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On a Sufficient Condition for Well-posedness in Gevrey Classes of Some Weakly Hyperbolic Cauchy Problems

By

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Introduction

In this paper we shall study well-posedness of the Cauchy problem for some weakly hyperbolic operators in Gevrey classes. That is to say, we consider whether we can determine a function space in which the Cauchy problem for given weakly hyperbolic operator is well-posed or not.

This question has been studied by several mathematicians.

The results independent of the lower order terms were obtained by Ohya [8], Leray-Ohya [6], Steinberg [9], Ivrii [3], Trepreau [10], Bronstein [1], Kajitani [5] and Nishitani [7], which show that the multiplicity of the characteristic roots determines the well-posed class.

On the other hand, in [4] Ivrii presented two interesting examples.

(I) Let $P=\partial_t^2-t^{2\mu}\partial_x^2+at^{\nu}\partial_x$, where μ, ν are non-negative integers and a is a non-zero constant. When $0\leqq\nu<\mu-1$, the Cauchy problem for P is $\gamma_{loc}^{(\kappa)}$-well-posed, if and only if $1\leqq\kappa<(2\mu-\nu)/(\mu-\nu-1)$.

(II) Let $P=\partial_t^2-x^{2\mu}\partial_x^2+ax^{\nu}\partial_x$, where μ, ν are non-negative integers and a is a non-zero constant. When $0\leqq\nu<\mu$, the Cauchy problem for P is $\gamma_{loc}^{(\kappa)}$-well-posed, if and only if $1\leqq\kappa<(2\mu-\nu)/(\mu-\nu)$. These two cases show that the lower order terms have a great effect on the well-posed class.

Igari [2] and Uryu [12] extended (I) for more general operators respectively and Uryu-Itoh [13] extended (II) for second order weakly hyperbolic operators.

In this article we shall consider the most general case of (II).

§1. Statement of the Result and Remarks

Let $(x, t)\in \boldsymbol{R}^n\times[0, T]$ and $(D_x, D_t)=(D_{x_1}, \cdots, D_{x_n}, D_t)=(-\sqrt{-1}\partial_{x_1}, \cdots, -\sqrt{-1}\partial_{x_n}, -\sqrt{-1}\partial_t)$. Let us denote by (ξ, τ) the dual variable of (x, t).

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Next we shall define the Gevrey classes.

Definition 1.1. ($\gamma_{loc}^{(\kappa)}, \gamma^{(\kappa)}$; $\kappa>1$) $f(x)\in\gamma_{loc}^{(\kappa)}$ implies that $f(x)\in C^\infty(\boldsymbol{R}^n)$ and for any compact set $K\subset\boldsymbol{R}^n$ there exist constants c, $R>0$ such that

$$(1.1)\qquad |D_x^\alpha f(x)|\leqq cR^{|\alpha|}|\alpha|!^\kappa,\ x\in K, \qquad \text{for any } \alpha.$$

$f(x)\in\gamma^{(\kappa)}$ implies that $f(x)\in C^\infty(\boldsymbol{R}^n)$ and (1.1) holds for any $x\in\boldsymbol{R}^n$.

Let $L=L(x, t, D_x, D_t)=L_0(x, t, D_x, D_t)+L_1(x, t, D_x, D_t)$, where

$$(1.2)\qquad L_0(x, t, D_x, D_t)=D_t^m+\sum_{\substack{|\alpha|+j=m\\ j\leqq m-1}}\sigma(x)^{|\alpha|\mu}a_{\alpha,j}(x, t)D_x^\alpha D_t^j$$

and

$$(1.3)\qquad L_1(x, t, D_x, D_t)=\sum_{|\alpha|+j\leqq m-1}\sigma(x)^{\nu_{\alpha,j}}a_{\alpha,j}(x, t)D_x^\alpha D_t^j.$$

We assume the following conditions on L.

(1.4) τ-roots of

$$\tilde{L}_0(x, t, \xi, \tau)=\tau^m+\sum_{\substack{|\alpha|+j=m\\ j\leqq m-1}}a_{\alpha,j}(x, t)\xi^\alpha\tau^j=0$$

are real and distinct.

$$(1.5)\qquad a_{\alpha,j}(x, t)\in\mathscr{B}([0, T], \gamma^{(\kappa)}).$$

(1.6) $\sigma(x)\in\gamma^{(\kappa)}$ and is a real-valued function.

(1.7) μ is a positive integer and $\nu_{\alpha,j}$ are non-negative integers.

Now we shall define ρ as follows.

$$(1.8)\qquad \rho=\max_{0\leqq|\alpha|+j\leqq m-1}\{(m-j-\nu_{\alpha,j}/\mu)/(m-j-|\alpha|), 1\}.$$

Then we have

Theorem 1.1. *Under* (1.4)-(1.7), *if* $1\leqq\kappa<\rho/(\rho-1)$, *the Cauchy problem for* L:

$$(1.9)\qquad \begin{cases} Lu(x, t)=f(x, t) & \text{in}\quad \boldsymbol{R}^n\times(0, T] \\ D_t^i u(x, t)|_{t=0}=u^i(x),\ i=0, \cdots, m-1 & \text{on}\quad \boldsymbol{R}^n \end{cases}$$

is $\gamma_{loc}^{(\kappa)}$*-well-posed, i. e. for any* $u^i(x)\in\gamma_{loc}^{(\kappa)}(i=0, \cdots, m-1)$ *and any* $f(x, t)\in\mathscr{B}([0, T], \gamma_{loc}^{(\kappa)})$, *there exists a unique solution* $u(x, t)\in\mathscr{B}([0, T], \gamma_{loc}^{(\kappa)})$ *of* (1.9).

Remark 1.1. When $\rho=1$, (1.9) is C^∞-well-posed.

Remark 1.2. In the case of the finite degeneracy our sufficient condition is best.

Remark 1.3. From Remark 1.1, we may only consider the case that $0\leqq\nu_{\alpha,j}\leqq|\alpha|\mu$.

§2. Proof of Theorem 1.1

In this section we shall reduce Theorem 1.1 to Theorem 2.1.

Definition 2.1. We say that $f(x)\in H^{\infty}$ belongs to $\Gamma^{(\kappa)}$ if there exist constants c, $R>0$ such that

$$\|D_x^{\alpha}f(x)\|\leqq cR^{|\alpha|}|\alpha|!^{\kappa} \quad \text{for any } \alpha, \tag{2.1}$$

where $\|\cdot\|$ denotes L^2-norm with respect to x.

Let $P=P(x, t, D_x, D_t)=P_0(x, t, D_x, D_t)+P_1(x, t, D_x, D_t)$.

$$P_0(x, t, \xi, \tau)=\prod_{j=1}^{m}(\tau-\sigma(x)^{\mu}\lambda_j(x, t, \xi)), \tag{2.2}$$

where $\lambda_j(x, t, \xi)\in\mathscr{B}([0, T], S^1(\kappa))$ are real-valued and $|(\lambda_i-\lambda_j)(x, t, \xi)|\geqq\delta\langle\xi\rangle$ for some constant $\delta>0$ if $i\neq j$. Further

$$P_1(x, t, \xi, \tau)=\sum_{k=0}^{m-1}\sum_{|\alpha|+j=k}\sigma(x)^{\nu_{\alpha,j}}a_{\alpha,j}(x, t, \xi)\tau^j, \tag{2.3}$$

where $a_{\alpha,j}(x, t, \xi)\in\mathscr{B}([0, T], S^{|\alpha|}(\kappa))$. Here $S^j(\kappa)$ are symbol classes defined in Appendix.

Then we get the following theorem.

Theorem 2.1. *Under* (1.4)-(1.7), *if* $1\leqq\kappa<\rho/(\rho-1)$, *the Cauchy problem for* P *is* $\Gamma^{(\kappa)}$*-well-posed.*

In order to prove Theorem 1.1, it is sufficient to show Theorem 2.1. For since an operator L is changed into above operator P by spacelike transformation, we can see that a domain of dependence is finite. Hence using a partition of unity Theorem 1.1 follows from Theorem 2.1.

