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Squaring Operations in the 4-Connective Fibre Spaces over the Classifying Spaces of the Exceptional Lie Groups

Dedicated to Professor Nobuo Shimada on his 60 th birthday

By

Akira KONO* and Kiminao ISHITOYA**

§ 1. Introduction

In this paper we calculate the squaring operations in the 4-connective fibre spaces over the classifying spaces of the exceptional Lie groups.

Let G be a compact, 1-connected and simple Lie group. As is well known, its classifying space BG is 3-connected and $H^4(BG;\boldsymbol{Z})\cong\boldsymbol{Z}$. Choose a generator $y_4\in H^4(BG;\boldsymbol{Z})$. Then the 4-connective fibre space $B\tilde{G}$ over BG is, by definition, the homotopy fibre of $y_4:BG\to K(\boldsymbol{Z},4)$. Note that $B\tilde{G}$ is a classifying space of $\tilde{G}$, the 3-connective fibre space over G. Here we quote the results in [2] and [4]. Define the sets J_l $(l=2,4,6,7,8)$ as follows:

$$J_2=\{9,10,12,2^i+1\,(i\geq 4)\},\quad J_4=J_2\cup\{16,24\},$$
$$J_6=\{10,12,16,18,\ 24,\ 33,34,\ 2^i+1\,(i\geq 6)\},$$
$$J_7=\{12,16,\ 20,24,28,\ 33,34,36,\ 2^i+1\,(i\geq 6)\},$$
$$J_8=\{16,\ 24,28,30,31,33,34,36,40,48,2^i+1\,(i\geq 6)\}.$$

Theorem 1.1. *Let G be one of G_2, F_4 and E_l $(l=6,7,8)$. Then*

$$H^*(B\tilde{G};\boldsymbol{F}_2)=\boldsymbol{F}_2[y_j;j\in J_l]\quad(l=\text{rank } G,\ \deg\ y_j=j)$$

where the generators can be taken so as to satisfy the following equalities whenever the suffixes in both sides appear in J_l:

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* Department of Mathematics, Kyoto University, Kyoto 606, Japan.
** Department of Mathematics, Aichi University of Education, Aichi 448, Japan.

$$(1.1)\qquad \begin{aligned} &Sq^1y_9=y_{10},\ Sq^2y_{10}=y_{12},\ Sq^8y_{10}=y_{18},\ Sq^8y_{12}=y_{20};\\ &Sq^8y_{16}=y_{24},\ Sq^4y_{24}=y_{28},\ Sq^2y_{28}=y_{30},\ Sq^1y_{30}=y_{31};\\ &Sq^1y_{33}=y_{34},\ Sq^2y_{34}=y_{36},\ Sq^4y_{36}=y_{40},\ Sq^8y_{40}=y_{48};\\ &Sq^{2^i}y_{2^i+1}=y_{2^{i+1}+1}. \end{aligned}$$

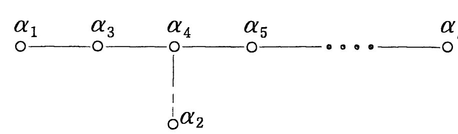
For the most part squaring operations on the y_j are determined from the data (1.1) by use of the Adem relations, but some remain undetermined. Our objective is to determine them completely.

In §2 we introduce a space $B\tilde{T}$ and a map $\tilde{\lambda}:B\tilde{T}\to B\tilde{G}$, where the induced homomorphism $\tilde{\lambda}^*$ is almost injective. In §§3 and 4 we investigate the action of the Weyl group $W(G)$ of G on $B\tilde{T}$, and the $\tilde{\lambda}^*(y_j)$ are determined. And in the final section we give the complete list of $Sq^{2^i}y_j$ and the correspondence of the generators between different groups.

Throughout this paper $H^*(\ )$ denotes the mod 2 cohomology ring, and $\rho:H^*(\ ;A)\to H^*(\ )$ denotes the mod 2 reduction for $A=\boldsymbol{Z}$ or $\boldsymbol{Z}_{(2)}$. $\sigma_i(x_1,\dots,x_n)$ denotes the i-th elementary symmetric polynomial in the x_i.

