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A Borel Parametrization of Polish Groups

By

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Abstract

This paper constructs a standard Borel space, PG, and a map $p \in PG \to G(p)$ such that each $G(p)$ is a Polish group, and such that every Polish group is isomorphic to at least one of the groups $G(p)$; PG thus serves as a parameter space for all Polish groups. We formulate the notion of a Borel map from a standard B Space to Polish groups, and that of a Borel functor from a standard Borel groupoid to Polish groups; both are defined in terms of the existence of Borel factorizations through PG. We apply these ideas to establish a general "Cohomology Lemma," asserting that cocycles, with values in Borel family of Polish groups, may be cobounded into a given family of dense, normal, Borel subgroups, whenever the underlying groupoid is a hyperfinite equivalence relation.

§ 0. Introduction

The purpose of this paper is to provide a parametrization, by a standard Borel space PG, for the space of Polish topological groups, i. e. those second countable topological groups whose underlying topology may be defined by a complete metric, and to present applications of this to the notion of "Borel functor" from a standard Borel groupoid to Polish groups. The need for such concepts became apparent during the course of joint work with M. Takesaki on the classification of the possible actions (up to cocycle conjugacy) of a discrete amenable group on a hyperfinite, semifinite injective von Neumann algebra [12], and the paper can be viewed as preparatory to this work. However, the point of view adopted also reveals a definition of A. Connes, [4], of the notion of Borel functor from a standard Borel groupoid to standard measure spaces, as being very natural.

A common situation in which the problems considered here arise is the following: if G is a Polish group and X a Polish G-space under

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the continuous map $(x, g)\in X\times G\to xg\in X$, the map $x\in X\to G_x=\{g\in G: xg=x\}$ carries X into the space $S(G)$ of closed subgroups of G. As in [1], one may endow $S(G)$ with a standard Borel structure, and the above map is then Borel. However, if for each x, $\Gamma(G_x)$ denotes the group of continuous homomorphisms from G_x to $\boldsymbol{T}$, the groups $\Gamma(G_x)$ do not appear naturally as closed subgroups of any particular Polish group, and the question of the nature of the map $x\to\Gamma(G_x)$ cannot be formulated without considerations at least similar to those given here.

The techniques of the paper may also be used to parametrize the Polish topological spaces. However, this may more easily be accomplished by observing that each Polish space X is homeomorphic with a closed subset of the Hilbert space $l^2(\boldsymbol{Z})$. One first uses the metric in X to embed X isometrically in the Banach space $C^b(X)$ of continuous bounded functions on X; since the image of X generates a separable Banach subspace of $C^b(X)$, one may apply the theorem of Kadec, [2], on the topological isomorphism of all separable, infinite dimensional Banach spaces—this argument was brought to my attention by E. Effros. It may be the case that there is a Polish group which is universal for all Polish groups in the same sense that $l^2(\boldsymbol{Z})$ is universal for Polish spaces; however, we have been unable to establish the existence of such a group.

The paper is organized as follows: §1 constructs the space PG and establishes that various naturally occurring subsets are Borel; in §2 we define and characterize the notion of a Borel map from a standard Borel space to Polish groups; in §3, various operations on Polish groups, such as taking quotients or duals, are shown to be Borel maps, and in §4 we introduce and examine the notion of Borel functor from a standard Borel groupoid to Polish groups. In particular, we show that the "dual" of a Borel functor to discrete abelian groups is a Borel functor to compact groups. Finally, in §5, we adapt the "Cohomology Lemma" of [8] to the context where the coefficient groups for cocycles on a groupoid are permitted to vary suitably from point to point along the unit space of the groupoid; in addition we show that the conclusion of the Cohomology Lemma of [8] characterizes hyperfinite equivalence relations. We have attempted full generality in §1 and §2, while in §3 and §4 we have proven only what is necessary for application in [12]; many open problems remain.

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§ 1. The Parameter Space PG

We let $N=\{0, 1, 2, \ldots\}$ throughout, and consider the set PG of pairs $(\mu, d)\in N^{N\times N}\times[0, 1]^{N\times N}$ satisfying

(A) μ is a "group product" on N, i. e.
- i) $\mu(\mu(m, n), p)=\mu(m, \mu(n, p))$ for all m, n, p;
- ii) $\mu(0, n)=\mu(n, 0)$ for all n;
- iii) $k\in N\to\mu(k, n)$ is surjective for each n, as is $k\to\mu(n, k)$

(B) d is a metric on N, i. e.
- i) $d(m, n)=d(n, m)$ for all m, n;
- ii) $d(m, n)=0$ if and only if $m=n$;
- iii) $d(m, p)\leq d(m, n)+d(n, p)$ for all m, n, p;

(C) (N, μ, d) is a topological group with d left invariant, i. e.
- i) if $\tau_{\mu}(n)$ is the unique solution to $\mu(k, n)=0$, then τ_{μ} is d-continuous;
- ii) μ is d-continuous;
- iii) $d(\mu(m, n), \mu(m, p))=d(n, p)$ for all m, n, p.

We give $N^{N\times N}$ and $[0, 1]^{N\times N}$ the product topologies; they are themselves Polish spaces.

Theorem 1.1. *The space PG of pairs* $(\mu, d)\in N^{N\times N}\times[0, 1]^{N\times N}$ *satisfying* (A), (B), (C) *above is a* $G_{\delta\sigma\delta}$ *subset, and hence a standard Borel space in the relative Borel structure.*

Proof. Since for each m, n, the evaluation map $(\mu, d)\to(\mu(m, n), d(m, n))$ is continuous on $N^{N\times N}\times[0, 1]^{N\times N}$, the conditions (A) i), (A) ii), (B) i), (B) iii) and (C) iii) each define closed subsets of the total space.

Since (μ, d) satisfies (A) iii) if and only if it belongs to

$\bigcap_n \bigcap_l \bigcup_k \{(\mu, d) : \mu(k, n) = l\}$, condition (A) iii) defines a G_δ set; since (μ, d) satisfies (B) ii) if and only if

$$(\mu, d) \in \bigcap_m \{(\mu, d) : d(m, m) = 0\} \cap \bigcap_{(m,n), m \neq n} \{(\mu, d) : \mu(m, n) > 0\},$$

condition (B) ii) also defines a G_δ set.

Note that $\{(\mu, d) : \tau_\mu(n) = m\} = \{(\mu, d) : \mu(m, n) = 0\}$, so that $(\mu, d) \to \tau_\mu(n)$ is continuous for each n. Thus if n_0, M and N are fixed,

$$E(n_0, N, M) = \bigcap_{n, d(n, n_0) < N^{-1}} \{(\mu, d) : d(\tau_\mu(n), \tau_\mu(n_0)) < M^{-1}\}$$

is a G_δ set. Since (μ, d) satisfies (C) i) if and only if it lies in $\bigcap_{(n_0, N)} \bigcup_M E(n_0, N, M)$, (C) i) defines a $G_{\delta\sigma\delta}$ set. Similarly, if m_0, n_0, M and N are given, $E(m_0, n_0, M, N) = \bigcap_{(m,n)} \{(\mu, d) : d(\mu(m, n), \mu(m_0, n_0)) < N^{-1}\}$, where we take the intersection over pairs (m, n) with $d(m, m_0) < M^{-1}$ and $d(n, n_0) < M^{-1}$, is G_δ. Since (μ, d) satisfies (C) ii) if and only if it lies in $\bigcap_{(m_0, n_0, N)} \bigcup_M E(m_0, n_0, M, N)$, (C) ii) defines a $G_{\delta\sigma\delta}$ set also. $\square$

Note that for $(\mu, d) \in PG$, the completion of the metric space $(\boldsymbol{N}, d_\mu^*)$, where $2d_\mu^*(m, n) = d(m, n) + d(\tau_\mu(m), \tau_\mu(n))$ is a Polish topological group, since this completion is homeomorphic with the uniform space completion of $(\boldsymbol{N}, \mu)$ in the two-sided uniformity defined by d; we shall denote this group by $G(\mu, d)$. Since every Polish group admits a countable dense subgroup and a left invariant, bounded, metric compatible with the topology (see, for example, [9]), every Polish group is isomorphic, as a Polish group, with some $G(\mu, d)$.