We shall prove Theorem 2.1 by the method of successive approximations. Therefore we decompose P as follows and consider the following scheme.

$$P(x, t, D_x, D_t)=Q_0(x, t, D_x, D_t)+Q_1(x, t, D_x, D_t),$$

where as $\nu_{\alpha,j}=|\alpha|\mu$

$$Q_0(x, t, D_x, D_t)=P_0(x, t, D_x, D_t)+\sum_{k=0}^{m-1}\sum_{|\alpha|+j=k}\sigma(x)^{\nu_{\alpha,j}}a_{\alpha,j}(x, t, D_x)D_t^j \tag{2.4}$$

and as $0\leqq\nu_{o,j}<|\alpha|\mu$

$$Q_1(x, t, D_x, D_t)=\sum_{k=1}^{m-1}\sum_{\substack{|\alpha|+j=k,\\ |\alpha|\neq 0}}\sigma(x)^{\nu_{\alpha,j}}a_{c,j}(x, t, D_x)D_t^j. \tag{2.5}$$

$$\begin{cases} Q_0u_0(x, t)=f(x, t) & \text{in } R^n\times(0, T] \\ D_t^iu_0(x, t)|_{t=0}=u^i(x),\ 0\leqq i\leqq m-1 & \text{on } R^n \end{cases} \tag{2.6$_0$}$$

and for $j\geqq 1$

$$(2.6)_j \quad \begin{cases} Q_0u_j(x, t)=-Q_1u_{j-1}(x, t) & \text{in} \quad \boldsymbol{R}^n\times(0, T] \\ D_t^iu_j(x, t)|_{t=0}=0, \ 0\leqq i\leqq m-1 & \text{on} \quad \boldsymbol{R}^n \end{cases}$$

Here we refer to Uryu [11].

Proposition 2.1. *The Cauchy problem for Q_0 is H^∞-well-posed.*

Since $\Gamma^{(\kappa)}\subset H^\infty$, $u_0(x, t)$ which is a solution of $(2.6)_0$ belongs to $\mathscr{B}([0, T], H^\infty(\boldsymbol{R}^n))$ by Proposition 2.1. If we note that Q_1 is a pseudo-differential operator in x, then we obtain that $Q_1u_0\in\mathscr{B}([0, T], H^\infty(\boldsymbol{R}^n))$. Hence it follows from $(2.6)_1$ and Proposition 2.1 that $u_1(x, t)\in\mathscr{B}([0, T], H^\infty(\boldsymbol{R}^n))$. Repeating these steps we get that for any $j\geqq 0$, $u_j(x, t)\in\mathscr{B}([0, T], H^\infty(\boldsymbol{R}^n))$. Therefore it is sufficient to show that the formal solution

$$(2.7) \qquad u(x, t)=\sum_{j=0}^{\infty}u_j(x, t)$$

converges in $\mathscr{B}([0, T], \Gamma^{(\kappa)})$.

Our plan is as follows. In § 3 we shall get an energy inequality for Q_0 in L^2. In § 4 we shall estimate derivatives of a solution of (2.8):

$$(2.8) \quad \begin{cases} Q_0v(x, t)=g(x, t) \\ D_t^iv(x, t)|_{t=0}=0, \quad 0\leqq i\leqq m-1 \end{cases}$$

where $g(x, t)\in\mathscr{B}([0, T], \Gamma^{(\kappa)})$ such that for any fixed integer $s\geqq 1$ $D_t^ig(x, t)|_{t=0}=0$, $0\leqq i\leqq s-1$. And in § 5 we shall obtain estimates of $Q_1v(x, t)$. Using the consequence in § 4 and § 5, we shall prove Theorem 2.1 in § 6.

§ 3. Energy Inequality for Q_0

The aim of this section is to show the following lemma.

Lemma 3.1. *Let $\Phi(t)=\sum_{k=0}^{m-1}a^{m-(k+1)}\sum_{i+j=k}\|\sigma(x)^{i\mu}\Lambda^iD_t^ju\|$, where Λ is the pseudo-differential operator with symbol $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$ and $a\geqq 1$. Then there exists a constant $c'>0$ such that*

$$(3.1) \qquad \Phi(t)\leqq c'\int_0^t\{\|Q_0u\|+a\Phi(\tau)\}\,d\tau$$

for $u(x, t)\in\mathscr{B}([0, T], H^\infty(\boldsymbol{R}^n))$, $D_t^iu|_{t=0}=0$, $0\leqq i\leqq m-1$.

In order to prove Lemma 3.1 we prepare several lemmas.

Let $\partial_j=D_t-\sigma(x)^\mu\lambda_j(x, t, D_x)$, $1\leqq j\leqq m$. We note that $\lambda_j\in\mathscr{B}([0, T], S^1)$ and there exists a constant $\delta>0$ such that $|(\lambda_i-\lambda_j)(x, t, \xi)|\geqq\delta\langle\xi\rangle$ if $i\neq j$.

Lemma 3.2. *For i, j with $1\leqq i$, $j\leqq m$, there exist pseudo-differential operators*

A_{ij}, B_{ij} and $C_{ij}\in\mathscr{B}([0, T], S^0)$ such that

$$[\partial_i, \partial_j]=A_{ij}\partial_i+B_{ij}\partial_j+C_{ij}, \tag{3.2}$$

where $[\cdot, \cdot]$ is the commutator.

Proof. Let $\sigma_0([\partial_i, \partial_j])$ be the principal symbol of $[\partial_i, \partial_j]$. Then by the product formula of pseudo-differential operators, we get

$$\begin{aligned}\sigma_0([\partial_i, \partial_j])&=\sum_{k=0}^{n}\{\partial_{\xi_k}(\xi_0-\sigma(x)^\mu\lambda_i)D_{x_k}(\xi_0-\sigma(x)^\mu\lambda_j)\\&\quad-\partial_{\xi_k}(\xi_0-\sigma(x)^\mu\lambda_j)D_{x_k}(\xi_0-\sigma(x)^\mu\lambda_i)\}\\&=\sigma(x)^\mu D_{ij}(x, t, \xi),\end{aligned}$$

where $D_{ij}\in\mathscr{B}([0, T], S^1)$. Here we use the notation $x_0=t$, $\xi_0=\tau$.

If we set $A_{ij}=D_{ij}/(\lambda_j-\lambda_i)$ and $B_{ij}=D_{ij}/(\lambda_i-\lambda_j)$, then A_{ij}, $B_{ij}\in\mathscr{B}([0, T], S^0)$ and $A_{ij}(\xi_0-\sigma(x)^\mu\lambda_i)+B_{ij}(\xi_0-\sigma(x)^\mu\lambda_j)=\sigma(x)^\mu D_{ij}$. Q. E. D.

Now we consider the modules $W_k(0\leqq k\leqq m-1)$ over the ring of pseudo-differential operators in x of order zero.

Let $\Pi_m=\partial_1\partial_2\cdots\partial_m$. Let W_{m-1} be the module generated by the monomial operators $\Pi_m/\partial_i=\partial_1\cdots\check{\partial}_i\cdots\partial_m$ of order $m-1$ and let W_{m-2} be the module generated by the operators $\Pi_m/\partial_i\partial_j(i\neq j)$ of order $m-2$ and so on.

Lemma 3.3. *For any monomial $\omega_k^\alpha\in W_k(0\leqq k\leqq m-1)$, there exist ∂_i and $\omega_{k+1}^\beta\in W_{k+1}$ such that*

$$\partial_i\omega_k^\alpha=\omega_{k+1}^\beta+\sum_{j=1}^{k+1}\sum_\gamma C_{\gamma j}\omega_{k+1-j}^\gamma, \tag{3.3}$$

where $C_{\gamma j}\in\mathscr{B}([0, T], S^0)$.

Proof. For any $\omega_k^\alpha=\partial_{j_1}\cdots\partial_{j_k}(j_1<\cdots<j_k)$, there exists some $j\notin\{j_1, \cdots, j_k\}$ with $1\leqq j\leqq m$. Hence if we use Lemma 3.2, we easily obtain (3.3). Q. E. D.

