Publ. RIMS, Kyoto Univ.
21 (1985), 1281-1297

Lower Bounds of Decay Order of Eigenfunctions of Second-order Elliptic Operators

Dedicated to Professor S. Mizohata on his 60th birthday

By

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§ 0. Introduction

Let real-valued function $v_{ij}(y)$ $(0\leq i<j\leq N)$ defined on $\mathbb{R}^3$ satisfy

$$\begin{aligned} &v_{ij}(y)\to 0 \text{ as } |y|\to\infty \text{ in } \mathbb{R}^3 && (0\leq i<j\leq N),\\ &v_{ij}(y)\geq 0 \text{ for } y\in\mathbb{R}^3 && (1\leq i<j\leq N),\\ &v_{ij}(y)\in L^2_{loc}(\mathbb{R}^3) && (0\leq i<j\leq N),\end{aligned}$$

and let

$$\begin{aligned} &H=-\sum_{i=1}^{N}\Delta_i+\sum_{i=1}^{N}v_{0i}(x^i)+\sum_{1\leq i<j\leq N}v_{ij}(x^i-x^j),\\ &D(H)=H^2(\mathbb{R}^{3N}),\\ &\Sigma=\inf\,\sigma_{ess}(H),\end{aligned}$$

where $x^i\in\mathbb{R}^3$, and Δ_i is a Laplacian in $\mathbb{R}^3$ with respect to x^i. Then by easylike application of Agmon [1, Theorem 5.2, p. 85] we have the following: If $u(x)$ satisfies

$$\begin{aligned} &(Hu)(x)=\lambda u(x) \text{ in } \mathbb{R}^{3N},\\ &u\in D(H), \text{ and } \lambda<\Sigma(\leq 0),\end{aligned}$$

then for any $\varepsilon>0$ there exists some constant $C>0$ such that for a. e. $x=(x^1,\dots,x^N)\in\mathbb{R}^{3N}$ we have

$$|u(x)|\leq Ce^{-(1-\varepsilon)\sqrt{\Sigma-\lambda}|x|}.$$

In this paper by Theorem 1.5 and 1.7 given in §1 we have: If, besides above conditions, $v_{ij}(y)$ satisfies

Communicated by S. Matsuura, June 13, 1985.

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$v_{0i}(y) \leq 0$ is a homogeneous function of degree $-(\gamma_0-\varepsilon_i)$ $(1 \leq i \leq N)$,

$v_{ij}(y) \geq 0$ is a homogeneous function of degree $-(\gamma_0+\varepsilon_{ij})$ $(1 \leq i < j \leq N)$,

where ε_i, $\varepsilon_{ij} \geq 0$ and $0 < \gamma_0 \leq 2$ (this case has been considered in Mochizuki-Uchiyama [8, p. 131～p. 133]), then, since

$\lim\sup_{r\to\infty} \{(r\partial_r+\gamma_0)(\sum_{0\leq i<j\leq N} v_{ij})\} \leq 0$, for u satisfying

$$-\sum_{i=1}^{N} \Delta_i u + \sum_{i=1}^{N} v_{0i}(x^i)u + \sum_{1\leq i<j\leq N} v_{ij}(x^i-x^j)u = \lambda u \text{ in } \boldsymbol{R}^{3N},$$

$u \in H^2_{loc}(\boldsymbol{R}^{3N})$,

supp $[u]$ is not a compact set in $\boldsymbol{R}^{3N}$,

λ is a real constant,

we have for any $\varepsilon > 0$

$$\lim_{R\to\infty} e^{2(1+\varepsilon)\sqrt{|\lambda|}R} \int_{R<|x|<R+1} |u|^2 dx = \infty, \quad \text{if } \lambda < 0,$$

$$\lim_{R\to\infty} e^{\varepsilon R} \int_{R<|x|<R+1} |u|^2 dx = \infty, \quad \text{if } \lambda \geq 0.$$

If v_{ij} is a Coulomb potential, namely,

$$v_{0i}(y) = -Z_i |y|^{-1} \quad (1 \leq i \leq N),$$
$$v_{ij}(y) = Z_{ij} |y|^{-1} \quad (1 \leq i < j \leq N),$$

where Z_i, Z_{ij} are non-negative constants, v_{ij} $(0 \leq i < j \leq N)$ satisfies the all conditions as mentioned above.

In this paper we also consider general second-order elliptic equations as following:

$$-\sum_{i,j=1}^{n} \left(\frac{\partial}{\partial x_i} + \sqrt{-1}\, b_i(x)\right) a_{ij}(x) \left(\frac{\partial}{\partial x_j} + \sqrt{-1}\, b_j(x)\right) u(x) + (q_1(x)+q_2(x))u(x) = \lambda u(x)$$

(in some neighborhood of infinity of $\boldsymbol{R}^n$).

Our main assumptions are the following:

curl $b(x) = o(r^{-1})$,

$q_1(x) = V_1(x) + V_2(x)$,

$V_1(x)$ satisfies the similar conditions as given for above v_{ij},

$V_2(x) = o(1)$ and $\partial_r V_2(x) = o(r^{-1})$

$q_2(x) = o(r^{-1/2})$

at infinity. More detailed conditions are stated in §1. Then u has a similar lower bound of decay order at infinity as given above.

Our results stated in §1 can be considered as a generalization of Bardos-Merigot [2, Theorem 2.2 (ii) and (iii), p. 329, which seems to need some modifications of statements of Theorem 2.2, and treats the case $a_{ij}(x)=\delta_{ij}$, $b_i(x)=q_2(x)=V_1(x)=0$]. Our treatment of $b_i(x)$ can be found in Ikebe-Uchiyama [5], Kalf [6], Mochizuki [7] and Eastham-Kalf [3, §6]. Our method of proofs is a short-cut of Roze [9] and Éidus [4]. Lastly we remark that if we assume a stronger condition $q_2(x)=o(r^{-1})$, then for λ satisfying $\gamma_0\lambda>\limsup_{r\to\infty}\{r\partial_r q_1(x)+\gamma_0 q_1(x)\}$ $(0<\gamma_0<2)$ we have

$$\lim_{R\to\infty} R^{(\gamma_0/2)-1+\varepsilon}\int_{R_0<|x|<R}|u|^2dx=\infty \quad \text{for any } \varepsilon>0.$$

(See, Uchiyama [10], Mochizuki [7] and Mochizuki-Uchiyama [8].)

