *Let Z be a complex manifold. Let φ and φ' be horizontal holomorphic maps from Z to D. Then $\delta(\varphi(z), \varphi'(z))$ is plurisubharmonic.*

(ii) *For any $C > 0$, $x_0 \in D$, $\{x \in D;\ \delta(x, x_0) \leq C\}$ is a compact set.*

Admitting this lemma for a while, we shall prove Proposition 4.1.1. By (4.1.8), for $0 < \varepsilon \ll \delta \ll 1$, we have

(4.2.6) $$\delta(g(\tau)\tilde{\Phi}_0(\tau),\ a) < \dim H_C + \delta$$

for $\tau \in E = \{\tau \in \boldsymbol{C}_+^l;\ (\sum \mathrm{Im}\ \tau_j)^\beta (\sum_j e^{-2\pi \mathrm{Im}\tau_j}) < \varepsilon,\ \mathrm{Im}\ \tau_j > M\}$. This implies

(4.2.7) $$\delta(\tilde{\Phi}_0(\tau), \Phi(\tau)) < \dim H_C + \delta \text{ for } \tau \in E.$$

By Lemma 4.2.1 (i), $\{\tau;\ \delta(\tilde{\Phi}_0(\tau),\ \Phi(\tau)) < a\}$ is pseudo-convex. Hence (4.1.10) implies

(4.2.8) $$\delta(\tilde{\Phi}_0(\tau),\ \Phi(\tau)) < \dim H_C + \delta \text{ for } \operatorname{Im} \tau_j > M'.$$

Now, it is enough to apply (4.2.5).

4.3. *Proof of Lemma* 4.2.1. (i) We may assume that Z is an open set of $\boldsymbol{C}$. We fix a reference point x_0 of X. Set $\varphi(z) = g(z) x_0$ with a C^∞-function $g(z) \in G_{\boldsymbol{R}}$. Set $\mathfrak{g}^{p,-p} = \{A \in \mathfrak{g};\ AF(x_0)^k \subset F(x_0)^{k+p}$ and $A\overline{F(x_0)}^k \subset \overline{F(x_0)}^{k-p}\}$. Then $\mathfrak{g} = \bigoplus \mathfrak{g}^{p,-p}$ and $\overline{\mathfrak{g}^{p,-p}} = \mathfrak{g}^{-p,p}$. That φ is holomorphic is equivalent to

(4.3.1) $$g^{-1} g_{\bar{z}} \in \bigoplus_{p \geq 0} \mathfrak{g}^{p,-p}.$$

Here $g_{\bar{z}}$ is the derivative of g with respect to $\bar{z}$. That φ is horizontal is equivalent to

(4.3.2) $$g^{-1} g_z \in \bigoplus_{p \geq -1} \mathfrak{g}^{p,-p}.$$

Hence $h = g^{-1} g_z \in \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{-1,1}$. Set $h = h_0 + h_{-1}$ with $h_0 \in \mathfrak{g}^{0,0}$ and $h_{-1} \in \mathfrak{g}^{-1,1}$. Then $\bar{h} = g^{-1} g_{\bar{z}} = \bar{h}_0 + h_1$ with $h_1 = \bar{h}_{-1} \in \mathfrak{g}^{1,-1}$. Then integrability condition implies

(4.3.3) $$h_z - \bar{h}_{\bar{z}} = [h, \bar{h}].$$

Hence we obtain

(4.3.4) $$h_{1z} = [h_1,\ h_0]$$
$$h_{-1\bar{z}} = [h_{-1},\ \bar{h}_0].$$

We have $C = C(\varphi(z)) = gC_0 g^{-1}$ where $C_0 = C(x_0)$. Hence

(4.3.5) $$C_z = g[h, C_0] g^{-1}.$$

Now, we have

(4.3.6) $$[h_0,\ C_0] = 0$$

(4.3.7) $$h_1 C_0 + C_0 h_1 = h_{-1} C_0 + C_0 h_{-1} = 0.$$

Hence we obtain

(4.3.8) $$C_z = 2g h_{-1} C_0 g^{-1}.$$

The easy calculation shows

(4.3.9) $$C_{z\bar{z}} = 2g(h_1 h_{-1} + h_{-1} h_1) C_0 g^{-1}.$$

We define $C'=C(\varphi'(z))$, g', h', h'_0, h'_1, h'_{-1} similarly. Then $u=\delta(\varphi(z), \varphi'(z))=(-)^n\mathrm{tr}CC'$, and we have, setting $\varphi=g^{-1}g'$,