§2. Cohomology of $B\tilde{T}$ and $B\tilde{C}$

In this and the following two sections G denotes the compact 1-connected exceptional Lie group of type E_l ($l=6, 7, 8$), and T a maximal torus of G. The Dynkin diagram of G is



where the α_i are the simple roots. Define a 1-dimensional torus T^1 by the equations $\alpha_i=0$ ($i\neq 2$), and let $C\subset G$ be the centralizer of T^1. Note that (see [1], for example)

$$(2.1)\qquad C=T^1\cdot SU(l).$$

The inclusions $T\subset C\subset G$ induce maps $\iota:BT\to BC$, $\kappa:BC\to BG$ and $\lambda=\kappa\circ\iota:BT\to BG$. Then the space $B\tilde{T}$ (resp. $B\tilde{C}$) is, by definition, the homotopy fibre of $y_4\circ\lambda:BT\to K(\boldsymbol{Z},4)$ (resp. $y_4\circ\kappa:BC\to K(\boldsymbol{Z},4)$). The maps ι, κ and λ induce maps $\tilde{\iota}:B\tilde{T}\to B\tilde{C}$, $\tilde{\kappa}:B\tilde{C}\to B\tilde{G}$ and $\tilde{\lambda}=\tilde{\kappa}\circ\tilde{\iota}:B\tilde{T}\to B\tilde{G}$, which make the following diagrams commutative:

$$
(2.2)\quad
\begin{array}{ccccc}
C/T & \xrightarrow{\tilde{i}} & B\tilde{T} & \xrightarrow{\tilde{\iota}} & B\tilde{C} \\
\| & & \downarrow & & \downarrow \\
C/T & \xrightarrow{i} & BT & \xrightarrow{\iota} & BC \\
 & & \downarrow & & \downarrow{\scriptstyle y_4\circ\kappa} \\
 & & K & = & K
\end{array}
\qquad
\begin{array}{ccccc}
G/T & \longrightarrow & B\tilde{T} & \xrightarrow{\tilde{\lambda}} & B\tilde{G} \\
\| & & \downarrow & & \downarrow \\
G/T & \longrightarrow & BT & \xrightarrow{\lambda} & BG \\
 & & \downarrow & & \downarrow{\scriptstyle y_4} \\
 & & K & = & K
\end{array}
\quad (K=K(\mathbf{Z},4))
$$

where rows and columns are fiberings. From the first diagram we have

Lemma 2.1. *ι and $\tilde{\iota}$ induce monomorphisms in $H^*(\ ;A)$ for any A.*

In fact, $H^{\mathrm{odd}}(C/T;A)=0$ by (2.1) and $H^{\mathrm{odd}}(BT;A)=0$, whence the Serre spectral sequence collapses for the lower row. Then i^* is onto, hence so is $\tilde{i}^*$, and the spectral sequence collapses for the upper row.

From now on by use of ι^* (resp. $\tilde{\iota}^*$) we regard $H^*(BC;A)$ (resp. $H^*(B\tilde{C};A)$) as a subalgebra of $H^*(BT;A)$ (resp. $H^*(B\tilde{T};A)$).

Recall that the fundamental weights w_i $(i=1,2,\ldots,l)$ form a basis of $H^2(BT;\mathbf{Z})$. For convenience of calculation we introduce t_i, t and $c_j\in H^*(BT;\mathbf{Z})$ in the following way. Let R_i be the reflection in the plane $\alpha_i=0$. After [5] and [3] we define

$$t_l=w_l,\quad t_i=R_{i+1}(t_{i+1}),\quad t_1=R_1(t_2)$$
$$c_j=\sigma_j(t_1,\ldots,t_l)\quad\text{and}\quad t=\frac{1}{3}c_1=w_2.$$

Then each R_i $(i\neq 2)$ acts on $\{t_j\}$ as a transposition, and

$$(2.3)\qquad R_2(t_j)=\begin{cases} t-b_1+t_j & (j\leq 3)\\ t_j & (j\geq 4)\end{cases}\qquad (b_1=t_1+t_2+t_3).$$

Since the Weyl group $W(G)$ (resp. $W(C)$) is generated by $\{R_i\}$ (resp. $\{R_i;\ i\neq 2\}$), we have from the data above that

$$(2.4)\qquad \begin{aligned} H^4(BT;\mathbf{Z})^{W(G)}&=\mathbf{Z}\cdot(c_2-4t^2),\\ H^*(BT;\mathbf{Z})^{W(C)}&=\mathbf{Z}[t,c_2,c_3,\ldots,c_l].\end{aligned}$$

Note that the Weyl group $W(X)$ acts trivially on the image of $H^*(BX;\mathbf{Z})\to H^*(BT;\mathbf{Z})$, and that $H^*(X/T;\mathbf{Z})$ is torsion free by the classical result of Bott. Consider the Poincaré polynomial of $H^*(BC;\mathbf{Z})$, which is obtained from (2.1). Then we have

Theorem 2.2. (i) $\lambda^*y_4=\pm(c_2-4t^2)$.