§1. Notations and Main Results

At first we shall list the notations which will be used freely in the sequel:

$\langle\xi,\eta\rangle=\xi_1\eta_1+\dots+\xi_n\eta_n$ for $\xi,\eta\in\boldsymbol{C}^n$;
$|\xi|=(\langle\xi,\bar{\xi}\rangle)^{1/2}$ for $\xi\in\boldsymbol{C}^n$;
$\hat{x}=x/|x|$ and $r=|x|$ for $x=(x_1,\dots,x_n)\in\boldsymbol{R}^n$;
$S(t)=\{x\mid |x|=t\}$ for $t>0$;
$B(s,t)=\{x\mid s<|x|<t\}$ for $t>s>0$;
$\partial_j=\partial/\partial x_j$ and $\partial_r=\partial/\partial r$;
$D_j=\partial_j+\sqrt{-1}\,b_j(x)$ and $D=(D_1,\dots,D_n)$;
grad $f=(\partial_1 f,\dots,\partial_n f)$ for scalar valued function $f(x)$;
div $g=\partial_1 g_1+\dots+\partial_n g_n$ for vector valued function $g(x)=(g_1(x),\dots,g_n(x))$;
$A=A(x)=(a_{ij}(x))$ is an $n\times n$ matrix;
$B=B(x)=\text{curl } b(x)=(\partial_i b_j(x)-\partial_j b_i(x))$ is an $n\times n$ matrix;
$(f)_-(x)=\max\{0,-f(x)\}\geq 0$ for a real valued function $f(x)$;
supp[f] denotes the closure of $\{x\mid f(x)\neq 0\}$;
$C^j(\Omega)$ denotes the class of j-times continuously differentiable functions;
$C_0^\infty(\Omega)=\{f(x)\mid$ for any $j=0,1,2,\dots$, $f\in C^j(\Omega)$ and supp[f] is a

compact set in $\Omega\}$;

$L^p(\Omega)=\{f(x)\mid\int_\Omega|f(x)|^p dx<\infty\}$ for $p\geq 1$;

$L^p_{loc}(\Omega)=\{f(x)\mid$ for any compact set $K\subset\Omega$, $\int_K|f(x)|^p dx<\infty\}$ for $p\geq 1$;

$H^m(\Omega)$ denotes the class of L^2-functions in Ω such that for all distribution derivatives up to m belongs to $L^2(\Omega)$;

$H^m_{loc}(\Omega)=\{f(x)\mid$ for any compact set $K\subset\Omega$, $f\in H^m(K)\}$;

$H^1_0(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ with respect to the norm $\left(\int_\Omega(|f|^2+|\mathrm{grad}\ f|^2)dx\right)^{1/2}$.

Next we shall state the conditions required in the theorems.

(A1) each $a_{ij}(x)\in C^2(\Omega)$ is a real-valued function;

(A2) $a_{ij}(x)=a_{ji}(x)$;

(A3) there exists a constant $C_1\geq 1$ such that for any $x\in\Omega$ and any $\xi\in\boldsymbol{C}^n$ we have

$$C_1^{-1}|\xi|^2\leq\langle A(x)\xi,\bar{\xi}\rangle\leq C_1|\xi|^2;$$

(A4) $a_{ij}(x)\to\delta_{ij}$ as $|x|\to\infty$;

(A5) $\partial_l a_{ij}(x)=o(r^{-1})$ as $|x|\to\infty$;

(A6) $\partial_m\partial_l a_{ij}(x)=o(r^{-1})$ as $|x|\to\infty$.

(B1) each $b_i(x)\in C^1(\Omega)$ is a real-valued function;

(B2) $|B(x)A(x)x|=O(1)$ as $|x|\to\infty$.

(C1) $q_1(x)$ is a real-valued function;

(C2) for any $w\in H^1_{loc}(\Omega)$ we have $q_1|w|^2\in L^1_{loc}(\Omega)$;

(C3) for any $w\in H^1_{loc}(\Omega)$ we have $|\mathrm{grad}\ q_1|\ |w|^2\in L^1_{loc}(\Omega)$;

(C4) there exist some real-valued function $\gamma(x)$ and some constant $0<\gamma_0<2$ such that

$$|\gamma(x)-\gamma_0|+r^{-1/2}|\partial_r\gamma(x)|+|\mathrm{grad}\ \gamma(x)-\hat{x}\partial_r\gamma(x)|=o(1)\ \text{as}\ r\to\infty,$$
$$\limsup_{r\to\infty}\{r\langle\hat{x},A(x)\mathrm{grad}\ q_1(x)\rangle+\gamma(x)q_1(x)\}<\infty.$$

(D1) $q_2(x)$ is a complex-valued function;

(D2) $q_2(x)=O(r^{-1/2})$ as $r\to\infty$.

(E) there exists some constant $R_0>0$ such that $\Omega\supset\{x\mid |x|>R_0\}$.

We shall consider

$$(*)\quad\begin{cases}-\langle D,ADu\rangle+(q_1+q_2)u=\lambda u\ \text{in}\ \Omega,\\ u\in H^2_{loc}(\Omega),\\ \mathrm{supp}[u]\ \text{is not a compact set in}\ \bar{\Omega}\ \text{(closure of}\ \Omega),\end{cases}$$

where λ is a real constant. Now we have

Theorem 1.1. *Let u satisfy (*), and let conditions (A), (B), (C), (D), (E) hold. Then for any $\mu>0$ satisfying*

$$(\#): \gamma_0(\mu^2+\lambda) > \limsup_{r\to\infty} \{r\langle\hat{x}, A(x)\operatorname{grad} q_1\rangle + \gamma(x) q_1(x) + (4\mu)^{-1} r |q_2(x)|^2 + (2-\gamma_0)^{-1} |B(x)A(x)x|^2\},$$

we have

$$\lim_{R\to\infty} e^{2\mu R}\int_{S(R)} [|\langle ADu, \hat{x}\rangle|^2 + (1+(q_1)_-)|u|^2] dS = \infty.$$

Theorem 1.2. *Besides the conditions assumed in Theorem 1.1, we assume that $q_1(x)$ satisfies the following:*

(*F*) *for any $\varepsilon>0$ there exists some $C_2>0$ such that for any $w\in H_0^1(\Omega)$ we have*

$$\int_\Omega (q_1)_- |w|^2 dx \le \varepsilon\int_\Omega |\operatorname{grad} w|^2 dx + C_2\int_\Omega |w|^2 dx.$$

Then we have for u and μ given in Theorem 1.1

$$\lim_{R\to\infty} e^{2\mu R}\int_{B(R,R+1)} |u|^2 dx = \infty,$$

and

$$\int_\Omega e^{2\mu r} |u|^2 dx = \infty.$$

Remark 1.1. Mochizuki [7] treated the case $\gamma(x) = \gamma_0\langle A\hat{x}, \hat{x}\rangle$.

