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Linear Radon-Nikodym Theorems for States on JBW and W* Algebras[1]

By

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Abstract

Linear Radon-Nikodym theorems for states on a von Neumann algebra are obtained in the context of a one parameter family of positive cones. Especially, necessary and sufficient conditions for existence of linear Radon-Nikodym derivatives are investigated. In the natural cone case we consider Jordan Banach algebras.

§ 1. Introduction

Let $\mathscr{M}$ be a von Neumann algebra on a Hilbert space H with a distinguished cyclic and separating vector ξ_0. Making use of the associated modular operator, [15], Araki [1], introduced a one parameter family of positive cones in the Hilbert space. Some linear Radon-Nikodym theorems for states are known in this context. More precisely, for a certain state φ in $\mathscr{M}_*^+$, the existence of a vector ξ (linear Radon-Nikodym derivative) satisfying $\varphi(x) = \langle x\xi_0, \xi\rangle + \langle x\xi, \xi_0\rangle$, $x \in \mathscr{M}$ in the cones was proved in [1], [11].

In the present work, we study necessary and sufficient conditions for a state to admit a linear Radon-Nikodym derivative in the cones. We begin with the natural cones [1], [3], [5], [9]. Here the Jordan algebra context is a "natural" setting, and several criteria are obtained. Then we consider all the cones in the von Neumann algebra context.

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§ 2. The Jordan Case

2. 1. Notations and Preliminaries

In the following H will be a real or complex Hilbert space, and let $L(H)$ be the bounded operators on H.

Let H^+ be a facially homogeneous selfdual cone in a real Hilbert space H (see [9]). Let $D(H^+)$ be the derivations of H^+ (i. e., $\{\delta\in L(H)$; $\exp(t\delta)H^+\subseteq H^+$ for all $t\in R\}$) and $M=D(H^+)_{s.a.}$ be the Jordan Banach algebra with predual (i. e., JBW algebra, [8]) of the selfadjoint derivations with product denoted by $\circ$ ([9], III. 2. 1). For $\xi\in H$, $\omega_\xi\in M_*^+$ is defined by $\omega_\xi(\delta)=\langle\delta\xi,\xi\rangle$, $\delta\in M$. For $\xi\in H^+$, $\langle\xi\rangle$ denotes the face generated by ξ, and for any face F in H^+ $F^\perp=\{\xi\in H^+;\langle\xi,\zeta\rangle=0$ for all $\zeta\in F\}$.

We here summarize results which will be needed later.

Theorem 2. 1. 1. ([9], III. 5. 2) *The map: $\xi\in H^+\to\omega_\xi\in M_*^+$ is a homeomorphism with respect to the norm topologies. Furthermore, if $\omega_{\xi_1}\leq\omega_{\xi_2}$ ($\xi_i\in H^+$) then $\xi_1\leq\xi_2$ (i. e., $\xi_2-\xi_1\in H^+$).*

As in [1], [5], we denote the unique vector in H^+ corresponding to $\varphi\in M_*^+$ by $\varphi^{1/2}$.

Lemma 2. 1. 2. ([9], I. 1. 4) *If $\{\xi_n\}_{n\in N_+}$ is a norm bounded monotone increasing sequence in H^+, then $\xi=\bigvee_n\xi_n\in H^+$ exists and $\lim_{n\to\infty}||\xi-\xi_n||=0$.*

2. 2. Linear Radon-Nikodym Theorems

Definition 2. 2. 1. *Let φ_0 and φ be in M_*^+. Then φ admits a linear Radon-Nikodym derivative ξ in H^+ with respect to φ_0 if $\varphi(\delta)=2\langle\delta\xi,\varphi_0^{1/2}\rangle$, $\delta\in M$. (The factor 2 is just a normalization constant, and will disappear in the W^*-case, see Theorem 2. 3. 2, i).) The cone of such φ is denoted by $LRND(\varphi_0)$.*

Remark 2. 2. 2. Note that ξ is unique if φ_0 is faithful. Indeed, if ξ' is another derivative, then for all δ in M we get

$$0=\langle\delta(\xi-\xi'),\ \varphi_0^{1/2}\rangle=\langle\xi-\xi',\ \delta\varphi_0^{1/2}\rangle$$

and $\xi=\xi'$ due to the lemma below and [9], II. 1. 5.

Theorem 2. 2. 3. *Let φ_0 and φ be in M_*^+. Then $\varphi\in LRND(\varphi_0)$ if and only if $\bigvee_n n[(\varphi_0+n^{-1}\varphi)^{1/2}-\varphi_0^{1/2}]$ is in H^+ (i.e., $\sup_{n\in N_+} n\|(\varphi_0+n^{-1}\varphi)^{1/2}-\varphi_0^{1/2}\|<\infty$, see Lemma 2. 2. 5). Moreover, if φ_0 is faithful the derivative of φ is exactly this vector.*

Lemma 2. 2. 4. *Let φ_0 be in M_*^+. Then φ_0 is faithful if and only if $\varphi_0^{1/2}$ is a quasi-interior vector in H^+ (i.e., $\langle\varphi_0^{1/2}\rangle^\perp=\{0\}$).*

Proof. By [9], II. 1. 5, a vector in H^+ is quasi-interior if and only if it is cyclic and separating for M. Thus if $\varphi_0^{1/2}$ is quasi-interior, then φ_0 is faithful. Conversely, if φ_0 is faithful, the face $\langle\varphi_0\rangle$ generated by φ_0 in M_*^+ is norm dense in M_*^+ ([9], Appendice 2, Lemma 9). Thus for $\xi\in H^+$ there exists a sequence $\{\omega_{\xi_n}\}$ in $\langle\varphi_0\rangle$ with $\xi_n\in H^+$ and $\lim_{n\to\infty}\|\omega_{\xi_n}-\omega_\xi\|=0$. Theorem 2. 1. 1 implies that $\xi_n\in\langle\varphi_0^{1/2}\rangle$ and $\lim_{n\to\infty}\|\xi-\xi_n\|=0$. Thus $\langle\varphi_0^{1/2}\rangle$ is dense in H^+ and $\varphi_0^{1/2}$ is quasi-interior.