(ii) $H^*(BC;\boldsymbol{Z})=\boldsymbol{Z}[t, c_2, c_3, \dots, c_l]$ *by means of* ι^*.

Now consider the fiberings derived from the columns in (2.2):

$$
(2.5)\qquad
\begin{array}{ccccc}
K(\boldsymbol{Z},3) & = & K(\boldsymbol{Z},3) & = & K(\boldsymbol{Z},3) \\
\downarrow f & & \downarrow f & & \downarrow f \\
B\tilde{T} & \xrightarrow{\tilde{\iota}} & B\tilde{C} & \xrightarrow{\tilde{\kappa}} & B\tilde{G} \\
\downarrow g & & \downarrow g & & \downarrow g \\
BT & \xrightarrow{\iota} & BC & \xrightarrow{\kappa} & BG
\end{array}
\qquad
\begin{array}{c}
(\tilde{\lambda}=\tilde{\kappa}\circ\tilde{\iota}) \\ \\ (\lambda=\kappa\circ\iota)
\end{array}
$$

where the cohomology of the common fibre is given by

$$H^*(K(\boldsymbol{Z},3))=\boldsymbol{F}_2[u_{2^i+1}\,;\,i\geq 1]\,(\deg\ u_j=j),\qquad Sq^{2^i}u_{2^i+1}=u_{2^{i+1}+1}.$$

By the definition of $B\tilde{T}$ the fundamental class u_3 transgresses to $\rho(\lambda^* y_4)=\rho(c_2)$, by (i) of Theorem 2.2. To avoid complexity we will omit the symbol ρ except in the case of emphasis. Then

Lemma 2.3. (i) *The transgression* τ *is given by*

$$\tau(u_3)=c_2,\ \tau(u_5)\equiv c_3,\ \tau(u_9)\equiv c_5',\ \tau(u_{17})\equiv c_9'$$

and $\qquad \tau(u_{2^i+1})\equiv 0\ \ (i\geq 5)$ *modulo the images in lower dimensions,*

where $c_5'=c_5+c_4c_1$, *and* $c_9'=c_8c_1+c_7c_1^2+c_6c_1^3$.

(ii) *The sequence* (c_2, c_3, c_5', c_9') *is regular in both* $H^*(BC)$ *and* $H^*(BT)$.

Proof. (i) This follows from the Wu formula and the commutativity of the transgression with Sq^i.

(ii) Clearly the sequence is regular in $H^*(BC)$. Then its regularity in $H^*(BT)$ follows from the fact that $H^*(BT)$ is a free $H^*(BC)$-module.

To simplify notation we will omit the symbol g^* in g^*x for $x\in H^*(BT;A)$. Define $J'=\{2^k+1;\ k\geq 5\}$. Then the main theorem in this section is stated as follows:

Theorem 2.4. *There exist* $\gamma_i\in H^{2i}(B\tilde{C})\subset H^{2i}(B\tilde{T})$ $(i=3, 5, 9, 17)$ *and* $v_j\in H^j(B\tilde{C})\subset H^j(B\tilde{T})$ $(j\in J')$ *such that*

$$H^*(B\tilde{C})=\boldsymbol{F}_2[c_1, c_2, \dots, c_l, \gamma_3, \gamma_5, \gamma_9, \gamma_{17}, v_j; j\in J']/(c_2, c_3, c_5', c_9'),$$
$$H^*(B\tilde{T})=\boldsymbol{F}_2[t_1, t_2, \dots, t_l, \gamma_3, \gamma_5, \gamma_9, \gamma_{17}, v_j; j\in J']/(c_2, c_3, c_5', c_9')$$

where the generators are related by

$$Sq^{2i-2}\gamma_i=\gamma_{2i-1}\ (i=3,5,9)\quad \textit{and}\quad Sq^{j-1}v_j=v_{2j-1}\ (j\in J').$$