Lemma 3.4. *Let $\Psi(t)$ be*

$$\Psi(t)=\sum_{k=0}^{m-1}\sum_\alpha a^{m-(k+1)}\|\omega_k^\alpha u\| \tag{3.4}$$

for $u(x, t)\in\mathscr{B}([0, T], H^\infty)$ and $a\geqq 1$. Then we have the following energy inequality

$$\frac{d}{dt}\Psi(t)\leqq c_1 a\Psi(t)+\|\Pi_m u\|. \tag{3.5}$$

Proof. By Lemma 3.3

$$\partial_i\omega_k^\alpha u=\omega_{k+1}^\beta u+\sum_{j=1}^{k+1}\sum_\gamma C_{ij}\omega_{k+1-j}^\gamma u\,.$$

If we set $v=\omega_k^\alpha u$ and $g=\omega_{k+1}^\beta u+\sum_{j=1}^{k+1}\sum_\gamma C_{\gamma j}\omega_{k+1-j}^\gamma u$, then

$$\frac{d}{dt}\|v\|^2=2\,\mathrm{Re}\Big(\frac{d}{dt}v,\ v\Big)$$
$$=2\,\mathrm{Re}\,(\sqrt{-1}\,g+\sqrt{-1}\,\sigma(x)^\mu\lambda_i v,\ v)$$
$$\leqq 2\|g\|\,\|v\|+c_2\|v\|^2.$$

Hence we get

$$\frac{d}{dt}\|v\|\leqq\|\omega_{k+1}^\beta u\|+c_3\Big\{\|\omega_k^\alpha u\|+\sum_{j=1}^{k+1}\sum_\gamma\|\omega_{k+1-j}^\gamma u\|\Big\}.$$

For any k with $0\leqq k\leqq m-2$, we have

$$a^{m-(k+1)}\frac{d}{dt}\|\omega_k^\alpha u\|$$
$$\leqq a^{m-(k+1)}\|\omega_{k+1}^\beta u\|+c_3\Big\{a^{m-(k+1)}\|\omega_k^\alpha u\|+\sum_{j=1}^{k+1}\sum_\gamma a^{m-(k+1)}\|\omega_{k+1-j}^\gamma u\|\Big\}$$
$$\leqq a\,a^{m-(k+2)}\|\omega_{k+1}^\beta u\|+c_3\Big\{a^{m-(k+1)}\|\omega_k^\alpha u\|+\sum_{j=1}^{k+1}\sum_\gamma a^{m-(k-j+2)}\|\omega_{k+1-j}^\gamma u\|\Big\}$$
$$\leqq a\Psi(t)+c_4\Psi(t)$$
$$\leqq c_5 a\Psi(t).$$

Similarly when $k=m-1$, we obtain

$$\frac{d}{dt}\|\omega_{m-1}^\alpha u\|\leqq\|\Pi_m u\|+c_6\Psi(t).$$ Q. E. D.

Lemma 3.5. *Let $\Pi_s=\partial_{i_1}\cdots\partial_{i_s}(1\leqq i_1<\cdots<i_s\leqq m)$. Then $\sigma(\Pi_s)$, the symbol of Π_s, is expressed in the form*

$$\sigma(\Pi_s)=\prod_{j=1}^{s}(\tau-\sigma(x)^\mu\lambda_{i_j})+R_{s-1}+\cdots+R_0, \tag{3.6}$$

where $R_{s-j}=\sum_{p+q=s-j}\sigma(x)^{p\mu}b_{pj}(x,t,\xi)\tau^q$ for some $b_{pj}\in\mathcal{B}([0,T],S^p)$, $j=1,\cdots,s$.

Proof. We carry out the proof by induction on s. When $s=1$, (3.6) is trivial. Suppose (3.6) holds for s. Since $\Pi_{s+1}=\Pi_s\partial_{i_{s+1}}$,

$$\sigma(\Pi_{s+1})=\sigma(\Pi_s)(\xi_0-\sigma(x)^\mu\lambda_{i_{s+1}})+\sum_{\alpha\neq0}\partial_\xi^\alpha\sigma(\Pi_s)D_x^\alpha(\xi_0-\sigma(x)^\mu\lambda_{i_{s+1}}).$$

Substituting the right hand side of (3.6) for $\sigma(\Pi_s)$, we have (3.6) with $s+1$. Q. E. D.

Corollary 3.1. *There exist pseudo-differential operators $C_{ij}(x,t,D_x)\in\mathcal{B}([0,T],S^i)$ such that*

$$Q_0-\Pi_m=\sum_{k=0}^{m-1}\sum_{i+j=k}\sigma(x)^{i\mu}C_{ij}(x,t,D_x)D_t^j. \tag{3.7}$$

Proof. From Lemma 3.5 with $s=m$ and the form of Q_0, (3.7) is verified. Q. E. D.

Lemma 3.6. *There exists a constant* $c_7>0$ *such that*

$$c_7^{-1}\Phi(t)\leqq\Psi(t)\leqq c_7\Phi(t). \tag{3.8}$$

Proof. In order to see that $\Phi(t)\leqq c_8\Psi(t)$, it is sufficient to show the following. There exist $A_j(x, t, \xi)\in\mathcal{B}([0, T], S^0)$ such that for $i'+j'=m-k$, $1\leqq k\leqq m$,

$$\sigma(x)^{i'\mu}\langle\xi\rangle^{i'}\tau^{j'}=\sum_{j=k}^{m}A_j(x, t, \xi)\prod_{\substack{i\neq j\\ i\geqq k}}(\tau-\sigma(x)^{\mu}\lambda_i). \tag{3.9}$$

Substituting $\sigma(x)^{\mu}\lambda_j$ for τ, then we obtain

$$a_j(x, t, \xi)\langle\xi\rangle^{m-k}=A_j(x, t, \xi)\prod_{\substack{i\neq j\\ i\geqq k}}(\lambda_j-\lambda_i),\quad \text{where } a_j\in\mathcal{B}([0, T], S^0).$$

Therefore we set

$$A_j(x, t, \xi)=a_j\langle\xi\rangle^{m-k}\{\prod_{\substack{i\neq j\\ i\geqq k}}(\lambda_j-\lambda_i)\}^{-1}.$$

On the other hand, it is trivial to see that $\Psi(t)\leqq c_9\Phi(t)$. Q. E. D.

Proof of Lemma 3.1. By Corollary 3.1 and Lemma 3.6, we have

$$\begin{aligned}\|\Pi_m u\|&\leqq\|(Q_0-\Pi_m)u\|+\|Q_0u\|\leqq c_{10}\Phi(t)+\|Q_0u\|\\&\leqq c_7c_{10}\Psi(t)+\|Q_0u\|.\end{aligned}$$

And from Lemma 3.4,

$$\Psi(t)\leqq c_{11}\int_0^t\{\|Q_0u\|+a\Psi(\tau)\}\,d\tau.$$

Using Lemma 3.6 again, we can obtain

$$\Phi(t)\leqq c'\int_0^t\{\|Q_0u\|+a\Phi(\tau)\}\,d\tau.$$

This completes the proof. Q. E. D.

§ 4. Estimate of $\Lambda^r v$

We assume the existence of solutions of the following Cauchy problem:

$$\begin{cases} Q_0v(x, t)=g(x, t)\\ D_t^iv(x, t)|_{t=0}=0,\quad 0\leqq i\leqq m-1\end{cases}$$

where $g(x, t)\in\mathcal{B}([0, T], \Gamma^{(\kappa)})$ such that for any fixed integer $s\geqq1$ $D_t^ig(x, t)|_{t=0}=0$, $0\leqq i\leqq s-1$.

Therefore we may assume that for any $r\geqq0$ there exist constants c, R, $M>0$ such that

$$\|\Lambda^r g(x, t)\|\leqq cR^r r!^{\kappa}t^s e^{Mrt}. \tag{4.1}$$

For simplicity we use the notation

$$w_r(s, t, R)=R^r r!^{\kappa} t^s e^{Mrt}.$$

We shall prove the basic lemma of this section.

Lemma 4.1. *Let $\Phi_r(t)$ be*

$$\Phi_r(t)=\sum_{k=0}^{m-1}(r+1)^{m-(k+1)}\sum_{i+j=k}\|\sigma(x)^{i\mu}\Lambda^{r+i}D_t^j v\|.$$

Then for any $r\geqq 0$ there exists a constant $A>0$ such that for sufficiently large R, M, s

$$\Phi_r(t)\leqq cAs^{-1}w_r(s, t, R). \tag{4.2}$$

Proof. We carry out the proof by induction on r.