Q. E. D.

Lemma 2. 2. 5. *Let φ and φ_0 be in M_*^+. Then $\xi_n=n[(\varphi_0+n^{-1}\varphi)^{1/2}-\varphi_0^{1/2}]$, $n\in N_+$. give rise to an increasing sequence in H^+.*

Proof. (Compare with [12].) Since $\varphi_0\leq\varphi_0+n^{-1}\varphi$, ξ_n is in H^+ by Theorem 2. 1. 1. Let $m\geq n$. We have to show

$$m^{-1}(\varphi_0+n^{-1}\varphi)^{1/2}\leq n^{-1}(\varphi_0+m^{-1}\varphi)^{1/2}-(n^{-1}-m^{-1})\varphi_0^{1/2}$$

(the vector on the right side is in H^+), or

$$\langle\delta[n^{-1}(\varphi_0+m^{-1}\varphi)^{1/2}-(n^{-1}-m^{-1})\varphi_0^{1/2}],\ n^{-1}(\varphi_0+m^{-1}\varphi)^{1/2}-(n^{-1}-m^{-1})\varphi_0^{1/2}\rangle$$
$$-\langle\delta[m^{-1}(\varphi_0+n^{-1}\varphi)^{1/2}],\ m^{-1}(\varphi_0+n^{-1}\varphi)^{1/2}\rangle\geq 0$$

for all δ in M^+. But the second expression is equal to

$$\begin{aligned}&n^{-2}(\varphi_0+m^{-1}\varphi)(\delta)-2n^{-1}(n^{-1}-m^{-1})\langle\delta(\varphi_0+m^{-1}\varphi)^{1/2},\ \varphi_0^{1/2}\rangle\\&\quad+(n^{-1}-m^{-1})^2\varphi_0(\delta)-m^{-2}(\varphi_0+n^{-1}\varphi)(\delta)\\&=n^{-1}(n^{-1}-m^{-1})[2\varphi_0(\delta)+m^{-1}\varphi(\delta)-2\langle\delta(\varphi_0+m^{-1}\varphi)^{1/2},\ \varphi_0^{1/2}\rangle]\\&=n^{-1}(n^{-1}-m^{-1})\langle\delta[(\varphi_0+m^{-1}\varphi)^{1/2}-\varphi_0^{1/2}],\ (\varphi_0+m^{-1}\varphi)^{1/2}-\varphi_0^{1/2}\rangle,\end{aligned}$$

and positive. Q. E. D.

(*Proof of Theorem* 2. 2. 3.) If $\varphi=2\langle\cdot\xi,\varphi_0^{1/2}\rangle$, we get for each δ in M^+

$$(\varphi_0+n^{-1}\varphi)(\delta)\le\langle\delta\varphi_0^{1/2},\varphi_0^{1/2}\rangle+2n^{-1}\langle\delta\xi,\varphi_0^{1/2}\rangle+n^{-2}\langle\delta\xi,\xi\rangle$$
$$=\langle\delta(\varphi_0^{1/2}+n^{-1}\xi),\ (\varphi_0^{1/2}+n^{-1}\xi)\rangle.$$

Thus $(\varphi_0+n^{-1}\varphi)^{1/2}\le\varphi_0^{1/2}+n^{-1}\xi$ by Theorem 2. 1. 1 and $n[(\varphi_0+n^{-1}\varphi)^{1/2}-\varphi_0^{1/2}]\le\xi$. The result follows from the previous Lemma. Conversely, let $\xi=\bigvee_n\xi_n\in H^+$ with $\xi_n=n[(\varphi_0+n^{-1}\varphi)^{1/2}-\varphi_0^{1/2}]$. For each $\delta\in M$, we compute

$$\begin{aligned}&\varphi(\delta)-2\langle\delta\xi_n,\varphi_0^{1/2}\rangle\\&=\varphi(\delta)-\langle\delta\xi_n,(\varphi_0+n^{-1}\varphi)^{1/2}\rangle-\langle\delta\xi_n,\varphi_0^{1/2}\rangle\\&\qquad+\langle\delta\xi_n,(\varphi_0+n^{-1}\varphi)^{1/2}-\varphi_0^{1/2}\rangle\\&=\langle\delta\xi_n,(\varphi_0+n^{-1}\varphi)^{1/2}-\varphi_0^{1/2}\rangle.\end{aligned}$$

Here, on the second line the first three terms sum up to 0 by the definition of the vector ξ_n. Since $\sup_n\|\xi_n\|=\|\xi\|<+\infty$ and $\lim_{n\to\infty}\|(\varphi_0+n^{-1}\varphi)^{1/2}-\varphi_0^{1/2}\|=0$ (2.1.1), $\lim_{n\to\infty}2\langle\delta\xi_n,\varphi_0^{1/2}\rangle=\varphi(\delta)$ and we have the result.

Q. E. D.

Corollary 2. 2. 6. *Let φ_0 be in M_*^+. The face generated by φ_0 is included in $LRND(\varphi_0)$.*

Proof. When $\varphi\le l\varphi_0$, $\varphi\in M_*^+$, we have

$$\varphi_0+n^{-1}\varphi\le(1+n^{-1}l)\varphi_0,\qquad n\in N_+,$$
$$(\varphi_0+n^{-1}\varphi)^{1/2}\le(1+n^{-1}l)^{1/2}\varphi_0^{1/2}.$$

Therefore, $\xi_n=n[(\varphi_0+n^{-1}\varphi)^{1/2}-\varphi_0^{1/2}]\le n[(1+n^{-1}l)^{1/2}-1]\varphi_0^{1/2}$. It is elementary to see $\sup_n n[(1+n^{-1}l)^{1/2}-1]=2^{-1}l$. Thus $\bigvee_n\xi_n\le2^{-1}l\varphi_0^{1/2}$ and the corollary follows from the previous theorem. Q. E. D.

It is also possible to characterize the existence of a derivative in terms of the Jordan product $\circ$.

Theorem 2. 2. 7. *Let φ and φ_0 be in M_*^+ with φ_0 faithful. Then $\varphi\in LRND(\varphi_0)$ if and only if $\varphi(\delta)^2\le c\varphi_0(\delta\circ\delta)$ for all δ in M and some $c>0$.*

Proof. Suppose $\varphi(\delta)^2\le c\varphi_0(\delta\circ\delta)$, $\delta\in M$. If $P(\delta)=2\delta^2-\delta\circ\delta$, then

$$\varphi(\delta)^2 \leq c(2\langle\delta\varphi_0^{1/2}, \delta\varphi_0^{1/2}\rangle - \langle P(\delta)\varphi_0^{1/2}, \varphi_0^{1/2}\rangle)$$
$$\leq 2c\|\delta\varphi_0^{1/2}\|^2$$

because $P(\delta)$ preserves the order ([9], III. 4. 5). Consider the linear map: $\delta\varphi_0^{1/2} \rightarrow \varphi(\delta)$. This is well-defined because $\varphi_0^{1/2}$ is separating (see the proof of Lemma 2. 2. 4). Since the map is bounded, the Riesz representation theorem asserts the existence of $\xi \in H$ with $\varphi(\delta) = 2\langle\xi, \delta\varphi_0^{1/2}\rangle$. Let $\xi = \xi^+ - \xi^-$ be the Jordan decomposition of ξ ([9], I. 1. 2), and let $\langle\xi^-\rangle$ be the face generated by ξ^- in H^+. Let $\delta_{<\xi^->} = 2^{-1}(1 + P_{<\xi^->} - P_{<\xi^->^\perp})$ be the associated positive facial derivation ([9], II. 2. 5). Here, P_F is the orthogonal projection onto the closure of $F-F$. Since $\delta_{<\xi^->}\xi = -\xi^-$, we get

$$\begin{aligned} 0 &\leq \varphi(\delta_{<\xi^->}) \qquad \text{(the positivity of } \varphi) \\ &= 2\langle\delta_{<\xi^->}\xi, \varphi_0^{1/2}\rangle \\ &= -2\langle\xi^-, \varphi_0^{1/2}\rangle \\ &\leq 0 \qquad \text{(the selfduality of } H^+). \end{aligned}$$

Therefore, $\langle\xi^-, \varphi_0^{1/2}\rangle = 0$, and consequently $\xi^- = 0$ due to the fact that $\varphi_0^{1/2}$ is quasi-interior (Lemma 2. 2. 4). We thus have shown $\xi = \xi^+ \in H^+$.

