Publ. RIMS, Kyoto Univ.
21 (1985), 1001-1030

Integration in Abelian C^*-Dynamical Systems

By

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Abstract

Let $\mathcal{A}=C_0(X)$ be an abelian C^*-algebra and $t\in\boldsymbol{R}\mapsto\sigma_t$ a strongly continuous group of $*$-automorphisms with generator δ_0. We consider derivations $\delta=\lambda\delta_0$, where λ is a multiplication operator on $C_0(X)$, and establish conditions on λ which ensure that δ has a unique generator extension. As a corollary we deduce that each derivation δ from $\bigcap_{n\geq 1}D(\delta_0^n)$ into $D(\delta_0)$ is closable and its closure is a generator. An analogous result is established for derivations defined on the smooth elements associated with the action of a compact Lie group on $\mathcal{A}$. Some results on local dissipations are also given.

§1. Introduction

Our aim is to analyze derivations δ defined on the smooth elements of an abelian C^*-dynamical system as generators of C_0-groups of $*$-automorphisms. The basic question is whether δ has a unique generator extension, and then a subsidiary problem is to relate this extension to δ. Since each abelian C^*-dynamical system determines a topological dynamical system, based on the spectrum of the abelian algebra, this question can be viewed as a problem of integration in topological dynamics. This approach appears particularly useful. In order to be more precise we introduce the following definition and notation.

Let $(\mathcal{A}, \boldsymbol{R}, \sigma)$ denote an abelian C^*-dynamical system. Thus $\mathcal{A}$ is an abelian C^*-algebra and $t\in\boldsymbol{R}\to\sigma_t\in\operatorname{Aut}\mathcal{A}$ is a strongly continuous one-parameter group of $*$-automorphisms of $\mathcal{A}$. Next let δ_0 denote the generator of σ and define $\mathcal{A}_n=D(\delta_0^n)$, and

$$\mathcal{A}_\infty=\bigcap_{n\geq 1}\mathcal{A}_n\,.$$

Communicated by H. Araki, April 1, 1985.
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It follows automatically that $\mathcal{A}_\infty$ is a norm dense *-subalgebra of $\mathcal{A}$. The principal object of this paper is a *-derivation δ on $\mathcal{A}$ with domain $D(\delta)=\mathcal{A}_\infty$. Our main result, Theorem 3.1, states that *if* $\delta:\mathcal{A}_\infty\to\mathcal{A}_1$ *then* δ *is closable and its closure* $\bar{\delta}$ *generates a strongly continuous one-parameter group* τ *of *-automorphisms of* $\mathcal{A}$. In particular $\bar{\delta}$ is the unique generator extension of δ. The group τ is closely related to σ but to describe this relationship it is necessary to introduce the underlying topological dynamical system.

Let X denote the spectrum of $\mathcal{A}$. Then $\mathcal{A}$ can be identified with $C_0(X)$, the continuous functions over X which vanish at infinity, and one can associate with σ a continuous one-parameter group S of homeomorphisms of X such that

$$(\sigma_t f)(\omega)=f(S_t\omega)$$

for all $t\in\boldsymbol{R}$, $\omega\in X$, and $f\in C_0(X)$. We note that strong continuity of σ is equivalent to joint continuity of the map $(t,\omega)\mapsto S_t\omega$ (See Lemma 2.2 below).

Now consider the topological dynamical system $(X, \boldsymbol{R}, S)$. To each point $\omega\in X$ we associate the orbit

$$S_{\boldsymbol{R}}\omega=\{S_t\omega\,;\,t\in\boldsymbol{R}\}$$

and use X_0 to denote the set of fixed points of S, i. e.,

$$X_0=\{\omega\,;\,S_t\omega=\omega \text{ for all } t\in\boldsymbol{R}\}$$

Next we associate to each $\omega\in X$ a period $p(\omega)$ by the definition

$$p(\omega)=\inf\{t>0\,;\ S_t\omega=\omega\}$$

and $p(\omega)=+\infty$ if there is no $t>0$ such that $S_t\omega=\omega$. Thus, for example, $\omega\in X_0$ if, and only if, $p(\omega)=0$. Finally we associate to each $\omega\in X$ a frequency $\nu(\omega)$ by setting $\nu(\omega)=0$ if $p(\omega)=+\infty$, $\nu(\omega)=1/p(\omega)$ if $0<p(\omega)<+\infty$, and $\nu(\omega)=+\infty$ if $p(\omega)=0$.

The starting point of our analysis is a result of [6] which states that δ is a derivation from $\mathcal{A}_\infty$ into $\mathcal{A}$ if, and only if,

$$\delta=\lambda\delta_0|_{\mathcal{A}_\infty}$$

where λ is a function which vanishes on X_0 and is continuous and polynomially bounded in the frequency on $X\backslash X_0$. By this last statement we mean that there is a $C>0$ and an integer $n\geqq 0$ such that

$$|\lambda(\omega)|\leqq C(1+\nu(\omega)^n)$$

for all $\omega\in X\backslash X_0$. (For earlier results of this nature see [3], [10], and for some further remarks on this result see [1].) Moreover δ maps $\mathcal{A}_\infty$ into $\mathcal{A}_1$ if, and only if, $\delta_0\lambda(\omega)=\lim_{t\to 0}(\lambda(S_t\omega)-\lambda(\omega))/t$ exists for $\omega\in X\backslash X_0$, $\delta_0\lambda$ is a continuous function on $X\backslash X_0$, and both λ and $\delta_0\lambda$ are polynomially bounded in the frequency. It is this special form $\delta=\lambda\delta_0$ which is crucial in our construction of an automorphism group with $\bar{\delta}$ as its generator. For example if $|\lambda|$ is bounded and bounded away from zero then for $\varepsilon>0$ (resp. $\varepsilon<0$) sufficiently small, $\delta_0-\varepsilon\delta$ is relatively bounded by δ_0 with relative bound $b<1$ on the component of X where $\lambda>0$ (resp. $\lambda<0$), and as $\pm\delta$ clearly are dissipative, δ is a generator by perturbation theory. The general situation is much more complicated. If λ is only polynomially bounded in the frequency then difficulties can occur at points with very large frequency, i.e., at fixed points of S. Different difficulties can occur if λ has zeros on $X\backslash X_0$. Note that if $\omega\in X\backslash X_0$ and $\lambda(\omega)=0$ then

$$(\delta f)(\omega)=\lambda(\omega)(\delta_0 f)(\omega)=0$$

for all $f\in\mathcal{A}_\infty$ and thus ω is a fixed point of δ but not for δ_0.

The key observation in our proof that $\bar{\delta}$ generates an automorphism group τ is the remark that each orbit of the associated homeomorphism group T on X should be contained in an orbit of S. Thus if T exists there should also exist functions $(t, \omega)\in\boldsymbol{R}\times X\mapsto x_\omega(t)\in\boldsymbol{R}$ such that

$$T_t\omega=S_{x_\omega(t)}\omega\,.$$

Then, formally.

$$\begin{aligned}\lambda(T_t\omega)(\delta_0 f)(T_t\omega)&=(\delta\tau_t f)(\omega\\&=\frac{d}{dt}(\tau_t f)(\omega\\&=\frac{d}{dt}f(S_{x_\omega(t)}\omega)\\&=x'_\omega(t)(\delta_0 f)(T_t\omega)\,.\end{aligned}$$

Consequently x_ω satisfies the first-order differential equation

$$x'_\omega(t)=\lambda_\omega(x_\omega(t))$$

where λ_ω is defined by

$$\lambda_\omega(t)=\lambda(S_t\omega)\,.$$

Our main technical result, in Section 2, is a version of the Picard-Lindelöf theorem of ordinary differential equations (see, for example, [12] Theorem 2.3.1). The latter theorem proves the existence of a flow T on $\boldsymbol{R}$ with the above structure whenever λ is a uniformly Lipschitz continuous function. In our version of the theorem the construction of T on each orbit follows from the classical Picard-Lindelof theorem but the new problem, which has some analogue to stability problems in ordinary differential equations, is the proof of joint continuity of the map $(t, \omega)\in\boldsymbol{R}\times X\mapsto T_t\omega$, which is necessary for the existence of τ. For this we need the Lipschitz constants for λ on the orbits to be uniformly bounded on each set of bounded frequency, and λ must satisfy a certain boundedness property at low-frequency points near X_0.

In Section 3 we derive our main theorem as a corollary of the technical results of Section 2, and in Section 4 we discuss the generator question for local dissipations (semi-derivations) defined on $\mathcal{A}_\infty$.

In Section 5 we derive the analogue of Theorem 3.1 with the action S of $\boldsymbol{R}$ on X replaced by an action of a compact Lie group. Let $\mathcal{A}_\infty$ (resp. $\mathcal{A}_1$) denote the algebra of infinitely (resp. once) differentiable elements for an action of a compact Lie group on $\mathcal{A}=C_0(X)$. If $\delta:\mathcal{A}_\infty\to\mathcal{A}_1$ is a *-derivation, then δ is closable and $\bar{\delta}$ is a generator.

§ 2. Picard-Lindelöf Theorems for Flows

In this section we establish the generalization of the Picard-Lindelöf theorem mentioned in the introduction and then characterize the generator of the flow constructed in this theorem.

Definition 2.1. *Let X be a locally compact Hausdorff space, and $t\in\boldsymbol{R}\mapsto S_t$ a one-parameter family of homeomorphisms of X. We define S to be a flow if*

a. *S is a group, i.e.*

$$S_tS_s=S_{t+s}, \qquad t, s\in\boldsymbol{R},$$

b. *The map*

$$(t, \omega)\in\boldsymbol{R}\times X\mapsto S_t\omega$$

is jointly continuous

Note that joint continuity does not generally follow from separate continuity, although it does if X is metrizable (see, for example, [17] Theorem 1.1). The relevance of joint continuity in the C^*-algebraic framework is a consequence of the following well-known lemma.

Lemma 2.2. *Let $\mathcal{A}=C_0(X)$ be an abelian C^*-algebra with spectrum X, $t\in\boldsymbol{R}\to\sigma_t$ a one-parameter group of *-automorphisms of $\mathcal{A}$, and $t\in\boldsymbol{R}\mapsto S_t$ the corresponding one-parameter group of homeomorphisms of X, i.e.*

$$(\sigma_tf)(\omega)=f(S_t\omega), \qquad f\in\mathcal{A},\ t\in\boldsymbol{R},\ \omega\in X.$$

The following conditions are equivalent:

1. *$t\in\boldsymbol{R}\mapsto\sigma_t$ is strongly continuous, i.e.*

$$\|\sigma_tf-f\|\xrightarrow[t\to0]{}0 \qquad \text{for all} \quad f\in\mathcal{A},$$

2. *$(t, \omega)\in\boldsymbol{R}\times X\mapsto S_t\omega$ is jointly continuous*

Proof. 1⇒2. Assume 1, and let $(t_\alpha, \omega_\alpha)$ be a net in $\boldsymbol{R}\times X$ converging to (t, ω). If $f\in C_0(X)$, we have

$$|f(S_{t_\alpha}\omega_\alpha)-f(S_t\omega)|\leqq\|\sigma_{t_\alpha}f-\sigma_tf\|+|(\sigma_tf)(\omega_\alpha)-(\sigma_tf)(\omega)|\xrightarrow[\alpha\to\infty]{}0+0$$

and as X is locally compact, it follows that

$$\lim_{\alpha} S_{t_\alpha}\omega_\alpha = S_t\omega$$

$2 \Rightarrow 1$. Assume that 1 is false, i. e. that there exists an $f \in C_0(X)$ such that

$$\overline{\lim_{t\to 0}} \|\sigma_t f - f\| = 2\varepsilon > 0$$

Then there exist sequences $t_n \neq 0$, $\omega_n \in X$ such that $t_n \to 0$ and

$$|f(S_{t_n}\omega_n) - f(\omega_n)| > \varepsilon$$

for all n. But there exists a compact set $K \subseteq X$ such that $|f(\omega)| < \varepsilon/2$ if $\omega \notin K$, thus it follows for each n that $S_{t_n}\omega_n \in K$ or $\omega_n \in K$. If $S_{t_n}\omega_n \in K$ then replacing the pair (t_n, ω_n) by $(-t_n, S_{t_n}\omega_n)$, we may assume $\omega_n \in K$ for all n. Since K is compact there exists a subnet $(t_\alpha, \omega_\alpha)$ of (t_n, ω_n) such that $t_\alpha \to 0$, $\omega_\alpha \to \omega \in K$. But as

$$|f(S_{t_\alpha}\omega_\alpha) - f(\omega_\alpha)| > \varepsilon$$

for all α, we cannot have

$$\lim_{\alpha\to\infty} S_{t_\alpha}\omega_\alpha = \omega$$

and 2 does not hold.