Now we shall consider a more special case:

$$(**): \begin{cases} -\langle D, Du\rangle + (q_1+q_2)u = \lambda u \text{ in } \Omega, \\ u\in H^2_{loc}(\Omega), \\ \operatorname{supp}[u] \text{ is not a compact set in } \bar{\Omega}, \end{cases}$$

where λ is a real constant. Here we can weaken (C3) as follows:

Theorem 1.3. *Let u satisfy (**). We assume conditions (B), (C), (D), (E) with $a_{ij}(x) = \delta_{ij}$ except for (C3). Instead of (C3) we assume*

$$(C3)': \text{for any } w\in H^1_{loc}(\Omega) \text{ we have } (\partial_r q_1)|w|^2 \in L^1_{loc}(\Omega).$$

Then for any $\mu>0$ satisfying

$$(\#\#): \gamma_0(\mu^2+\lambda) > \limsup_{r\to\infty} \{r\partial_r q_1(x) + \gamma(x) q_1(x)$$

$$+(4\mu)^{-1}r\,|q_2(x)|^2+(2-\gamma_0)^{-1}|B(x)x|^2\}$$

we have

$$\lim_{R\to\infty} e^{2\mu R}\int_{S(R)}[\,|\langle Du,\hat{x}\rangle|^2+(1+(q_1)_-)\,|u|^2]dS=\infty.$$

Theorem 1.4. *Besides the conditions given in Theorem* 1.3, *we assume* (F), *which is given in Theorem* 1.2. *Then for* u *and* μ *given in Theorem* 1.3 *we have*

$$\lim_{R\to\infty} e^{2\mu R}\int_{B(R,R+1)}|u|^2dx=\infty,$$

and

$$\int_{\Omega} e^{2\mu r}|u|^2dx=\infty.$$

As a special case of Theorem 1.4, we have

Theorem 1.5. *We assume that* $q_1(x)$ *satisfies* $(C1)$, $(C2)$, $(C3)'$, (F) *and* $\limsup_{r\to\infty}(r\partial_r q_1+\gamma_0 q_1)\leq 0$ *for some constant* $0<\gamma_0<2$, $q_2(x)$ *is a complex-valued function satisfying* $q_2(x)=o(r^{-1/2})$ *as* $r\to\infty$, $b_i(x)\in C^1(\Omega)$ *is a real-valued function* $(1\leq i\leq n)$ *satisfying* $\partial_i b_j(x)-\partial_j b_i(x)=o(r^{-1})$ *as* $r\to\infty$ *and* (E) *holds. Then for* u *satisfying* (**) *and for any* $\varepsilon>0$ *we have*

$$\lim_{R\to\infty} e^{2(1+\varepsilon)\sqrt{|\lambda|}R}\int_{B(R,R+1)}|u|^2dx=\infty,\quad if\ \lambda<0,$$

$$\lim_{R\to\infty} e^{\varepsilon R}\int_{B(R,R+1)}|u|^2dx=\infty,\qquad if\ \lambda\geq 0.$$

Lastly we shall consider the most special case:

$$(***):\begin{cases}-\Delta u+(q_1+q_2)u=\lambda u \text{ in } \Omega,\\ u\in H^2_{loc}(\Omega),\\ \text{supp}[u] \text{ is not a compact set in } \bar{\Omega},\end{cases}$$

where λ is a real constant and Δ is a Laplacian in $\boldsymbol{R}^n$. Now we can also weaken (C3) and (C4) as follows:

Theorem 1.6. *Let* u *satisfy* (***). *We assume conditions* (C), (D), (E) *except for* $(C3)$ *and* $(C4)$. *Instead of* $(C3)$ *and* $(C4)$ *we assume* $(C3)'$ *and*

(C4)′: there exist some real-valued function $\gamma(x)$ *and some constants* $0<\gamma_0\leq 2$, $0<\alpha<2$ *such that*

$$\gamma(x)+|\mathrm{grad}\ \gamma(x)-\hat{x}\partial_r\gamma(x)|^{\alpha}\leq 2 \ \textit{for}\ r>R_0,$$

$$|\gamma(x)-\gamma_0|+r^{-1/2}|\partial_r\gamma(x)|+|\mathrm{grad}\ \gamma(x)-\hat{x}\partial_r\gamma(x)|=o(1) \ \textit{as}\ r\to\infty,$$

$$\lim_{r\to\infty}\sup\{r\partial_r q_1(x)+\gamma(x)q_1(x)\}<\infty.$$

Then for any $\mu>0$ *satisfying*

$$(\#\#\#):\ \gamma_0(\mu^2+\lambda)>\lim_{r\to\infty}\sup\{r\partial_r q_1(x)+\gamma(x)q_1(x)+(4\mu)^{-1}r|q_2(x)|^2\},$$

we have

$$\lim_{R\to\infty} e^{2\mu R}\int_{S(R)}[|\partial_r u|^2+(1+(q_1)_-)|u|^2]dS=\infty.$$

Theorem 1.7. *Besides the conditions given in Theorem* 1.6, *we assume (F). Then for u and* μ *given in Theorem* 1.6 *we have*

$$\lim_{R\to\infty} e^{2\mu R}\int_{B(R,R+1)}|u|^2dx=\infty$$

and

$$\int_{\Omega} e^{2\mu r}|u|^2dx=\infty.$$

Remark 1.2. If $0<\gamma_0<2$ and $a_{ij}(x)=\delta_{ij}$, (C4)′ is the same condition as (C4).

§2. Proof of Theorem 1.1

We shall begin to give the following definition of which meaning is shown in Lemma 2.1. In §2 we assume (A), (B), (C), (D) and (E).

Definition 2.1. Let $u(x)$ satisfy (*). For smooth real-valued functions $\rho=\rho(r)$, $f=f(r)$ and $g=g(x)$, let

$$v(x)=e^{\rho(r)}u(x),$$
$$k_1(x)=-\{\rho'(r)\}^2\langle A(x)\hat{x},\hat{x}\rangle,$$
$$k_2(x)=\rho''(r)\langle A(x)\hat{x},\hat{x}\rangle+\rho'(r)\ \mathrm{div}\{A(x)\hat{x}\}$$

and

$$F(t;\rho,f,g)=\int_{S(t)}[f\{2|\langle\hat{x},ADv\rangle|^2-\langle\hat{x},A\hat{x}\rangle(\langle Dv,A\overline{Dv}\rangle + (q_1-\lambda+k_1)|v|^2)\}+g\ \mathrm{Re}[\langle\hat{x},A\overline{Dv}\rangle v]]dS,$$

where Re[w] means the real part of w.

