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On Z_q-Equivariant Immersions for $q=2^r$

Dedicated to Professor Nobuo Shimada on his 60th birthday

By

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§ 1. Introduction

Let Z_q be the cyclic group of order q, where q is an integer >1. A C^∞-differentiable map f of a Z_q-manifold in another Z_q-manifold is called a Z_q-equivariant immersion (or simply a Z_q-immersion) if f is an immersion and a Z_q-equivariant map.

Let m and k be non-negative integers. Euclidean $(m+2k)$-space R^{m+2k} has a structure of a Z_q-manifold (R^{m+2k}, Z_q) defined by the action: $Z_q \times R^{m+2k} \to R^{m+2k}$;

$$(T, (t_1, \cdots, t_m, z_{m+1}, \cdots, z_{m+k})) \longmapsto (t_1, \cdots, t_m, Tz_{m+1}, \cdots, Tz_{m+k}),$$

where $T=\exp(2\pi i/q)$ is the generator of Z_q $(\subset S^1)$, $t_1, \cdots, t_m$ are real numbers $(\in R)$ and $z_{m+1}, \cdots, z_{m+k}$ are complex numbers $(\in C=R^2)$. This Z_q-manifold is also written by $R^{m,2k}$.

The unit $(2n+1)$-sphere S^{2n+1} in complex $(n+1)$-space C^{n+1} has a structure of a Z_q-manifold (S^{2n+1}, Z_q) defined by the action: $Z_q \times S^{2n+1} \to S^{2n+1}$;

$$(T, (z_0, \cdots, z_n)) \longmapsto (Tz_0, \cdots, Tz_n),$$

where $z_0, \cdots, z_n$ are complex numbers with $\sum_{j=0}^n |z_j|^2=1$. This action is free and differentiable of class C^∞. The orbit differentiable manifold S^{2n+1}/Z_q is the mod q standard lens space $L^n(q)$. As is easily seen, there is a Z_q-immersion of (S^{2n+1}, Z_q) in $R^{m,0}$ if and only if there is an immersion of $L^n(q)$ in R^m.

A. Jankowski obtained in [1] some non-existence theorems for Z_2-immersions. In [2] we considered Z_{p^r}-immersions, where p is an odd prime. In this note we prove some non-existence theorems for Z_{2^r}-immersions.

§ 2. Statements of Results

Theorem 1. *Let r be an integer >1, and n and k be integers with $0 \leqq k \leqq n$. Assume that there is an integer m satisfying the following conditions*:

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(i) $0<k+m\leqq n/2$,

(ii) $\binom{n+m}{k+m}\equiv(-1)^{k+m}(2s+1)^2 \bmod 2^r$ *for some integer* s,

(iii) $n+m+1\not\equiv 0 \bmod 2^{n-m-k-1}$.

Then there does not exist a Z_{2^r}*-immersion of* (S^{2n+1}, Z_{2^r}) *in* $(R^{2n+2m+2k+1}, Z_{2^r})=R^{2n+2m+1,2k}$.

If $k=0$, we have a new result on the non-existence of an immersion of $L^n(2^r)$ in $R^{2n+2m+1}$.

Corollary 2. *Let* r *be an integer* >1. *Assume that there is an integer* m *satisfying the following conditions*:

(i) $0<m\leqq n/2$,

(ii) $\binom{n+m}{m}\equiv(-1)^{m}(2s+1)^2 \bmod 2^r$ *for some integer* s,

(iii) $n+m+1\not\equiv 0 \bmod 2^{n-m-1}$.

Then there does not exist an immersion of $L^n(2^r)$ *in* $R^{2n+2m+1}$.

For integers r, n and k such that $r>1$ and $0\leqq k\leqq n$, define the integer $L(r, n, k)$ as follows:

$$L(r, n, k)=\max\left\{j\in Z \mid 1\leqq j\leqq n/2, \binom{n-k+j}{j}\not\equiv 0 \bmod 2^{r+n-2j+1-\varepsilon}\right\},$$

where $\varepsilon=0$ or 1 according to n being even or odd respectively. Then we have

Theorem 3. *There does not exist a* Z_{2^r}*-immersion of* (S^{2n+1}, Z_{2^r}) *in* $(R^{2n+2L}, Z_{2^r})=R^{2n+2L-2k,2k}$, *where* $L=L(r, n, k)$.