Let S be a flow on X. If $\omega \in X$ recall that the period of ω is defined as

$$p(\omega) = \inf\{t > 0;\ S_t\omega = \omega\}$$

and the frequency of ω is

$$\nu(\omega) = 1/p(\omega)$$

In particular $\nu(\omega) = \infty$ if ω is a fixed point, and $\nu(\omega) = 0$ if $S_t\omega \neq \omega$ for all $t \neq 0$.
We shall need the following continuity property of the map $\omega \mapsto \nu(\omega)$.

Lemma 2.3. *The map $\omega \mapsto \nu(\omega)$ is upper semicontinuous, i. e. if $\omega_\alpha \to \omega$ then*

$$\nu(\omega) \geqq \overline{\lim_{\alpha}}\, \nu(\omega_\alpha).$$

Furthermore, if in this situation $\nu(\omega) < \infty$ and $\bar{\nu} = \overline{\lim_{\alpha}}\, \nu(\omega_\alpha) > 0$, then

$$\frac{\nu(\omega)}{\bar{\nu}} \in \{1, 2, 3, \cdots\}$$

Remark. Simple examples, like doubling an 8 into a 0, i. e. the standard flow on the Möbius strip, or the flow generated by $y(d/dx)$ on $\boldsymbol{R}^2$, show that $\omega \to \nu(\omega)$ is not continuous in general.

Proof. If $\underline{p} = \underline{\lim}_{\alpha}\, p(\omega_\alpha)$, it is enough to show that $p(\omega) \leqq \underline{p}$, and that $\underline{p}/p(\omega)$ is an integer provided $p(\omega) > 0$ and $\underline{p} < \infty$. Assume first that $\underline{p} < \infty$. Then it is enough to show that $S_{\underline{p}}\omega = \omega$. We may assume that $\lim_{\alpha} p(\omega_\alpha) = \underline{p}$ exists by passing to a subnet. But as

$$S_{p(\omega_\alpha)}\omega_\alpha=\omega_\alpha$$

for all α, it follows by limiting, and by joint continuity of S, that

$$S_{\underline{p}}\omega=\omega .$$

If $\underline{p}=\infty$, the claim is trivial.

The following characterization of the generator of a flow will be useful, and is an elaboration of the well-known equivalence of weak and strong generators, [7], Corollary 3.1.8.

Lemma 2.4. *Let S be a flow on X, σ the corresponding automorphism group of $C_0(X)$ and δ the generator of σ, i. e. $f\in D(\delta)$ if, and only if, $\lim_{t\to 0}\|(\sigma_t(f)-f)/t-\delta(f)\|=0$ for some $\delta(f)\in C_0(X)$.*

The following conditions are equivalent;

1. $f\in D(\delta)$
2. *The limit*

$$g(\omega)\equiv\lim_{t\to 0}(f(S_t\omega)-f(\omega))/t$$

exists for each $\omega\in X$, and $g\in C_0(X)$. In this situation $g=\delta(f)$.

Proof. 1⇒2 and the last statement are trivial. Assume 2. Then

$$f(S_t\omega)-f(\omega)=\int_0^t ds\ \ g(S_s\omega)$$

and hence

$$|(f(S_t\omega)-f(\omega))/t-g(\omega)|=\left|\frac{1}{t}\int_0^t ds\ \ (g(S_s\omega)-g(\omega))\right|$$
$$\leqq\sup_{|s|\leqq|t|}\|\sigma_s(g)-g\| .$$

But the last number is independent of ω and tends to zero as $t\to 0$ by strong continuity. Thus $f\in D(\delta)$.

For completeness, we state the following well known result on existence and uniqueness of solutions of first order ordinary differential equations.

Lemma 2.5. (*Picard-Lindelöf*) *Let $\lambda: \boldsymbol{R}\mapsto\boldsymbol{R}$ be a function which is Lipschitz continuous in the sense that there exists a constant $K>0$ such that*

$$|\lambda(x)-\lambda(y)|\leqq K|x-y|$$

for all $x, y\in\boldsymbol{R}$. Then the initial-value problem

$$\frac{d}{dx}f(x)=\lambda(f(x)) ,$$

$$f(0)=y ,$$

has a unique solution $f_y : \boldsymbol{R} \mapsto \boldsymbol{R}$ *for each* $y \in \boldsymbol{R}$. *These solutions satisfy the group property*

$$f_y(t+s) = f_{f_y(t)}(s)$$

and hence if we define

$$T_t y = f_y(t)$$

for $y, t \in \boldsymbol{R}$ *then* T *is a flow on* $\boldsymbol{R}$.

Remark. If τ is the one-parameter group of *-automorphisms of $C_0(\boldsymbol{R})$ defined by T, it is easy to see that the generator of τ is an extension of $\delta = \lambda(d/dx)$. We will show in Theorem 2.6 that this generator is the unique generator extension of δ. The Lipschitz condition on λ near the zeros of λ is of prime importance for this proof. If λ is merely continuous and bounded, δ may have a continuum of generator extensions, or none at all, see, for example, [4].

In Theorem 2.12 we show that if, in addition, λ is continuously differentiable, then the closure of δ is the generator of τ. A necessary and sufficient condition for a closed derivation on $C([0, 1])$ to be a generator has been given in [15].

Proof. The existence and uniqueness of the solution f_y follows by the method of successive approximations (see [12]). Since K is independent of x, y, the solution exists on all of $\boldsymbol{R}$. Since $\lambda(f)$ does not depend explicitly on x, the group property follows because

$$s \longrightarrow f_y(t+s)$$

$$s \longrightarrow f_{f_y(t)}(s)$$

are both solutions to the initial value problem

$$g'(s) = \lambda(g(s))$$

$$g(0) = f_y(t).$$

The proof of the joint continuity of $(t, y) \mapsto T_t y$ will be shown in a more general setting in the proof of Theorem 2.6 (see Observation 1). But as $T_{-t} = (T_t)^{-1}$, each T_t is a homeomorphism.

The main theorem of this section is the following.

Theorem 2.6. *Let* σ *be a strongly continuous one-parameter group of *-automorphisms of an abelian C*-algebra* $\mathcal{A} = C_0(X)$ *with generator* δ_0 *and associated flow* S *on* X, *and let* $X_0 \subseteqq X$ *denote the fixed points of* S. *Assume that* λ *is a real continuous (not necessarily bounded) function on* $X \backslash X_0$ *which satisfies bounds of the type*

$$|\lambda(S_t\omega) - \lambda(\omega)| \leqq K(\nu(\omega))|t|,$$

where $K : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ *is a function which is bounded on bounded intervals. Assume* *also that for any compact subset $C\subseteqq X$ there exists an $\varepsilon>0$ such that λ is uniformly bounded on*

$$C\cap\{\omega\in X;\ \nu(\omega)<\varepsilon\}.$$

It follows that the derivation δ defined by

$$\delta f(\omega)=\begin{cases}\lambda(\omega)\delta_0 f(\omega) & \text{if } \omega\in X\backslash X_0\\ 0 & \text{if } \omega\in X_0\end{cases}$$

on

$$D(\delta)=\{f\in D(\delta_0);\ \text{the right hand function above is in } C_0(X)\}$$

*is densely defined and has a unique extension to a generator of a strongly continuous one-parameter group τ of *-automorphisms of $\mathcal{A}$.*

Remark. The low frequency boundedness of λ is necessary because of examples such as $X=\boldsymbol{R}^2$, $\delta_0=y(d/dx)$, $\lambda(x, y)=1/y$.

Proof. We prove the existence part of this theorem by explicitly constructing the flow T corresponding to τ.

If $\omega\in X\backslash X_0$ define a function $\lambda_\omega:\boldsymbol{R}\mapsto\boldsymbol{R}$ by

$$\lambda_\omega(t)=\lambda(S_t\omega).$$

Then

$$|\lambda_\omega(t)-\lambda_\omega(s)|=|\lambda(S_{t-s}S_s\omega)-\lambda(S_s\omega)|$$
$$\leqq K(\nu(S_s\omega))|t-s|=K(\nu(\omega))|t-s|$$

because ν is constant along S-orbits. It follows from Lemma 2.5 that the initial value problem

$$x'_\omega(t)=\lambda_\omega(x_\omega(t))$$
$$x_\omega(0)=0$$

has a unique solution $t\in\boldsymbol{R}\mapsto x_\omega(t)$. Define

$$T_t\omega=S_{x_\omega(t)}\omega$$

for all $t\in\boldsymbol{R}$, and $\omega\in X\backslash X_0$. If $\omega\in X_0$, define $T_t\omega=\omega$. Then T is the candidate for the new flow generated by an extension of $\delta=\lambda\delta_0$. The group properties $T_tT_s=T_{t+s}$, $T_0=1$ follow immediately from Lemma 2.5. (Note that if $\nu(\omega)>0$, λ_ω is periodic with period $1/\nu(\omega)$, and hence the function $f_y(t)$ of Lemma 2.5 has the property $f_{y+1/\nu(\omega)}(t)=f_y(t)+1/\nu(\omega)$. Thus the definition of T is consistent).

Observation 1. *The map*

$$(t, \omega)\in\boldsymbol{R}\times X\longrightarrow T_t\omega\in X$$

is jointly continuous.

Proof. Let $(t_\alpha, \omega_\alpha)\to(t, \omega)$ in $\boldsymbol{R}\times X$. We have to show that $T_{t_\alpha}\omega_\alpha\to T_t\omega$ in X, i.e.

$$S_{x_{\omega_\alpha}(t_\alpha)}\omega_\alpha \longrightarrow S_{x_\omega(t)}\omega .$$

We divide the discussion into two cases.

Case 1. $\nu(\omega)<\infty$.

Since S is jointly continuous, by Lemma 2.2, it suffices to show that $x_{\omega_\alpha}(t_\alpha)\to x_\omega(t)$. But as

$$|x_\omega(t)-x_{\omega_\alpha}(t_\alpha)|\leqq|x_\omega(t)-x_\omega(t_\alpha)|+|x_\omega(t_\alpha)-x_{\omega_\alpha}(t_\alpha)|$$

and $x_\omega(t_\alpha)\to x_\omega(t)$ it suffices to show that

$$x_\omega(t_\alpha)-x_{\omega_\alpha}(t_\alpha) \longrightarrow 0 .$$

For this we first fix a $t_0>0$ and seek an estimate for $|x_\omega(t)-x_{\omega_\alpha}(t)|$ on the interval $[0, t_0]$. Now it follows from the differential equation for x that

$$(*) \qquad |x_\omega(t)-x_{\omega_\alpha}(t)|\leqq\int_0^t ds\ |\lambda_\omega(x_\omega(s))-\lambda_{\omega_\alpha}(x_{\omega_\alpha}(s))| , \qquad t\geqq 0 .$$

We are assuming that $\nu(\omega)<\infty$, and as $\nu(\omega)\geqq\overline{\lim_\alpha}\,\nu(\omega_\alpha)$ by Lemma 2.3, it follows that the frequencies $\nu(\omega_\alpha)$ are uniformly bounded. Hence it follows from the hypothesis of the theorem that there exists a constant K such that

$$|\lambda_{\omega_\alpha}(x)-\lambda_{\omega_\alpha}(y)|\leqq K|x-y|$$

for all x, $y\in\boldsymbol{R}$ and all α, and the same estimates hold for λ_ω as well. Also, since $\lambda_{\omega_\alpha}(0)=\lambda(\omega_\alpha)\to\lambda(\omega)$, we can choose the constant K so large that

$$|\lambda_{\omega_\alpha}(0)|\leqq K$$

for all α, and then it follows from the previous estimate that

$$|\lambda_{\omega_\alpha}(x)|\leqq(|x|+1)K$$

for all $x\in\boldsymbol{R}$ and all α.