To show the converse we need the operator inequality $\delta^2 \leq \delta \circ \delta$ (as operators in $L(H)$). Here δ^2 is the square of δ as an operator in $L(H)$. To show this, we may assume $\delta \geq 0$. Indeed, let $\delta = \delta^+ - \delta^-$ be the decomposition in [9], III. 2. 3 ($\delta^\pm \in M_+$, $\delta^+ \circ \delta^- = 0$, $[\delta^+, \delta^-] = 0$). If $(\delta^\pm)^2 \leq \delta^\pm \circ \delta^\pm$ is known, one gets

$$\begin{aligned} \delta^2 &= (\delta^+)^2 + (\delta^-)^2 - 2\delta^+\delta^- \\ &\leq (\delta^+)^2 + (\delta^-)^2 \qquad \text{(since } \delta^+\delta^- \geq 0) \\ &\leq \delta^+ \circ \delta^+ + \delta^- \circ \delta^- \\ &= (\delta^+ - \delta^-) \circ (\delta^+ - \delta^-) = \delta \circ \delta. \end{aligned}$$

Let $\delta = \int_0^a \lambda d\delta_{F(\lambda)}$ (≥ 0) be the facial decomposition (see [9], II. 2. 6). Then, for each $\varepsilon > 0$ we have

$$\delta + \varepsilon = \int_0^a (\lambda + \varepsilon)\, d\delta_{F(\lambda)},$$

$$P(\delta + \varepsilon)\ (= 2(\delta + \varepsilon)^2 - (\delta + \varepsilon) \circ (\delta + \varepsilon))$$
$$= \exp\{2\int_0^a \log(\lambda + \varepsilon)\, d\delta_{F(\lambda)}\}.$$

(see the proof of [9], III. 4. 5.) The operator concavity of $\lambda(\geq 0)$ $\rightarrow\log(\lambda+\varepsilon)$ implies

$$\int_0^a \log(\lambda+\varepsilon)\,d\delta_{F(\lambda)} \leq \log\{\int_0^a (\lambda+\varepsilon)\,d\delta_{F(\lambda)}\}.$$

Since the involved operators commute, we get

$$P(\delta+\varepsilon) \leq (\delta+\varepsilon)^2,$$

that is, $(\delta+\varepsilon)^2 \leq (\delta+\varepsilon)\circ(\delta+\varepsilon)$. The desired inequality $\delta^2 \leq \delta\circ\delta$ can be obtained by letting $\varepsilon\searrow 0$.

When $\varphi(\delta)=2\langle\delta\xi, \varphi_0^{1/2}\rangle$, $\delta\in M$, we estimate

$$\begin{aligned}\varphi(\delta)^2 &\leq c\langle\delta\varphi_0^{1/2}, \delta\varphi_0^{1/2}\rangle && \text{(Cauchy-Schwarz)}\\ &\leq c\langle\delta\circ\delta\varphi_0^{1/2}, \varphi_0^{1/2}\rangle && (\text{by } \delta^2\leq\delta\circ\delta)\\ &= c\varphi(\delta\circ\delta)\end{aligned}$$

Q. E. D.

Let φ and φ_0 be in M_*^+ with φ_0 faithful. If $\varphi \leq l\varphi_0$ for some $l>0$, there exists δ_φ in M^+ such that $\varphi^{1/2}=\delta_\varphi\varphi_0^{1/2}$ and $\|\delta_\varphi\|\leq l^{1/2}$. In fact, since $\varphi^{1/2}\leq l^{1/2}\varphi_0^{1/2}$ (Theorem 2. 1. 1), the assertion follows from [9], III. 5. 4. In particular, we have

$$\varphi(\delta)=\langle\delta_\varphi\delta\delta_\varphi\varphi_0^{1/2}, \varphi_0^{1/2}\rangle, \quad \delta\in M.$$

Conversely, if this is satisfied, then we compute

$$\begin{aligned}\varphi(\delta) &= \langle\delta\delta_\varphi\varphi_0^{1/2}, \delta_\varphi\varphi_0^{1/2}\rangle\\ &= \langle\delta[(\delta_\varphi\varphi_0^{1/2})^+ - (\delta_\varphi\varphi_0^{1/2})^-], (\delta_\varphi\varphi_0^{1/2})^+ - (\delta_\varphi\varphi_0^{1/2})^-\rangle && ([9], \text{I. 1. 2})\\ &= \langle\delta[(\delta_\varphi\varphi_0^{1/2})^+ + (\delta_\varphi\varphi_0^{1/2})^-], (\delta_\varphi\varphi_0^{1/2})^+ + (\delta_\varphi\varphi_0^{1/2})^-)\rangle && ([9], \text{I. 2. 3})\\ &= \langle\delta\,|\delta_\varphi\varphi_0^{1/2}|, |\delta_\varphi\varphi_0^{1/2}|\rangle.\end{aligned}$$

Therefore, by uniqueness, we get $\varphi^{1/2}=|\delta_\varphi\varphi_0^{1/2}|$. (Notice that if $\varphi^{1/2}=|\delta_\varphi\varphi_0^{1/2}|$ we can reverse the above computation.)

We emphasize the fact that φ is in general different from $\varphi_0\circ U_{\delta_\varphi}$, where

$$U_\delta\delta' = 2\delta\circ(\delta\circ\delta') - (\delta\circ\delta)\circ\delta'.$$

Even in the von Neumann algebra case, we get (with the notation of 2. 3)

$$\begin{aligned}\langle xyx\varphi_0^{1/2}, \varphi_0^{1/2}\rangle &= \varphi_0(xyx) = \tilde{\varphi}_0\circ i(xyx) \quad \text{(see the beginning of 2. 3.)}\\ &= \tilde{\varphi}_0\circ i(U_x y) = \tilde{\varphi}_0(U_{i(x)}i(y))\\ &= \tilde{\varphi}_0(U_{\delta_x}(\delta_y)) = \langle U_{\delta_x}(\delta_y)\varphi_0^{1/2}, \varphi_0^{1/2}\rangle\\ &\neq \langle\delta_x\delta_y\delta_x\varphi_0^{1/2}, \varphi_0^{1/2}\rangle.\end{aligned}$$

Definition 2.2.8. Let φ and φ_0 be in M_*^+. We say that φ has a quadratic Radon-Nikodym derivative with respect to φ_0 if there exists $\delta_\varphi \in M^+$ such that

$$\varphi(\delta) = \langle \delta_\varphi \delta \delta_\varphi \varphi_0^{1/2}, \varphi_0^{1/2} \rangle, \qquad \delta \in M.$$

The set of such φ is denoted by $QRND(\varphi_0)$.