When $r=0$, it follows from Lemma 3.1 and (4.1) that

$$\Phi_0(t)\leqq c'\int_0^t cw_0(s, \tau, R)d\tau+c'\int_0^t\Phi_0(\tau)d\tau.$$

By Gronwall's inequality, we get

$$\Phi_0(t)\leqq cAs^{-1}w_0(s, t, R)$$

if we choose A such that $A\geqq c'Te^{c'T}$.

We assume that (4.2) is valid for any r such that $0\leqq r\leqq n$.

Let us show that (4.2) is valid for $r=n+1$. For $r>0$, operating the pseudo-differential operator Λ^r on both sides of $Q_0v=g$, we get

$$Q_0\Lambda^r v=\Lambda^r g+[Q_0, \Lambda^r]v.$$

We shall estimate the commutator $[Q_0, \Lambda^r]v$. We note that

$$Q_0(x, t, \xi, \tau)=\tau^m+\sum_{\substack{i+j\leqq m\\ j\leqq m-1}}\sigma(x)^{i\mu}a_i(x, t, \xi)\tau^j,$$

where $a_i\in\mathscr{B}([0, T], S^i(\kappa))$. Therefore we have

$$[Q_0, \Lambda^r]=\sum_{\substack{i+j\leqq m\\ j\leqq m-1}}[\sigma(x)^{i\mu}a_i, \Lambda^r]D_t^j.$$

By the fomula of pseudo-differential operator, we obtain

$$\sigma([\sigma(x)^{i\mu}a_i, \Lambda^r])=\sum_{k=1}^{r+i-1}\sum_{|\alpha|=k}\frac{1}{\alpha!}\partial_\xi^\alpha\langle\xi\rangle^r D_x^\alpha\{\sigma(x)^{i\mu}a_i\}+r_i(x, t, \xi).$$

It follows from Lemma A.3 in Appendix and $\mu\geqq 1$ that

$$\begin{aligned}&\|[\sigma(x)^{i\mu}a_i, \Lambda^r]D_t^j v\|\\ &\quad\leqq \hat{c}r\|\sigma(x)^{i\mu-1}\Lambda^{r+i-1}D_t^j v\|+\cdots+\hat{c}r^i\|\sigma(x)^{i\mu-i}\Lambda^r D_t^j v\|\\ &\qquad+\sum_{k=i+1}^{r}\hat{c}\hat{R}^{k-i}(k-i)!^{\kappa}\binom{r}{k}\|\Lambda^{r+i-k}D_t^j v\|\end{aligned}$$

$$+\sum_{k=r+1}^{r+i-1}\hat{c}\hat{R}^{k-i}(k-i)!^{\kappa}\|\Lambda^{r+i-k}D_t^j v\|$$

$$+\hat{c}\hat{R}^r r!^{\kappa}\|D_t^j v\|$$

$$\leqq c_{12}r\{\|\sigma(x)^{\rho'(i-1)}\Lambda^{r+i-1}D_t^j v\|+\cdots+r^{i-1}\|\Lambda^r D_t^j v\|\}$$

$$+\sum_{k=i+1}^{r}\left(\frac{\hat{R}}{R}\right)^{k-i}\binom{r}{k}\binom{r}{k-i}^{-\kappa}(r+i-k+1)^{-(m-j-1)}cAs^{-1}w_r(s,\ t,\ R)$$

$$+\sum_{k=r+1}^{r+i-1}\left(\frac{\hat{R}}{R}\right)^{k-i}\binom{r}{k-i}^{-\kappa}cAs^{-1}w_r(s,\ t,\ R)$$

$$+\left(\frac{\hat{R}}{R}\right)^{r}cAs^{-1}w_r(s,\ t,\ R).$$

Hence we make $R\geqq 2\hat{R}$, and get

$$\|[Q_0,\ \Lambda^r]v\|\leqq c_{13}\{r\Phi_r(t)+rcAs^{-1}w_r(s,\ t,\ R)+cAs^{-1}w_r(s,\ t,\ R)\}$$

$$\leqq c_{14}\{r\Phi_r(t)+rcAs^{-1}w_r(s,\ t,\ R)\}.$$

We note that c_{14} is independent of r.

From Lemma 3.1,

$$\Phi_r(t)\leqq c'\int_0^t\{\|\Lambda^r g\|+\|[Q_0,\ \Lambda^r]v\|+(r+1)\Phi_r(\tau)\}\,d\tau$$

$$\leqq c'\int_0^t\{cw_r(s,\ \tau,\ R)+c_{15}(r+1)\Phi_r(\tau)+c_{14}rcAs^{-1}w_r(s,\ \tau,\ R)\}\,d\tau.$$

Let $f(t)=c'\int_0^t\{cw_r(s,\ \tau,\ R)+c_{14}rcAs^{-1}w_r(s,\ \tau,\ R)\}\,d\tau$, then

$$\Phi_r(t)\leqq f(t)+c_{16}r\int_0^t\Phi_r(\tau)d\tau.$$

Therefore we obtain

$$\Phi_r(t)\leqq f(t)+c_{16}r\int_0^t f(\tau)e^{c_{16}r(t-\tau)}d\tau.$$

Now we calculate $f(t)$.

$$f(t)=c'c\int_0^t R^r r!^{\kappa}\tau^s e^{Mr\tau}d\tau+rc'c_{14}cAs^{-1}\int_0^t R^r r!^{\kappa}\tau^s e^{Mr\tau}d\tau$$

$$\leqq c'cTs^{-1}w_r(s,\ t,\ R)+c'c_{14}cAs^{-1}M^{-1}w_r(s,\ t,\ R)$$

$\leqq (cA/2)s^{-1}w_r(s,\ t,\ R)$, if we make $A\geqq 4c'T$ and $M\geqq 4c'c_{14}$.

Hence we get

$$\Phi_r(t)\leqq cA2^{-1}s^{-1}w_r(s,\ t,\ R)+c_{16}r\int_0^t cA2^{-1}s^{-1}w_r(s,\ \tau,\ R)e^{c_{16}r(t-\tau)}d\tau$$

$$\leqq cA2^{-1}s^{-1}w_r(s,\ t,\ R)+cA2^{-1}s^{-1}c_{16}(M-c_{16})^{-1}w_r(s,\ t,\ R)$$

$$\leqq cAs^{-1}w_r(s,\ t,\ R),$$

where M is a sufficiently large number such that $c_{16}(M-c_{16})^{-1}\leqq 1$. Q. E. D.

Lemma 4.2. *For any* $r\geqq 0$ *there exists a constant* $A>0$ *such that for* $i+j=k$ *with* $k=0, \cdots, m-1$

$$\|\sigma(x)^{i\mu}\Lambda^{r+i}D_t^j v\|\leqq cAs^{-(m-k)}w_r(s+m-k-1, t, R).$$

Proof. It follows from Lemma 4.1 that

$$\|\sigma(x)^{i\mu}\Lambda^{r+i}D_t^{j+m-k-1}v\|\leqq cAs^{-1}w_r(s, t, R).$$

Hence we get that

$$\begin{aligned}\|\sigma(x)^{i\mu}\Lambda^{r+i}D_t^j v\| &\leqq\int_0^t\cdots\int_0^{\tau_2}\|\sigma(x)^{i\mu}\Lambda^{r+i}D_t^{j+m-k-1}v\|d\tau_1\cdots d\tau_{m-k-1}\\ &\leqq cAs^{-1}R^r r!^{\kappa}e^{Mrt}\int_0^t\cdots\int_0^{\tau_2}\tau_1^s d\tau_1\cdots d\tau_{m-k-1}\\ &\leqq cAs^{-(m-k)}w_r(s+m-k-1, t, R).\end{aligned}$$

Q. E. D.

§5. Estimate of $\Lambda^r Q_1 v$

We begin with the following lemma.