Now as x_{ω_α} satisfies the differential inequality

$$\begin{aligned}|x'_{\omega_\alpha}(t)|&=|\lambda_{\omega_\alpha}(x_{\omega_\alpha}(t))|\\&\leqq K(1+|x_{\omega_\alpha}(t)|)\end{aligned}$$

and the unique solution of the differential equation

$$y'=K(1+y)$$

with $y(0)=0$ is $y(t)=e^{Kt}-1$, we obtain the estimates

$$|x_{\omega_\alpha}(t)|\leqq e^{Kt}-1, \qquad |x_\omega(t)|\leqq e^{Kt}-1$$

Hence we have the crude first estimate

$$|x_{\omega_\alpha}(t)-x_\omega(t)|\leqq 2(e^{Kt}-1).$$

We also have

$$(**)\quad |\lambda_{\omega_\alpha}(x_{\omega_\alpha}(s))-\lambda_\omega(x_\omega(s))|\leqq|\lambda_{\omega_\alpha}(x_{\omega_\alpha}(s))-\lambda_{\omega_\alpha}(x_\omega(s))|+|\lambda_{\omega_\alpha}(x_\omega(s))-\lambda_\omega(x_\omega(s))|$$
$$\leqq K|x_{\omega_\alpha}(s)-x_\omega(s)|+M_\alpha$$

where

$$M_\alpha=\sup\{|\lambda_{\omega_\alpha}(x)-\lambda_\omega(x)|\,;\ |x|\leqq e^{Kt_0}-1\}.$$

Now, let M be a constant such that $e^{Ks}-1\leqq Ms$ for $0\leqq s\leqq t_0$. Inserting the crude first estimate into (**) and using $K\leqq M$ we find

$$|\lambda_{\omega_\alpha}(x_{\omega_\alpha}(s))-\lambda_\omega(x_\omega(s))|\leqq 2M^2s+M_\alpha,$$

inserting this into (*) we then obtain

$$|x_{\omega_\alpha}(t)-x_\omega(t)|\leqq M^2t^2+M_\alpha t,$$

and inserting this improved estimate into (**) gives

$$|\lambda_{\omega_\alpha}(x_{\omega_\alpha}(s))-\lambda_\omega(x_\omega(s))|\leqq M^3s^2+M_\alpha Ms+M_\alpha.$$

Reinserting this into (*) and iterating we finally find

$$|x_{\omega_\alpha}(t)-x_\omega(t)|\leqq 2\frac{M^nt^n}{n!}+\frac{M_\alpha}{M}\Big(Mt+\frac{M^2t^2}{2!}+\cdots+\frac{M^{n-1}t^{n-1}}{(n-1)!}\Big),$$

and

$$|\lambda_{\omega_\alpha}(x_{\omega_\alpha}(s))-\lambda_\omega(x_\omega(s))|\leqq 2\frac{M^{n+1}s^n}{n!}+M_\alpha\Big(1+Ms+\frac{M^2s^2}{2!}+\cdots+\frac{M^{n-1}s^{n-1}}{(n-1)!}\Big),$$

for all n. Thus

$$|x_{\omega_\alpha}(t)-x_\omega(t)|\leqq M_\alpha(e^{Mt}-1)/M,$$

where M and M_α are independent of $t\in[0, t_0]$. It follows that if $0<t<t_0$ and $\lim t_\alpha=t$, so that $t_\alpha\in[0, t_0]$ for $\alpha\geqq\alpha_0$, then

$$|x_{\omega_\alpha}(t_\alpha)-x_\omega(t_\alpha)|\leqq M_\alpha(e^{Mt_0}-1)/M\qquad\text{for}\quad \alpha\geqq\alpha_0.$$

But since $\lambda_{\omega_\alpha}(s)=\lambda(S_s\omega_\alpha)$ converges pointwise in s to $\lambda_\omega(s)=\lambda(S_s\omega)$, and these functions are uniformly Lipschitz continuous, it follows that the convergence is uniform in compacts, and hence

$$M_\alpha=\sup\{|\lambda_{\omega_\alpha}(x)-\lambda_\omega(x)|\,;\ |x|\leqq e^{Kt_0}-1\}\longrightarrow 0.$$

Consequently $|x_{\omega_\alpha}(t_\alpha)-x_\omega(t_\alpha)|\to 0$ and thus

$$\lim_\alpha x_{\omega_\alpha}(t_\alpha)=x_\omega(t).$$

The case $t(=\lim t_\alpha)\leqq 0$ is treated similarly, and thus the proof of Case 1 is complete.

Case 2. $\nu(\omega)=\infty$, i.e. $\omega\in X_0$.

By passing to subnets it suffices to consider the following two situations:

Case 2.1. $\nu(\omega_\alpha)\to 0$.

By the low frequency boundedness of λ it follows that $\lambda(\omega_\alpha)$ is bounded in α, and combining this with the Lipschitz condition we deduce as in Case 1 that the functions x_{ω_α} are uniformly bounded on compact intervals, e. g.

$$|x_{\omega_\alpha}(t_\alpha)|\leqq K \quad \text{for all } \alpha.$$

Again passing to a subnet, we can assume that $x_{\omega_\alpha}(t_\alpha)\to x$, and then it follows from the joint continuity of S that

$$T_{t_\alpha}\omega_\alpha=S_{x_{\omega_\alpha}(t_\alpha)}\omega_\alpha \longrightarrow S_x\omega=\omega.$$

Case 2.2. There is an $\varepsilon>0$ such that $\varepsilon\leqq\nu(\omega_\alpha)$ for all α.

Then

$$T_{t_\alpha}\omega_\alpha\subseteqq S_{\boldsymbol{R}}\omega_\alpha=S_{[0,1/\varepsilon]}\omega_\alpha$$

for all α. Proceeding as in case 2.1 we deduce that

$$T_{t_\alpha}\omega_\alpha \longrightarrow \omega$$

This ends the proof of Observation 1, and it now follows from Lemma 2.2 that T determines a strongly continuous one-parameter group τ of *-automorphisms of $C_0(X)$ through

$$(\tau_t f)(\omega)=f(T_t\omega),$$

for $f\in C_0(X)$, $t\in\boldsymbol{R}$, $\omega\in X$. Let δ_T denote the generator of this group.

Observation 2. *δ_T is an extension of δ.*

Proof. By Lemma 2.4 it suffices to prove that

$$\lim_{t\to 0}(f(T_t\omega)-f(\omega))/t=\lambda(\omega)(\delta_0 f)(\omega)$$

for all $\omega\in X\backslash X_0$ and all $f\in D(\delta_0)$. But

$$\begin{aligned}(f(T_t\omega)-f(\omega))/t&=(f(S_{x_\omega(t)}\omega)-f(\omega))/t \longrightarrow (\delta_0 f)(\omega)\cdot x'_\omega(0)\\ &=\lambda(\omega)(\delta_0 f)(\omega)\end{aligned}$$

by the differential equation for x_ω and the chain rule.

Observations 1 and 2 establish the existence of the generator extension of δ. It remains to prove density of $D(\delta)$ and the uniqueness of the flow T. To do this and subsequently to discuss the generator δ_T of the flow T, we introduce "high frequency cutoffs", that is, algebras of functions which are constant on orbits of sufficiently high frequency. Recalling that $\mathcal{A}_n=D(\delta_0^n)$ and $\mathcal{A}_\infty=\bigcap_{n\geq 1}\mathcal{A}_n$, we define

$$\mathcal{D}_n=\{f\in\mathcal{A}_n\,;\,f \text{ has compact support and there exists an } M>0 \text{ such that } f(S_t\omega)=f(\omega) \text{ whenever } \nu(\omega)\geqq M \text{ and } t\in\boldsymbol{R}\}$$

where $n=0, 1, 2, \cdots, \infty$. Then $\mathcal{D}_n$ is clearly a *-algebra which is invariant under σ. Moreover $\mathcal{D}_0$ is also invariant under τ.

Lemma 2.7.

(a) *If $f\in\mathcal{A}_\infty$ has compact support in $X\backslash X_0$, then $f\in\mathcal{D}_\infty\cap D(\delta)$.*

(b) *If $f\in C_0(X_0)$ has compact support and Δ is a compact subset of $X\backslash X_0$, then there is a function $g\in\mathcal{D}_\infty\cap D(\delta)$ with compact support such that $g|_{X_0}=f$ and $g|\Delta=0$.*

(c) *$\mathcal{D}_\infty\cap D(\delta)$ is dense in $\mathcal{A}$.*

(d) *If $\omega\in X\backslash X_0$ and $f\in C(S_{[-1,1]}\omega)$ is a function such that $t\to f(S_t\omega)$ is infinitely often differentiable on $[-1, 1]$, then there is a $g\in\mathcal{D}_\infty\cap D(\delta)$ with compact support in $X\backslash X_0$ such that g extends f.*

Proof.

(a) It follows from the upper semi-continuity of ν, Lemma 2.3, that ν is bounded on the compact set $\operatorname{supp}(f)$. Hence f is zero on orbits of sufficiently high frequency. Also $\lambda\delta_0(f)$ is continuous with compact support on $X\backslash X_0$. Hence $f\in\mathcal{D}_\infty\cap D(\delta)$.

(b) Let V be an open neighbourhood of $\operatorname{supp}(f)$ with compact closure. By hypothesis there is an $\varepsilon_0>0$ such that λ is bounded on

$$\overline{V}\cap\{\omega\,;\,\nu(\omega)<\varepsilon_0\}.$$

Let W be an open neighbourhood of $\operatorname{supp}(f)$ such that

$$\overline{W}\subseteqq V\backslash S_{[-2,\,(1/\varepsilon_0)+2]}\Delta,$$

and let h be a continuous extension of f to the closed set

$$X_{\varepsilon_0}=\{\omega\,;\,\nu(\omega)\geqq\varepsilon_0\}$$

with compact support in $W\cap X_{\varepsilon_0}$. Then define a function k on X_{ε_0} by taking the mean over each orbit, i. e.

$$k(\omega)=\begin{cases}\nu(\omega)\displaystyle\int_0^{1/\nu(\omega)}dt\;h(S_t\omega) & \text{if}\quad \omega\in X_{\varepsilon_0}\backslash X_0\\ h(\omega) & \text{if}\quad \omega\in X_0.\end{cases}$$

Since $\nu(\omega)\geqq\varepsilon_0$ on X_{ε_0}, it follows that k is well-defined, k has compact support in $S_{[-(1/\varepsilon_0),0]}(\operatorname{supp}(h))\subseteqq S_{[-(1/\varepsilon_0),0]}W\cap X_{\varepsilon_0}$, and k is constant on orbits in X_{ε_0}. We now argue that k is continuous.