We have thus proved:

Proposition 2.2.9. *Let φ and φ_0 be in M_*^+ with φ_0 faithful.*

i) *The face generated by φ_0 is included in $QRND(\varphi_0)$.*

ii) *$\varphi \in QRND(\varphi_0)$ if and only if $\varphi^{1/2} = |\delta_\varphi \varphi_0^{1/2}|$ for $\delta_\varphi \in M_+$.*

We remark that both of the inclusion $LRND(\varphi_0) \subseteq QRND(\varphi_0)$ and $QRND(\varphi_0) \subseteq LRND(\varphi_0)$ are false (even in the commutative case).

2.3. Connection with the von Neumann Case

Let $\mathcal{M}$ be a von Neumann algebra, and φ_0 be a faithful normal state on $\mathcal{M}$ with the standard modular object Δ, J ([15]). Then $H^+ = \mathcal{P}^\natural_{\mathcal{M}, \varphi_0^{1/2}}$ $(= \mathcal{P}^\natural)$ is a facially homogeneous selfdual cone (see [3] or [9], VI. 1). There is a Jordan isomorphism i between $\mathcal{M}_{s.a.}$ $(x \circ y = 2^{-1}(xy + yx))$ and $D(H^+)_{s.a.}$ given by $i(x) = 2^{-1}(x + JxJ)$ (see [3] or [9], VI. 2. 3). To each $\psi \in \mathcal{M}_*^+$ we can associate $\tilde{\psi} \in (D(H^+)_{s.a.})_*^+$ in such a way that ψ is just the complex extension of $\tilde{\psi} \circ i$. Therefore, for $x = x^*$ in $\mathcal{M}$, we get

$$\begin{aligned} \varphi_0(x) &= \tilde{\varphi}_0 \circ i(x) = 2^{-1} \langle (x + JxJ) \varphi_0^{1/2}, \varphi_0^{1/2} \rangle \\ &= 2^{-1} (\langle x\varphi_0^{1/2}, \varphi_0^{1/2} \rangle + \langle \varphi_0^{1/2}, x\varphi_0^{1/2} \rangle) \quad (\text{since } J\xi = \xi, \xi \in H^+) \\ &= \langle x\varphi_0^{1/2}, \varphi_0^{1/2} \rangle. \end{aligned}$$

Similarly, for $\xi \in H^+$ and $x \in \mathcal{M}_{s.a.}$, we compute

$$\begin{aligned} \langle i(x)\xi, \varphi_0^{1/2} \rangle &= 2^{-1} \langle (x + JxJ)\xi, \varphi_0^{1/2} \rangle \\ &= 2^{-1} (\langle x\xi, \varphi_0^{1/2} \rangle + \langle x\varphi_0^{1/2}, \xi \rangle) \end{aligned}$$

Thus it is natural to introduce:

Definition 2.3.1. *We say that $\varphi \in \mathcal{M}_*^+$ has a linear Radon-Nikodym derivative ξ (with respect to φ_0) in the natural cone $\mathcal{P}^\natural$ if $\varphi(x) = \langle x\xi, \varphi_0^{1/2} \rangle + \langle x\varphi_0^{1/2}, \xi \rangle$, $x \in \mathcal{M}$. The cone of such φ is again denoted by $LRND(\varphi_0)$.*

The results in 2.2 read:

Theorem 2.3.2. *Let φ and φ_0 be in $\mathcal{M}_*^+$ with φ_0 faithful.*

i) $\varphi \in LRND(\varphi_0)$ *if and only if* $\bigvee_{n \in N^+} n[(\varphi_0+n^{-1}\varphi)^{1/2}-\varphi_0^{1/2}]$ *exists in* $\mathcal{P}^\natural$. *In this case, the vector* $\xi = \bigvee_{n \in N^+} n[(\varphi_0+n^{-1}\varphi)^{1/2}-\varphi_0^{1/2}]$ *satisfies* $\varphi(x) = \langle x\xi, \varphi_0^{1/2}\rangle + \langle x\varphi_0^{1/2}, \xi\rangle$, $x \in \mathcal{M}$.

ii) *If* $\varphi \leq l\varphi_0$ *for some* $l>0$ *then* $\varphi \in LRND(\varphi_0)$ *(cf. [1], Theorem 5).*

iii) $\varphi \in LRND(\varphi_0)$ *if and only if*

$$|\varphi(x)|^2 \leq c\varphi_0(x^*x+xx^*)$$

for all x *in* $\mathcal{M}$ *(or equivalently for all* x *in* $\mathcal{M}_{s.a.}$*) and some* $c>0$.

Proof. i) and ii) follow from 2.2.3 and 2.2.6.
iii) The condition $|\varphi(x)|^2 \leq c2^{-1}\varphi_0(x^*x+xx^*)$, $x \in \mathcal{M}$ is equivalent to $\tilde{\varphi}(\delta)^2 \leq c\tilde{\varphi}_0(\delta\circ\delta)$, $\delta \in D(\mathcal{P}^\natural)_{s.a.}$. In fact, the latter implies

$$\tilde{\varphi}(i(x))^2 \leq c\tilde{\varphi}_0(i(x)\circ i(x)) = c\tilde{\varphi}_0(i(x^2))$$

for all $x \in \mathcal{M}_{s.a.}$. Since $\phi = \tilde{\phi}\circ i$, we get

$$\varphi(x)^2 \leq c\varphi_0(x^2),\ x = x^* \in \mathcal{M}.$$

For an arbitrary x in $\mathcal{M}$, we apply this to $x+x^*$ and $\sqrt{-1}(x-x^*)$. Adding the two resulting inequalities, we get

$$4\varphi(x)\varphi(x^*) \leq 2c\varphi_0(x^*x+xx^*).$$

The converse implication is trivial. Therefore, the result follows from 2.2.7. Q.E.D.

Finally, applying this result to a factor of type I_∞, we obtain

Corollary 2.3.3. *Let h_0 be a non-singular positive trace class operator on a Hilbert space H, and h be a positive trace class operator. The following three conditions are equivalent:*

i) *there exists a (unique) positive Hilbert-Schmidt class operator k such that*

$$h = h_0^{1/2}k + kh_0^{1/2}.$$

ii) $\sup_{n \in N_+} n\|(h_0+n^{-1}k)^{1/2}-h_0^{1/2}\|_2 < \infty$, *where* $\|\ \|_2$ *denotes the Hilbert-*

Schmidt norm.

iii) *there exists a positive constant c such that*

$$(Tr(hx))^2 \leq cTr(h_0x^2)$$

for all $x=x^* \in L(H)$.