Corollary 4. *There does not exist an immersion of* $L^n(2^r)$ *in* R^{2n+2L} *where* $L=L(r, n, 0)$.

This corollary is known (cf. [3, Corollary 1.5] or [4, Chapter 6, Proposition 4.16]).

Corollary 2 is very restricted. But, in some cases, this gives better results than Corollary 4. For example, $L^{21}(4)$ (resp. $L^{36}(4)$) is not immersible in R^{62} (resp. R^{108}) by Corollary 4, but $L^{21}(4)$ (resp. $L^{36}(4)$) is not immersible in R^{63} (resp. R^{109}) by Corollary 2.

§ 3. Preliminaries

In this section we recall some known results according to [2, Lemmas 2.1–2.3 and Proposition 2.4].

For a Z_q-space (X, Z_q), let $\theta(X, Z_q)$ denote a Z_q-vector bundle $(X\times R^2, X, \pi, R^2)$ defined as follows:

(1) $\pi: X\times R^2\to X$ is the projection onto the first factor.

(2) Z_q acts on $X\times R^2$ diagonally; $T(x, z)=(Tx, Tz)$, where $x\in X$, $z\in R^2$ and $T=\exp(2\pi i/q)$.

Lemma 3.1. *If X and Y are Z_q-spaces and $f: X\to Y$ is a Z_q-map, then $f^*\theta(Y, Z_q)=\theta(X, Z_q)$.*

A G-vector bundle $E\to X$ determines a vector bundle $E/G\to X/G$ and this correspondence induces a homomorphism $\rho: KO_G(X)\to KO(X/G)$.

Let $r\eta$ be the real restriction of the canonical complex line bundle η over $L^n(q)$. Then we see

Lemma 3.2. $\rho(\theta(S^{2n+1}, Z_q))=r\eta$.

Define the action of Z_q on the total space of the Whitney sum $m\oplus k\theta(R^{m+2k}, Z_q)$ of the m-dimensional trivial bundle m over $R^{m\ 2k}$ and $k\theta(R^{m+2k}, Z_q)$ by

$$\begin{aligned}&T((u, t_1), \cdots, (u, t_m), (u, z_{m+1}), \cdots, (u, z_{m+k}))\\&=((Tu, t_1), \cdots, (Tu, t_m), (Tu, Tz_{m+1}), \cdots, (Tu, Tz_{m+k})),\end{aligned}$$

where $u\in R^{m,2k}$, $t_i\in R$ $(i=1, \cdots, m)$, $z_{m+j}\in R^2$ $(j=1, \cdots, k)$ and T is the generator of Z_q. Then we have

Lemma 3.3. *There is a Z_q-bundle isomorphism of the tangent Z_q-bundle $\tau(R^{m,2k})$ onto the Z_q-bundle $m\oplus k\theta(R^{m+2k}, Z_q)$.*

Using γ-operations, we obtain

Proposition 3.4. *Let n and k be integers with $0\leqq k\leqq n$, and put*

$$L=\max\left\{j\,\middle|\,\binom{n-k+j}{j}(r\eta-2)^j\neq 0\right\}.$$

Then there does not exist a Z_q-immersion of (S^{2n+1}, Z_q) in $(R^{2n+2L}, Z_q)=R^{2n+2L-2k,2k}$.

§4. Proofs of Theorems 1 and 3

Two spaces X and Y are said to be mod q S-related, if there are non-negative integers m and n and a map $f: S^mX\to S^nY$ which induces isomorphisms of all homology groups with Z_q-coefficients, where S^kZ denotes the k-fold suspension of a space Z. The following is proved in the line of the proof of Proposition 3.1 of [2].

Proposition 4.1. *Let r be a positive integer, and l and n be integers with $0<l\leqq n/2$. Assume that there is a positive integer t satisfying the following conditions:*

(i) *$(l+t)r\eta$ has linearly independent $2t$ cross-sections, where $r\eta$ is the real restriction of the canonical complex line bundle η over $L^n(2^r)$.*

(ii) $\binom{l+t}{l}\equiv(2s+1)^2 \bmod 2^r$ *for some integer s.*

Then the stunted lens spaces $L^n(2^r)/L^{l-1}(2^r)$ aud $L^{n+t}(2^r)/L^{l-1+t}(2^r)$ are $\bmod 2^r$ *S-related.*

Combining this proposition with Proposition 3.2 in [2], we have

Proposition 4.2. *Let r be an integer >1. Then, under the assumption of Proposition* 4.1, $t\equiv 0 \bmod 2^{n-l-1}$.