Lemma 5.1. *If* $\sigma(x)\in\mathscr{B}(\boldsymbol{R}^n)$ *and* $0\leqq\nu<\mu$, *then*

$$\|\sigma(x)^{\nu}u\|\leqq\|u\|^{1-\nu/\mu}\|\sigma(x)^{\mu}u\|^{\nu/\mu}. \tag{5.1}$$

Proof. By Hölder's inequality,

$$\begin{aligned}\|\sigma(x)^{\nu}u\|^2 &=\int|\sigma(x)^{\nu}u|^2dx=\int|u|^{2(1-\nu/\mu)}|\sigma(x)^{\mu}u|^{2\nu/\mu}dx\\ &\leqq\left(\int|u|^2dx\right)^{1-\nu/\mu}\left(\int|\sigma(x)^{\mu}u|^2dx\right)^{\nu/\mu}\\ &=\|u\|^{2(1-\nu/\mu)}\|\sigma(x)^{\mu}u\|^{2\nu/\mu}.\end{aligned}$$

Q. E. D.

Lemma 5.2. *Let* $\rho(\alpha, j)=m-j-\nu_{\alpha,j}/\mu$, *then*

$$\begin{aligned}&\sum_{k=1}^{m-1}\sum_{\substack{|\alpha|+j=k\\ |\alpha|\neq 0}}\|\sigma(x)^{\nu_{\alpha,j}}\Lambda^{r+|\alpha|}D_t^j v\|\\ &\leqq\sum_{k=1}^{m-1}\sum_{\substack{|\alpha|+j=k\\ |\alpha|\neq 0}}cAs^{-\rho(\alpha,j)}\{(r+|\alpha|)\cdots(r+1)\}^{-\nu_{\alpha,j}\kappa/\mu}\\ &\quad\times w_{r+|\alpha|}(s+\rho(\alpha, j)-1, t, R).\end{aligned} \tag{5.2}$$

Proof. From Lemma 4.2 and Lemma 5.1, we obtain that for $|\alpha|+j=k$ and $|\alpha|\neq 0$

$$\begin{aligned}&\|\sigma(x)^{\nu_{\alpha,j}}\Lambda^{r+|\alpha|}D_t^j v\|\\ &\leqq\|\Lambda^{r+|\alpha|}D_t^j v\|^{1-\nu_{\alpha,j}/|\alpha|\mu}\|\sigma(x)^{|\alpha|\mu}\Lambda^{r+|\alpha|}D_t^j v\|^{\nu_{\alpha,j}/|\alpha|\mu}\end{aligned}$$

$$=\{cAs^{-(m-j)}w_{r+|\alpha|}(s+m-j-1,\ t,\ R)\}^{1-\nu_{\alpha,j}/|\alpha|\mu}$$

$$\times\{cAs^{-(m-k)}w_r(s+m-k-1,\ t,\ R)\}^{\nu_{\alpha,j}/|\alpha|\mu}$$

$$=cAs^{-\rho(\alpha,j)}R^{r+|\alpha|(1-\nu_{\alpha,j}/|\alpha|\mu)}(r+|\alpha|)!^{\kappa}\{(r+|\alpha|)\cdots(r+1)\}^{-\kappa\nu_{\alpha,j}/|\alpha|\mu}$$

$$\times t^{s+\rho(\alpha,j)-1}e^{M\{r+|\alpha|(1-\nu_{\alpha,j}/|\alpha|\mu)\}t}$$

$$\leqq cAs^{-\rho(\alpha,j)}\{(r+|\alpha|)\cdots(r+1)\}^{-\kappa\nu_{\alpha,j}/|\alpha|\mu}w_{r+|\alpha|}(s+\rho(\alpha,\ j)-1,\ t,\ R).$$

Q. E. D.

We note that

(5.3) $\nu_{\alpha,j}=0$ or there exists a non-negative integer $p_{\alpha,j}$ such that $p_{\alpha,j}\mu<\nu_{\alpha,j}\leqq(p_{\alpha,j}+1)\mu$.

Lemma 5.3. *For any* $r\geqq 0$, *the following estimate holds.*

(5.4) $$\|\Lambda^rQ_1v\|\leqq\tilde{c}cA\sum_{k=1}^{m-1}\sum_{\substack{|\alpha|+j=k\\|\alpha|\neq 0}}K_j^{\alpha}(s,\ r)w_{r+|\alpha|}(s+\rho(\alpha,\ j)-1,\ t,\ R),$$

where

$$K_j^{\alpha}(s,\ r)=s^{-\rho(\alpha,j)}\{(r+|\alpha|)\cdots(r+1)\}^{-\kappa\nu_{\alpha,j}/|\alpha|\mu}$$

$$+\sum_{i=1}^{p_{\alpha,j}+1}s^{-(m-j-p_{\alpha,j}+i-1)}\{(r+|\alpha|)\cdots(r+|\alpha|-i+1)\}^{1-\kappa}$$

$$\times\{(r+|\alpha|-i)\cdots(r+|\alpha|-p_{\alpha,j})\}^{-\kappa}.$$

Proof.

$$\|\Lambda^rQ_1v\|\leqq\sum_{k=1}^{m-1}\sum_{\substack{|\alpha|+j=k\\|\alpha|\neq 0}}\|\Lambda^r(\sigma(x)^{\nu_{\alpha,j}}a_{\alpha,j}(x,\ t,\ D_x)D_t^jv)\|$$

$$\leqq\sum_{k=1}^{m-1}\sum_{\substack{|\alpha|+j=k\\|\alpha|\neq 0}}\{\|\sigma(x)^{\nu_{\alpha,j}}a_{\alpha,j}\Lambda^rD_t^jv\|+\|[\Lambda^r,\ \sigma(x)^{\nu_{\alpha,j}}a_{\alpha,j}]D_t^jv\|\}$$

$$=I_1+I_2.$$

Since

$$I_1\leqq c_{18}\sum_{k=1}^{m-1}\sum_{\substack{|\alpha|+j=k\\|\alpha|\neq 0}}\{\|\sigma(x)^{\nu_{\alpha,j}}\Lambda^{r+|\alpha|}D_t^jv\|+\|\Lambda^{r+|\alpha|-1}D_t^jv\|\},$$

the first term has been estimated by Lemma 5.2 and the second term will be estimated within I_2.

The estimate of I_2 is similar to the proof of Lemma 4.1. In fact since

$$\sigma([\Lambda^r,\ \sigma(x)^{\nu_{\alpha,j}}a_{\alpha,j}])=\sum_{i=1}^{r+|\alpha|-1}\sum_{|\beta|=i}\frac{1}{\beta!}\partial_{\xi}^{\beta}\langle\xi\rangle^rD_x^{\beta}\{\sigma(x)^{\nu_{\alpha,j}}a_{\alpha,j}\}+r(x,\ t,\ \xi),$$

if we note that $\nu_{\alpha,j}-|\beta|=(p_{\alpha,j}+1-|\beta|)\mu+(\nu_{\alpha,j}-p_{\alpha,j}\mu-1)+(|\beta|-1)(\mu-1)$ and use Lemma A.3 and Lemma 4.2, then we have

$$I_2 \leqq c_{19} \sum_{k=1}^{m-1} \sum_{\substack{|\alpha|+j=k \\ |\alpha|\neq 0}} \sum_{i=1}^{p_{\alpha,j}+1} cAs^{-(m-j-p_{\alpha,j}+i-1)}$$

$$\times \{(r+|\alpha|) \cdots (r+|\alpha|-i+1)\}^{1-\kappa}\{(r+|\alpha|-i) \cdots (r+|\alpha|-p_{\alpha,j})\}^{-\kappa}$$

$$\times w_{r+|\alpha|}(s+m-j-p_{\alpha,j}+i-2, t, R).$$

And if we note that $m-j-p_{\alpha,j}+i-1 \geqq \rho(\alpha, j)$, then we can get (5.4).

Q. E. D.

§ 6. Proof of Theorem 2.1

In order to prove Theorem 2.1, we prepare several lemmas.