Let ω_α be a net in X_{ε_0} such that $\omega_\alpha\to\omega'$ in X_{ε_0}. Then

$$\nu(\omega')\geqq\overline{\lim}\,\nu(\omega_\alpha)\geqq\varepsilon_0.$$

By passing to subsets we may assume that $\nu_0=\lim_\alpha \nu(\omega_\alpha)$ exists as a finite positive number or as $+\infty$. We therefore divide the discussions into these two cases.

Case 1. $\varepsilon_0 \leqq \nu_0 < \infty$.

Since $1/\nu_0$ is a period for $t\rightarrow h(S_t\omega')$ by Lemma 2.3, we have

$$k(\omega')=\nu_0\int_0^{1/\nu_0} dt\ \ h(S_t\omega'),$$

and

$$k(\omega_\alpha)=\nu(\omega_\alpha)\int_0^{1/\nu(\omega_\alpha)} dt\ \ h(S_t\omega_\alpha).$$

Since the family of functions $t\rightarrow h(S_t\omega_\alpha)$ converges pointwise to $t\rightarrow h(S_t\omega')$ and this family is uniformly equicontinuous by the strong continuity of σ, it follows that the convergence is uniform on $[0, 1/\nu_0]$ and hence

$$k(\omega')=\lim_\alpha k(\omega_\alpha).$$

Case 2. $\nu_0=\infty$.

In this case $\nu(\omega')=\infty$, i.e. $\omega'\in X_0$, and $k(\omega')=h(\omega')$. Hence

$$|k(\omega_\alpha)-k(\omega')| \leqq |k(\omega_\alpha)-h(\omega_\alpha)|+|h(\omega_\alpha)-h(\omega')|.$$

The latter term converges to zero and the former is dominated by

$$\sup\{\|\sigma_t h-h\|\ ;\ 0\leqq t\leqq 1/\nu_\alpha\},$$

which also converges to zero by strong continuity of σ.

This completes the proof that k is continuous on X_{ε_0}.

Next extend k to a continuous function on X, also denoted k, with compact support in the open set $S_{[-(1/\varepsilon_0),0]}W$. Let $\phi\in C^\infty(\boldsymbol{R})$ be a positive function with support in $[-1, 1]$ and with total integral one, and define

$$g(\omega)=\int dt\ \ \phi(t)\ \ k(S_t\omega).$$

Clearly $g\in\mathcal{D}_\infty$, $g|X_0=f$, and

$$\operatorname{supp}(g)\subseteqq S_{[-1,1]}\operatorname{supp}(k)\subseteqq S_{[-(1/\varepsilon_0)-1,1]}W,$$

which is disjoint from Δ. It remains to show that $g\in D(\delta)$.

Given a net ω_α in $X\backslash X_0$ converging to a point $\omega_0\in X_0$, we have to show that $\lim_\alpha \lambda(\omega_\alpha)\delta_0(g)(\omega_\alpha)=0$. It suffices to show that any subnet of this net has in turn a subnet such that $\lambda\delta_0(g)$ has limit zero over the subnet. If $\delta_0(g)(\omega_\alpha)$ is eventually zero, there is nothing to do; otherwise we can extract a subnet (also called ω_α) such that $\delta_0(g)(\omega_\alpha)\neq 0$ for all α. It then follows that $\nu(\omega_\alpha)<\varepsilon_0$ and $\omega_\alpha\in\operatorname{Supp}(\delta_0(g))\subseteqq\operatorname{Supp}(g)\subseteqq S_{[-(1/\varepsilon)-1\ 1]}W$. Hence for each α there is

$t_\alpha \in [-1, (1/\varepsilon_0)+1]$ such that $S_{t_\alpha}\omega_\alpha \in W$, and passing again to a subnet, we can arrange that $t_0 = \lim_\alpha t_\alpha$ exists, and therefore

$$S_{t_\alpha}\omega_\alpha \longrightarrow S_{t_0}\omega_0 = \omega_0 .$$

But this means that $\omega_0 \in \overline{W} \subseteq V$, so that $\omega_\alpha \in V$ eventually. We can assume (by again going to a subnet) that for all α

$$\omega_\alpha \in V \cap \{\omega ; \nu(\omega) < \varepsilon_0\} .$$

Since λ is bounded on this set and $\delta_0(g)(\omega_\alpha) \to 0$, we obtain the desired conclusion.

(c) It follows from (a) and (b) that the *-algebra $\mathcal{D}_\infty \cap D(\delta)$ separate points of X and is zero at no point of X. Thus (c) follows from the Stone-Weierstrass theorem.

(d) It follows from [9], Théorème 3.1, that there exists a finite number m of functions $\psi_i \in C^\infty(\boldsymbol{R})$ with $\text{supp}(\psi_i) \subseteq [-1, 1]$ and functions $h_i \in C^\infty(S_{[-2,2]}\omega)$ such that

$$f(S_t\omega) = \sum_{i=1}^{m} \int_{-1}^{1} ds\ \psi_i(s)\ h_i(S_{s+t}\omega)$$

for $|t| < 1$. Extend each h_i to a continuous function on X with compact support in $X \backslash X_0$ and define

$$g = \sum_{i=1}^{m} \int_{-1}^{1} ds\ \psi_i(s)\ \sigma_s(h_i) .$$

Then $g(S_t\omega) = f(S_t\omega)$ for $|t| \leqq 1$, $g \in \mathcal{A}_\infty$, and g has compact support in $X \backslash X_0$. By part (a), $g \in \mathcal{D}_\infty \cap D(\delta)$.

This ends the proof of Lemma 2.7, and also establishes that $D(\delta)$ is dense in Theorem 2.6. Our next task is to prove uniqueness of the flow T constructed above. Therefore, let U be another flow on X such that the generator δ_U of the associated automorphism group extends δ. In order to show $U = T$ we need the following general lemma.

Lemma 2.8. *If U and S are general flows on a locally compact Hausdorff space X, and for each $\omega \in X$ there exists an $\varepsilon > 0$ such that $U_{\langle -\varepsilon, \varepsilon \rangle}\omega \subseteq S_{\boldsymbol{R}}\omega$, then $U_{\boldsymbol{R}}\omega \subseteq S_{\boldsymbol{R}}\omega$ for all $\omega \in X$.*

Proof. Let $t \in \boldsymbol{R}$, we have to show $U_t\omega \subseteq S_{\boldsymbol{R}}\omega$ for all $\omega \in X$. By compactness of $[-|t|, |t|]$ there is a finite subset $t_1, \cdots, t_n$ of $[-|t|, |t|]$ and positive constant $\varepsilon_1, \cdots, \varepsilon_n$ such that

$$[-|t|, |t|] \subseteq \bigcup_k \langle t_k - \varepsilon_k, t_k + \varepsilon_k \rangle$$

and

$$U_{\langle t_k - \varepsilon_k, t_k + \varepsilon_k \rangle}\omega \subseteq S_{\boldsymbol{R}} U_{t_k}\omega$$

But as two S-orbits are either equal or disjoint, it follows that if two of the intervals $\langle t_k-\varepsilon_k, t_k+\varepsilon_k\rangle$ and $\langle t_m-\varepsilon_m, t_m+\varepsilon_m\rangle$ overlap, $U_{\langle t_k-\varepsilon_k, t_k+\varepsilon_k\rangle}\omega$ and $U_{\langle t_m-\varepsilon_m, t_m+\varepsilon_m\rangle}\omega$ are contained in the same S-orbit. Since any two elements in $[-|t|, |t|]$ can be connected by a finite number of the intervals $\langle t_k-\varepsilon_k, t_k+\varepsilon_k\rangle$, it follows that all $U_{\langle t_k-\varepsilon_k, t_k+\varepsilon_k\rangle}\omega$ are contained in one S-orbit, namely $S_{\boldsymbol{R}}\omega$. Thus $U_{[-|t|, |t|]}\omega\subseteqq S_{\boldsymbol{R}}\omega$ and hence $U_{\boldsymbol{R}}\omega\subseteqq S_{\boldsymbol{R}}\omega$.

Returning to the proof of Theorem 2.6, let U be the flow on X whose generator δ_U extends δ.

Observation 3. *$U_{\boldsymbol{R}}\omega\subseteqq S_{\boldsymbol{R}}\omega$ for all $\omega\in X$.*

Proof. We argue by contradiction. If the statement is false, it follows from Lemma 2.8 that there exist an $\omega\in X$ and a sequence $t_n\in\boldsymbol{R}\backslash\{0\}$ such that $t_n\to 0$ but $U_{t_n}\omega\notin S_{\boldsymbol{R}}\omega$ for $n=1, 2, \cdots$. But then $S_{\boldsymbol{R}}U_{t_n}\omega\cap S_{\boldsymbol{R}}\omega=\emptyset$ for all n since two S-orbits are either equal or disjoint. Next set $\omega_n=U_{t_n}\omega$. Since $\omega_n\to\omega$ one sees by induction that there is a subsequence of ω_n, which we also denote by ω_n, such that

$$\omega_n\notin\left(\bigcup_{k=1}^{n-1} S_{[-3, 3]}\omega_k\right)\cup S_{[-4, 4]}\omega\,.$$

But then all the closed sets $S_{[-1, 1]}\omega_n$ are disjoint and we may define a function f on

$$C=(\bigcup_n S_{[-1, 1]}\omega_n)\cup S_{[-2, 2]}\omega$$

by $f(\omega')=t_n$ if $\omega'\in S_{[-1, 1]}\omega_n$ and $f(\omega')=0$ if $\omega'\in S_{[-2, 2]}\omega$. Since $\omega_n\to\omega$, the set C is compact and the function f is continuous on C. Again there are two cases.

Case 1. $\omega\in X\backslash X_0$.

Then $\overline{\lim}\,\nu(\omega_n)\leqq\nu(\omega)<\infty$ and hence ν is uniformly bounded on C. By upper semi-continuity of ν and compactness of C it follows that ν is uniformly bounded in a neighbourhood of C. Let g be a continuous extension of f which vanishes outside this neighbourhood; then ν is uniformly bounded on supp(g). Next let φ be a positive C^1-function on $\boldsymbol{R}$ with support in $[-1, 1]$ and total integral one. Define

$$h=\int dt\ \varphi(t)\sigma_t(g)\,.$$

Then $h\in D(\delta_0)$ and ν, and hence λ, is uniformly bounded on supp(h). Thus $h\in D(\delta)$. Also

$$h(\omega_n)=\int dt\ \varphi(t)g(S_t\omega_n)$$

$$=\int dt\ \varphi(t)t_n=t_n$$

and as $h(S_t\omega)=0$ for $|t|<1$ we have

$$h(U_t\omega)=h(\omega)+t(\delta h)(\omega)+o(t)$$
$$=o(t).$$

Since $h(U_{t_n}\omega)=t_n$ this is a contradiction. Thus $U_R\omega\subseteq S_R\omega$ if $\omega\in X\backslash X_0$.

Case 2. $\omega\in X_0$.

Let D be a neighbourhood of C with compact closure $\bar{D}$. By the assumption on λ, there exists an $\varepsilon>0$ such that λ is uniformly bounded on

$$S_{[-1,1]}\bar{D}\cap\{\omega'\,;\,\nu(\omega')<\varepsilon\}$$

By passing to subsequences of ω_n, we may consider three subcases.

Case 2.1. $\nu(\omega_n)<\varepsilon$ for all n.

In this case we extend f by first defining $f=0$ on the closed sets $\{\omega'\,|\,\nu(\omega')\geqq\varepsilon\}$ and $X\backslash D$, and then extending f arbitrarily to a function in $C_0(X)$. Let ϕ be a positive C^1-function on $\boldsymbol{R}$ with support in $[-1, 1]$ and total integral one. Define

$$h=\int dt\ \phi(t)\ \sigma_t(g)$$

Then $h\in D(\delta_0)$, $\operatorname{supp} h\subseteq S_{[-1,1]}\bar{D}$ and $h=0$ on $\{\omega'\,;\,\nu(\omega')\geqq\varepsilon\}$, and thus $\delta_0 h=0$ on the latter S-invariant set. It follows that λ is uniformly bounded on $\{\omega'\,;\,(\delta_0 h)(\omega')\neq 0\}$, and thus $h\in D(\delta)$. As $h(\omega_n)=t_n$, we deduce a contradiction as in Case 1.