Although the previous result was obtained without using the Tomita-Takesaki theory for von Neumann algebras, we shall use it to get a generalization in the next section. However, such a theory exists even in the non-commutative framework of the Jordan algebras (see [7]). For instance, using [7] we can easily generalize [11], Theorem 1.6 as follows:

Proposition 2.3.4. *Let M be a JBW algebra and* φ, φ_0 *normal states on M with* φ_0 *faithful. Then* $\varphi=\varphi_0(\cdot\circ h)$ *for* $h \in M^+$ *if and only if* $\tilde{\varphi}(\delta)=\int_{-\infty}^{\infty}\varphi(\theta_t(\delta))\ (\cosh(\pi t))^{-1}dt$ *satisfies* $\tilde{\varphi} \leq l\varphi_0$ *for some* $l>0$. *Here,* $\{\theta_t\}$ *is the cosine family associated with* φ_0.

§3. The von Neumann Case

3.1. Notations and Preliminaries

Let $\mathcal{M}$ be a (σ-finite) von Neumann algebra with a standard form $(\mathcal{M}, H, J, \mathcal{P}^\natural)$, [1], [3], [5], and ξ_0 be a distinguished cyclic and separating vector in the natural cone $\mathcal{P}^\natural$ with $\varphi_0=\omega_{\xi_0} \in \mathcal{M}_*^+$ (i.e., $\xi_0=\varphi_0^{1/2}$). Fixing these throughout, we denote the corresponding modular objects by Δ, J, and the modular automorphism group on $\mathcal{M}$ by σ_t $(=Ad\Delta^{it})$, $t \in \mathbf{R}$ [15]. We also set

$$\mathcal{M}_0=\{x \in \mathcal{M};\ t \in \mathbf{R} \to \sigma_t(x) \in \mathcal{M} \text{ extends to an } \mathcal{M}\text{-valued entire function}\},$$

which is σ-weakly dense in $\mathcal{M}$.

Definition 3.1.1. ([1]) *For each* $0 \leq \alpha \leq 1/2$, P^α $(=P^\alpha_{\varphi_0})$ *denotes the closure of the positive cone* $\Delta^\alpha\mathcal{M}_+\xi_0$ *in H.*

It is well known that $P^{1/4}$ is exactly the natural cone $\mathcal{P}^\natural$. We here summarize results on the cones which will be needed later.

Proposition 3. 1. 2. ([1], [3], [5])

i) *The map*: $\xi\in\mathscr{P}^{\natural}\to\omega_{\xi}\in\mathscr{M}_{*}^{+}$ *is a homeomorphism with respect to the norm topologies. Furthermore, if* $\omega_{\xi_1}\leq\omega_{\xi_2}$ $(\xi_1\in\mathscr{P}^{\natural})$, *then* $\xi_2-\xi_1\in\mathscr{P}^{\natural}$.

ii) $P^{\alpha}=JP^{1/2-\alpha}$, *and it is the dual cone* $(P^{1/2-\alpha})'=\{\xi\in H;\ \langle\xi,\zeta\rangle\geq 0$ *for all* $\zeta\in P^{1/2-\alpha}\}$ *of* $P^{1/2-\alpha}$ (*In particular*, $\mathscr{P}^{\natural}$ *is selfdual*).

iii) $P^{\alpha}\subseteq\mathscr{D}(\Delta^{1/2-2\alpha})$, *the domain of* $\Delta^{1/2-2\alpha}$, *and* $\Delta^{1/2-2\alpha}\xi=J\xi$ *if* $\xi\in P^{\alpha}$.

As in [11], we denote the function $(2\cosh(\pi t))^{-1}$, $t\in\boldsymbol{R}$, by $F(t)$, and recall

Lemma 3. 1. 3. (Lemma 1. 4, [11]) *If* $f(z)$ *is a bounded continuous function on the strip* $0\leq\mathrm{Re}\,z\leq 1$ *which is analytic in the interior, then we have*

$$f(1/2)=\int_{-\infty}^{\infty}\{f(it)+f(1+it)\}F(t)\,dt.$$

Lemma 3. 1. 4. *Let* $\beta>0$ *and* $\varphi\in\mathscr{M}_{*}^{+}$. *There exists* $l>0$ *such that* $\int_{-\infty}^{\infty}\varphi(\sigma_{\beta t}(x))F(t)\,dt\leq l\varphi_0(x)$, $x\in\mathscr{M}_{+}$, *if and only if for some* (*or equivalently all*) $\varepsilon>0$ *there exists* $l_{\varepsilon}>0$ *such that* $\int_{-\varepsilon}^{\varepsilon}\varphi(\sigma_t(x))\,dt\leq l_{\varepsilon}\varphi_0(x)$, $x\in\mathscr{M}_{+}$ (*thus the condition does not depend on a value of* β).

We notice that

$$\int_{-\infty}^{\infty}\varphi(\sigma_{\beta t}(x))F(t)\,dt=\int_{-\infty}^{\infty}\varphi(\sigma_t(x))F(\beta^{-1}t)\beta^{-1}dt.$$

Thus, similar arguments as Lemma 4. 1, [11], imply this result, and full details are left to the reader.

3. 2. Linear Radon-Nikodym Theorems

Here we obtain some necessary and sufficient conditions for a state to admit a linear Radon-Nikodym derivative in the cones.

Theorem 3. 2. 1. *Let* φ *be an element in* $\mathscr{M}_{*}^{+}$ *and* $0\leq\alpha\leq 1/2$. *The following conditions are equivalent*:

i) $\varphi(x)=\langle x\xi_0,\zeta\rangle+\langle x\zeta,\xi_0\rangle$, $x\in\mathscr{M}$, *for a vector* ζ *in* P^α, *that is*, ζ *is a linear Radon-Nikodym derivative of* φ *in* P^α (*with respect to* φ_0),

ii) $\varphi(x)=\langle(1+\Delta^{1-2\alpha})x\xi_0,\zeta\rangle$, $x\in\mathscr{M}_0$, *for a vector* ζ,

iii) *There exists a constant* $c>0$ *such that*

$$|\varphi(x)|\leq c\|(1+\Delta^{1-2\alpha})x\xi_0\|,\qquad x\in\mathscr{M}_0,$$

vi) $\int_{-\infty}^{\infty}\varphi\circ\sigma_{(1-2\alpha)t}(x)F(t)\,dt=\langle\Delta^{1/2-\alpha}x\xi_0,\zeta\rangle$, $x\in\mathscr{M}$, *for a vector* ζ.