Lemma 2.1. *For $t>s>R_0$ we have*

$$\begin{aligned}&F(t;\rho,f,g)-F(s;\rho,f,g)\\&=\int_{B(s,t)}[2(2\rho'f+f'-r^{-1}f)\ |\langle ADv,\hat{x}\rangle|^2+(2r^{-1}f+g\\&-f\,\mathrm{div}(A\hat{x})-\langle\hat{x},A\hat{x}\rangle f')\langle ADv,\overline{Dv}\rangle+2r^{-1}f(|ADv|^2\\&-\langle ADv,\overline{Dv}\rangle)+2f\ \mathrm{Re}[\langle\hat{x},(\langle ADv,\mathrm{grad}\rangle A)\overline{Dv}\rangle]\\&-f\langle(\langle\hat{x},A\ \mathrm{grad}\rangle A)Dv,\overline{Dv}\rangle+2\mathrm{Re}[(f(q_2+k_2)+g\rho'\\&\quad+2^{-1}\partial_r g)<\hat{x},A\overline{Dv}>v]\\&-2f\ \mathrm{Im}[\langle BA\hat{x},A\overline{Dv}\rangle v]+\mathrm{Re}[<(\mathrm{grad}\ g-\hat{x}\partial_r g),A\overline{Dv}\rangle v]\\&+\{(q_1-\lambda+k_1)(g-f\ \mathrm{div}(A\hat{x})-f'\langle\hat{x},A\hat{x}\rangle)\\&-f\langle\hat{x},A\ \mathrm{grad}(q_1+k_1)\rangle+g(\mathrm{Re}[q_2]+k_2)\}\ |v|^2]dx,\end{aligned}$$

where Im[w] *means the imaginary part of* w.

Proof. By Definition 2.1 $v(x)\in H^2_{loc}(\Omega)$ and $v(x)$ satisfies for $r>R_0$

$$-\langle D,ADv\rangle+2\rho'\langle\hat{x},ADv\rangle+\{(q_1-\lambda+k_1)+(q_2+k_2)\}v=0.$$

Then we have

$$\mathrm{Re}\int_{B(s,t)}[-\langle D,ADv\rangle+2\rho'\langle\hat{x},ADv\rangle+\{(q_1-\lambda+k_1)+(q_2+k_2)\}v]\times[2f(r)\langle\hat{x},A\overline{Dv}\rangle+g(x)\bar{v}]dx=0.$$

In the left side of the above equation we use the integration by parts such as

$$\int_{B(s,t)}\partial_i w\ dx=\int_{S(t)}\hat{x}_i w\ dS-\int_{S(s)}\hat{x}_i w\ dS\qquad \text{for } w,\ \partial_i w\in L^1_{loc}(\Omega)$$

(Cf. Lemma 4.1). Noting

$$D_iD_j-D_jD_i=\sqrt{-1}(\partial_i b_j(x)-\partial_j b_i(x)),$$
$$2\ \mathrm{Re}[\langle D_iDv,A\overline{Dv}\rangle]=\partial_i(\langle ADv,\overline{Dv}\rangle)-\langle(\partial_i A)Dv,\overline{Dv}\rangle,$$

we have the assertion. □

Lemma 2.2. *Let $1-2^{-1}\gamma_0<l<1$. Then for any $\mu>0$ satisfying* (#)

there exists some $R_1 \geq R_0$ *such that for any* $t > s > R_1$ *and any* $m \geq 1$, *we have*

$$F(t; \mu r + mr^l, 1, \operatorname{div}(A\hat{x}) - \gamma(x) r^{-1})$$
$$\geq F(s; \mu r + mr^l, 1, \operatorname{div}(A\hat{x}) - \gamma(x) r^{-1}).$$

Proof. Let $\mu > 0$ satisfy (♯). By (B2) we can find some $0 < \delta < 2 - \gamma_0$ such that

$$\gamma_0(\mu^2 + \lambda) > \limsup_{r \to \infty} \{r \langle \hat{x}, A \operatorname{grad} q_1 \rangle + \gamma q_1 + (4\mu)^{-1} r |q_2|^2 + \delta^{-1} |BAx|^2\}.$$

Let

$$\rho(r) = \mu r + mr^l,\ f(r) = 1 \text{ and } g(x) = \operatorname{div}(A(x)\hat{x}) - \gamma(x) r^{-1},$$

where $m \geq 1$ and $1 - 2^{-1}\gamma_0 < l < 1$ are constants. Let each $\varepsilon_i(r)$ $(i = 1, 2, \dots)$ be a positive function for $r > 0$ which tends to 0 as $r \to \infty$. Noting

$$|\langle A\hat{x}, \hat{x} \rangle - 1| + r |\langle \hat{x}, A \operatorname{grad}(\langle A\hat{x}, \hat{x} \rangle) \rangle| + r^{1/2} |\partial_r g| + r |\operatorname{grad} g - \hat{x} \partial_r g| \leq \varepsilon_1(r),$$
$$\operatorname{div}(A\hat{x}) = O(r^{-1}) \quad \text{as } r \to \infty,$$

we have the following by direct calculation.