Case 2.2. $\infty>\nu(\omega_n)>\varepsilon$ for all n.

Note that the argument in Case 1 actually established that if $\omega'\in X\backslash X_0$, there exists an $\varepsilon>0$ such that $U_{\langle-\varepsilon,\varepsilon\rangle}\omega'\subseteq S_R\omega'$. Thus $\{t\,;\,U_t\omega'\subseteq S_R\omega'\}$ is open. But if $\nu(\omega')\neq 0$, i.e. $S_R\omega'$ is closed and homeomorphic to a circle, then the set is also closed, and hence it is equal to $\boldsymbol{R}$. We have thus shown that $U_R\omega'\subseteq S_R\omega'$ if $\infty>\nu(\omega')>0$. Hence Case 2.2. cannot occur, as ω is contained in the U_R-orbit through ω_n for any n.

Case 2.3. $\nu(\omega_n)=\infty$ for all n.

Then $\omega_n\in X_0$ for all n, and f extends to a function h in $D(\delta)$ by Lemma 2.7. b. As $h(\omega_n)=t_n$, we obtain a contradiction as in Case 1.

To finish the proof of $U=T$ we need another general lemma, which is known (see Appendix 2 in [13]).

Lemma 2.9. *If U and S are flows on X and $U_R\omega\subseteq S_R\omega$ for all $\omega\in X$, then* *there exists for each $\omega \in X$ a continuous function $y_\omega: \boldsymbol{R} \to \boldsymbol{R}$ such that $y_\omega(0)=0$ and $U_t\omega = S_{y_\omega(t)}\omega$ for all $t \in \boldsymbol{R}$. The function y_ω is uniquely determined by the continuity requirement if $\omega \in X \setminus X_0$, and if $\omega \in X_0$ we may put $y_\omega(t)=0$ for all t, where X_0 denotes the set of fixed points for the flow S.*

Proof. If $\nu(\omega)>0$, then $S_{\boldsymbol{R}}\omega$ is closed in X and is homeomorphic to the circle $\boldsymbol{T}$, and the restriction of U to $S_{\boldsymbol{R}}\omega$ is a one-parameter group of homeomorphisms of the circle, from which the existence and uniqueness of y_ω is immediate.

If $\nu(\omega)=0$, then the map $t \to S_t\omega$ is one to one, and thus there exists a unique function $y_\omega(t)$ such that $U_t\omega = S_{y_\omega(t)}$ for all $t \in \boldsymbol{R}$. Clearly $y_\omega(0)=0$ and it remains to show that y_ω is continuous. But continuity is clear once we can show that for any $T>0$ there is an $N>0$ such that $U_{[-T,T]}\omega \subseteq S_{[-N,N]}\omega$, because the map $t \in [-N, N] \to S_t\omega$ is a homeomorphism (although the map $t \in \boldsymbol{R} \to S_t\omega$ is not necessarily a homeomorphism if the orbit $S_{\boldsymbol{R}}\omega$ is not closed in X). The proof of the existence of N does not follow from straightforward Baire category arguments, but reduces to the following topological lemma: The interval $[0, 1]$ is not a countable union of disjoint non-empty closed sets. See the proof of Theorem 2.50 in [13] for the complete argument.

We now finish the proof that $U=T$. Let $y_\omega(t)$ be the functions defining the flow U by Observation 3 and Lemma 2.9. It suffices to show that $y_\omega(t)=x_\omega(t)$ for all $\omega \in X$ and $t \in \boldsymbol{R}$. This is trivial if $\omega \in X_0$. If $\omega \in X \setminus X_0$, define

$$V=\{t \in \boldsymbol{R};\ y_\omega(t)=x_\omega(t)\}$$

Then $0 \in V$ and V is closed, so if we can prove that V is open, then $V=\boldsymbol{R}$ and the theorem is proved. So let $t_0 \in V$ and put $\omega_0 = S_{x_\omega(t_0)}\omega = T_{t_0}\omega = U_{t_0}\omega$. We have to show that $T_t\omega_0 = U_t\omega_0$ for t in a neighbourhood of zero. But as $\omega_0 \in X \setminus X_0$, $t \to S_t\omega_0$ is $1-1$ in a neighbourhood $[-\varepsilon, \varepsilon]$ of zero, and by Lemma 2.7. d, there exists a function $g \in D(\delta)$ such that $g(S_t\omega_0)=t$ for $|t| \leqq \varepsilon$. Choose $\eta>0$ so that $|y_{\omega_0}(t)| \leqq \varepsilon$, $|x_{\omega_0}(t)| \leqq \varepsilon$ for $|t|<\eta$. Then

$$y_{\omega_0}(t)=g(S_{y_{\omega_0}(t)}\omega_0)=g(U_t\omega_0)$$

and $x_{\omega_0}(t)=g(T_t\omega_0)$ for $|t|<\eta$. Since $g \in D(\delta) \subseteq D(\delta_U)$, it follows that y_{ω_0} is differentiable for $|t| \leqq \eta$ and

$$\begin{aligned} y'_{\omega_0}(t) &= (\delta_U g)(U_t\omega_0) = (\delta g)(U_t\omega_0) \\ &= \lambda(U_t\omega_0)(\delta_0 g)(U_t\omega_0) \\ &= \lambda_{\omega_0}(y_{\omega_0}(t)) \end{aligned}$$

since $(\delta_0 g)(S_s\omega_0)=1$ for $|s| \leqq \varepsilon$. But the unique solution of this equation with $y_{\omega_0}(0)=0$ is

$$y_{\omega_0}(t)=x_{\omega_0}(t)$$

by Lemma 2.5. Thus $U_t\omega_0=T_t\omega_0$ for $|t|<\eta$, and thus $U=T$.

This concludes the proof of Theorem 2.6.

Although Theorem 2.6 establishes that δ has a unique generator extension δ_T and Lemma 2.7 gives some rudimentary information about $D(\delta_T)$, it is unclear whether the assumptions are sufficient to ensure that $\delta_T=\bar{\delta}$, the closure of δ. This stronger form of uniqueness follows, however, if one assumes a stronger smoothness property for λ. As a preliminary to deriving this result we prove the following lemmas.

Lemma 2.10. *Adopt the assumptions of Theorem 2.6, and let δ_T denote the unique generator extension of δ. Then for all $g\in D(\delta_T)$ and all $\omega\in X\backslash X_0$ such that $\lambda(\omega)\neq 0$, the limit*

$$\delta_0(g)(\omega)\equiv\lim_{t\to 0}(g(S_t\omega)-g(\omega))/t$$

exists, and furthermore

$$\delta_T(g)(\omega)=\lambda(\omega)\delta_0(g)(\omega).$$

Proof. Let g and ω be as in the statement of the lemma, and define $g_\omega(t)=g(S_t\omega)$ and $\lambda_\omega(t)=\lambda(S_t\omega)$. As in the proof of Theorem 2.6 let x_ω be the unique solution of the initial value problem

$$x'_\omega(t)=\lambda_\omega(x_\omega(t)), \qquad x_\omega(0)=0.$$

Then x_ω is a C^1-function with non-zero derivative at $t=0$ and therefore has a C^1-inverse y_ω in a neighbourhood of $t=0$. Now since $g\in D(\delta_T)$, $g_\omega(x_\omega(s))=g(T_s\omega)$ is a C^1-function of s and $(g_\omega\circ x_\omega)'(0)=\delta_T(g)(\omega)$. Therefore $g_\omega=(g_\omega\circ x_\omega)\circ y_\omega$ is a C^1-function near zero whose derivative at zero is

$$g'_\omega(0)=\delta_T(g)(\omega)\, y'_\omega(0)=\delta_T(g)(\omega)/\lambda(\omega).$$

But $g'_\omega(0)=d/ds|_{s=0}g(S_s\omega)$, so both the existence of this derivative and the formula for $\delta_T(g)(\omega)$ are established.

Lemma 2.11. *Adopt the assumption of Theorem 2.6, but further assume that λ is differentiable in the sense that*

$$\delta_0(\lambda)\equiv\lim_{t\to 0}(\lambda(S_t\omega)-\lambda(\omega))/t$$

exists pointwise and is a continuous function of $\omega\in X\backslash X_0$.

(a) *The formula*

$$\lambda(T_t\omega)=\exp\left\{\int_0^t ds\ (\delta_0(\lambda))(T_s\omega)\right\}\lambda(\omega)$$

is valid for $\omega\in X\backslash X_0$ and $t\in\boldsymbol{R}$.

(b) *If $f\in\mathscr{D}(\delta)$, then*

$$(*)\qquad \delta_0(f\circ T_t)(\omega)=\exp\left\{\int_0^t ds\ (\delta_0\lambda)(T_s\omega)\right\}(\delta_0 f)(T_t\omega)$$

for all $\omega\in X\backslash X_0$, *where the* δ_0 *on the left side is defined as in the heading of the lemma.*

(c) *If* $f\in\mathcal{D}_1\cap D(\delta)$, *then* $(f\circ T_t)\in\mathcal{D}_1\cap D(\delta)$ *for all* $t\in\boldsymbol{R}$.

Remark. Our main purpose here is to establish that $\mathcal{D}_1\cap D(\delta)$ is invariant under the group τ_t, and for this we need to show that for $f\in\mathcal{D}_1\cap D(\delta)$ and $t\in\boldsymbol{R}$, $(f\circ T_t)\in D(\delta_0)$. But by Lemma 2.4 it suffices to show that the pointwise derivative $\delta_0(f\circ T_t)(\omega)\equiv\lim_{s\to 0}((f\circ T_t)(S_s\omega)-f\circ T_t(\omega))/s$ exists and defines a continuous function on X. Since $f\in D(\delta)\subseteqq D(\delta_T)$, $f\circ T_t$ also lies in $D(\delta_T)$ and Lemma 2.10 already shows that $\delta_0(f\circ T_t)(\omega)=\lambda(\omega)^{-1}\delta_T(f\circ T_t)(\omega)$ exists and is continuous on $Y=\{\omega\in X\backslash X_0;\ \lambda(\omega)\neq 0\}$. But in order to prove the existence and continuity of $\delta_0(f\circ T_t)(\omega)$ on all of X we have to establish the formula (*) in the statement of the lemma.

Proof.

(a) The hypothesis on λ implies that λ_ω is a C^1-function with derivative $\lambda'_\omega=(\delta_0\lambda)_\omega$. Thus by the chain rule,

$$\frac{d}{dt}\lambda_\omega(x_\omega(t))=\lambda'_\omega(x_\omega(t))x'_\omega(t)=\lambda'_\omega(x_\omega(t))\lambda_\omega(x_\omega(t))$$

The unique solution of this equation with $\lambda_\omega(x_\omega(0))=\lambda_\omega(0)$ is

$$\lambda_\omega(x_\omega(t))=\exp\left\{\int_0^t ds\ \lambda'_\omega(x_\omega(s))\right\}\lambda_\omega(0).$$

On the space X the relation reads

$$\lambda(T_t\omega)=\exp\left\{\int_0^t ds\ (\delta_0\lambda)(T_s\omega)\right\}\lambda(\omega),$$

which proves the first formula in the statement of the lemma.