Furthermore, the vector in i), ii), *and* iv) *are identical.*

Proof. i)$\Rightarrow$ii) Because of $J\Delta^{1/2-2\alpha}\zeta=\zeta$ (3.1.2, iii)), for each $x\in\mathscr{M}_0$ (hence $x\xi_0\in\bigcap_{n\in Z}\mathscr{D}(\Delta^n)$) we compute

$$\begin{aligned}\langle x\zeta,\xi_0\rangle&=\langle\zeta,x^*\xi_0\rangle=\langle J\Delta^{1/2-2\alpha}\zeta,J\Delta^{1/2}x\xi_0\rangle\\&=\langle\Delta^{1/2}x\xi_0,\Delta^{1/2-2\alpha}\zeta\rangle=\langle\Delta^{1-2\alpha}x\xi_0,\zeta\rangle,\\\varphi(x)&=\langle x\xi_0,\zeta\rangle+\langle x\zeta,\xi_0\rangle\\&=\langle(1+\Delta^{1-2\alpha})x\xi_0,\zeta\rangle.\end{aligned}$$

ii)$\Rightarrow$iii) This is just the Cauchy Schwarz inequality.

iii)$\Rightarrow$ii) At first we claim that $\mathscr{M}_0\xi_0$ is a core for $\Delta^{1-2\alpha}$ (It is obvious if $1-2\alpha\leq1/2$.) If one sets $\mathscr{M}_{exp}=\{x\in\mathscr{M}_0$ there exist $\beta=\beta_x$ and $\gamma=\gamma_x$ such that $\|\sigma_{-in}(x)\|\leq\beta\exp(\gamma n)$ for all $n\in N_+\}$, $\mathscr{M}_{exp}\xi_0$ ($\subseteq\mathscr{M}_0\xi_0$) is dense in H ([6], Lemma 4.2). For each $x\in\mathscr{M}_{exp}$ we estimate

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{n!}\|\Delta^n x\xi_0\|&\leq\sum_{n=0}^{\infty}\frac{1}{n!}\beta\exp(\gamma n)\\&=\beta\exp(\exp\gamma)<+\infty.\end{aligned}$$

Thus Δ (hence $\Delta^{1-2\alpha}$) is essentially self-adjoint on $\mathscr{M}_{exp}\xi_0$ thanks to Nelson's analytic vector theorem ([14], p. 202). Hence, $\mathscr{M}_0\xi_0$ is a core for $\Delta^{1-2\alpha}$, equivalently, $(1+\Delta^{1-2\alpha})\mathscr{M}_0\xi_0$ is a dense subspace in H. We consider the linear map: $(1+\Delta^{1-2\alpha})x\xi_0\in(1+\Delta^{1-2\alpha})\mathscr{M}_0\xi_0\to\varphi(x)\in\boldsymbol{C}$. This densely defined (and well-defined) functional is bounded by the assumption. Thus, ii) follows from the Riesz representation theorem (applied to the extension of this bounded functional).

ii)$\Rightarrow$iv) The both sides of iv) define elements in $\mathscr{M}_*$ as seen easily. It, thus, suffices to check iv) for each $x\in\mathscr{M}_0$. For such an x, we compute

$$\int_{-\infty}^{\infty}\varphi\circ\sigma_{(1-2\alpha)t}(x)F(t)\,dt$$

$$=\int_{-\infty}^{\infty}\langle(1+\Delta^{1-2\alpha})\Delta^{(1-2\alpha)it}x\xi_0,\zeta\rangle F(t)\,dt$$
$$=\int_{-\infty}^{\infty}\langle\Delta^{1/2-\alpha}\{\Delta^{-(1/2-\alpha)+(1-2\alpha)it}+\Delta^{(1/2-\alpha)+(1-2\alpha)it}\}x\xi_0,\zeta\rangle F(t)\,dt$$
$$=\langle\Delta^{1/2-\alpha}x\xi_0,\zeta\rangle.$$

Here, on the last line, we used Lemma 3. 1. 3 for

$$f(z)=\langle\Delta^{1/2-\alpha}\Delta^{(1-2\alpha)(z-1/2)}x\xi_0,\zeta\rangle.$$

iv)$\Rightarrow$i) the left side of iv) being an element in $\mathcal{M}_*^+$, iv) shows that ζ belongs to P^α (3. 2. 1, ii)). To show i), we may and do assume $x\in\mathcal{M}_0$. For such an x, based on iv) we compute

$$\langle x\xi_0,\zeta\rangle=\langle\Delta^{1/2-\alpha}\sigma_{-i(\alpha-1/2)}(x)\,\xi_0,\zeta\rangle$$
$$=\int_{-\infty}^{\infty}\varphi\circ\sigma_{(1-2\alpha)t-i(\alpha-1/2)}(x)\,F(t)\,dt,$$
$$\langle x\zeta,\xi_0\rangle=\langle\Delta^{1/2-\alpha}\sigma_{i(\alpha-1/2)}(x)\,\xi_0,\zeta\rangle\ \text{(see the computation in i)}\Rightarrow\text{ii))}$$
$$=\int_{-\infty}^{\infty}\varphi\circ\sigma_{(1-2\alpha)t+i(\alpha-1/2)}(x)\,F(t)\,dt.$$

Thus, Lemma 3. 1. 3 applied to

$$f(z)=\varphi\circ\sigma_{-(1-2\alpha)(z-1/2)i}(x)$$

implies that

$$\langle x\xi_0,\zeta\rangle+\langle x\zeta,\xi_0\rangle=f(1/2)=\varphi(x).$$

Q. E. D.

Corollary 3. 2. 2. (Uniqueness) *Assume that $\varphi\in\mathcal{M}_*^+$ and $0\leq\alpha\leq 1/2$. If φ admits a linear Radon-Nikodym derivative ζ in P^α, then it is uniquely determined by φ (and α).*

Proof. This follows from Theorem 3. 2. 1, iv). Q. E. D.

Corollary 3. 2. 3. *Assume that $\varphi\in\mathcal{M}_*^+$ and $0\leq\alpha\leq\alpha'\leq 1/2$. If φ admits a linear Radon-Nikodym derivative in $P^{\alpha'}$, then so does it in P^α.*

Proof. This follows from Theorem 3. 2. 1, ii), and the boundedness of the operator $(1+\Delta^{1-2\alpha'})(1+\Delta^{1-2\alpha})^{-1}$. Q. E. D.

Lemma 3. 2. 4. *For each $x\in\mathcal{M}_0$ and $0\leq\alpha\leq 1/2$, we get*

$$\|(1+\Delta^{2-4\alpha})^{1/2}x\xi_0\| \leq \|(1+\Delta^{1-2\alpha})x\xi_0\| \\ \leq \sqrt{2}\|(1+\Delta^{2-4\alpha})^{1/2}x\xi_0\|.$$

This is a consequence of the spectral decomposition theorem, and used to rewrite Theorem 3. 2. 1 in a "modular operator free" form.

Corollary 3. 2. 5. *Assume that* $\varphi \in \mathcal{M}_*^+$ *and* $0 \leq \alpha \leq 1/2$. *There exists a constant* $c > 0$ *such that*

$$|\varphi(x)| \leq c\{\varphi_0(x^*x) + \varphi_0(\sigma_{i(2\alpha-1/2)}(x)\sigma_{i(2\alpha-1/2)}(x)^*)\}^{1/2},$$

$x \in \mathcal{M}_0$, *if and only if* φ *admits a linear Radon-Nikodym derivative in* P^α. (*When* $\alpha = 1/4$, *the theorem corresponds to Theorem* 2. 2. 7 *and Theorem* 2. 3. 2, iii). *When* $\alpha = 1/2$, *the right side is* $\sqrt{2}c\varphi_0(x^*x)^{1/2}$.)