We recall the following construction of the completion of a metric space (X, δ). We may assume $\delta(x, y) \leq 1$ for all $x, y \in X$, and embed (X, δ) isometrically in $C^b(X)$, the space of bounded continuous functions on X with the uniform norm, via $x \to \varphi_x^\delta$, where $\varphi_x^\delta(y) = \delta(x, y)$. The completion of (X, δ) is isometrically isomorphic with the uniform closure of $\{\varphi_x^\delta : x \in X\}$ in $C^b(X)$. In our context, we will regard $(\boldsymbol{N}, \mu, d)$ as embedded in $[0, 1]^{\boldsymbol{N}}$ via the map $n \to \varphi_n^{(\mu, d)} = \varphi_n^{d_\mu^*}$; we write $\Phi(\mu, d) = \{\varphi_n^{(\mu, d)} : n \in \boldsymbol{N}\}$, and $\overline{\Phi(\mu, d)}$ for the uniform closure of $\Phi(\mu, d)$. Note that $[0, 1]^{\boldsymbol{N}}$ is a Polish space in the product topology, although not in the uniform topology since this is not separable; however, both topologies generate the same standard Borel structure.

We define

$$\mathscr{P}\mathscr{G} = \{(\mu, d, \varphi) \in PG \times [0, 1]^N : \varphi \in \overline{\Phi(\mu, d)}\}.$$

Theorem 1.2. a) *$\mathscr{P}\mathscr{G}$ is a standard Borel space in the relative Borel structure;*

b) *$\mathscr{P}\mathscr{G}$ has the properties*

i) *the projection $\pi : \mathscr{P}\mathscr{G} \to PG$ is Borel;*

ii) *the relative Borel structure on $G(\mu, d) = \pi^{-1}(\mu, d)$ coincides with that generated by its topology;*

iii) *the maps*

$$(\mu, d, \phi, \mu, d, \psi) \in \mathscr{P}\mathscr{G} * \mathscr{P}\mathscr{G} \to (\mu, d, \mu(\phi, \psi)) \in \mathscr{P}\mathscr{G} \text{ and}$$
$$(\mu, d, \phi) \in \mathscr{P}\mathscr{G} \to (\mu, d, \tau_\mu(\phi)) \in \mathscr{P}\mathscr{G}$$

are Borel, where $\mathscr{P}\mathscr{G} * \mathscr{P}\mathscr{G}$ is the set of all $(\Phi, \Psi) \in \mathscr{P}\mathscr{G} \times \mathscr{P}\mathscr{G}$ with $\pi(\Phi) = \pi(\Psi)$.

iv) *there is a countable family, $\{\phi_k\}$, of Borel maps from PG to $\mathscr{P}\mathscr{G}$ with $\phi_k(\mu, d) \in G(\mu, d)$, with*

a) *$\{\phi_k(\mu, d) : k \in N\}$ a dense subgroup of $G(\mu, d)$ for each $(\mu, d) \in PG$, and*

b) *for each k, the map $\phi \in \mathscr{P}\mathscr{G} \to \|\phi - \phi_k(\pi(\phi))\|$ is Borel.*

In the statement of iii), *μ and τ_μ have been extended from $\Phi(\mu, d)$ to $\overline{\Phi(\mu, d)}$.*

Proof. a) Note $(\mu, d, \varphi) \in \mathscr{P}\mathscr{G}$ if and only if $(\mu, d, \varphi) \in \bigcap_N \bigcup_n \bigcap_m E(n, m, N)$, where $E(n, m, N) = \{(\mu, d, \varphi) : |d_\mu^*(n, m) - \varphi(m)| < N^{-1}\}$; since each $E(n, m, N)$ is open, $\mathscr{P}\mathscr{G}$ is $G_{\delta\sigma\delta}$.

b) Properties i) and ii) are obvious. To show iv) a) we define $\varphi_k(\mu, d) = \varphi_k^{(\mu, d)}$, and the claim is clear. To show iv) b), note that the map $(\mu, d, \varphi) \in PG \times [0, 1]^N \to \|\varphi - \varphi_k(\mu, d)\|$ is Borel in (μ, d) for fixed φ, and continuous in φ (with respect to the norm topology) for fixed (μ, d), and hence Borel as a function of two variables; iv) b) follows immediately.

To show iii), it suffices to show that the functions $(\mu, d, \varphi, \psi) \to \mu(\varphi, \psi)(n)$ and $(\mu, d, \varphi) \to \tau_\mu(\varphi)(n)$ are Borel on the appropriate domains, where (μ, d, ϕ, ψ) denotes the element $(\mu, d, \phi, \mu, d, \psi)$ in $\mathscr{P}\mathscr{G} * \mathscr{P}\mathscr{G}$. But if $U \subseteq [0, 1]$ is open, $\mu(\varphi, \psi)(n) \in U$ if and only if there is an open set V with $\mu(\varphi, \psi)(n) \in V \subseteq \overline{V} \subseteq U$, and hence if and only if $(\mu, d, \varphi, \psi) \in \bigcap_k \bigcup_j \bigcup_l E(k, j, l, V)$ where $E(k, j, l, V) = \{(\mu, d, \varphi, \psi) : \|\varphi - \varphi_j(\mu, d)\| < 2^{-k}, \|\psi - \varphi_l(\mu, d)\| < 2^{-k}$ and $\varphi_{\mu(j, l)}^{(\mu, d)}(n) \in V\}$. Since each

$E(k,l,j,V)$ is Borel (using iv)), and V may be chosen to run through a fixed countable basis for the topology of $[0,1]$, $(\mu,d,\varphi,\psi)\to\mu(\varphi,\psi)(n)$ is indeed Borel. A similar argument shows that $(\mu,d,\varphi)\to\tau_\mu(\varphi)(n)$ is also Borel.

Remark 1.3. The argument in the proof of iii) above is easily modified to show that if $(\mu,d)\to g(\mu,d)$ is a map from PG to $\mathscr{P}\mathscr{G}$ with $g(\mu,d)\in G(\mu,d)$ for each (μ,d) (i.e. g is a section for π), then g is Borel if and only if $(\mu,d)\to\|g(\mu,d)-\varphi_k(\mu,d)\|$ is Borel for each k. Also, if g, g' are two Borel sections for π, $(\mu,d)\to\|g(\mu,d)-g'(\mu,d)\|$ is Borel.

Theorem 1.4. *Let* (*) *denote any of the following properties which a Polish group might have: discrete, abelian, compact, locally compact. Then* $\{(\mu,d)\in PG\colon G(\mu,d)$ *satisfies* $(*)\}$ *is Borel.*

Proof. Since $G(\mu,d)$ is discrete if and only if $d(m,n)>N^{-1}$ for some N and all m, n with $m\neq n$, and since $G(\mu,d)$ is abelian if and only if $\mu(m,n)=\mu(n,m)$ for all m, n, both these sets are Borel.