(b) If $f\in D(\delta)\subseteqq D(\delta_T)$, $f\circ T_t$ lies in $D(\delta_T)$ and

$$\delta_T(f\circ T_t)=(\delta_T f)\circ T_t.$$

For $\omega\in Y=\{\omega'\in X\backslash X_0;\ \lambda(\omega')\neq 0\}$ we evaluate the left-hand side using Lemma 2.10 and obtain the equation

$$\begin{aligned}\lambda(\omega)\delta_0(f\circ T_t)(\omega)&=\lambda(T_t\omega)(\delta_0 f)(T_t\omega)\\&=\lambda(\omega)\exp\left\{\int_0^t ds\ (\delta_0\lambda)(T_s\omega)\right\}(\delta_0 f)(T_t\omega).\end{aligned}$$

Cancelling the factor $\lambda(\omega)$ gives the desired formula (*) for all $\omega\in Y$.

Next suppose that $\omega\in X\backslash X_0$, but $\lambda(\omega)=0$. Then $T_t\omega=\omega$ for all t. Let V be the flow on $\boldsymbol{R}$ corresponding to the flow T on $S_{\boldsymbol{R}}\omega$, i.e. $V_t(y)$ is determined by the usual continuity requirements and

$$S_{V_t(y)}\omega=T_tS_y\omega, \qquad t\in\boldsymbol{R},\ y\in\boldsymbol{R}.$$

Then

$$V_t(y)=x_{S_y(\omega)}(t)+y$$

and since V is the flow on $\boldsymbol{R}$ determined by the vector field $\lambda_\omega(y)d/dy$, and $\lambda_\omega(0)=0$, we have

$$\frac{d}{dt}V_t(y)=\lambda_\omega(V_t(y))=\lambda'_\omega(0)V_ty+o(V_ty)$$

where we used Taylors formula, and the fact that λ_ω is continuously differentiable. This formula can be written

$$\frac{d}{dt}(e^{-t\lambda'_\omega(0)}V_t(y))=e^{-t\lambda'_\omega(0)}o(V_t(y))$$

where $o(\cdot)$ is a function depending only on λ_ω such that $\lim_{h\to 0}o(h)/h=0$. Integrating, and using $V_0(y)=y$, we get

$$\begin{aligned}V_t(y)&=e^{t\lambda'_\omega(0)}y+e^{t\lambda'_\omega(0)}\int_0^t ds\ e^{-s\lambda'_\omega(0)}\ o(V_s(y))\\&=e^{t\lambda'_\omega(0)}\left[y+\int_0^t ds\ e^{-s\lambda'_\omega(0)}\left[\frac{o(V_sy)}{V_sy}\right]V_sy\right].\end{aligned}$$

Feeding the expression for V_sy back into the integral and iterating we arrive finally at the expression

$$V_ty=e^{t\lambda'_\omega(0)}y(1+O(t,\ y))$$

where $O(\cdot)$ satisfies $\lim_{y\to 0}O(t,\ y)=0$ for all t. Hence $\partial/\partial y|_{y=0}\ V_ty=e^{t\lambda'_\omega(0)}$. For $f\in D(\delta)$,

$$(f\circ T_t)(S_s\omega)=f(T_tS_s\omega)=f_\omega(V_ts),$$

where $f_\omega(x)=f(S_x\omega)$ as usual. Therefore by the chain rule

$$\begin{aligned}\delta_0(f\circ T_t)(\omega)&=\frac{d}{ds}\bigg|_{s=0}(f\circ T_t)(S_s\omega)\\&=f'_\omega(V_t0)\frac{\partial}{\partial s}\bigg|_{s=0}V_ts\\&=f'_\omega(0)\exp\{t\lambda'_\omega(0)\}\\&=(\delta_0f)(\omega)\exp\{t\delta_0(\lambda)(\omega)\},\end{aligned}$$

using the facts that $f\in D(\delta_0)$ and $V_t0=0$ for all t. This establishes that $\delta_0(f\circ T_t)(\omega)$ exists, and since $T_s\omega=\omega$ for all s, $\delta_0(f\circ T_t)(\omega)$ is given by the formula (*) in this case as well.

(c) Let $f\in\mathcal{D}_1\cap D(\delta)$ and $t\in\boldsymbol{R}$. First we show that $f\circ T_t\in D(\delta_0)$. By part (b), $\delta_0(f\circ T_t)(\omega)$ exists, as a pointwise derivative, at all $\omega\in X$ and is given by the formula

$$(*)\qquad \delta_0(f\circ T_t)(\omega)=\begin{cases}\exp\left\{\int_0^t ds\ (\delta_0\lambda)(T_s\omega)\right\}(\delta_0 f)(T_t\omega) & \text{if}\quad \omega\in X\backslash X_0,\\ 0 & \text{if}\quad \omega\in X_0.\end{cases}$$

Since $f\in\mathcal{D}_1$, f has compact support, thus $f\circ T_t$ and then $\delta_0(f\circ T_t)$ have compact supports. Thus by Lemma 2.4, to show that $f\circ T_t\in D(\delta_0)$, it suffices to show the continuity of the right hand side of (*). As f is constant on orbits of frequency larger than a certain M, $(\delta_0 f)(T_t\omega)=0$ if $\nu(\omega)\geqq M$. Thus it suffices to verify

$$\lim_\alpha \delta_0(f\circ T_t)(\omega_\alpha)=\delta_0(f\circ T_t)(\omega)$$

for convergent nets $\omega_\alpha\to\omega$ such that $\nu(\omega_\alpha)<M$ for all α. The other cases are trivial due to the upper semicontinuity of ν. Thus two cases remain.

Case 1. $\nu(\omega)<\infty$.

Because of the formula (*) and the continuity of $(\delta_0 f)\circ T_t$ it suffices to show

$$\int_0^t ds\ (\delta_0\lambda)(T_s\omega_\alpha)\longrightarrow\int_0^t ds\ (\delta_0\lambda)(T_s\omega)$$

The integrands converge pointwise on $[0, t]$ by continuity of $\delta_0\lambda$ on $X\backslash X_0$. Let ψ be a continuous function with compact support in $X\backslash X_0$ such that

$$\delta_0\lambda(T_s\omega')=\psi(T_s\omega')=\tau_s(\psi)(\omega')$$

for all $\omega'\in\{\omega_\alpha\}\cup\{\omega\}$ and $s\in[0, t]$. Then the norm continuity of $s\to\tau_s\psi$ implies that the integrands are uniformly equicontinuous and therefore converge uniformly.

Case 2. $\nu(\omega)=\infty$.

Then $\omega\in X_0$ and $\delta_0(f\circ T_t)(\omega)=0$. Note that

$$|\delta_0\lambda(\omega')|\leqq K(\nu(\omega'))$$

for $\omega'\in X\backslash X_0$ by the Lipschitz continuity of λ, and thus $\delta_0\lambda$ is uniformly bounded on

$$\{\omega'\in X:\ \nu(\omega')<M\}$$

The uniform boundedness of $\exp\left\{\int_0^t ds\ \delta_0\lambda(T_s\omega_\alpha)\right\}$ together with the continuity of $(\delta_0 f)\circ T_t$ implies that

$$\lim_\alpha \delta_0(f\circ T_t)(\omega_\alpha)=0=\delta_0(f\circ T_t)(\omega).$$

This establishes that $\delta_0(f\circ T_t)\in C_0(X)$, and $(f\circ T_t)\in D(\delta_0)$. Since clearly $f\circ T_t\in\mathcal{D}_0$, it follows that

$$f\circ T_t\in\mathcal{D}_0\cap D(\delta_0)=\mathcal{D}_1.$$

Finally we prove that $f\circ T_t\in D(\delta)$. We already know that $f\circ T_t\in D(\delta_0)$, and by parts (a) and (b)

$$(***)\qquad \lambda(\omega)\ \delta_0(f\circ T_t)(\omega)=\lambda(\omega)\ \exp\left\{\int_0^t ds\ (\delta_0\lambda)(T_s\omega)\right\}(\delta_0 f)(T_t\omega)$$

$$=\lambda(T_t\omega)\ (\delta_0 f)(T_t\omega)=\delta(f)(T_t\omega)$$

for all $\omega\in X$. But as T_t is a homeomorphism of X and $f\in D(\delta)$, the map

$$\omega\longrightarrow T_t\omega\longrightarrow\delta(f)(T_t\omega)$$

is in $C_0(X)$, and thus $f\circ T_t\in D(\delta)$. (Note that the formula (***) also formally follows by noting that the generator δ_T of the automorphism group τ defined by the flow T extends δ and commutes with τ, and hence

$$\delta_T(\tau_t f)(\omega)=\tau_t(\delta_T f)(\omega)=\tau_t(\delta f)(\omega)=\lambda(T_t\omega)\ (\delta_0 f)(T_t\omega).)$$

This completes the proof of Lemma 2.11.

The following result is now easily established.

Theorem 2.12. *Let σ be a strongly continuous one-parameter group of *-automorphisms on an abelian C^*-algebra $\mathcal{A}=C_0(X)$ with generator δ_0 and associated flow S on X, and let $X_0\subseteq X$ denote the fixed points of S. Assume that λ is a continuous (not necessarily bounded) function on $X\backslash X_0$ such that*

$$(\delta_0\lambda)(\omega)=\frac{d}{dt}\bigg|_{t=0}\lambda(S_t\omega)$$

exists and is a continuous function of ω which is bounded on the sets $\{\omega\in X;\ \nu(\omega)<M\}$ for all $M>0$. Assume also that for any compact subset $C\subseteq X$ there exists an $\varepsilon>0$ such that λ is uniformly bounded on

$$C\cap\{\omega\in X;\ \nu(\omega)<\varepsilon\}.$$

*It follows that the derivation $\delta=\lambda\delta_0$ is closable, and its closure generates a one-parameter group τ of *-automorphisms of $\mathcal{A}$.*

Proof. By Theorem 2.6 it suffices to show that $D(\delta)$ is a core for δ_T. But according to Lemmas 2.7 and 2.11, $\mathcal{D}_1\cap D(\delta)$ is dense in $\mathcal{A}$ and $\tau_t(\mathcal{D}_1\cap D(\delta))=\mathcal{D}_1\cap D(\delta)$ for all $t\in\mathbf{R}$. Therefore it follows from [7], Corollary 3.1.7, that $\mathcal{D}_1\cap D(\delta)$ is a core for δ_T.

If λ and $\delta_0\lambda$ are polynomially bounded in the frequency, the assumptions of Theorem 2.12 are automatically fulfilled. In this case $\mathcal{A}_\infty\subseteq D(\delta)$, but although $\mathcal{A}_\infty$ is a common core for $\bar{\delta}$ and δ_0, the domains of these derivations can be quite different. It is also possible that $\mathcal{A}_\infty$, and even $\mathcal{A}_1$ fails to be invariant under the group τ generated by $\bar{\delta}$. The following example shows that one can have $\tau(\mathcal{A}_\infty)\subsetneqq\mathcal{A}_1$ even if $\delta(\mathcal{A}_\infty)\subset\mathcal{A}_\infty$.

Example 2.13. Let $\mathcal{A}=C_0(\boldsymbol{R}^2)$ and σ the group defined by

$$(\sigma_t f)(r, \theta)=f(r, \theta+t/r).$$

Thus the orbits of the associated flow are concentric circles centred at the origin and the orbit of radius r has frequency $(2\pi r)^{-1}$; the origin is a fixed point. It follows that $\mathcal{A}_\infty$ consists of those $f\in\mathcal{A}$ which are infinitely often differentiable in θ and such that the partial derivatives of f with respect to θ go to zero faster than any power of r as $r\to 0$, uniformly in θ. Next define δ on $\mathcal{A}_\infty$ by

$$(\delta f)(r, \theta)=\frac{\sin\theta}{r^n}\frac{\partial f(r, \theta)}{\partial\theta}$$

for some $n>1$. A simple calculation then shows that

$$(\tau_t f)(r, \theta)=f(r, 2\tan^{-1}(e^{t/r^n}\tan\theta/2)).$$

In particular

$$\frac{\partial(\tau_t f)}{\partial\theta}(r, 0)=\left(\frac{\partial f}{\partial\theta}\right)(r, 0)e^{t/r^n}.$$

It follows that $\delta(\mathcal{A}_\infty)\subset\mathcal{A}_\infty$ but $\tau(\mathcal{A}_\infty)\subsetneqq\mathcal{A}_1$. Moreover $D(\bar{\delta})\subsetneqq D(\delta_0)$ and $D(\delta_0)\subsetneqq D(\bar{\delta})$.