$$2(2\rho' f + f' - r^{-1} f) |\langle ADv, \hat{x} \rangle|^2 = r^{-1}(4\mu r + 4mlr^l - 2) |\langle ADv, \hat{x} \rangle|^2,$$
$$(2r^{-1} f + g - f \operatorname{div}(A\hat{x}) - \langle \hat{x}, A\hat{x} \rangle f') \langle ADv, \overline{Dv} \rangle$$
$$\geq r^{-1}(2 - \gamma_0 - \varepsilon_2(r)) \langle ADv, \overline{Dv} \rangle,$$
$$2r^{-1} f(|ADv|^2 - \langle ADv, \overline{Dv} \rangle) \geq -\varepsilon_3(r) r^{-1} \langle ADv, \overline{Dv} \rangle,$$
$$2f \operatorname{Re}[\langle \hat{x}, (\langle ADv, \operatorname{grad} \rangle A) \overline{Dv} \rangle] \geq -\varepsilon_4(r) r^{-1} \langle ADv, \overline{Dv} \rangle,$$
$$-f \langle (\langle \hat{x}, A \operatorname{grad} \rangle A) Dv, \overline{Dv} \rangle \geq -\varepsilon_5(r) r^{-1} \langle ADv, \overline{Dv} \rangle,$$
$$2 \operatorname{Re}[\{f(q_2 + k_2) + g\rho' + 2^{-1} \partial_r g\} \langle \hat{x}, A\overline{Dv} \rangle v]$$
$$\geq -r^{-1}(4\mu r + mlr^l) |\langle \hat{x}, ADv \rangle|^2 - r^{-1}\{(4\mu)^{-1} r |q_2|^2 + mlr^{l-1}\varepsilon_6(r) + \varepsilon_7(r)\} |v|^2,$$
$$-2f \operatorname{Im}[\langle BA\hat{x}, A\overline{Dv} \rangle v] \geq -\delta r^{-1} \langle ADv, \overline{Dv} \rangle - r^{-1}(\delta^{-1} |BAx|^2 + \varepsilon_8(r)) |v|^2,$$
$$\operatorname{Re}[\langle (\operatorname{grad} g - \hat{x} \partial_r g), A\overline{Dv} \rangle v] \geq -\varepsilon_9(r) r^{-1}(\langle ADv, \overline{Dv} \rangle + |v|^2),$$
$$(q_1 - \lambda + k_1)(g - f \operatorname{div}(A\hat{x}) - f' \langle \hat{x}, A\hat{x} \rangle) - f \langle \hat{x}, A \operatorname{grad}(q_1 + k_1) \rangle$$
$$= r^{-1}(\gamma \rho'^2 \langle A\hat{x}, \hat{x} \rangle + 2r\rho' \rho'' \langle A\hat{x}, \hat{x} \rangle^2 + r\rho'^2 \langle \hat{x}, A \operatorname{grad}(\langle A\hat{x}, \hat{x} \rangle) \rangle + \gamma\lambda - r \langle \hat{x}, A \operatorname{grad} q_1 \rangle - \gamma q_1)$$
$$\geq r^{-1}[\{\gamma_0(\mu^2 + \lambda) - (r \langle \hat{x}, A \operatorname{grad} q_1 \rangle + \gamma q_1) - \varepsilon_{10}(r)\}$$

$$+m^2l^2(2l+\gamma_0-2-\varepsilon_{11}(r))r^{2l-2}$$
$$+2\mu ml(l+\gamma_0-1-\varepsilon_{12}(r))r^{l-1}],$$
$$g(\mathrm{Re}[q_2]+k_2)\geq -\varepsilon_{13}(r)r^{-1}(1+2\mu mlr^{l-1}).$$

Then by Lemma 2. 1, we have

$$\begin{aligned} &F(t\,;\ \mu r+mr^l,\ 1,\ \mathrm{div}(A\hat{x})-\gamma(x)r^{-1}) \\ &\quad -F(s\,;\ \mu r+mr^l,\ 1,\ \mathrm{div}(A\hat{x})-\gamma(x)r^{-1}) \\ &\quad \geq\int_{B(s,t)} r^{-1}[(3mlr^l-2)\ |\langle ADv, \hat{x}\rangle|^2 \\ &\quad +(2-\gamma_0-\delta-\varepsilon_{14}(r))\langle ADv, \overline{Dv}\rangle \\ &\quad +\{\gamma_0(\mu^2+\lambda)-(r\langle\hat{x}, A\ \mathrm{grad}\ q_1\rangle \\ &\quad +\gamma(x)q_1+(4\mu)^{-1}r\,|q_2|^2+\delta^{-1}|BAx|^2)-\varepsilon_{14}(r)\}\ |v|^2 \\ &\quad +m^2l^2(2l+\gamma_0-2-\varepsilon_{14}(r))r^{2l-2}|v|^2 \\ &\quad +2\mu ml(l+\gamma_0-1-\varepsilon_{14}(r))r^{l-1}|v|^2]dx. \end{aligned}$$

Noting

$$l+\gamma_0-1=2^{-1}(2l+\gamma_0-2)+2^{-1}\gamma_0>0,$$

there exists some $R_1\geq R_0$ such that for any $r\geq R_1$ and $m\geq 1$, we have

$$\begin{aligned} &3mlR_1^l-2\geq 0, \\ &2-\gamma_0-\delta-\varepsilon_{14}(r)\geq 0, \\ &\gamma_0(\mu^2+\lambda)-(r\langle\hat{x}, A\ \mathrm{grad}\ q_1\rangle+\gamma(x)q_1+(4\mu)^{-1}r\,|q_2|^2 \\ &\qquad +\delta^{-1}|BAx|^2)-\varepsilon_{14}(r)\geq 0, \\ &2l+\gamma_0-2-\varepsilon_{14}(r)\geq 0, \\ &l+\gamma_0-1-\varepsilon_{14}(r)\geq 0. \end{aligned}$$

Therefore we have the assertion. □

Proof of Theorem 1. 1. If $\limsup_{r\to\infty}\{r\langle\hat{x}, A\ \mathrm{grad}\ q_1\rangle+\gamma q_1\}>-\infty$, then by (B2) and (D2)

$$\gamma_0(\mu^2+\lambda)-\limsup_{r\to\infty}\{r\langle\hat{x}, A\ \mathrm{grad}\ q_1\rangle+\gamma q_1+(4\mu)^{-1}r\,|q_2|^2 + (2-\gamma_0)^{-1}|BAx|^2\}$$

is a continuous function of $\mu>0$. Therefore for any $\mu>0$ satisfying (#), we can find μ' such that $0<\mu'<\mu$ and μ' satisfies (#) also. This conclusion also holds in case $\limsup_{r\to\infty}\{r\langle\hat{x}, A\ \mathrm{grad}\ q_1\rangle+\gamma q_1\}=-\infty$. Noting the fact mentioned above, we have only to show that for any $\mu'>0$ satisfying (#) we have

$$\liminf_{R\to\infty} e^{2\mu' R}\int_{S(R)}[\,|\langle ADu, \hat{x}\rangle|^2+(1+(q_1)_-)\ |u|^2]dS>0.$$

We shall prove above by contradiction. So we assume that this is not true. Then there exists some $\mu_0>0$ satisfying (#) such that

$$\liminf_{R\to\infty} e^{2\mu_0 R}\int_{S(R)}[|\langle ADu, \hat{x}\rangle|^2+(1+(q_1)_-|u|^2]dS=0.$$