Lemma 6.1. *For any* $u^i(x) \in \Gamma^{(\kappa)} (0 \leqq i \leqq m-1)$ *and any* $g(x, t) \in \mathcal{B}([0, T], \Gamma^{(\kappa)})$, *there exists a unique solution* $u(x, t) \in \mathcal{B}([0, T], \Gamma^{(\kappa)})$ *of the equation*:

$$\text{(6.1)} \qquad \begin{cases} Q_0(x, t, D_x, D_t)u(x, t) = g(x, t) \\ D_t^i u(x, t)|_{t=0} = u^i(x), \quad 0 \leqq i \leqq m-1. \end{cases}$$

And especially, if $u^i(x) \equiv 0 \, (0 \leqq i \leqq m-1)$ *and* $D_t^i g(x, t)|_{t=0} = 0 \, (0 \leqq i \leqq s-1)$, *then we obtain that* $D_t^i u(x, t)|_{t=0} = 0 \, (0 \leqq i \leqq s+m-1)$, *where s is a positive integer.*

Proof. It follows from Proposition 2.1 that there exists a unique solution $u(x, t) \in \mathcal{B}([0, T], H^\infty)$ of (6.1). Therefore let us show that $u(x, t) \in \mathcal{B}([0, T], \Gamma^{(\kappa)})$.

For any fixed integer $s \geqq 1$, let $u_s(x, t)$ be

$$u_s(x, t) = u(x, t) - \sum_{j=0}^{s+m-1} \frac{t^j}{j!} \partial_t^j u(x, t)|_{t=0},$$

then $u_s(x, t)$ satisfies the equation

$$Q_0 u_s(x, t) = g(x, t) - Q_0\Big(\sum_{j=0}^{s+m-1} \frac{t^j}{j!} \partial_t^j u(x, t)|_{t=0}\Big)$$

$$\equiv g_s(x, t).$$

Therefore we get that $g_s(x, t) \in \mathcal{B}([0, T], \Gamma^{(\kappa)})$ such that $D_t^i g_s(x, t)|_{t=0} = 0$, $0 \leqq i \leqq s-1$. From the consequence of § 4, it is easily seen that $u_s(x, t) \in \mathcal{B}([0, T], \Gamma^{(\kappa)})$. Hence we obtain that $u(x, t) \in \mathcal{B}([0, T], \Gamma^{(\kappa)})$.

As to the latter, since $D_t^m u = (D_t^m u - Q_0 u) + g$, we can get that $D_t^m u|_{t=0} = 0$. And since $D_t^{m+1} u = D_t(D_t^m u - Q_0 u) + D_t g$, we get that $D_t^{m+1} u|_{t=0} = 0$. Hence if we repeat these steps, we have that $D_t^i u|_{t=0} = 0$, $0 \leqq i \leqq s+m-1$. Q. E. D.

Lemma 6.2. *Let* $u_j(x, t)$ *be the solution of* $(2.6)_j$, *then* $u_j(x, t) \in \mathcal{B}([0, T], \Gamma^{(\kappa)})$ *for* $j \geqq 0$. *Moreover for* $j \geqq 1$, $D_t^i u_j(x, t)|_{t=0} = 0$, $0 \leqq i \leqq m+2j-3$.

Proof. It follows from the first assersion of Lemma 6.1 that $u_0(x, t) \in$

$\mathcal{B}([0, T], \Gamma^{(\kappa)})$. Since $Q_1u_0 \in \mathcal{B}([0, T], \Gamma^{(\kappa)})$, if we use Lemma 6.1 once more, we can get that $u_1(x, t) \in \mathcal{B}([0, T], \Gamma^{(\kappa)})$. Therefore repeating these steps we have $u_j(x, t) \in \mathcal{B}([0, T], \Gamma^{(\kappa)})$ for $j \geqq 0$.

Next from the second assersion of Lemma 6.1 and the form of $(2.6)_1$, $D_t^i u_1|_{t=0}=0$, $0 \leqq i \leqq m-1$. Since $D_t^i Q_1 u_1|_{t=0}=0$ for $i=0, 1$, we obtain that $D_t^i u_2|_{t=0}=0$ for $0 \leqq i \leqq m+1$. Similarly, we conclude the second assersion of Lemma 6.2.

Q. E. D.

From Lemma 6.2, for any fixed integer $s \geqq 1$, there exists $N=N(s) \in \boldsymbol{N}$ such that for any $j \geqq N-1$, $D_t^i u_j(x, t)|_{t=0}=0$, $0 \leqq i \leqq s-1$. Therefore we may assume that for any $r \geqq 0$

$$\|\Lambda^r Q_1 u_{N-1}\| \leqq c w_r(s, t, R), \tag{6.2}$$

where c and R are positive constants.

Lemma 6.3. *Under* (6.2), *if* $1 \leqq \kappa < \rho/(\rho-1)$, *there exist constants* $\tilde{A}$, B, $\gamma > 0$ *which are independent of* r *such that*

$$\|\Lambda^r u_{N+n}\| \leqq c\tilde{A} B^n n^{-\gamma n} w_r(s, t, 2^\kappa R) \tag{6.3}$$

for $n=0, 1, 2, \cdots$.

Proof. From (6.2) and Lemma 4.2, we get that

$$\|\sigma(x)^{|\alpha_1|\mu} \Lambda^{r+|\alpha_1|} D_t^{j_1} u_N\| \leqq cAs^{-(m-j-|\alpha_1|)} T^{m-1} w_r(s, t, R).$$

It follows from Lemma 5.3 that

$$\|\Lambda^r Q_1 u_N\| \leqq \tilde{c} cA \sum_{k_1=1}^{m-1} \sum_{\substack{|\alpha_1|+j_1=k_1 \\ |\alpha_1| \neq 0}} K_{j_1}^{\alpha_1}(s, r) w_{r+|\alpha_1|}(s+\rho(\alpha_1, j_1)-1, t, R).$$

If we use Lemma 4.2, we have that

$$\|\sigma(x)^{|\alpha_2|\mu} \Lambda^{r+|\alpha_2|} D_t^{j_2} u_{N+1}\| \leqq c_T c A^2 \sum_{k_1=1}^{m-1} \sum_{\substack{|\alpha_1|+j_1=k_1 \\ |\alpha_1| \neq 0}}$$
$$(s+\rho(\alpha_1, j_1)-1)^{-(m-j_2-|\alpha_2|)} K_{j_1}^{\alpha_1}(s, r) w_{r+|\alpha_1|}(s+\rho(\alpha_1, j_1)-1, t, R),$$

where $c_T = \tilde{c} T^{m-1}$.

Applying Lemma 5.3 again, we obtain, that

$$\|\Lambda^r Q_1 u_{N+1}\| \leqq \tilde{c} c_T c A^2 \sum_{k_1, k_2=1}^{m-1} \sum_{\substack{|\alpha_1|+j_1=k_1 \\ |\alpha_1| \neq 0}} \sum_{\substack{|\alpha_2|+j_2=k_2 \\ |\alpha_2| \neq 0}}$$
$$K_{j_1}^{\alpha_1}(s, r) K_{j_2}^{\alpha_2}(s+\rho(\alpha_1, j_1)-1, r+|\alpha_1|)$$
$$\times w_{r+|\alpha_1|+|\alpha_2|}(s+\rho(\alpha_1, j_1)-1+\rho(\alpha_2, j_2)-1, t, R)$$

From Lemma 4.2, we get that

$$\|\sigma(x)^{|\alpha_3|\mu}\Lambda^{r+|\alpha_3|}D_t^{j_3}u_{N+2}\| \leqq cA(c_TA)^2\sum\sum\sum K_{j_1}^{\alpha_1}(s, r)K_{j_1,j_2}^{\alpha_1,\alpha_2}(s, r)$$
$$\times(s+\rho(\alpha_1, j_1)-1+\rho(\alpha_2, j_2)-1)^{-(m-j_3-|\alpha_3|)}$$
$$\times w_{r+|\alpha_1|+|\alpha_2|}(s+\rho(\alpha_1, j_1)-1+\rho(\alpha_2, j_2)-1, t, R),$$

where $K_{j_1,j_2}^{\alpha_1,\alpha_2}(s, r)=K_{j_2}^{\alpha_2}(s+\rho(\alpha_1, j_1)-1, r+|\alpha_1|)$.