We also remark that the hypotheses in Theorem 2.12 do not generally imply that $\mathcal{D}_1\subseteqq D(\delta)$. An example is obtained by modifying the definition of σ and δ above as follows

$$(\sigma_t f)(r, \theta)=f(r, \theta+t),$$

and

$$(\delta f)(r, \theta)=\frac{1}{r}(\delta_0 f)(r, \theta)=\frac{1}{r}\frac{\partial f(r, \theta)}{\partial\theta}.$$

Then $\lambda(r, \theta)=1/r$ is constant on S-orbits, so $\delta_0\lambda=0$, and λ is bounded on $\{\omega\,|\,\nu(\omega)<1/2\pi\}=\phi$, so all the hypotheses of Theorem 2.12 are satisfied. In this case $\mathcal{D}_1=D(\delta_0)$, but $\mathcal{D}_1\subsetneqq D(\delta)$, since for example, $f(r, \theta)=r\sin\theta$ is in $D(\delta_0)$, but not in $D(\delta)$.

Note that if the last assumption in Theorem 2.12 is replaced by "For any compact subset $C\subseteqq X$ and any $M>0$, λ is uniformly bounded on $C\cap\{\omega\in X;\ \nu(\omega)<M\}$", then a simple argument establishes that $\mathcal{D}_1\subseteqq D(\delta)$. This is used in the proof of Theorem 3.1.

Finally we emphasize that the smoothness assumptions on λ adopted in Theorems 2.6 and 2.12 are only essential in a neighbourhood of the zeros of λ. These results can be easily generalized by use of the perturbation result mentioned in the introduction, e.g. if λ_1 satisfies the assumptions of Theorem 2.12 and λ_2 is bounded continuous on $X\backslash X_0$ and bounded away from zero then $\overline{\lambda_1\lambda_2\delta_0}$ is a generator.

§ 3. Smooth Derivations

In this section we prove the theorem on derivations stated in the introduction.

Theorem 3.1. *Let* $(\mathcal{A}, \boldsymbol{R}, \sigma)$ *be an abelian* C^**-dynamical system and denote the generator of* σ *by* δ_0. *Define* $\mathcal{A}_n = D(\delta_0^n)$ *and* $\mathcal{A}_\infty = \bigcap_{n\geq 1}\mathcal{A}_n$.

If $\delta : \mathcal{A}_\infty \to \mathcal{A}_1$ *is a *-derivation then* δ *is closable and its closure* $\bar{\delta}$ *generates a strongly continuous one-parameter group* τ *of *-automorphisms of* $\mathcal{A}$.

Proof. It follows from [6] Theorems 1.2 and 4.2 that the condition on δ implies that δ has the form $\delta = \lambda\delta_0$, where λ is a once differentiable function on $X\backslash X_0$ such that λ and $\delta_0\lambda$ are continuous and polynomially bounded on $X\backslash X_0$. Thus λ satisfies the hypotheses of Theorem 2.12.

The polynomial growth of λ in the frequency implies that the natural domain of δ, the $D(\delta)$ defined in the statement of Theorem 2.6, contains $\mathcal{A}_\infty$. This follows because for $f \in \mathcal{A}_\infty$ and $n \in \boldsymbol{N}$ there exists a $K_n > 0$ such that

$$|\delta_0(f)(\omega)| \leqq K_n\left(\frac{1}{1+\nu(\omega)}\right)^n$$

by Observation 6 of [6].

Now the extension of δ to $D(\delta)$ has a unique generator extension δ_T, and it follows from the argument used in the proof of Theorem 2.12 that $\mathcal{D}_1 \subseteq D(\delta)$ is a core for δ_T. It remains only to show that $\mathcal{A}_\infty$ is a core as well.

To this end, fix $f \in \mathcal{D}_1$ and let $h \in C_0(\boldsymbol{R})$ be a positive, infinitely often differentiable, function with support in $[-1, 1]$ and total integral one. Then defining

$$f_n = n\int dt\ h(nt)\ \sigma_t f$$

$$= \int dt\ h(t)\ \sigma_{t/n} f$$

one has $f_n \in \mathcal{A}_\infty \cap \mathcal{D}_1$. Moreover, $f_n \to f$ and $\delta_0 f_n \to \delta_0 f$ by strong continuity of σ. But as f is constant on S-orbits of frequency larger than a certain M, the f_n are also constant on these orbits, and $(\delta_0 f_n)(\omega) = 0 = (\delta_0 f)(\omega)$ if $\nu(\omega) \geqq M$. But λ is bounded on the subsets $\{\omega ; \nu(\omega) \leqq M\}$ and hence

$$\delta f_n = \lambda\delta_0 f_n \longrightarrow \lambda\delta_0 f = \delta f.$$

Consequently $\mathcal{A}_\infty \cap \mathcal{D}_1$ is a core for δ_T or, equivalently $\delta_T = \bar{\delta} =$ the closure of δ.

§ 4. Local Dissipations

In this section we discuss various aspects of the generator problem for more general operators associated with $(\mathcal{A}, \boldsymbol{R}, \sigma)$, local dissipations.

An operator H from $\mathcal{A}_\infty$ into $\mathcal{A}$ is defined to be local if

$$\mathrm{Supp}\,(Hf)\subseteq \mathrm{Supp}\,(f)$$

for all $f\in\mathcal{A}_\infty$ and it is defined to be a dissipation, or semi-derivation, if $H\bar{f}=\overline{Hf}$ and

$$(H\bar{f}f)\leqq(H\bar{f})f+\bar{f}(Hf)$$

for all $f\in\mathcal{A}_\infty$. In [6] Theorem 1.2B it was established that $H:\mathcal{A}_\infty\to\mathcal{A}$ is a local dissipation if, and only if,

$$H=-\lambda_2\delta_0^2+\lambda_1\delta_0+\lambda_0$$

where the real functions λ_0, λ_1, λ_2 satisfy the following: λ_0 is a bounded continuous function over X, λ_1 and λ_2 vanish on X_0 and are continuous functions on $X\backslash X_0$ polynomially bounded in the frequency, and λ_0, $\lambda_2\geqq 0$. We now argue that each local dissipation H is dissipative, hence closable, and then discuss properties of its closure $\bar{H}$ as a generator.

Lemma 4.1. *Let $H:\mathcal{A}_\infty\to\mathcal{A}$ be a local dissipation. Then H is dissipative, hence closable.*

Proof. The operator H is defined to be dissipative if for each $f\in\mathcal{A}_\infty$

$$\mathrm{Re}\,\mu(Hf)\geqq 0$$

for at least one $\mu\in\mathcal{A}^*$ with $\|\mu\|=1$ such that $\mu(f)=\|f\|$, or, equivalently, if

$$\|(1+\alpha H)f\|\geqq\|f\|$$

for all $f\in\mathcal{A}_\infty$ and for all small $\alpha>0$. (See for example, [2] Section 2.1). But by an argument of Kishimoto [14] (see [7], page 230, line 9-14) it suffices, for dissipations, to verify this condition for positive f. Now if $f\in\mathcal{A}_\infty$ and f is positive one can find $\omega\in X$ such that

$$f(\omega)=\|f\|$$

and choose μ to be the point measure at ω. Then $t\mapsto(\sigma_t f)(\omega)=f(S_t\omega)$ must attain its maximum at $t=0$. Therefore $(\delta_0 f)(\omega)=0$ and $-(\delta_0^2 f)(\omega)\geqq 0$. But then

$$\mu(Hf)=\lambda_2(\omega)(-\delta_0^2 f)(\omega)+\lambda_1(\omega)(\delta_0 f)(\omega)+\lambda_0(\omega)\geqq 0\,,$$

and H is dissipative. Finally it is a standard result, [7], Lemma 3.1.14, that a dissipative operator H is closable, and its closure $\bar{H}$ is dissipative.

Next for each real $f\in\mathcal{A}$ define the positive and negative parts $f_\pm=(\pm f)\vee 0$ and introduce the half-norm N by $N(f)=\|f_+\|$. Now, a real operator K on $\mathcal{A}$ is defined to be N-dissipative or dispersive it for each real $f\in D(K)$

$$\mu(Kf)\geqq 0$$

for at least one $\mu\in\mathcal{A}^*$ with $\|\mu\|=1$ such that $\mu(f)=N(f)$ or, equivalently, if

$$N(1+\alpha K)f)\geqq N(f)$$

for all $f\in D(K)$ and all small $\alpha>0$. ([2], Section 2.1)

Lemma 4.2. *Let $H:\mathcal{A}_\infty\to\mathcal{A}$ be a local dissipation. Then H is N-dissipative.*

Proof. Let $f\in\mathcal{A}_\infty$ be real. If $f\leqq 0$ then $N(f)=0$ and $N(1+\alpha H)f)\geqq 0$. Therefore we can assume $f_+\neq 0$. Now choose $\omega\in X$ such that $f(\omega)=f_+(\omega)=\|f_+\|$ and again note that this implies that $t\mapsto f(S_t\omega)$ has a maximum at $t=0$, hence $(\delta_0 f)(\omega)=0$ and $-(\delta_0^2 f)(\omega)\geqq 0$. Then if μ is the point measure at ω

$$\mu(Hf)=\lambda_2(\omega)(-(\delta_0^2 f)(\omega))+\lambda_1(\omega)(\delta_0 f)(\omega)+\lambda_0(\omega)\geqq 0\,.$$

But since

$$\mu(Hf)\leqq(N((1+\alpha H)f)-N(f))/\alpha\,,\qquad \alpha>0$$

this proves that $N((1+\alpha H)f)\geqq N(f)$ for all real $f\in\mathcal{A}_\infty$ and all $\alpha>0$.

The following result is now a consequence of these dissipation properties and standard semigroup theory, see [8].

Proposition 4.3. *Let $H:\mathcal{A}_\infty\to\mathcal{A}$ be a local dissipation with closure $\bar{H}$. Then the following conditions are equivalent;*

1. $\overline{(1+\alpha H)(\mathcal{A}_\infty)}=\mathcal{A}$ *for all small* $\alpha>0$.
2. $\bar{H}$ *generates a* C_0*-semigroup,*
3. $\bar{H}$ *generates a positive* C_0*-semigroup of contractions.*

Remark. It is unclear under what conditions the closure $\bar{H}$ of a local dissipation is again a dissipation, e. g. it is not evident that $D(\bar{H})$ is an algebra. Note, however, that if H is any operator such that $\bar{H}$ generates a positive C_0-semigroup of contractions τ, then τ is strongly positive in the sense

$$\tau_t(\bar{f}f)\geqq\tau_t(\bar{f})\tau_t(f)$$

for all $f\in\mathcal{A}$ and $t\in\boldsymbol{R}$, by the generalized Schwarz's inequality for abelian algebras, [7].

Finally we make some remarks on criteria for $\bar{H}$ to be a generator. It is natural to conjecture, in analogy with Theorem 3.1, that if H is a local dissipation which maps $\mathcal{A}_\infty$ into $\mathcal{A}_2$ then $\bar{H}$ is a generator. But we have not been able to prove this.