By the remarks given at the beginning of this proof, we can find $0<\mu<\mu_0$ satisfying (#). By Definition 2.1, for given $m\geq 1$ and $1-2^{-1}\gamma_0<l<1$ there exists some constant $C_3>0$ such that for any $t>R_0$ we have

$$\begin{aligned}
&F(t;\ \mu r+mr^l,\ 1,\ \mathrm{div}(A\hat{x})-\gamma r^{-1})\\
&\quad=e^{2\rho(t)}\int_{S(t)}[2|\langle\hat{x}, ADu\rangle|^2+\langle\hat{x}, A\hat{x}\rangle\{-\langle Du, A\overline{Du}\rangle+(\lambda-q_1)|u|^2\\
&\quad+(\mathrm{div}(A\hat{x})-\gamma r^{-1})\rho'|u|^2\}+2\rho'^2\langle\hat{x}, A\hat{x}\rangle^2|u|^2\\
&\quad+(2\rho'\langle\hat{x}, A\hat{x}\rangle+\mathrm{div}(A\hat{x})-\gamma r^{-1})\mathrm{Re}[\langle\hat{x}, A\overline{Du}\rangle u]]dS\\
&\quad\leq C_3e^{2\mu t+2mt^l}\int_{S(t)}[|\langle\hat{x}, ADu\rangle|^2+(1+(q_1)_-)|u|^2]dS,
\end{aligned}$$

where $\rho=\mu r+mr^l$. Then noting $l<1$ and $\mu<\mu_0$ we have

$$\liminf_{t\to\infty} F(t;\ \mu r+mr^l,\ 1,\ \mathrm{div}(A\hat{x})-\gamma r^{-1})\leq 0.$$

So letting $t\to\infty$ along suitable sequence in Lemma 2.2, we have for any $s\geq R_1$ and any $m\geq 1$

$$F(s;\ \mu r+mr^l,\ 1,\ \mathrm{div}(A\hat{x})-\gamma r^{-1})\leq 0.$$

On the other hand, since supp$[u]$ is not a compact set in $\bar{\Omega}$, there exists some $R_2\geq R_1$ such that

$$\int_{S(R_2)}|\langle\hat{x}, A\hat{x}\rangle|^2|u|^2dS>0.$$

Since $e^{-2\mu R_2-2mR_2^l}F(R_2;\ \mu r+mr^l,\ 1,\ \mathrm{div}(A\hat{x})-\gamma r^{-1})$ is a quadratic in m of which coefficient of m^2 is $2l^2R_2^{2(l-1)}\int_{S(R_2)}|\langle\hat{x}, A\hat{x}\rangle|^2|u|^2\ dS>0$, then there exists some $m_0\geq 1$ such that

$$F(R_2;\ \mu r+m_0r^l,\ 1,\ \mathrm{div}(A\hat{x})-\gamma r^{-1})>0,$$

which leads to the contradiction. □

§3. Proof of Theorem 1.2

Lemma 3.1. *Let $\Omega\supset\{x|\ |x|>R_0\}$ be a domain in $\mathbf{R}^n$, and each $b_i(x)$ $(i=1,\dots,n)$ be real-valued function on Ω. Let $V(x)$ satisfy the following:*

for any $\varepsilon>0$ there exists some $C_4>0$ such that for any $w\in H_0^1(\Omega)$

$$\int_\Omega |V(x)||w(x)|^2dx \le \varepsilon\int_\Omega |\mathrm{grad}\ w|^2dx + C_4\int_\Omega |w(x)|^2dx.$$

Let $\zeta(t)$ be a smooth function on $(-\infty,\infty)$ satisfying $0\le\zeta(t)\le 1$ for $-\infty<t<\infty$, $\zeta(t)=1$ for $1/3\le t\le 2/3$ and $\mathrm{supp}[\zeta]\subset(0,1)$. *And let $\zeta_R(t)=\zeta(t-R)$. Then for any $\varepsilon>0$ there exists some constant $C_5>0$ such that for any $w\in H_{loc}^1(\Omega)$ and any $R>R_0$, we have*

$$\int_\Omega \zeta_R^2(|x|)\,|V(x)||w(x)|^2dx \le \varepsilon\int_\Omega \zeta_R^2|Dw|^2dx + C_5\int_{B(R,R+1)}|w|^2dx.$$

Proof. Let $w^{(\eta)}(x)=\{|w(x)|^2+\eta^2\}^{1/2}$ for $\eta>0$ and $w\in H_{loc}^1(\Omega)$. Then $w^{(\eta)}\in H_{loc}^1(\Omega)$ and $|\mathrm{grad}\ w^{(\eta)}|\le|Dw|$, because of $w^{(\eta)}$ grad $w^{(\eta)}=\mathrm{Re}[\bar{w}\ \mathrm{grad}\ w]=\mathrm{Re}[\bar{w}Dw]$ and $|w|\le|w^{(\eta)}|$. Then by $\zeta_R w^{(\eta)}\in H_0^1(\Omega)$ for $R>R_0$, we have

$$\int_\Omega \zeta_R^2|v||w^{(\eta)}|^2dx \le \varepsilon\int_\Omega |\mathrm{grad}(\zeta_R w^{(\eta)})|^2dx + C_4\int_\Omega |\zeta_R w^{(\eta)}|^2dx$$

$$\le 2\varepsilon\int_\Omega \zeta_R^2|Dw|^2dx + (C_4+2\varepsilon\max_{0\le t\le 1}(\zeta'(t))^2)\int_{B(R,R+1)}|w^{(\eta)}|^2dx.$$

Letting $\eta\to 0$, we have the assertion. (This proof is the same one given by Eastham-Kalf [3, p. 249].) □

Lemma 3.2. *Let $\Omega\supset\{x\,|\,|x|>R_0\}$ be a domain in $\boldsymbol{R}^n$, and let $\zeta_R(t)$ be the same as given in Lemma 3.1. Let $A(x)=(a_{ij}(x))$, $b_i(x)$ $(i=1,\dots,n)$ and $V(x)$ satisfy the following:*

(*1*): *there exists some constant $C_6\ge 1$ such that for any $x\in\Omega$ and any $\xi\in\boldsymbol{C}^n$*

$$C_6^{-1}|\xi|^2\le\langle A(x)\xi,\bar{\xi}\rangle\le C_6|\xi|^2,$$

(*2*): $\mathrm{div}(A\hat{x})$ *is bounded on Ω,*

(*3*): *each $b_i(x)\in C^1(\Omega)$ is a real-valued function,*

(*4*): *$V(x)$ is a complex-valued function on Ω,*

(*5*): *for any $\varepsilon>0$ there exists some $C_7>0$ such that for any $w\in H_0^1(\Omega)$*

$$\int_\Omega (\mathrm{Re}V)_-(x)\,|w(x)|^2dx \le \varepsilon\int_\Omega |\mathrm{grad}\ w|^2dx + C_7\int_\Omega |w|^2dx.$$

Let $u\in H_{loc}^2(\Omega)$ satisfy

$$-\langle D,ADu\rangle+Vu=0\ \ in\ \ \Omega.$$

Then there exists some constant $C_8>0$ such that for any $R>R_0$ we have

$$\int_{\Omega}\zeta_R^2\{|Du|^2+(\mathrm{Re}\ V)_-|u|^2\}dx\leq C_8\int_{B(R,R+1)}|u|^2dx.$$