Proof. For each $x \in \mathcal{M}_0$, we compute

$$\begin{aligned} &\|(1+\Delta^{2-4\alpha})^{1/2}x\xi_0\|^2 \\ &= \langle (1+\Delta^{2-4\alpha})x\xi_0, x\xi_0\rangle \\ &= \varphi_0(x^*x) + \|\Delta^{1-2\alpha}x\xi_0\|^2 \\ &= \varphi_0(x^*x) + \|\Delta^{1-2\alpha}J\Delta^{1/2}x^*\xi_0\|^2 \\ &= \varphi_0(x^*x) + \|\Delta^{2\alpha-1}\Delta^{1/2}x^*\xi_0\|^2 \\ &= \varphi_0(x^*x) + \|\Delta^{2\alpha-1/2}x^*\xi_0\|^2 \\ &= \varphi_0(x^*x) + \|\sigma_{-i(2\alpha-1/2)}(x^*)\xi_0\|^2 \\ &= \varphi_0(x^*x) + \varphi_0(\sigma_{-i(2\alpha-1/2)}(x^*)^*\sigma_{-i(2\alpha-1/2)}(x^*)) \\ &= \varphi_0(x^*x) + \varphi_0(\sigma_{i(2\alpha-1/2)}(x)\sigma_{i(2\alpha-1/2)}(x)^*). \end{aligned}$$

On the last line we used the following easy consequence of uniqueness of analytic continuation:

$$\sigma_z(x)^* = \sigma_{\bar{z}}(x^*), \quad x \in \mathcal{M}_0, \quad z \in \mathbf{C}.$$

Now the corollary follows from Theorem 3. 2. 1, iii), and Lemma 3. 2. 4. Q. E. D.

Proposition 3. 2. 6. *Assume that* $\varphi \in \mathcal{M}_*^+$ *and* $0 \leq \alpha < 1/2$. *There exists a (unique)* h_α *in* $\mathcal{M}_+$ *such that* $\varphi(x) = \langle x\xi_0, \Delta^\alpha h_\alpha \xi_0\rangle + \langle x\Delta^\alpha h_\alpha \xi_0, \xi_0\rangle$, $x \in \mathcal{M}$, *if and only if for some (or equivalently all)* $\varepsilon > 0$, *there exists* $l_\varepsilon > 0$ *such that*

$$\int_{-\varepsilon}^{\varepsilon} \varphi \circ \sigma_t dt \leq l_\varepsilon \varphi_0.$$

In particular, (although h_α *does depend on* α*) the existence of* h_α *does not* *depend on* $\alpha\in[0,1/2)$. (*cf.* [11], *Theorem* 1.6.)

Proof. If $\zeta=\Delta^{\alpha}h_{\alpha}\xi_0\in P^{\alpha}$ is a linear Radon-Nikodym derivative, then Theorem 3.2.1, iv), implies that

$$\begin{aligned}\varphi^{\alpha}(x)&=\langle\Delta^{1/2-\alpha}x\xi_0,\Delta^{\alpha}h_{\alpha}\xi_0\rangle\\&=\langle x\xi_0,\Delta^{1/2}h_{\alpha},\xi_0\rangle\\&=\langle x\xi_0,Jh_{\alpha}J\xi_0\rangle,\end{aligned}$$

where we define $\varphi^{\alpha}=\int_{-\infty}^{\infty}\varphi\circ\sigma_{(1-2\alpha)t}F(t)\,dt$. Due to $Jh_{\alpha}J\in\mathcal{M}'$, we get

$$\varphi^{\alpha}\leq\|h_{\alpha}\|\varphi_0.$$

Conversely, if $\varphi^{\alpha}\leq l\varphi_0$ for some $l>0$, then the Radon-Nikodym cocycle: $t\in\boldsymbol{R}\rightarrow(D\varphi^{\alpha};\ D\varphi_0)_t\in\mathcal{M}$, [4], extends to a bounded σ-weakly continuous function on $-1/2\leq Im\ z\leq 0$ which is analytic in the interior ([6], Lemma 3.3 for example). Setting $h_{\alpha}=(D\varphi^{\alpha};\ D\varphi_0)^{*}_{-i/2}(D\varphi^{\alpha};\ D\varphi_0)_{-i/2}\in\mathcal{M}_+$, we compute

$$\begin{aligned}&\langle\Delta^{1/2-\alpha}x\xi_0,\Delta^{\alpha}h_{\alpha}\xi_0\rangle\\&\quad=\langle x\xi_0,Jh_{\alpha}J\xi_0\rangle\\&\quad=\langle xJ(D\varphi^{\alpha};\ D\varphi_0)_{-i/2}\xi_0,J(D\varphi^{\alpha};\ D\varphi_0)_{-i/2}\xi_0\rangle\\&\quad=\varphi^{\alpha}(x)\end{aligned}$$

since $J(D\varphi^{\alpha};\ D\varphi_0)_{-i/2}\xi_0=(D\varphi^{\alpha};\ D\varphi_0)_{-i/2}\xi_0$ is the unique implementing vector for φ^{α} in $\mathcal{P}^{\natural}$. We thus have proved that

$\varphi^{\alpha}\leq l\varphi_0\Leftrightarrow$there exists a linear Radon-Nikodym derivative of the form $\Delta^{\alpha}h_{\alpha}\xi_0$.

Now the proposition follows from Lemma 3.1.4. Q. E. D.

The case $\alpha=1/2$ is excluded from the above result. But this is a trivial case (and, in fact, corresponds to the "most elementary" Radon-Nikodym theorem). In fact, we get

$$\begin{aligned}\varphi(x)&=\langle x\xi_0,\Delta^{1/2}h\xi_0\rangle+\langle x\Delta^{1/2}h\xi_0,\xi_0\rangle\\&=2\langle xJhJ\xi_0,\xi_0\rangle\end{aligned}$$

for some $h\in\mathcal{M}_+$ if and only if $\varphi\leq l\varphi_0$ for some $l>0$.

Finally we relate Proposition 3.2.6 to Sakai's Radon-Nikodym theorem.

Proposition 3.2.7. *If the condition in Proposition* 3.2.6 *is satisfied,* *then there exists a (unique) positive operator* h *in* $\mathcal{M}_+$ *such that*

$$\varphi(x)=\varphi_0(hxh), \quad x\in\mathcal{M}.$$

Among other things, its proof will be given in the appendix.

Appendix

As before, let $\varphi_0=\omega_{\xi_0}$ ($\xi_0\in\mathcal{P}^\natural$) be a fixed faithful normal state on a von Neumann algebra $\mathcal{M}$. In the main part of the article we studied linear Radon-Nikodym theorems. Here we investigate when $\varphi\in\mathcal{M}_*^+$ admits its Sakai Radon-Nikodym derivative (with respect to φ_0) in a quadratic form.