For the compact and locally compact cases, let $B_r(\varphi)$ denote $\{\psi\in[0,1]^N\colon \|\psi-\varphi\|<r\}$, and let $\overline{B_r(\varphi)}$ denote its uniform closure. Note that $G(\mu,d)$ is compact if and only if $G(\mu,d)\cap B_2(\varphi_0^{(\mu,d)})$ is totally bounded, while $G(\mu,d)$ is locally compact if and only if $G(\mu,d)\cap B_{(M^{-1})}(\varphi_0^{(\mu,d)})$ is totally bounded for some integer M. But $G(\mu,d)\cap B_r(\varphi_0^{(\mu,d)})$ is totally bounded if and only if for each integer $R>0$, there is an integer N with

$$G(\mu,d)\cap B_r(\varphi_0^{(\mu,d)})\subseteq\overline{\bigcup_{n=1}^{N}B_{(R^{-1})}(\varphi_n^{(\mu,d)})}.$$

But this occurs if and only if

$$(\mu,d)\in\bigcap_l\bigcup_{n=1}^{N}E(R,l,n),$$

where the intersection is over those l's with $\|\varphi_l^{(\mu,d)}-\varphi_0^{(\mu,d)}\|<r$, and

$$E(R,l,n)=\{(\mu,d)\colon \|\varphi_l^{(\mu,d)}-\varphi_n^{(\mu,d)}\|<R^{-1}\}.$$

Since $(\mu,d)\to\|\varphi_l^{(\mu,d)}-\varphi_n^{(\mu,d)}\|$ is Borel, the desired conclusion follows. □

Remark 2.5. Using the Følner condition, [6], it is routine to verify that $\{(\mu, d):\ G(\mu, d)$ is discrete and amenable$\}$ is Borel in PG. We lack the necessary measure theoretic considerations to handle the case of locally compact amenable groups. In general, algebraically defined classes of groups are easily seen to be Borel, while topologically or analytically defined groups are much more difficult to handle.

§2. Borel Maps to Polish Groups

In this section, X denotes a standard Borel space — implicitly, we assume X is uncountable.

Definition 2.1. If, for each $x\in X$, G_x is a Polish group, we say the map $x\to G_x$ is Borel if there is a Borel map $f: X\to PG$, and maps $\{\theta_x:\ x\in X\}$, with $\theta_x:\ G_x\to G(f(x))$ an isomorphism of Polish groups for each $x\in X$.

Remark 2.2. If $x\in X\to G_x$ is a Borel map to Polish groups, the map f of Definition 2.1 may be chosen injective, as follows. If $f(x)=(\mu_x, d_x)$ and θ_x are as in the definition, we may suppose N has d_x-diameter 1, and define $d'_x(m,n)=d_x(m,n)\,(\alpha(x)+d_x(m,n))^{-1}$, where α is a fixed Borel isomorphism of X with $\left[\frac{1}{2}, 1\right]$. If now $f'(x)=(\mu_x, d'_x)$, f' is injective, and f' and $\{\theta_x\}$ satisfy Definition 2.1. When appropriate, we shall assume f has been made injective via this device.

Theorem 2.3. *A map $x\to G_x$ from a standard Borel space X to Polish groups is Borel if and only if $Y=\dot{\cup}\, G_x$ admits a standard Borel structure such that*

i) *the projection $\pi: Y\to X$ is Borel,*

ii) *the relative Borel structure on $G_x=\pi^{-1}(x)$ coincides with that generated by the topology,*

iii) *the maps $(y,y')\in Y^*Y=\{(y,y')\in Y\times Y:\ \pi(y)=\pi(y')\}\to yy'\in Y$, and $y\in Y\to y^{-1}\in Y$ are Borel,*

iv) *there are countably many Borel maps g_k: $X\to Y$ with $g_k(x)\in G_x$ for all x, and metrics δ_x on G_x, compatible with the topology, such that*

   a) *$\{g_k(x):\ k\in N\}$ is dense in G_x for $x\in X$;*

   b) *the map $y\in Y\to\delta_{\pi(y)}(y,\ g_k(\pi(y)))$ is Borel for each $k\in N$.*

Proof. Suppose $x \to G_x$ is Borel, and f, θ are as in definition 2.1, with f injective. Since $f(X)$ is thus Borel in PG, $\pi^{-1}(f(X)) \subseteq \mathscr{P}\mathscr{G}$ is Borel, where π is the projection of $\mathscr{P}\mathscr{G}$ onto PG.

But $\dot{\cup} G_x$ and $\pi^{-1}(f(X))$ are in bijective correspondence via the map Θ, $\Theta(y) = (f(\pi(y)), \theta_{\pi(y)}(y))$, and we endow $\dot{\cup} G_x$ with the unique Borel structure for which Θ is a Borel isomorphism. The properties i) -iv) follow from the correspoding properties of $\mathscr{P}\mathscr{G}$ established in Theorem 1.2.

Conversely, suppose that $\dot{\cup} G_x$ admits a standard Borel structure, sections $\{g_k\}$ and metrics δ_x satisfying i) -iv). In order to mimic the structure in $\mathscr{P}\mathscr{G}$, we must

a) modify the functions g_k so that $g_k(x) \neq g_l(x)$ for all x and $k \neq l$, and so that $\{g_k(x) : k \in N\}$ is a group, and

b) modify the metrics δ_x so as to be left invariant.

To achieve a), define $h_0(x) = 1_x$, the unit element of G_x.

By considering finite products (in any order) of the $g_k(x)$ and their inverses, we find a sequence $\{g'_k\}$ of Borel sections such that $\{g'_k(x) : k \in N\}$ is a dense subgroup. Now define h_k inductively by $h_k(x) = g'_l(x)$ if l is the smallest index for which $g_l(x) \notin \{h_0(x), \dots, h_{k-1}(x)\}$. The sections h_k are Borel and have the desired properties.

To achieve b), let $\mathscr{L}_x$ be the left uniformity of δ_x. The proof of the Metrization Lemma [9] produces metrics δ'_x on G_x which are left invariant with uniformity $\mathscr{L}_x$. Further, since it is possible to choose bases $U_n(x)$ for $\mathscr{L}_x$ in such a way that $U_n(x) \circ U_n(x) \circ U_n(x) \subseteq U_{n-1}(x)$ and $\{(x,y,y') \in X \times Y \times Y : (y,y') \in U_n(x)\}$ Borel, the resulting metrics δ'_x still satisfy condition iv) b). (Our notation is as in [9, p. 186]).

With these modifications, define μ_x and d_x on $N \times N$ by $g_m(x) g_n(x) = g_{\mu_x(m,n)}(x)$ and $d_x(m,n) = \delta_x(g_m(x), g_n(x))$. Note $(\mu_x, d_x) \in PG$, and $x \to (\mu_x, d_x)$ is Borel. The completion $G(\mu_x, d_x)$ of (N, μ_x, d_x^*) is evidently isomorphic as a Polish group with G_x, and the result follows. $\square$

Under some conditions one may conclude that $x \to G_x$ is Borel under somewhat different hypotheses.

Proposition 2.4. *Suppose $x \to G_x$ is a map from a standard Borel space X to Polish groups, and that there are metrics δ'_x on G_x, and countably many sections g'_k: $X \to \dot{\cup} G_x$ with*

a) δ'_x *left invariant and compatible with the topology on* G_x,

b) $\{g'_k(x): k\in N\}$ *is a dense subgroup of* G_x *for all* x,

c) *the functions* $x\to\delta'_x(g'_k(x), g'_l(x))$ *and* $x\to\delta'_x(g'_k(x)g'_l(x), g'_m(x))$ *are Borel for all* k, l, *and* m.

Then there is a unique Borel structure $\mathscr{B}$ *on* $\dot{\cup} G_x$ *for which the sections* $g_k=g'_k$ *are Borel and such that* $\mathscr{B}$, g_k *and* $\delta_x=\delta'_x$ *satisfy the conditions* i) –iv) *of Theorem* 2. 3.

Proof. Since $g'_k(x)=g'_l(x)$ if and only if $\delta'_x(g'_k(x), g'_l(x))=0$, we may assume by "cutting and pasting" that $g'_k(x)\neq g'_l(x)$ for all x wherever $k\neq l$.

Define μ_x and d_x on $N\times N$ by

$$g'_m(x)g'_n(x)=g'_{\mu_x(m,n)}(x), \text{ and}$$

$$d_x(m,n)=\delta'_x(g'_m(x), g'_n(x)).$$

As in the proof of Theorem 2. 3, we may suppose that $x\to f(x)=(\mu_x, d_x)\in PG$ is injective and Borel. If $\theta_x: G_x\to G(f(x))$ is determined by $\theta_x(g'_k(x))=\varphi_k(\mu_x, d_x)$ the map $\Theta: \dot{\cup} G_x\to\pi^{-1}(f(X))$ given by $\Theta(y)=(f(\pi(y)), \theta_{\pi(y)}(y))$ is injective, and, as in the proof of Theorem 2. 2, we give $\dot{\cup} G_x$ the Borel structure $\mathscr{B}$ for which Θ is an isomorphism.