We set

$$K_{j_1,\cdots,j_l}^{\alpha_1,\cdots,\alpha_l}(s, r)$$
$$=K_{j_l}^{\alpha_l}(s+\rho(\alpha_1, j_1)-1+\cdots+\rho(\alpha_{l-1}, j_{l-1})-1, r+|\alpha_1|+\cdots+|\alpha_{l-1}|).$$

Inductively we obtain that for any $n\geqq 0$

$$\|\Lambda^r u_{N+n}\|\leqq cA(c_TA)^n\sum\cdots\sum K_{j_1}^{\alpha_1}\cdots K_{j_1,\cdots,j_n}^{\alpha_1,\cdots,\alpha_n}$$
$$\times w_{r+|\alpha_1|+\cdots+|\alpha_n|}(s+\rho(\alpha_1, j_1)-1+\cdots+\rho(\alpha_n, j_n)-1, t, R).$$

By the way,

$$K_{j_1}^{\alpha_1}\cdots K_{j_1,\cdots,j_n}^{\alpha_1,\cdots,\alpha_n}$$
$$=\sum\cdots\sum s^{-a_1}(r+1)^{-b_1^1}\cdots(r+|\alpha_1|)^{-b_1^{|\alpha_1|}}$$
$$\times(s+\rho(\alpha_1, j_1)-1)^{-a_2}(r+|\alpha_1|+1)^{-b_2^1}\cdots(r+|\alpha_1|+|\alpha_2|)^{-b_1^{|\alpha_2|}}$$
$$\times\cdots\times(s+\rho(\alpha_1, j_1)-1+\cdots+\rho(\alpha_{n-1}, j_{n-1})-1)^{-a_n}$$
$$\times(r+|\alpha_1|+\cdots+|\alpha_{n-1}|+1)^{-b_n^1}\cdots(r+|\alpha_1|+\cdots+|\alpha_n|)^{-b_n^{|\alpha_n|}},$$

where $a_k\in\{\rho(\alpha_k, j_k), m-j_k-p_{\alpha_k,j_k}+i_k-1\}$ and $b_k^l\in\{\kappa\nu_{\alpha_k,j_k}/|\alpha_k|\mu, \kappa-1, \kappa, 0\}$.

We note the following.

(6.4) If $a_k=\rho(\alpha_k, j_k)$, then $b_k^1, \cdots, b_k^{|\alpha_k|}=\kappa\nu_{\alpha_k,j_k}/|\alpha_k|\mu$.

(6.5) If $a_k=m-j_k-p_{\alpha_k,j_k}+i_k-1$, then $b_k^1, \cdots, b_k^{|\alpha_k-p_{\alpha_k,j_k}-1}=0$,

$b_k^{|\alpha_k|-p_{\alpha_k,j_k}}, \cdots, b_k^{|\alpha_k|-i_k}=\kappa$ and $b_k^{|\alpha_k|-i_k+1}, \cdots, b_k^{|\alpha_k|}=\kappa-1$.

Let $s\geqq\max\{\rho(\alpha, j)-1\}$, $\omega=\min\{\rho(\alpha, j)-1\}$ and $a=\min\{a_k\}$ and if we use Lemma A.4, then we have

$$s^{-a_1}\cdots(s+\rho(\alpha_1, j_1)-1+\cdots+\rho(\alpha_{n-1}, j_{n-1})-1)^{-a_n}$$
$$\leqq(\rho(\alpha_n, j_n)-1)^{-a_1}\cdots(\rho(\alpha_1, j_1)-1+\cdots+\rho(\alpha_n, j_n)-1)^{-a_n}$$
$$\leqq\omega^{-(a_1+\cdots+a_n)}1^{-a_1}\cdots n^{-a_n}$$
$$\leqq\omega^{-an}A_1R_1^n n^{-(a_1+\cdots+a_n)}.$$

Let $r=0$, them by Lemma A.4 again,

$$(r+1)^{-b_1^1}\cdots(r+|\alpha_1|)^{-b_1^{|\alpha_1|}}$$
$$\times\cdots\times(r+|\alpha_1|+\cdots+|\alpha_{n-1}|+1)^{-b_n^1}\cdots(r+|\alpha_1|+\cdots+|\alpha_n|)^{-b_n^{|\alpha_n|}}$$

$$\leqq A_1R_1^n(|\alpha_1|+\cdots+|\alpha_n|)^{-(b_1^1+\cdots+b_n^{|\alpha_n|})}$$

$$\leqq A_1R_1^n n^{-(b_1^1+\cdots+b_n^{|\alpha_n|})}.$$

Further we estimate

$$w_{r+|\alpha_1|+\cdots+|\alpha_n|}(s+\rho(\alpha_1, j_1)-1+\cdots+\rho(\alpha_n, j_n)-1, t, R).$$

$$R^{r+|\alpha_1|+\cdots+|\alpha_n|}\leqq R^r R^{(m-1)n},$$

$$t^{s+\rho(\alpha_1, j_1)-1+\cdots+\rho(\alpha_n, j_n)-1}\leqq t^s T^{\varpi n}, \text{ where } \tilde{\omega}=\max\{\rho(\alpha, j)-1\},$$

$$e^{M(r+|\alpha_1|+\cdots+|\alpha_n|)t}\leqq e^{Mrt}e^{M(m-1)Tn}.$$

Using Lemma A.5,

$$(r+|\alpha_1|+\cdots+|\alpha_n|)!^{\kappa}\leqq 2^{(r+|\alpha_1|+\cdots+|\alpha_n|)\kappa}r!^{\kappa}(|\alpha_1|+\cdots+|\alpha_n|)!^{\kappa}$$

$$\leqq 2^{r\kappa}2^{(m-1)n\kappa}r!^{\kappa}A_2R_2^n n^{(|\alpha_1|+\cdots+|\alpha_n|)\kappa}.$$

Therefore we obtain that

$$\|\Lambda^r u_{N+n}\|$$

$$\leqq cAA_1^2A_2(c_TAR_1^2R_2\omega^{-a}R^{m-1}2^{(m-1)\kappa}T^{\varpi}e^{M(m-1)T})^n w_r(s, t, 2^{\kappa}R)$$

$$\times\sum\cdots\sum n^{(|\alpha_1|+\cdots+|\alpha_n|)\kappa-(a_1+\cdots+a_n)-(b_1^1+\cdots+b_n^{|\alpha_n|})}.$$

Let q be the number of $\rho(\alpha_k, j_k)$'s in $\{a_k\}_{1\leq k\leq n}$ and if we remember (6.4), (6.5) and (5.3), then we obtain that

$$(a_1+\cdots+a_n)+(b_1^1+\cdots+b_n^{|\alpha_n|})-(|\alpha_1|+\cdots+|\alpha_n|)\kappa$$

$$=\rho(\alpha_1, j_1)+\cdots+\rho(\alpha_q, j_q)$$

$$+(m-j_{q+1}-p_{\alpha_{q+1}, j_{q+1}}+i_{q+1}-1)+\cdots+(m-j_n-p_{\alpha_n, j_n}+i_n-1)$$

$$+\kappa\nu_{\alpha_1, j_1}/\mu+\cdots+\kappa\nu_{\alpha_q, j_q}/\mu$$

$$+\kappa(p_{\alpha_{q+1}, j_{q+1}}+1-i_{q+1})+\cdots+\kappa(p_{\alpha_n, j_n}+1-i_n)$$

$$+(\kappa-1)i_{q+1}+\cdots+(\kappa-1)i_n-(|\alpha_1|+\cdots+|\alpha_n|)\kappa$$

$$=(\rho(\alpha_1, j_1)+\kappa\nu_{\alpha_1, j_1}/\mu-|\alpha_1|\kappa)+\cdots+(\rho(\alpha_q, j_q)+\kappa\nu_{\alpha_q, j_q}/\mu-|\alpha_q|\kappa)$$

$$+(m-j_{q+1}-p_{\alpha_{q+1}, j_{q+1}}-1+p_{\alpha_{q+1}, j_{q+1}}\kappa+\kappa-|\alpha_{q+1}|\kappa)$$

$$+\cdots+(m-j_n-p_{\alpha_n, j_n}-1+p_{\alpha_n, j_n}\kappa+\kappa-|\alpha_n|\kappa)$$

$$\geqq(m-j_1-\nu_{\alpha_1, j_1}/\mu+\kappa\nu_{\alpha_1, j_1}/\mu-|\alpha_1|\kappa)$$

$$+\cdots+(m-j_n-\nu_{\alpha_n, j_n}/\mu+\kappa\nu_{\alpha_n, j_n}/\mu-|\alpha_n|\kappa)$$

$$=(m-j_1-|\alpha_1|)\{(m-j_1-\nu_{\alpha_1, j_1}/\mu)/(m-j_1-|\alpha_1|$$

$$-((m-j_1-\nu_{\alpha_1,j_1}/\mu)/(m-j_1-|\alpha_1|)-1)\kappa\}+\cdots$$
$$+(m-j_n-|\alpha_n|)\{(m-j_n-\nu_{\alpha_n,j_n}/\mu)/(m-j_n-|\alpha_n|)$$
$$-((m-j_n-\nu_{\alpha_n,j_n}/\mu)/(m-j_n-|\alpha_n|)-1)\kappa\}$$
$$\geqq n\{\rho-(\rho-1)\kappa\}.$$

This completes the proof. Q.E.D.