Proof of Theorem 1. Put $q=2^r$, $r>1$. Suppose that there exists a Z_q-immersion $f:(S^{2n+1}, Z_q)\to R^{2n+2m+1,2k}$. Let ν be the normal Z_q-bundle of f. Then we have

$$\tau(S^{2n+1}, Z_q)\oplus\nu=f^*\tau(R^{2n+2m+1,2k}).$$

Since $\rho(\tau(S^{2n+1}, Z_q))=\tau(L^n(q))$ (=the usual tangent bundle of $L^n(q)$), we have, by Lemmas 3.1-3.3,

$$\begin{aligned}\tau(L^n(q))\oplus\rho\nu&=\rho f^*((2n+2m+1)\oplus k\theta(R^{2n+2m+1+2k}, Z_q))\\&=(2n+2m+1)\oplus k\rho\theta(S^{2n+1}, Z_q)=(2n+2m+1)\oplus kr\eta\,.\end{aligned}$$

It is well-known that $\tau(L^n(q))\oplus 1=(n+1)r\eta$. Thus

$$(n+1-k)r\eta+\rho\nu=2n+2m+2\,.$$

Let $A=u\cdot 2^{r+n-1}$, where u is some positive integer. Then $A(r\eta-2)=0$, because $r\eta-2$ $(\in\widetilde{KO}(L^n(q)))$ is of order $2^{r+n-1-\varepsilon}$, where $\varepsilon=0$ or 1 according to n being even or odd respectively (cf. [3, Theorem 1.4]). Hence, if we take u such that $2A-2n-2+2k>2n+1$, we have

$$(A-n-1+k)r\eta=(2A-2n-2m-2)\oplus\rho\nu\,.$$

Put $l=k+m$ and $t=A-n-m-1$. Then the above equality implies that $(l+t)r\eta$ has linearly independent $2t$ cross-sections. Since we may choose u so that $\binom{A-n-1+k}{k+m}\equiv\binom{-n-1+k}{k+m} \bmod 2^r$, we have, by (ii),

$$\binom{l+t}{l}=\binom{A-n-1+k}{k+m}\equiv\binom{-n-1+k}{k+m}=(-1)^{k+m}\binom{n+m}{k+m}\equiv(2s+1)^2 \bmod 2^r.$$

We therefore see, by Proposition 4.2, that $t\equiv 0 \bmod 2^{n-m-k-1}$, and hence $n+m+1 \equiv 0 \bmod 2^{n-m-k-1}$. But this contradicts (iii). q. e. d.

There are errors in [2]. As is seen in the proof of Theorem 1, we must correct them as follows:

Line 14 in p. 344 should be replaced by

(ii) $\binom{n+m}{n-k} \equiv (-1)^{k+m}(ap+b)^2 \bmod p^r$ for some integers a and b with $(b, p)=1$

Line 14 in p. 347 should be replaced by

(ii) $\binom{l+t}{l} \equiv (ap+b)^2 \bmod p^r$ for some integers a and b with $(b, p)=1$.

Proof of Theorem 3. For $1 \leqq j \leqq n/2$, the order of $(r\eta-2)^j$ $(\in \widetilde{KO}(L^n(2^r)))$ is equal to $2^{r+n-2j+1-\varepsilon}$, where $\varepsilon=0$ or 1 according to n being even or odd respectively (cf. [3, Theorem 1.4]). Thus the result follows from Proposition 3.4.

q. e. d.

References

[1] Jankowski, A., Note on Z_2-equivariant immersions, *Ann. Soc. Math. Polonae, Comment. Math. Prace Mat.* **17** (1973), 81–85.

[2] Kobayashi, T., Some Z_q-equivariant immersions, *Hiroshima Math. J.* **6** (1976), 343–351.

[3] Kobayashi, T. and Sugawara, M., Note on *KO*-rings of lens spaces mod 2^r, *Hiroshima Math. J.* **8** (1978), 85–90.

[4] Mahammed, N., Piccinini, R. and Suter, U., Some applications of topological K-theory, *North-Holland Math. Studies* Vol. **45**, North-Holland, Amsterdam, 1980.