Proof. By Lemma 3. 1, for any $\varepsilon>0$ there exists some $C_5>0$ such that

$$\int_{\Omega}\zeta_R^2(\mathrm{Re}V)_-|u|^2dx\leq\varepsilon\int_{\Omega}\zeta_R^2|Du|^2dx+C_5\int_{B(R,R+1)}|u|^2dx.$$

So we have only to show that there exists some constant $C_9>0$ such that for any $R>R_0$

$$\int_{\Omega}\zeta_R^2|Du|^2dx\leq C_9\int_{B(R,R+1)}|u|^2dx.$$

By integration by parts we have

$$\begin{aligned}
0&=\mathrm{Re}\int_{\Omega}\zeta_R^2[-\langle D,ADu\rangle+Vu]\bar{u}\ dx\\
&=\int_{\Omega}[\zeta_R^2\{\langle ADu,\overline{Du}\rangle+(\mathrm{Re}V)\ |u|^2\}-\{\zeta_R\zeta_R'\ \mathrm{div}(A\hat{x})\\
&\quad+(\zeta_R\zeta_R')'\langle\hat{x},A\hat{x}\rangle\}\ |u|^2]dx\\
&\geq\int_{\Omega}\zeta_R^2[C_6^{-1}|Du|^2-(\mathrm{Re}V)_-|u|^2]dx-C_{10}\int_{B(R,R+1)}|u|^2dx\\
&\geq(C_6^{-1}-\varepsilon)\int_{\Omega}\zeta_R^2|Du|^2dx-(C_5+C_{10})\int_{B(R,R+1)}|u|^2dx,
\end{aligned}$$

where we also use Lemma 3. 1. So if we choose ε to satisfy $0<\varepsilon<C_6^{-1}$, we have the assertion. □

Now we prove Theorem 1. 2.

Proof of Theorem 1. 2. By Theorem 1. 1, for any $\mu>0$ satisfying (#) we have

$$\lim_{R\to\infty}e^{2\mu R}\int_{S(R)}[|\langle ADu,\hat{x}\rangle|^2+(1+(q_1)_-)\ |u|^2]dS=\infty.$$

Fix $\mu>0$ satisfying (#). Then for any $M>0$ there exists $R_3>R_0$ such that for any $t>R_3$

$$\int_{S(t)}[|\langle ADu,\hat{x}\rangle|^2+(1+(q_1)_-)\ |u|^2]dS\geq Me^{-2\mu t}.$$

So we have for $R>R_3$

$$M\int_0^1\zeta^2(t)dt\circ e^{-2\mu(R+1)}\leq M\int_R^{R+1}\zeta_R^2(t)e^{-2\mu t}dt$$

$$\leq\int_{B(R,R+1)}\zeta_R^2[|\langle ADu, \hat{x}\rangle|^2+(1+(q_1)_-)|u|^2]dx$$

$$\leq C_{11}\Big(\int_{\Omega}\zeta_R^2\{|Du|^2+(q_1)_-|u|^2\}dx+\int_{B(R,R+1)}|u|^2dx\Big)$$

$$\leq C_{12}\int_{B(R,R+1)}|u|^2dx,$$

where we use Lemma 3.2 with $V(x)=q_1+q_2-\lambda$, which satisfies the condition given in Lemma 3.2. This means the former half of the statement of Theorem 1.2 holds. The latter half is easily obtained from

$$\int_{\Omega}e^{2\mu r}|u|^2dx\geq e^{2\mu R}\int_{B(R,R+1)}|u|^2dx \text{ for } R>R_0. \qquad \square$$

§4. Proofs of Theorems 1.3~1.7

If $a_{ij}(x)=\delta_{ij}$ in Ω, we can weaken (C3) as (C3)′. (The author [10] and Mochizuki [7] neglected this condition, but Kalf [6] pointed out this in application of the Gauss's theorem (integration by parts).) We prepare the following:

Lemma 4.1. *Let* $0<a<b$. *We assume* $w\in L^1(B(a,b))$ *and* $\partial_r w\in L^1(B(a,b))$. *Then for any* $a<t<b$, $\int_{S(t)}w\,dS$ *exists (by choosing suitable representative for* w, *if necessary) and for any* $a<s<t<b$

$$\int_{B(s,t)}\partial_r w\,dx=\int_{S(t)}w\,dS-\int_{S(s)}w\,dS-\int_{B(s,t)}(n-1)r^{-1}w\,dx.$$

Proof. Let $j(\sigma)\in C_0^\infty(-\infty,\infty)$ satisfy the following: $j(\sigma)=0$ for $|\sigma|\geq 1$, $j(\sigma)\geq 0$ for $\sigma\in(-\infty,\infty)$ and $\int_{-\infty}^{\infty}j(\sigma)d\sigma=1$. Take a', b' arbitrarily to satisfy $a<a'<b'<b$, and let

$$h(r)=\int_{S(r)}w(x)dS \qquad \text{for } a<r<b,$$

$$h_\varepsilon(\rho)=\varepsilon^{-1}\int_{B(a,b)}j(\varepsilon^{-1}(\rho-|x|))w(x)dx \qquad \text{for } a'<\rho<b',$$

where $0<\varepsilon<\min\{a'-a,b-b'\}$. Since $h(r)\in L^1(a,b)$, for any $\eta>0$ there exists some $0<\delta<\min\{a'-a,b-b'\}$ such that for any ξ satisfying $|\xi|<\delta$ we have

$$\int_{a'}^{b'} |h(r)-h(r+\xi)|dr<\eta.$$

We have for $a'<\rho<b'$

$$h_\varepsilon(\rho)=\int_{\rho-\varepsilon}^{\rho+\varepsilon}\varepsilon^{-1}j(\varepsilon^{-1}(\rho-r))h(r)dr=\int_{-1}^{1}j(\sigma)h(\rho-\varepsilon\sigma)d\sigma.$$

Then for any $\eta>0$ there exists some $0<\delta<\min\{a'-a, b-b'\}$ such that for any $0<\varepsilon<\delta$

$$\begin{aligned}\int_{a'}^{b'}|h_\varepsilon(\rho)-h(\rho)|d\rho&\leq\int_{a'}^{b'}d\rho\int_{-1}^{1}j(\sigma)|h(\rho-\varepsilon\sigma)-h(\rho)|d\sigma\\&=\int_{-1}^{1}j(\sigma)d\sigma\int_{a'}^{b'}|h(\rho-\varepsilon\sigma)-h(\rho)|d\rho<\eta.\end{aligned}$$