To show that $\mathscr{B}$ is unique subject to conditions i) –iv) of Theorem 2. 3 we note that $\theta_x(g_k(x))(m)=\varphi_k^{(\mu_x, d_x)}(m)=d_x^*(k,m)=\delta_x^*(g_k(x), g_m(x))$; in view of the density of $\{g_k(x): k\in N\}$ in G_x, we conclude that $\theta_x(y)(m)=\delta_x^*(y, g_m(x))$ for all $y\in G_x$. Thus Θ is a Borel isomorphism of $\dot{\cup} G_x$ with $\pi^{-1}(f(X))$ if and only if π is Borel and $y\to\theta_{\pi(y)}(y)$ is Borel, and hence if and only if π is Borel and $y\to\theta_{\pi(y)}(y)(m)=\delta^*_{\pi(y)}(y, g_m(\pi(y)))$ is Borel for each m. Thus any other standard Borel structure on $\dot{\cup} G_x$ satisfying i) –iv) coincides with $\mathscr{B}$. □

Remark 2. 5. If $f': X\to PG$ is a Borel map, and $f: X\to PG$ is the injective Borel map associated to f as in Remark 2. 2, the Borel structure which $\pi^{-1}\{x: f'(x)=f'(x_0)\}$ inherits as a Borel subset of $\pi^{-1}(f(X))$ is just the product structure on $f'^{-1}(f'(x_0))\times G_{x_0}$. In particular, it is independent of the map $\alpha: X\to\left[\frac{1}{2}, 1\right]$ used in the construction of f.

We now proceed to compare our notion of Borel map with the more familiar notion of "Effros Borel" map, as in [3] and [1]. Recall that if H is a fixed Polish group, the space $S(H)$ of closed subgroups of H has a standard Borel structure generated by the sets $\{G\in S(H): G\subseteq A\}$, where A runs through the closed subsets of H; we refer to this as the Effros Borel structure on $S(H)$.

Theorem 2.5. *Suppose $x\to G_x$ is a map from a standard Borel space to $S(H)$. Then $x\to G_x$ is Effros Borel if and only if $\dot{\cup} G_x=\{(x,g):g\in G_x\}\subseteq X\times H$ is a Borel subset of $X\times H$ satisfying condition* iv) a) *of Theorem* 2.3, *(existence of enough Borel sections). If $x\to G_x$ is Effros Borel, then it is Borel, i.e. $\dot{\cup} G_x\subseteq X\times H$ also satisfies conditions* i), ii), iii), *and* iv) b) *of Theorem* 2.3.

Proof. Suppose $x\to G_x$ is Effros Borel. Note that for fixed $g\in H$, $x\to d(G_x,g)$ is Borel in x, since $d(G_x,g)>\in$ if and only if $G_x\subseteq H-B_{\in+n^{-1}}(g)$ for some n, where $B_r(g)=\{h\in H:d(h,g)<r\}$ and d is a metric on G compatible with the topology. Since $(x,g)\to d(G_x,g)$ is also continuous in g for fixed x, $(x,g)\to d(G_x,g)$ is Borel in (x,g) and $\{(x,g):\ g\in G_x\}=\{(x,g):\ d(G_x,g)=0\}$ is Borel. The relative Borel structure on $\dot{\cup} G_x$ evidently satisfies conditions i), ii), and iii) of Theorem 2.3, since the maps $(x,h)\to(x,h^{-1})$ and $(x,h,h')\to(x,hh')$ are Borel on $X\times H$ and $X\times H\times H$. Condition iv) a) follows from [3, p. 82], and condition iv) b) follows since, for a fixed Borel section $x\to g(x)\in H$, the map $(x,g)\to d(g,g(x))$ is Borel, being Borel in x for fixed g and continuous in g for fixed x.

Conversely, if $\{(x,g):\ g\in G_x\}\subseteq X\times H$ is Borel and g_k are Borel functions on X with $\{g_k(x):\ k\in N\}$ dense in G_x for all x, then, if $A\subseteq H$ is closed, $\{x:\ G_x\subseteq A\}=\bigcap_k\{x:\ g_k(x)\in A\}$; this set is thus Borel, since g_k are also Borel as maps into H. □

§3. Operations on Polish Groups

Throughout this section, X denotes a standard Borel space.

Theorem 3.1. *Let $x\to G_x$ be a Borel map to Polish groups. Then*

i) *if $[G_x,G_x]$ denotes the closure of the commutator subgroup of G_x,*

$x \to [G_x, G_x]$ is Borel;

ii) *if $x \to N_x$ is a Borel map to Polish groups with N_x a closed normal subgroup of G_x for each x, and $\dot{\cup} N_x$ a Borel subset of $\dot{\cup} G_x$, then $x \to G_x/N_x$ is a Borel map.*

Proof. Let $\{g_k: k \in N\}$ be the Borel sections of $\dot{\cup} G_x$ provided by Theorem 2.3. If $h_{k,l}(x) = g_k(x) g_l(x) g_k(x)^{-1} g_l(x)^{-1}$, then each $h_{k,l}$ is a Borel section of $\dot{\cup} G_x$ with $h_{k,l}(x) \in [G_x, G_x]$ for each $x \in X$. Taking products and inverses, we find a countable family $\{g'_k: k \in N\}$ of Borel sections of $\dot{\cup} G_x$ whose values at each x form a group dense in $[G_x, G_x]$. If $\{d_x: x \in X\}$ are the metrics on G_x provided by Theorem 2.3, then $x \to d_x(g'_k(x), g'_l(x))$ and $x \to d_x(g'_k(x) g'_l(x), g'_m(x))$ are Borel for each k, l, m, so that $x \to [G_x, G_x]$ is Borel by Proposition 2.4.

If N_x is as described in ii), let $\{g_k^N\}$ be the countable family of Borel sections for $\dot{\cup} N_x$, and let d_x be as above. Let δ'_x be the quotient metric on $H_x = G_x/N_x$, so $\delta'_x(h, h') = \inf\{d_x(g, g'): g \in h, g' \in h'\}$, and let $h_k(x)$ be the image of $g_k(x)$ in H_x. Note that for each x, $\{h_k(x): k \in N\}$ is a dense subgroup of H_x and $\delta'_x(h_k(x), h_l(x)) = \inf_{m,n} d_x(g_k(x) g_m^N(x), g_l(x) g_n^N(x))$ is a Borel function of x since the functions $x \to g_m^N(x)$ are Borel as maps to $\dot{\cup} G_x$, $\dot{\cup} N_x$ being a Borel subset of $\dot{\cup} G_x$. Again, Proposition 2.3 shows that $x \to H_x = G_x/N_x$ is Borel. $\square$

We now turn to the formation of dual groups. We believe that the map $(\mu, d) \to G(\mu, d)^\wedge$ is a Borel map from $LCAG = \{(\mu, d): G(\mu, d)$ is locally compact abelian$\}$ to Polish groups, although we have been unable to prove this. Any proof of this conjecture would presumably involve a suitable "Borel choice" of Haar measures for $G(\mu, d)$ — note that the proof in Theorem 1.4 that $LCPG$ is Borel shows that one may choose a Borel set K in $\{(\mu, d, \varphi) : (\mu, d) \in LCPG$ and $\varphi \in G(\mu, d)\}$ such that for each (μ, d), $K(\mu, d) = K \cap \{(\mu, d, \varphi) : \varphi \in G(\mu, d)\}$ is a precompact neighbourhood of the identity in $G(\mu, d)$. Presumably, if one chooses Haar measures $m_{(\mu,d)}$ on $G(\mu, d)$ with $m_{(\mu,d)}(K(\mu, d)) = 1$, then $(\mu, d) \to m_{(\mu,d)}$ is a suitable choice. Even if this is the case, however, completing the proof that $(\mu, d) \to G(\mu, d)^\wedge$ is Borel on $LCAG$ would seem to require a considerable amount of auxillary machinery. For this reason we will consider here only the case of discrete abelian groups, where more elementary techniques suffice.