Corollary 6.1. *If* $1\leqq\kappa<\rho/(\rho-1)$, *the formal solution*

$$u(x, t)=\sum_{j=0}^{\infty} u_j(x, t)$$

converges in $\mathcal{B}([0, T], \Gamma^{(\kappa)})$.

Proof. If we devide $u(x, t)$ as

$$u(x, t)=\sum_{j=0}^{N-1} u_j(x, t)+\sum_{j=N}^{\infty} u_j(x, t),$$

then this Corollary immidiately follows from Lemma 6.2 and Lemma 6.3.

Q.E.D.

Hence we obtain the existence of solutions.

Next we shall show the uniqueness of solutions.

Lemma 6.4. *If* $u(x, t)\in\mathcal{B}([0, T], \Gamma^{(\kappa)})$ *is a solution of the Cauchy problem*:

$$\begin{cases} P(x, t, D_x, D_t)u(x, t)=0 \\ D_t^i u(x, t)|_{t=0}=0, \quad 0\leqq i\leqq m-1, \end{cases}$$

where $1\leqq\kappa<\rho/(\rho-1)$, *then* $u(x, t)\equiv 0$.

Proof. We may assume that for sufficiently large s there exist constants c, $R>0$ such that

$$\|\Lambda^r u\|\leqq c w_r(s, t, R) \qquad \text{for any} \quad r\geqq 0.$$

Therefore similar to the proof of Lemma 6.3, we can obtain that

$$\|\Lambda^r u\|\leqq c\tilde{A}B^n n^{-\gamma n} w_r(s, t, \tilde{R}).$$

Let n be infinity, then we have that $u(x, t)\equiv 0$. Q.E.D.

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Appendix

Following Igari [2] and Uryu [12], we introduce a certain class of pseudo-differential operators.

Definition A.1. 1) For any $m\in\boldsymbol{R}$ and $\kappa>1$, we denote by $S^{m}(\kappa)$ the set of functions $h(x, \xi)\in C^{\infty}(\boldsymbol{R}^{n}\times\boldsymbol{R}^{n})$ satisfying the property that for any α, β, there exist constants c_{α} and R such that

$$|\partial_{\xi}^{\alpha}D_{x}^{\beta}h(x, \xi)|\leqq c_{\alpha}R^{|\beta|}|\beta|!^{\kappa}\langle\xi\rangle^{m-|\alpha|} \quad \text{for} \quad (x, \xi)\in\boldsymbol{R}^{n}\times\boldsymbol{R}^{n}.$$

2) For any $h(x, \xi)\in S^{m}(\kappa)$, we shall define a semi-norm of $h(x, \xi)$ such that for any integer $l\geqq 0$

$$|h(x, \xi)|_{l}=\max_{|\alpha+\beta|\leqq l}\sup_{(x,\xi)\in\boldsymbol{R}^{n}\times\boldsymbol{R}^{n}}|\partial_{\xi}^{\alpha}D_{x}^{\beta}h(x, \xi)|\langle\xi\rangle^{-m+|\alpha|}.$$

Now we can define a pseudo-differential operator with a symbol $h(x, \xi)\in S^{m}(\kappa)$ as follows.

$$H(x, D_{x})u(x)=(2\pi)^{-n}\int e^{ix\cdot\xi}h(x, \xi)\hat{u}(\xi)d\xi.$$

Lemma A.1 (*see Igari* [2]). *Let* $h(x, \xi)\in S^{m}(\kappa)$ *and* r *be non-negative integers. Then*

$$\sigma(\Lambda^{r}H)=\sum_{j=1}^{N-1}\sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_{\xi}^{\alpha}\langle\xi\rangle^{r}D_{x}^{\alpha}h(x, \xi)+r_{N}(x, \xi),$$

where $N=r+m$. *And for any integer* $l\geqq 0$, *there exist constants* c_{l}, $R>0$ *such that*

$$|D_{x}^{\alpha}h(x, \xi)\langle\xi\rangle^{-m}|_{l}\leqq c_{l}R^{|\alpha|-m}(|\alpha|-m)!^{\kappa}$$

and

$$|r_{N}(x, \xi)|_{l}\leqq c_{l}R^{r}r!^{\kappa}.$$

The following lemma is well-known.

Lemma A.2. *For any* $h(x, \xi)\in S^{0}$, *there exist a constant* c *and non-negative integer* l *dependent only on dimension* n *such that*

$$\|H(x, D_{x})u\|\leqq c|h(x, \xi)|_{l}\|u\|.$$

Lemma A.3 (*see Uryu* [12]). *Under the assumptions of Lemma A.1, if we denote* $h_{j}(x, \xi)$ *by*

$$h_{j}(x, \xi)=\sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_{\xi}^{\alpha}\langle\xi\rangle^{r}D_{x}^{\alpha}h(x, \xi),$$

then there exist $\hat{c}$, $\hat{R}>0$ *such that*

$$\|H_{j}(x, D_{x})u\|\leqq\hat{c}\hat{R}^{j-m}(j-m)!^{\kappa}\binom{r}{j}\|\Lambda^{m+r-j}u\| \quad \textit{for} \quad 1\leqq j\leqq r,$$

$$\|H_{j}(x, D_{x})u\|\leqq\hat{c}\hat{R}^{j-m}(j-m)!^{\kappa}\|\Lambda^{m+r-j}u\| \quad \textit{for} \quad r+1\leqq j\leqq N-1,$$

$$\|R_{N}(x, D_{x})u\|\leqq\hat{c}\hat{R}^{r}r!^{\kappa}\|u\|.$$

Lemma A.4. *Let* $\{i_1, \cdots, i_n\}$ *be a subset of* $\{a_1, \cdots, a_m\}$, *then there exist constants* A_1, $R_1>0$ *such that*

$$n^{i_1+\cdots+i_n} \leqq A_1 R_1^n 1^{i_1} 2^{i_2} \cdots n^{i_n}.$$

Proof. Set $S=n^{i_1+\cdots+i_n}/1^{i_1}\cdots n^{i_n}$. Then

$$\begin{aligned} S&=(n/1)^{i_1}(n/2)^{i_2}\cdots(n/n)^{i_n}\\ &\leqq(n/1)^a(n/2)^a\cdots(n/n)^a\\ &=(n^n/n!)^a, \quad \text{where } a=\max\{a_1, \cdots, a_m\}. \end{aligned}$$

Using Stirling's formula, we can get the desired inequality. Q. E. D.

Lemma A.5. *Let* $\{i_1, \cdots, i_n\}\subset\{1, \cdots, m-1\}$, *then there exist constants* A_2, $R_2>0$ *such that*

$$(i_1+\cdots+i_n)! \leqq A_2 R_2^n n^{i_1+\cdots+i_n}.$$

Proof. By Stirling's formula, we obtain that

$$\begin{aligned} (i_1+\cdots+i_n)! &\leqq A_2(i_1+\cdots+i_n)^{(i_1+\cdots+i_n)}\\ &\leqq A_2\{n(m-1)\}^{(i_1+\cdots+i_n)}\\ &\leqq A_2(m-1)^{n(m-1)} n^{i_1+\cdots+i_n}. \end{aligned}$$

Q. E. D.

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