Theorem A. *Let* $\varphi=\omega_{\xi_\varphi}$ ($\xi_\varphi\in\mathcal{P}^\natural$) *be an element in* $\mathcal{M}_*^+$. *The following two conditions are equivalent*:

i) *there exists a (unique) positive* h *in* $\mathcal{M}$ *such that* $\varphi(x)=\varphi_0(hxh)$, $x\in\mathcal{M}$,

ii) *the positive part* $|\chi_\varphi|$ *of the polar decomposition of* $\chi_\varphi=\langle\cdot\,\xi_\varphi, \xi_0\rangle \in\mathcal{M}_*$ *satisfies* $|\chi_\varphi|\leq l\varphi_0$ *for some* $l>0$.

Furthermore, in this case, the quadratic Radon-Nikodym derivative h *in* i) *is exactly* $|(D|\chi_\varphi|;\ D\varphi_0)_{-i/2}|^2$.

In [13], the $L(H)$-version of the theorem was proved. It is possible to generalize their arguments to an arbitrary von Neumann algebra by making use of the non-commutative L^p-theory. But here we present a self-contained proof based on our approach.

Proof. Let $\chi_\varphi=u_\varphi|\chi_\varphi|$ be the polar decomposition. For $x\in\mathcal{M}$ we compute

$$\begin{aligned}\langle u_\varphi u_\varphi^*\xi_\varphi, x\xi_0\rangle &=\langle x^*u_\varphi u_\varphi^*\xi_\varphi, \xi_0\rangle\\ &=\chi_\varphi(x^*u_\varphi u_\varphi^*)=(u_\varphi|\chi_\varphi|)(x^*u_\varphi u_\varphi^*)\\ &=|\chi_\varphi|(x^*u_\varphi u_\varphi^*u_\varphi)=|\chi_\varphi|(x^*u_\varphi)\\ &=(u_\varphi|\chi_\varphi|)(x^*)=\chi_\varphi(x^*)\\ &=\langle\xi_\varphi, x\xi_0\rangle.\end{aligned}$$

Since $\mathcal{M}\xi_0$ is dense, the above computations show $u_\varphi u_\varphi^*\xi_\varphi=\xi_\varphi$. We claim that the unique implementing vector $\xi_\varphi^\#$ of φ in $P^0=(\mathcal{M}_+\xi_0)^-$

$(\varphi=\langle\cdot\xi_\varphi^\sharp,\xi_\varphi^\sharp\rangle)$ is $Ju_\varphi^*\xi_\varphi$, hence, $|\chi_\varphi|=\langle\cdot J\xi_\varphi^\sharp,\xi_0\rangle$. At first, since

$$\langle u_\varphi^*\xi_\varphi, x\xi_0\rangle=\langle xu_\varphi^*\xi_\varphi,\xi_0\rangle=|\chi_\varphi|(x)\geq 0$$

for any $x\in\mathscr{M}_+$, we know

$$Ju_\varphi^*\xi_\varphi\in J(P^0)'=JP^{1/2}=P^0$$

(3.1.2, ii)). Also, for each $x\in\mathscr{M}$, we compute

$$\begin{aligned}\langle xJu_\varphi^*\xi_\varphi, Ju_\varphi^*\xi_\varphi\rangle &= (xJu_\varphi^*J\xi_\varphi, Ju_\varphi^*J\xi_\varphi\rangle\\ &=\langle xJu_\varphi u_\varphi^*\xi_\varphi,\xi_\varphi\rangle \qquad (Ju_\varphi J\in\mathscr{M}')\\ &=\langle x\xi_\varphi,\xi_\varphi\rangle \qquad (u_\varphi u_\varphi^*\xi_\varphi=\xi_\varphi\in\mathscr{P}^\natural)\\ &=\varphi(x).\end{aligned}$$

Therefore, we have shown $\xi_\varphi^\sharp=Ju_\varphi^*\xi_\varphi$.

To prove i)⇒ii), let us assume $\varphi=h\varphi_0h$, $h\in\mathscr{M}_+$. This means $\xi_\varphi^\sharp=h\xi_0$, and for each $x\in\mathscr{M}_+$ we estimate

$$\begin{aligned}|\chi_\varphi|(x)&=\langle xJ\xi_\varphi^\sharp,\xi_0\rangle=\langle xJhJ\xi_0,\xi_0\rangle\\ &\leq\|h\|\langle x\xi_0,\xi_0\rangle=\|h\|\varphi_0(x).\end{aligned}$$

Conversely, let us assume $|\chi_\varphi|\leq l\varphi_0$. Then $k=(D|\chi_\varphi|\,;\,D\varphi_0)_{-i/2}$ makes sense as an element in $\mathscr{M}$. Notice that $Jk\xi_0=k\xi_0$ is the unique implementing vector of $|\chi_\varphi|$ in $\mathscr{P}^\natural$. For each $x\in\mathscr{M}$, we compute

$$\begin{aligned}\langle xJ\xi_\varphi^\sharp,\xi_0\rangle&=|\chi_\varphi|(x)\\ &=\langle xJk\xi_0, Jk\xi_0\rangle=(xJk^*k\xi_0,\xi_0\rangle.\end{aligned}$$

The density of $\mathscr{M}\xi_0$ shows that $\xi_\varphi^\sharp=k^*k\xi_0$, that is, $h=k^*k=|k|^2$ is the quadratic Radon-Nikodym derivative. Q. E. D.

Lemma B. *If* φ,ψ *in* $\mathscr{M}_*^+$ *satisfy* $\varphi\leq\psi$, χ_φ,χ_ψ *in the Theorem A satisfy* $|\chi_\varphi|\leq|\chi_\psi|$.

Proof. Here we have to use the non-commutative L^1-theory. We use the approach in [2], [10], where a relationship between the L^p-spaces and the cones are clarified. All the necessary definitions and facts can be found in these articles. The L^1-space can be identified with the predual $\mathscr{M}_*$. Then the element $\Delta_{\varphi\varphi_0}^{1/2}\Delta^{1/2}$ in the L^1-space corresponds to χ_φ, and $|\chi_\varphi|$ corresponds to the absolute value part $|\Delta_{\varphi\varphi_0}^{1/2}\Delta^{1/2}|=(\Delta^{1/2}\Delta_{\varphi\varphi_0}\Delta^{1/2})^{1/2}$ of the polar decomposition (as an operator). Therefore, the lemma follows from the operator monotonicity of the square root function. Q. E. D.

(*Proof of Proposition* 3.2.7.) Let us assume $\varphi(x)=\langle xh_0\xi_0, \xi_0\rangle+\langle x\xi_0, h_0\xi_0\rangle$, $x\in\mathscr{M}$, for $h_0\in\mathscr{M}_+$ as in Proposition 3.2.6 ($\alpha=0$). We observe that

$$\varphi\leq\varphi_0+\varphi+h_0\varphi_0h_0=\omega_{\xi_0}+\varphi+\omega_{h_0\xi_0}$$
$$=\omega_{(1+h_0)\xi_0}.$$

Since $\psi=\omega_{(1+h_0)\xi_0}$ admits the quadratic Radon-Nikodym derivative $1+h_0$, the result follows from Theorem A and Lemma B.

Q. E. D.

We remark that the converse of Proposition 3.2.7 is false. A counterexample in the $L(H)$ situation can be found in [12].

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