Theorem 3.2. *Let $DAG=\{(\mu,d)\in PG: G(\mu,d)$ is discrete and abelian$\}$. Then $(\mu,d)\in DAG\to G(\mu,d)^\wedge$ is Borel.*

Proof. Let $\{g_k: k\in N\}$ be Borel sections for $\dot{\cup}\{G(\mu,d): (\mu,d)\in DAG\}$, as provided by Theorem 2.3. Let $\boldsymbol{Z}^\infty$ denote the countable direct sum of copies of $\boldsymbol{Z}$, and define for each $(\mu,d)\in DAG$,

$$\alpha_{(\mu,d)}(e_k)=g_k(\mu,d),$$

where $e_k\in\boldsymbol{Z}^\infty$ is the element all of whose entries are 0, except the k^{th}, which is 1. Note $\alpha_{(\mu,d)}$ extends to a surjective homomorphism, also denoted by $\alpha_{(\mu,d)}$. Let $H(\mu,d)=\ker\alpha_{(\mu,d)}$, so that $G(\mu,d)\simeq\boldsymbol{Z}^\infty/H(\mu,d)$, and $G(\mu,d)^\wedge\simeq H(\mu,d)^\perp$, where $H(\mu,d)^\perp$ is those elements of $(\boldsymbol{Z}^\infty)^\wedge=\prod_0^\infty\boldsymbol{T}$ which are one on $H(\mu,d)$. It suffices to show that $\{(\mu,d,\chi): \chi\in H(\mu,d)^\perp\}$ is a Borel subset of $DAG\times\boldsymbol{T}^N$, and that this subset possesses a sufficiently large family of Borel sections— it is clear that the conditions i), ii), iii), and iv) b) of Theorem 2.3 are satisfied, where we use (the restrictions of) an invariant metric on $\boldsymbol{T}^N$ which is compatible with the product topology.

Note that if $n=(n_0,n_1,\dots,n_m,0,\dots)\in\boldsymbol{Z}^\infty$, then $n\in H(\mu,d)$ if and only if we have $\mu(n_0,\mu(n_1,\mu(\dots,\mu(n_{m-1},n_m))\dots)=0$, so that $\{(\mu,d,n): n\in H(\mu,d)\}$ is Borel in $DAG\times\boldsymbol{Z}^\infty$. Thus there are countably many Borel maps $\{h_k: k\in N\}$ from DAG to $\boldsymbol{Z}^\infty$ with

$$H(\mu,d)=\{h_k(\mu,d): k\in N\}.$$

We claim that $(\mu,d)\to H(\mu,d)^\perp$ is Borel from DAG to the space of closed subgroups of $\boldsymbol{T}^N$ equipped with the Effros Borel structure. For if $V\subseteq\boldsymbol{T}$ is open, and $U_j(V)=(\prod_1^{j-1}\boldsymbol{T})\times V\times(\prod_{j+1}^\infty\boldsymbol{T})$, then $H(\mu,d)^\perp\cap U_j(V)\neq\emptyset$ if and only if $1\in\bigcap_k\langle V,h_k(\mu,d)_j\rangle$, where $h_k(\mu,d)_j$ is the j^{th} coordinate of $h_k(\mu,d)$, and $\langle\ ,\ \rangle$ is the pairing of $\boldsymbol{T}$ with $\boldsymbol{Z}$. But for fixed V, the set of (μ,d)'s satisfying this condition is Borel; since every open set in $\boldsymbol{T}^N$ is a countable union of finite intersections of sets of the form $U_j(V)$, for V chosen from a fixed countable basis for the topology of $\boldsymbol{T}$, we see that $\{(\mu,d): H(\mu,d)^\perp\cap W\neq\emptyset\}$ is Borel for each open W in $\boldsymbol{T}^N$.

The existence of countably many Borel functions $\chi_k: DAG\to\boldsymbol{T}^N$ with $\{\chi_k(\mu,d): k\in N\}$ dense in $H(\mu,d)^\perp$ now follows from [3, p. 82], and $(\mu,d)\to G(\mu,d)^\wedge$ is Borel as required. □

§4. Borel Functors to Polish Groups

In this section, $\mathscr{G}$ will denote a standard Borel groupoid, with range and source maps r and s; thus $\mathscr{G}$ is a standard Borel space which is also a small category with inverses, and for which the maps $\gamma\in\mathscr{G}\to\gamma^{-1}\in\mathscr{G}$ and $(\gamma_1,\gamma_2)\in\mathscr{G}^{(2)}\to\gamma_1\gamma_2\in\mathscr{G}$ are Borel. Here $\mathscr{G}^{(2)}$ denotes the set of composable pairs, $\mathscr{G}^{(2)}=\{(\gamma_1,\gamma_2): r(\gamma_2)=s(\gamma_1)\}$; $\mathscr{G}^{(2)}$ is by assumption a Borel subset of $\mathscr{G}\times\mathscr{G}$. The space of units (objects) of $\mathscr{G}$ will be denoted X (for further discussion, the reader may consult [10], [4]).

We shall consider the space of Polish groups as a category; Hom (G_1,G_2) will be the set of homeomorphic isomorphisms of G_1 with G_2. In considering functors F, covariant or contravariant, from $\mathscr{G}$ to the category of Polish groups, we shall use the notation (F_x,F_γ) to distinguish groups (F_x) from morphisms (F_γ).

Definition 4.1. *A covariant functor $F=(F_x,F_\gamma)$ from $\mathscr{G}$ to Polish groups is Borel if*

i) *$x\to F_x$ is a Borel map to Polish groups;*

ii) *the map $(\gamma,g)\in\mathscr{G}*F\to F_\gamma(g)\in\dot{\cup}F_x$ is Borel, where $\mathscr{G}*F=\{(\gamma,g)\in\mathscr{G}\times\dot{\cup}F_x\colon \pi(g)=s(\gamma)\}$, and $\dot{\cup}F_x$ is given the Borel structure described in the proof of Theorem* 2.3.

Thus we require that the action be compatible with the canonical Borel structure on $\dot{\cup}F_x$. A similar definition applies to contravariant functors.

Example 4.2. Suppose G is a fixed Polish group, and X a Polish G-space under the action $(g,x)\in G\times X\to gx\in X$. The space $G\times X$ becomes a standard Borel groupoid with unit space $\{e\}\times X$, or just X, under the product $(g,g'x)(g',x)=(gg',x)$. If $F_x=\{g\in G\colon gx=x\}$, then as is well known, $x\to F_x$ is a Borel map to Polish groups; if $\gamma=(g,x)$, the map F_γ: $h\in F_x\to ghg^{-1}\in F_{gx}$ is an isomorphism of Polish groups, and $F=(F_x,F_\gamma)$ is a covariant functor. In this case, we can identify $\dot{\cup}F_x$ with $\{(x,g): g\in F_x\}$, and $\mathscr{G}*F$ is identified with $\{(h,x,g): g\in F_x\}\subseteq G\times X\times G$; the above "action map" is precisely $(h,x,g)\to(hx,hgh^{-1})$. Thus F is in fact a Borel functor.