It does follow that if $H:\mathcal{A}_\infty\to\mathcal{A}_2$ then the coefficients λ_i in the representation

$$H=-\lambda_2\delta_0^2+\lambda_1\delta_0+\lambda_0$$

satisfy the smoothness properties $\lambda_i\in D(\delta_0^2)$, and the derivatives $\delta_0\lambda_i$, $\delta_0^2\lambda_i$, are polynomially bounded in the frequency for $i=1$, 2 and bounded for $i=0$. This follows from [6], Theorem 4.2. These conditions then imply, by an argument which we sketch below, that the closure of $-\lambda_2\delta_0^2+\lambda_0$ is a generator. Moreover the closure of $\lambda_1\delta_0$ is a generator by Theorem 2.12. Nevertheless it is unclear whether the sum of these two generators is a generator without additional assumptions. One such assumption which is sufficient to guarantee that $\bar{H}$ is a generator is the condition

$$\lambda_2 \geqq \varepsilon \lambda_1^2$$

for some $\varepsilon > 0$ or, equivalently, $\|\lambda_1/\sqrt{\lambda_2}\|_\infty < +\infty$. We also sketch the proof of this below.

Let us first consider the proof that the closure of $-\lambda_2\delta_0^2 + \lambda_0$ is a generator, but for simplicity assume that λ_0, λ_2, $\delta_0\lambda_2$, $\delta_0^2\lambda_2$, are bounded. The general case can then be handled by approximation arguments. But in this case it suffices to prove that the closure of $-\lambda_2\delta_0^2 = -(\sqrt{\lambda_2}\delta_0)^2 + (1/2)(\delta_0\lambda_2)\delta_0$ is a generator. Now $\overline{\sqrt{\lambda_2}\delta_0}$ is the generator of a group of *-automorphisms τ by Theorem 2.12 and hence $-(\overline{\sqrt{\lambda_2}\delta_0})^2$ is the generator of a contraction semigroup ρ constructed by the algorithm

$$\rho_t = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} ds\ e^{-s^2/4t}\ \tau_s .$$

Now consider the term $(\delta_0\lambda_2)\delta_0$. Since $\lambda_2 \in D(\delta_0^2)$ and $\|\delta_0^2\lambda_2\| < +\infty$ by our simplifying assumption one has

$$0 \leqq \lambda_2(S_t\omega) = \lambda_2(\omega) + t(\delta_0\lambda_2)(\omega) + \int_0^t dt_1 \int_0^{t_1} dt_2\ (\delta_0^2\lambda_2)(S_{t_2}\omega)$$

and hence

$$0 \leqq \lambda_2(\omega) + t(\delta_0\lambda_2)(\omega) + (t^2/2)\|\delta_0^2\lambda_2\| .$$

Consequently

$$|(\delta_0\lambda_2)(\omega)|^2 \leqq 2\lambda_2(\omega)\|\delta_0^2\lambda_2\| .$$

Moreover if τ denotes the group generated by the derivation $\delta = -\overline{\sqrt{\lambda_2}\delta_0}$ then for $f \in \mathcal{A}_\infty$

$$(\tau_t f)(\omega) = f(\omega) + t(\delta f)(\omega) + \int_0^t dt_1 \int_0^{t_1} dt_2\ \tau_{t_2}((\delta^2 f)(\omega))$$

and hence

$$\|\delta f\| \leqq (t/2)\|\delta^2 f\| + (2/t)\|f\|$$

by the triangle inequality. Combining these estimates gives

$$\begin{aligned} \|(\delta_0\lambda_2)(\delta_0 f)\| &\leqq \|(\delta_0\lambda_2)/\sqrt{\lambda_2}\|\,\|(\sqrt{\lambda_2}\delta_0)(f)\| \\ &\leqq \sqrt{2\|\delta_0^2\lambda_2\|}(\varepsilon\|(\sqrt{\lambda_2}\delta_0)^2(f)\| + (1/\varepsilon)\|f\|) \end{aligned}$$

for all $f \in \mathcal{A}_\infty$ and any $\varepsilon > 0$. Thus $(\delta_0\lambda_2)\delta_0$ is relatively bounded with respect to $-(\sqrt{\lambda_2}\delta_0)^2$ with relative bound zero. Hence the closure of the sum $-(\sqrt{\lambda_2}\delta_0)^2 + (1/2)(\delta_0\lambda_2)\delta_0 = -\lambda_2\delta_0^2$ is a generator.

Note that the same argument also shows that if $\lambda_2 \geqq \varepsilon\lambda_1^2$, then $\lambda_1\delta_0$ is relatively bounded with respect to $-(\sqrt{\lambda_2}\delta_0)^2$ with relative bound zero. Hence the closure of $-(\sqrt{\lambda_2}\delta_0)^2+((1/2)(\delta_0\lambda_2)\delta_0+\lambda_1\delta_0)=-\lambda_2\delta_0^2+\lambda_1\delta_0$ is a generator.

§ 5. Compact Group Actions

In this section we prove an analogue of Theorem 3.1, with the action S of $\boldsymbol{R}$ on X replaced by an action of a *compact* Lie group. Thus we consider a locally compact Hausdorff space X, a compact group $\boldsymbol{G}$, and a topological transformation group $(X, \boldsymbol{G}, S)$, i. e., $g\mapsto S_g$ is a homomorphism of $\boldsymbol{G}$ into the group of homeomorphisms of X such that $(g, x)\mapsto S_g(x)$ is jointly continuous. The corresponding automorphic action σ of $\boldsymbol{G}$ on $\mathcal{A}=C_0(X)$ defined by $\sigma_g f(\omega)=f(S_{g^{-1}}\omega)$ is strongly continuous. For $n\geqq 1$, $\mathcal{A}_n$ denotes the algebra of functions $f\in\mathcal{A}$ such that $g\mapsto\sigma_g(f)$ is a C^n function from $\boldsymbol{G}$ to $\mathcal{A}$, and $\mathcal{A}_\infty=\bigcap_{n\geqq 1}\mathcal{A}_n$. If $X_1, \cdots X_d$ is a basis for the Lie algebra of $\boldsymbol{G}$, then $f\in\mathcal{A}_n$ if, and only if, f is in the domain of $\sigma(X_{i_1}\cdots X_{i_n})$ for all choices of $i_1, \cdots, i_n$, and $\mathcal{A}_n$ is a Banach space with the norm

$$\|f\|_n=\sup\{\|\sigma(X_{i_1}\cdots X_{i_k})f\|\,;\ 0\leqq k\leqq n \text{ and } 1\leqq i_j\leqq d\}.$$

$\mathcal{A}_n$ is a Banach algebra in an equivalent norm and $\mathcal{A}_\infty$ is a Frechet algebra, with topology generated by the norms $\|\ \|_n$, $n\geqq 1$.

Theorem 5.1. *Let X be a locally compact Hausdorff space, $\mathcal{A}=C_0(X)$ and $\alpha:\boldsymbol{G}\to\mathrm{Aut}(\mathcal{A})$ a strongly continuous action of a compact Lie group $\boldsymbol{G}$ on $\mathcal{A}$. If $\delta:\mathcal{A}_\infty\to\mathcal{A}_1$ is a *-derivation, then δ is closable and its closure $\bar{\delta}$ generates a strongly continuous action of $\boldsymbol{R}$ on $\mathcal{A}$. Furthermore, the orbits of the corresponding flow on X are contained in the orbits of the action of $\boldsymbol{G}$ on X.*

Proof. Fix a $\boldsymbol{G}$-orbit M in X. Any function $f\in\mathcal{A}_\infty$ satisfying $f|_M=0$ can be approximated in the $\mathcal{A}_\infty$-topology by functions of the form $f\varphi$, where $\varphi:X\to[0, 1]$ is an element of $C_0(X)$ which is constant on $\boldsymbol{G}$-orbits and zero in a neighbourhood of M. Then $\delta(f\varphi)|_M=0$ because of the locality property of derivations defined on a dense domain with a C^∞-functional calculus [3]. Consequently $\delta(f)|_M=0$ because δ is automatically continuous with respect to the $\mathcal{A}_\infty$-topology [16, 5].

Thus δ restricts to M; that is the formula $\delta_M(f|_M)=\delta(f)|_M$ defines a *-derivation δ_M in $C(M)$ with domain $\{f|_M\,;f\in\mathcal{A}_\infty\}$. It is convenient here to refer again to the theorem of Dixmier and Malliavin [9] which states that $\mathcal{A}_\infty$ is the linear span of functions of the form

$$\varphi*f=\int_G\varphi(g)\,\sigma_g(f)\,dg$$

where $\varphi\in C^\infty(\boldsymbol{G})$, and $f\in\mathcal{A}$. Applying this result also to the action of $\boldsymbol{G}$ on $C(M)$, we see that $\{f|_M\,;f\in\mathcal{A}_\infty\}$ is all of the algebra of C^∞-elements for the action of G on $C(M)$. Now M can be identified with some coset space G/K via a G-equivariant homeomorphism. Under this identification, the algebra of C^∞-elements for G acting on $C(M)$ is identified with $C^\infty(G/K)$ and δ_M becomes a derivation on $C^\infty(G/K)$. Since the range of δ is contained in $\mathcal{A}_1$, it follows that δ_M maps $C^\infty(G/K)$ into $C^1(G/K)$, and δ_M is given by a C^1-vectorfield on G/K. Now the basic existence and uniqueness theorem for ordinary differential equations implies that $\bar{\delta}_M$ generates a one-parameter group of automorphisms of $C(M)$, see [13], Theorem 3.43.

Rather than explicitly patching together the flows on the G-orbits generated by the vector fields δ_M, as in the proof of Theorem 2.6, we can use global criteria for $\bar{\delta}$ to be a generator. First it is easy to see that $\pm\delta$ are dissipative operators; in fact if f is real valued and achieves its maximum modulus at a point ω, then $\delta(f)(\omega)=0$, since $\delta_{G(\omega)}$ is given by a vector field. It remains to check that $(\mathrm{id}\pm\delta)\mathcal{A}_\infty$ are dense subspaces of $\mathcal{A}$. For this, one can follow the argument in [11], Theorem 3.2. Since this argument involves a partition of unity on the locally compact Hausdorff space X/G, it is essential here again that G is compact.

Finally the statement regarding orbits: Let $\{\beta_t : t\in \mathbf{R}\}$ be the one-parameter group of homeomorphisms of X corresponding to the automorphism group $\exp(t\bar{\delta})$. Suppose that for some $\omega\in X$ the orbit $\{\beta_t(\omega)\}$ does not lie in the G-orbit $G(\omega)$. In this case there is a function $f\in\mathcal{A}_\infty$ which is constant on each G-orbit but not constant on $\{\beta_t(\omega)\}$. But by the first part of the proof, $\delta(f)|_M=0$ for each orbit M, $f\in\ker(\delta)$. Then f is analytic for δ and

$$\begin{aligned} f(\beta_t\omega) &= (e^{t\bar{\delta}}f)(\omega) \\ &= \Big(f+\sum_{k\geq 1}\frac{t^k\delta^k(f)}{k!}\Big)(\omega) \\ &= f(\omega), \end{aligned}$$

a contradiction. (The proof of the corresponding statement concerning orbits in [11], Theorem 3.2 was incorrect and should be replaced by the present proof.)

Acknowledgements

This paper was written whilst the first three authors were visiting the Australian National University with the support of the Mathematical Sciences Research Centre. O. B. and T. D. were partially supported by the NAVF, and F. G. was partially supported by the NSF.

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Note added in proof: Some of the results of this paper have been refined and extended in:

[18] Robinson, D. W., Smooth derivations on abelian D^*-dynamical systems, preprint (1985).
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[20] Batty, C. J. K. and Robinson, D. W., Commutators and geverators, preprint (1985).
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The paper [18] contains a conuterexample to the conjecture that if H is a local dissipation from A_∞ into A_2, then $\overline{H}$ is a generator.