Next we have

$$\begin{aligned}\frac{d}{d\rho}h_\varepsilon(\rho)&=\varepsilon^{-1}\int_{B(a,b)}\partial_\rho\{j(\varepsilon^{-1}(\rho-|x|))\}w(x)dx\\&=-\varepsilon^{-1}\int_{B(a,b)}\partial_r\{j(\varepsilon^{-1}(\rho-r))\}w(x)dx\\&=\varepsilon^{-1}\int_{B(a,b)}j(\varepsilon^{-1}(\rho-r))(\partial_r w+(n-1)r^{-1}w)dx,\end{aligned}$$

where we use the definition of distribution derivative $\partial_r w$. So by $\partial_r w+(n-1)r^{-1}w\in L^1(B(a,b))$ we have similarly

$$\int_{a'}^{b'}|\frac{d}{d\rho}h_\varepsilon(\rho)-\int_{S(\rho)}(\partial_r w+(\mathrm{n}-1)r^{-1}w)dS|d\rho\to0 \text{ as } \varepsilon\to0.$$

By $h_\varepsilon(\rho)\in C^1[a',b']$, for any b'' satisfying $a'<b''<b'$ there exists some $C_{13}>0$ such that for any ρ satisfying $a'<\rho<b''$ we have

$$\begin{aligned}|h_\varepsilon(\rho)-h_{\varepsilon'}(\rho)|&=(b'-\rho)^{-1}|\int_\rho^{b'}\frac{d}{d\sigma}\{(b'-\sigma)(h_\varepsilon(\sigma)-h_{\varepsilon'}(\sigma)\}d\sigma|\\&\leq C_{13}\int_{a'}^{b'}\{|\frac{d}{d\sigma}h_\varepsilon(\sigma)-\frac{d}{d\sigma}h_{\varepsilon'}(\sigma)|+|h_\varepsilon(\sigma)-h_{\varepsilon'}(\sigma)|\}d\sigma\to0\end{aligned}$$

as $\varepsilon, \varepsilon'\to0$. Because b'' is arbitrary and $\int_{a'}^{b'}|h_\varepsilon(\rho)-h(\rho)|d\rho\to0$ as $\varepsilon\to0$, we have $\lim_{\varepsilon\to0}h_\varepsilon(\rho)=h(\rho)$ for any $\rho\in(a',b')$, where suitable representative of w is chosen, if necessary. Since for any s, t satisfying $a'<s<t<b'$

$$\int_s^t\frac{d}{d\rho}h_\varepsilon(\rho)d\rho=h_\varepsilon(t)-h_\varepsilon(s),$$

we have the assertion for $a'<s<t<b'$ by letting $\varepsilon\to0$. Because a'

and b' are arbitrary, we have the assertion. □

Now we can show the proofs of Theorem 1.3∼1.7.

Proof of Theorem 1.3. Under our weak condition (C3)′, Lemma 2.1 is also true by replacing $a_{ij}(x)$ with δ_{ij}. In fact in the proof of Lemma 2.1 the term related to $q_1(x)$ is

$$\int_{B(s,t)} (2fq_1 \ \mathrm{Re}[\langle \hat{x}, \overline{Dv} \rangle v] + gq_1 |v|^2)\,dx$$
$$= \int_{B(s,t)} (fq_1 \partial_r(|v|^2) + gq_1 |v|^2)\,dx,$$

which, noting Lemma 4.1, can be integrated by parts. Then Lemma 2.2 and the proof of Theorem 1.1 is also true, if we replace $a_{ij}(x)$ with δ_{ij}. □

Proof of Theorem 1.4. Noting Theorem 1.3 we have the assertion by the same way as the proof of Theorem 1.2. □

Proof of Theorem 1.5. For any $\varepsilon > 0$ let

$$\mu = \begin{cases} (1+\varepsilon)\sqrt{|\lambda|}, & \text{if } \lambda < 0, \\ \varepsilon/2, & \text{if } \lambda \geq 0. \end{cases}$$

Then this $\mu > 0$ satisfies (##). So by Theorem 1.4 we have the assertion. □

Proof of Theorem 1.6. Replacing $a_{ij}(x)$ with δ_{ij} and $b_i(x)$ with 0, Lemma 2.1 is also true. Then we have

$$\begin{aligned} &F(t\,; \rho, f, g) - F(s\,; \rho, f, g) \\ &= \int_{B(s,t)} [2(2\rho' f + f' - r^{-1} f)\, |\partial_r v|^2 + (2r^{-1} f + g - f \ \mathrm{div}(\hat{x}) \\ &\quad - f')\, |\mathrm{grad}\ v|^2 + 2\mathrm{Re}[(f(q_2 + k_2) + g\rho' + 2^{-1}\partial_r g)\,(\overline{\partial_r v})\, v] \\ &\quad + \mathrm{Re}[\langle (\mathrm{grad}\ g - \hat{x}\partial_r g), \overline{\mathrm{grad}\ v} \rangle v] + \{(q_1 - \lambda + k_1)\,(g - f\ \mathrm{div}(\hat{x}) - f') \\ &\quad - f\partial_r(q_1 + k_1) + g(\mathrm{Re}[q_2] + k_2)\}\, |v|^2]\,dx. \end{aligned}$$

In the above let

$$f(r) = 1,\ g(x) = f\,\mathrm{div}(\hat{x}) + f' - \gamma(x) r^{-1} = (n - 1 - \gamma(x)) r^{-1},$$
$$\rho(r) = \mu r + m r^l.$$

Then we have

$$\begin{aligned}&(2r^{-1}f+g-f\ \mathrm{div}(\hat{x})-f')\,|\mathrm{grad} v|^2\\&\qquad+\mathrm{Re}[\langle(\mathrm{grad}\ g-\hat{x}\partial_r g),\ \overline{\mathrm{grad}\ v}\rangle v]\\&\geq r^{-1}(2-\gamma(x)-|\mathrm{grad}\ \gamma(x)-\hat{x}\partial_r\gamma(x)|^{\alpha})\,|\mathrm{grad}\ v|^2\\&\qquad-(4r)^{-1}|\mathrm{grad}\ \gamma(x)-\hat{x}\partial_r\gamma(x)|^{2-\alpha}|v|^2\\&\geq -r^{-1}\varepsilon_{15}(r)\,|v|^2.\end{aligned}$$

So replacing $a_{ij}(x)$ with δ_{ij} and $b_i(x)$ with 0, Lemma 2.2 and the proof of Theorem 1.1 are also true under our weak conditions (C3)′ and (C4)′. □

Proof of Theorem 1.7. By the same way as the proof of Theorem 1.2 we have the assertion. □

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