Theorem 4.3. *Let $F=(F_x, F_\gamma)$ be a Borel functor from the standard Borel groupoid $\mathscr{G}$ to discrete (Polish) groups. For $x\in X$ and $\gamma\in\mathscr{G}$, let $\hat{F}_x$ be the homomorphisms from F_x to $\mathbf{T}$, and let $\hat{F}_\gamma$ be the transpose of F_γ. Then $\hat{F}=(\hat{F}_x, \hat{F}_\gamma)$ is a Borel functor (with variance opposite that of F).*

Proof. We treat the case where F is covariant. Since $\hat{F}_x$ is the Pontrjagin dual of the abelian group $F_x/[F_x, F_x]$, $x\to\hat{F}_x$ is Borel by Theorems 3.1 and 3.2. If $\{h_k\colon k\in\mathbf{N}\}$ are Borel maps from X to $\dot{\cup} F_x/[F_x, F_x]$ with $\{h_k(x)\colon k\in\mathbf{N}\}=F_x/[F_x, F_x]$ for each x, then the functions f_k defined on $\dot{\cup}\hat{F}_x$ by

$$f_k(\chi)=(\pi(\chi),\ \langle\chi, h_k(\pi(\chi))\rangle)$$

are Borel and separate points on $\dot{\cup}\hat{F}_x$. Thus the functions f_k generate the Borel structure in $\dot{\cup}\hat{F}_x$. Also, for each k, the map $(\gamma,\chi)\in\mathscr{G}*\hat{F}\to f_k(\hat{F}_\gamma(\chi))$ is Borel, since

$$\begin{aligned} f_k(\hat{F}_\gamma(\chi)) &= (r(\gamma),\ \langle\hat{F}_\gamma(\chi),\ h_k(r(\gamma))\rangle) \\ &= (r(\gamma),\ \langle\chi, F_{(\gamma^{-1})}(h_k(r(\gamma)))\rangle); \end{aligned}$$

since $\gamma\to(\gamma^{-1}, h_k(r(\gamma)))\in\mathscr{G}*F$ is Borel, $\gamma\to F_{(\gamma^{-1})}(h_k(r(\gamma)))\in\dot{\cup} F_x$ is also Borel. Thus $(\gamma,\chi)\in\mathscr{G}*\hat{F}\to\hat{F}_\gamma(\chi)\in\dot{\cup}\hat{F}_x$ is Borel. $\square$

The need for this result in [12] is one of the principal stimuli for the present work; the other follows.

§5. The Cohomology Lemma; Converse and Extension

In this section, $\mathscr{K}$ will denote a standard Borel groupoid for which $r^{-1}(x)\cap s^{-1}(x)=\{x\}$ for each unit $x\in X$, and for which $r^{-1}(x)$ is countable for each $x\in X$; thus $\mathscr{K}$ may be viewed as an equivalence relation on X with countable equivalence classes and Borel graph. Also, we will denote by m a measure on X which is non-singular for $\mathscr{K}$, i.e. such that $m(\mathscr{K}(E))=0$ whenever $m(E)=0$, where $\mathscr{K}(E)=\{x\in X\colon$ for some $y\in E$, $(x,y)\in\mathscr{K}\}$. We will refer to $(\mathscr{K}, m)$ as a measured discrete equivalence relation, as in [5].

If G is a Polish group, a G-cocycle on $(\mathscr{K}, m)$ is a Borel map $\rho\colon\mathscr{K}\to G$ with $\rho(\gamma_1\gamma_2)=\rho(\gamma_1)\rho(\gamma_2)$ a.e on $\mathscr{K}^{(2)}$—see [5] for the precise meaning of "a.e on $\mathscr{K}^{(2)}$" above.

Also, we shall say that $(\mathscr{K}, m)$ is hyperfinite if there are Borel equivalence relations $\mathscr{K}_n$ on X with finite equivalence classes, with $\mathscr{K}_n \subseteq \mathscr{K}_{n+1}$ for each n, and $\mathscr{K} \cap (X_0 \times X_0) = \bigcup_1^\infty \mathscr{K}_n \cap (X_0 \times X_0)$, where X_0 is some conull set in X.

The following "Cohomology Lemma" is proven in [8], which corrects an error in [11].

Theorem 5.1. ([8; *Appendix*]). *Let* $(\mathscr{K}, m)$ *be a hyperfinite measured discrete equivalence relation. Let* G *be a Polish group with* H *a normal Borel subgroup of* G, *and let* ρ_1, ρ_2 *be* G-*cocycles on* $(\mathscr{K}, \mu)$ *with* $\rho_1 = \rho_2$ *mod* $\bar{H}$ *a.e on* $\mathscr{K}$. *Then there is a Borel map* $P: X \to \bar{H}$ *such that*

$$\rho_1^P = \rho_2 \text{ mod } H \text{ a.e.}$$

Here $\bar{H}$ *is the closure of* H *in* G, *and* $\rho_1^P(\gamma) = P(r(\gamma))\rho_1(\gamma)P(s(\gamma))^{-1}$.

Here, we shall present a generalization of Theorem 5.1 (see Theorem 5.5) and a converse (Theorem 5.2). The existence of such a converse was asserted by A. Connes at an informal conference held in Oslo in July, 1978, and is doubtless known to many; we include a proof for completeness sake. Notation will be as above.

Theorem 5.2. *Let* $(\mathscr{K}, m)$ *be a measured discrete equivalence relation with the property that whenever* G *is a Polish group,* H *is a normal Borel subgroup of* G, *and* ρ_1, ρ_2 *are* G-*cocycles on* $(\mathscr{K}, m)$ *with* $\rho_1 = \rho_2$ *mod* $\bar{H}$ *a.e, then there is a Borel map* $P: X \to \bar{H}$ *with* $\rho_1^P = \rho_2$ *mod* H *a.e. Then* $(\mathscr{K}, m)$ *is hyperfinite.*

Proof. We assume each equivalence class in $\mathscr{K}$ is infinite. Let $\mathscr{A} = l^\infty(\mathbb{Z})$, and $G = \mathrm{Aut}(\mathscr{A})$ be the group of automorphisms of $\mathscr{A}$ in its usual standard Borel structure, [7]. Note that any $\theta \in G$ is of the form $(\theta(f))(n) = f(\pi_\theta^{-1}(n))$ for some permutation π_θ of $\mathbb{Z}$. If H_n is the subgroup of G corresponding to the permutations with support in $\{-n, -n+1, \dots, n-1, n\}$, $H = \bigcup_1^\infty H_n$ is normal, Borel and dense in G.

For each $x \in X$, let $\mathscr{A}(x) = l^\infty(r^{-1}(x))$, and for $(x,y) \in \mathscr{K}$, define $L(x,y): \mathscr{A}(y) \to \mathscr{A}(x)$ by $(L(x,y)f)(z) = f(z)$, for $f \in \mathscr{A}(y)$. Since $x \to \mathscr{A}(x)$ is a Borel field of von Neumann algebras, and each $\mathscr{A}(x)$ is isomorphic with $\mathscr{A}$, we may choose a Borel field $x \to \alpha_x$ of isomorphisms $\alpha_x: \mathscr{A}(x) \to \mathscr{A}$. Thus if $\rho(x,y) = \alpha_x \circ L(x,y) \circ \alpha_y^{-1}$, ρ is a G-cocycle.

By our assumption, there is a Borel map $x \to P(x) \in G$ such that $\sigma(x,y) = \rho^P(x,y) \in H$ *a. e.* But then, if $\mathscr{K}_n = \sigma^{-1}(H_n)$, $\mathscr{K}_n$ has finite equivalence classes and, up to a null set, $\mathscr{K} = \bigcup_1^\infty \mathscr{K}_n$ as required. $\square$

We now turn to a generalization of Theorem 5. 1. Our objective is to allow the groups G, H of the theorem to vary in a suitable manner from point to point over X. Let F be a (covariant) functor from $\mathscr{K}$ to Polish groups, and let $\dot{\cup} F_x$ have the standard Borel structure provided by Theorem 2. 3.

Definition 5. 3. If for each $x \in X$, N_x is a normal Borel subgroup of F_x, we write $N \lhd F$ in case $\dot{\cup} N_x$ is a Borel subset of $\dot{\cup} F_x$, and $F_\gamma(N_{s(\gamma)}) = N_{r(\gamma)}$ for all $\gamma \in \mathscr{K}$.

Note if $N \lhd F$, we may consider $N = (N_x, N_\gamma)$ as a functor by taking $N_\gamma = F_\gamma$ restricted to $N_{s(\gamma)}$.