$$u_{z\bar{z}}=2(-)^n\mathrm{tr}((h_1h_{-1}+h_{-1}h_1)C_0\varphi C_0\varphi^{-1} + (h'_1h'_{-1}+h'_{-1}h'_1)C_0\varphi^{-1}C_0\varphi+2h_{-1}C_0\varphi h'_1C_0\varphi^{-1} +2h_1C_0\varphi h'_{-1}C_0\varphi^{-1}). \tag{4.3.10}$$

If we denote by * the adjoint with respect to the Hermitian form $S(C_0u, \bar{v})$, then we have $\mathrm{tr}AA^*\geq 0$ for $A\in\mathrm{End}(H_C)$. If we define ${}^tA$ by $S({}^tAu, v)=S(u, Av)$, then $A^*=C_0{}^t\bar{A}C_0^{-1}$. Hence we obtain

$$\mathrm{tr}AC_0{}^t\bar{A}C_0^{-1}\geq 0. \tag{4.3.11}$$

By setting

$$A=\varphi^{-1}h_{\pm 1}+h'_{\pm 1}\varphi^{-1},$$

and using ${}^tY=-Y$ for $Y\in\mathfrak{g}$, we obtain

$$\mathrm{tr}AC_0{}^t\bar{A}C_0^{-1}=(-)^n\mathrm{tr}(h_{\pm 1}h_{\mp 1}C_0\varphi C_0\varphi^{-1}+h'_{\mp 1}h'_{\pm 1}C_0\varphi^{-1}C_0\varphi +h_1C_0\varphi h'_{-1}C_0\varphi^{-1}+h_{-1}C_0\varphi h'_1C_0\varphi^{-1})\geq 0.$$

This shows $u_{z\bar{z}}\geq 0$.

4.4. *Proof of Lemma* (ii). In order to prove this, let $\{x_n\}$ be a sequence in D such that $\{\delta(x_n, x_0)\}_n$ is bounded and that x_n converges to a point $x_\infty\in\check{D}$. It is enough to show $x_\infty\in D$. Then, $\langle\,|\rangle_{x_n}$ tends to a positive definite Hermitian form $\langle\,|\rangle_\infty$.

Let $H_C=\bigoplus H_n^{p,q}$ be the Hodge decomposition at x_n. We may assume $H_n^{p,q}\to H_\infty^{p,q}$. Since $H_n^{p,q}$'s are orthogonal to each other with respect to $\langle\,|\rangle_{x_n}$, $\{H_\infty^{p,q}\}$ is orthogonal to $\langle\,|\rangle_\infty$. Hence $\bigoplus H_\infty^{p,q}\to H_C$ is injective. Comparing the dimension, we have $H_C=\bigoplus H_\infty^{p,q}$. Therefore this is the Hodge decomposition at x_∞ and $S(C(x_\infty)u, \bar{v})=\langle u\,|v\rangle_\infty$. Thus x_∞ belongs to D.

References

[C-K] E. Cattani and A. Kaplan, Polarized mixed Hodge structures and the local monodromy of a variation of Hodge structure, *Inv. Math.*, **67** (1982), 101-115.

[D] P. Deligne, Théorie de Hodge I, II and III, *Actes, Congrès int. math.*, 1970, Tome **1**, 425-430, *Publ. Math. I.H.E.S.*, **40** (1972), 5-57 and *ibid.*, **44** (1974), 5-77.

[G-S] P. Griffiths and W. Schmid, Locally homogeneous complex manifolds, *Acta Math.*, **123** (1969), 253-302.

[S] W. Schmid, Variation of Hodge structure : the singularities of the period mappings, *Inv. Math.*, **22** (1973), 211–319.
[S–Z] J. Steenbrink and S. Zucker, Variation of mixed Hodge structure I, *preprint.*
[Z] S. Zucker, Hodge theory with degenerating coefficients : L_2 cohomology in the Poincaré metric, *Annals of Math.*, **109** (1979), 415–476.