Definition 5. 4. If F is a covariant Borel functor from $\mathscr{K}$ to Polish groups, an F-cocycle on $(\mathscr{K}, m)$ is a Borel map $\rho: \gamma \in \mathscr{K} \to \rho(\gamma) \in F_{r(\gamma)}$ with $\rho(\gamma_1\gamma_2) = \rho(\gamma_1) F_{\gamma_1}(\rho(\gamma_2))$ a. e on $\mathscr{K}^{(2)}$.

Theorem 5. 5. *Let $(\mathscr{K}, m)$ be a hyperfinite discrete measured equivalence relation, let F be a covariant Borel functor from $\mathscr{K}$ to Polish groups, and let ρ_1, ρ_2 be F-cocycles on $(\mathscr{K}, m)$. Suppose that for each $\gamma \in \mathscr{K}$, $\rho_1(\gamma) = \rho_2(\gamma)$ mod $\bar{N}_{r(\gamma)}$, a. e. $\mathscr{K}^{(2)}$ where $N \lhd F$. Then there is a Borel function $P: X \to \dot{\cup} \bar{N}_x$ with $P(x) \in \bar{N}_x$ and*

$$\rho_2(\gamma) = \rho_1^P(\gamma) \ \ mod\ N_{r(\gamma)}\ \ a.\,e,$$

where

$$\rho_1^P(\gamma) = P(r(\gamma)) \rho_1(\gamma) F_\gamma(P(s(\gamma)))^{-1}.$$

The proof is modelled on the proof given in [8] of Theorem 5. 1, and ultimately depends on remarks of A. Connes made during a conference held in Kingston, Canada, in July 1975. Throughout, d_x are the complete metrics on G_x provided by Theorem 2. 3; note that $\dot{\cup} \bar{N}_x \subseteq \dot{\cup} G_x$ is a Borel set.

Lemma 5. 6. *Let F, N, ρ_1, ρ_2 be as in Theorem 5. 5, let $x \to \in (x) > 0$ be a Borel function on X, and let $\mathscr{L} \subseteq \mathscr{K}$ be an equivalence relation with* *finite equivalence classes. Then there is a Borel function* $Q: X \to \dot{\cup} \bar{N}_x$ *with* $Q(x) \in \bar{N}_x$ *and a Borel function* $n: \mathscr{L} \to \dot{\cup} N_x$ *with*

i) $\rho_2(\gamma) = n(\gamma)\rho_1^Q(\gamma)$ *for* $\gamma \in \mathscr{L}$;

ii) $d_x(Q(x), 1_x) < \in (x)$ *for* $x \in X$;

iii) $Q(x) = 1_x$ *on some section for* $\mathscr{L}$.

Proof. We may suppose that each equivalence class under $\mathscr{L}$ has l points, and choose a Borel partition, $\{L_j, j=0, 1, \dots, l-1\}$ of X such that each L_j is a section for $\mathscr{L}$, i. e. meets each $\mathscr{L}$-equivalence class precisely once.

Define $Q(x) = 1_x$ for $x \in L_0$.

For $\gamma \in s^{-1}(L_0)$, define

$$D_\in(\gamma) = \{g \in N_{r(\gamma)} : d_{r(\gamma)}(g^{-1}\rho_2(\gamma)\rho_1(\gamma)^{-1}, \ 1_{r(\gamma)}) < \in (r(\gamma))\}$$

and note that

$$D_\in = \{(\gamma, g) : g \in D_\in(\gamma)\}$$

is a Borel subset of $s^{-1}(L_0) \times (\dot{\cup} N_x)$ whose projection on $s^{-1}(L_0)$ is all of $s^{-1}(L_0)$. By the von Neumann measurable selection theorem, there is a measurable map $n: \gamma \in s^{-1}(L_0) \to n(\gamma)$ with $n(\gamma) \in D_\in(\gamma)$ for each γ. We may assume n is a Borel function after deletion of a suitable null set, and that $n(x, x) = 1_x$ for $x \in L_0$ since $\rho_1(x, x) = \rho_2(x, x) = 1_x$ for $x \in L_0$.

Define Q on $X - L_0$ by

$$Q(y) = n(y, x)^{-1}\rho_2(y, x)\rho_1(y, x)^{-1},$$

where $x \in L_0$ is the unique element with $(y, x) \in \mathscr{L}$. Note Q is Borel on X, and satisfies conditions ii), iii) of the Lemma. Also, if $(y, z) \in \mathscr{L}$ is arbitrary and $x \in L_0$ is the unique element with $(y, z) = (y, x)(z, x)^{-1}$, a routine calculation using the (a. e) identities $\rho_j(y, z) = \rho_j(y, x)F_{(y,x)}(\rho_j((z, x)^{-1}))$ for $j=1, 2$ shows that $\rho_2(y, z) = n(y, x)\rho_1^Q(y, z)$ $F_{(y,z)}(n(z, x)^{-1})$. In view of the normality of N_x in F_x, and the invariance under F i. e. $F_\gamma(N_{s(\gamma)}) = N_{r(\gamma)}$, the Lemma follows. □

Lemma 5. 7. *Let* $\mathscr{K}$, F, N, ρ_1, ρ_2, $\mathscr{L}$ *and* $\in$ *be as in Lemma* 5. 6, *and let* Q, n *be as in the conclusion of Lemma* 5. 6. *Let* $\mathscr{M}$ *be an equivalence relation on* X *with finite equivalence classes with* $\mathscr{L} \subseteq \mathscr{M} \subseteq \mathscr{K}$, *and let* $\delta > 0$ *be given. Then there are Borel maps* $n': \mathscr{M} \to \dot{\cup} N_x$, $R: X \to \dot{\cup} \bar{N}_x$ *with*

i) $\rho_2(\gamma) = n'(\gamma)\rho_1^R(\gamma)$ *for* $\gamma \in \mathscr{M}$;

ii) $d_x(Q(x), R(x)) < \delta$ *for* $x \in X$;

iii) $R(x) = 1_x$ *on some section for* $\mathscr{M}$;

iv) $n'(\gamma) = n(\gamma)$ *for* $\gamma \in \mathscr{L}$.

Proof. We may assume that the equivalence classes under $\mathscr{L}$ and $\mathscr{M}$ are of constant size l and m respectively. Let $\{L_j, j=0, 1, \ldots, l-1\}$ be as in Lemma 5.6, and set $\mathscr{M}^0 = \mathscr{M} \cap (L_0 \times L_0)$; $\mathscr{M}^0$ has equivalence classes of size $m^0 = ml^{-1}$.

For $x_0 \in L_0$, define

$$B(x_0) = \{g \in F_{x_0} : \max_{(x, x_0) \in \mathscr{L}} d_x(Q(x)\rho_1(x, x_0)F_{(x, x_0)}(g)\rho_1(x, x_0)^{-1}, Q(x)) \geq \delta\}$$

and $\in_0(x) = d_x(B(x), 1_x)$ for $x \in L_0$. By our hypothesis, $\in_0$ is Borel, $\in_0(x) > 0$, and we may apply Lemma 5.6 to $\mathscr{M}^0$, $\rho_1|\mathscr{M}^0$, $\rho_2|\mathscr{M}^0$ and $\in_0$; thus there are Borel functions n_0, Q_0 with

i) $\rho_2(\gamma) = n_0(\gamma)\rho_1^{Q_0}(\gamma)$ on $\mathscr{M}^0$,

ii) $d_x(Q_0(x), 1_x) < \in_0(x)$ on L_0;

iii) $Q_0(x) = 1_x$ on a section for $\mathscr{M}^0$.

We also have $n_0(\gamma) \in N_{r(\gamma)}$ and $Q_0(x) \in \bar{N}_x$. for $\gamma \in \mathscr{M}^0$ and $x \in L_0$.

Note that for any $x \in X$, there is a unique $x_0 \in L_0$ with $(x, x_0) \in \mathscr{L}$; we define then

$$R(x) = Q(x)\rho_1(x, x_0)F_{(x, x_0)}(Q_0(x_0))\rho_1(x, x_0)^{-1},$$

so that R is Borel. Also, $d_{x_0}(Q_0(x_0), 1_{x_0}) < \in_0(x_0) = d_{x_0}(B(x_0), 1_{x_0})$, so $Q_0(x_0) \notin B(x_0)$; thus

$$d_x(R(x), Q(x)) = d_x(Q(x)\rho_1(x, x_0)F_{(x, x_0)}(Q_0(x_0))\rho_1(x, x_0)^{-1}, Q(x))$$

is less than δ, and condition ii) is satisfied. Also, if $x \in L_0$, $x = x_0$ and $R(x) = Q_0(x)$, so that condition iii) is also satisfied.

If $(x, y) \in \mathscr{M}$ is arbitrary, there are unique elements x_0, $y_0 \in L_0$ with $(x, x_0) \in \mathscr{L}$, $(y_0, y) \in \mathscr{L}$ and $(x_0, y_0) \in \mathscr{M}^0$, such that $(x, y) = (x, x_0)(x_0, y_0)(y_0, y)$. Using the cocycle identity for ρ_1 and ρ_2, one obtains by laborious calculation,

$$\begin{aligned} \rho_2(x, y)\rho_1^R(x, y)^{-1} &= n'(x, y) \\ &= n_1(x, y)n_2(x, y)n(x, x_0), \end{aligned}$$

where

$$\begin{aligned} n_1(x, y) &= \rho_2(x, y_0)F_{(x, y_0)}(n(y_0, y))\rho_2(x, y_0)^{-1}, \\ n_2(x, y) &= \rho_2(x, x_0)F_{(x, x_0)}(n_0(x_0, y_0))\rho_2(x, x_0)^{-1}. \end{aligned}$$

The calculation above may be done as follows:

$$\begin{aligned}&\rho_2(x,y)\rho_1^R(x,y)^{-1}\\&\quad=\rho_2(x,y_0)F_{(x,y_0)}(\rho_2(y_0,y)F_{(y_0,y)}(R(y)))\rho_1(x,y)^{-1}R(x)^{-1}\end{aligned}$$

which, on substitution for $R(y)$, replacing $\rho_2(y_0,y)$ by $n(y_0,y)Q(y_0)$ $\rho_1(y_0,y)Q(y)^{-1}$, and remembering $Q(y_0)=1=\rho_1(y_0,y)F_{(y_0,y)}(\rho_1(y,y_0))$ yields

$$n_1(x,y)\rho_2(x,y_0)F_{(x,y_0)}(Q_0(y_0)\rho_1(y_0,y))\rho_1(x,y)^{-1}R(x)^{-1}.$$

Replacing $\rho_2(x,y_0)$ by $\rho_2(x,x_0)F_{(x,x_0)}(\rho_2(x_0,y_0))$ and using $\rho_2(\gamma)=n_0(\gamma)\rho_1^{Q_0}(\gamma)$ on $\mathscr{M}^0$ now yields

$$\begin{aligned}&n_1(x,y)n_2(x,y)\rho_2(x,x_0)F_{(x,x_0)}(Q_0(x_0)\rho_1(x_0,y_0)),\\&\qquad F_{(x,y_0)}(\rho_1(y_0,y))\rho_1(x,y)^{-1}R(x)^{-1}\end{aligned}$$

and further substitution of $(n\rho_1^Q)(x,x_0)$ for $\rho_2(x,x_0)$ yields

$$\begin{aligned}&n_1(x,y)n_2(x,y)n(x,x_0)Q(x)\rho_1(x,x_0)F_{(x,x_0)}(Q_0(x_0))\rho_1(x,x_0)^{-1}\\&\qquad\rho_1(x_1x_0)F_{(x,x_0)}(\rho_1(x_0,y_0))F_{(x,y_0)}(\rho_1(y_0,y))\rho_1(x,y)^{-1}R(x)^{-1}.\end{aligned}$$

But the last four terms which involve ρ_1 disappear on application of the cocycle identity for ρ_1, and substitution for $R(x)$ then reduces the whole expression to $n_1(x,y)n_2(x,y)n(x,x_0)$ as claimed. Clearly n_1 and n_2 take values in $\dot\cup N_x$, and if $(x,y)\in\mathscr{L}$ we have $x_0=y_0$ so that

$$n'(x,y)=\rho_2(x,x_0)F_{(x,x_0)}(n(x_0,y))\rho_2(x,x_0)^{-1}n(x,x_0).$$

However, the fact that $\gamma\to n(\gamma)^{-1}\rho_2(\gamma)=\rho_1^Q(\gamma)$ is an F-cocycle implies that this last quantity is nothing but $n(x,y)$, and n' extends n as required. $\square$

Proof of Theorem 5. 5. Since $(\mathscr{K},m)$ is hyperfinite, we may, after deleting a null set, assume that $\mathscr{K}=\bigcup_1^\infty\mathscr{K}_n$, where $\mathscr{K}_n\subseteq\mathscr{K}_{n+1}$ for each n, and each $\mathscr{K}_n$ has finite equivalence classes.

Applying Lemma 5. 6, and then Lemma 5. 7 inductively, we find sequences $\{n_k\}$ and $\{P_k\}$ of Borel maps, $n_k:\mathscr{K}_k\to\dot\cup N_x$ and $P_k:X\to\dot\cup\bar N_x$, such that

i) $\rho_2(\gamma)=n_k(\gamma)\rho_1^{P_k}(\gamma)$ on $\mathscr{K}_k$;

ii) $d_x(P_k(x),\ P_{k+1}(x))<2^{-k}$ on X;

iii) n_{k+1} on $\mathscr{K}_{k+1}$ extending n_k on $\mathscr{K}_k$.

Thus we may define $P:X\to\dot\cup\bar N_x$ by $P(x)=\lim_k P_k(x)$, and $n:\mathscr{K}\to\dot\cup N_x$ by $n(\gamma)=n_k(\gamma)$ for $\gamma\in\mathscr{K}_k$, clearly, P and n are well defined and

Borel, and

$$\rho_2(\gamma) = n(\gamma)\rho_1^p(\gamma) \text{ a.e on } \mathscr{K}. \qquad \square$$

References

[1] Auslander, L. and Moore, C. C., Unitary representations of solvable Lie groups, *Memoirs of A. M. S.*, **62** (1966).

[2] Bessaga, C. and Pelczynski, A., *Selected Topics in Infinite Dimensional Topology*, Warsaw, Monografie Matematyczne, 1975.

[3] Christensen, J. P. R., *Topology and Borel Structure*, North Holland Mathematics Studies, **10**, Amsterdam, 1974.

[4] Connes, A., Sur la theorie non-commutative de l'integration, *Springer Lecture Notes in Mathematics*, **725** (1979), 19-143.

[5] Feldman, J. and Moore, C. C., Ergodic equivalence relations, cohomology, and von Neumann algebras I, *T. A. M. S.*, **236** (1977), 289-324.

[6] Greenleaf, F. P., *Invariant Means on Topological Groups*, Van Nostrand Mathematical Studies, 1969.

[7] Haagerup, U., The standard form of von Neumann algebras, *Math. Scand.*, **37** (1975), 271-283.

[8] Jones, V. and Takesaki, M., Actions of compact Abelian groups on semi-finite injective factors, to appear in *Acta Math.*

[9] Kelley, J., *General Topology*, University Series in Higher Mathematics, van Nostrand, 1975.

[10] Ramsay, A., Virtual groups and group actions, *Advances in Mathematics*, **6** (1971), 253-322.

[11] Sutherland, C., *Notes on Orbit Equivalence; Krieger's Theorem*, Lecture Note Series No. 23, Oslo, 1976.

[12] Sutherland, C. and Takesaki, M., Actions of amenable groups and groupoids on semi-finite injective von Neumann algebras, *in preparation*.