Proof. Consider the Serre spectral sequence for the middle column in (2.5). By (i) of Lemma 2.3 there exist $\gamma_3\in H^6(B\tilde{C})$ and $v_{33}\in H^{33}(B\tilde{C})$ such that $f^*\gamma_3=u_3^2$ and $f^*v_{33}=u_{33}$. Define $\gamma_{2i+1}=Sq^{2i}\gamma_{i+1}$ $(i=2,4,8)$ and $v_{2j-1}=Sq^{j-1}v_j$ $(j\in J')$, so that $f^*\gamma_i=u_i^2$ $(i=3,5,9,17)$ and $f^*v_j=u_j$ $(j\in J')$. Then by (ii) of the lemma we have

$$H^*(B\tilde{C})=H^*(BC)/(c_2,c_3,c_5',c_9')\otimes \boldsymbol{F}_2[\gamma_i,v_j\,;\,i=3,5,9,17,j\in J']$$

and the same with C replaced by T.

§3. The Action of the Weyl Group on $H^*(B\tilde{T})$

Recall that $y_4\circ\lambda=\lambda^*y_4$ is $W(G)$-invariant. Thus the action of $W(G)$ on BT lifts to $B\tilde{T}$ in such a way that

(3.1) the canonical map $g:B\tilde{T}\to BT$ is equivariant, and

(3.2) $W(X)$ acts trivially on the image of $H^*(B\tilde{X};A)\to H^*(B\tilde{T};A)$ where $X=C$ or G.

By (3.1) the action of $W(G)$ on $\{t_i\}$ in $H^*(B\tilde{T})$ is the same as that in $H^*(BT)$. In order to determine the action on $\{\gamma_i\}$ we consider the cohomology with coefficients $\boldsymbol{Z}_{(2)}$.

By Theorem 2.4 $H^*(B\tilde{C};\boldsymbol{Z}_{(2)})$ is torsion free for $*\leq 32$. Thus we can define $g_i\in H^{2i}(B\tilde{C};\boldsymbol{Z}_{(2)})\subset H^{2i}(B\tilde{T};\boldsymbol{Z}_{(2)})$ $(i=3,5,9)$ by

$$2g_3=c_3,\ 2g_5=c_5'=c_5+c_4c_1\ \text{and}\ 2g_9=c_9'=c_8c_1+c_7c_1^2+c_6c_1^3$$

since $\rho(c_3)=\rho(c_5')=\rho(c_9')=0$. As a corollary to 2.2 in [3]

Lemma 3.1. *g_3 is not divisible by 2.*

Therefore $f^*:H^*(B\tilde{T})\to H^*(K(\boldsymbol{Z},3))$ sends $\rho(g_3)$ to u_3^2, and so we may take $\gamma_3=\rho(g_3)$ in Theorem 2.4.

Now we shall determine the action of $W(G)$ on $\{\rho(c_i)\}$, $\{\rho(g_j)\}$ and $\{\gamma_k\}$. Each R_i $(i\neq 2)$ acts trivially on them by (3.2) with $X=C$. Our objective in this section is to determine the action of R_2. From now on we will exclusively use the notations

$$R=R_2\quad\text{and}\quad \bar{R}=R-1.$$

Define $b_i=\sigma_i(t_1,t_2,t_3)$ and $a_i=\sigma_i(t_4,t_5,\dots,t_l)\in H^{2i}(B\tilde{T};\boldsymbol{Z})$, so that

(3.3)
$$c_n = \sum_{i+j=n} b_i a_j.$$

By (2.3) $R(a_j) = a_j$ for any j, and

$$\sum \mathrm{R}(b_i) = R(\sum b_i) = R(\prod_{i=1}^{3}(1+t_i)) = \prod_{i=1}^{3}(1+R(t_i))$$
$$= \prod_{i=1}^{3}(1+t-b_1+t_i) = \sum (1+t-b_1)^{3-i} b_i.$$

Substituting $c_1 = 3t$, $c_2 = 4t^2$, $c_3 = 2g_3$ and $c_5 = 2g_5 - c_4c_1$ into (3.3), the b_i and a_5 are expressed in terms of g_3, g_5, a_j $(1 \leq j \leq 4)$ and t. Then so are the $\bar{R}(b_i)$:

$$\bar{R}(b_1) \equiv -a_1,\ \bar{R}(b_2) \equiv a_1^2 \text{ and } \bar{R}(b_3) \equiv -a_2a_1 + a_1^3 \bmod (4, 2t),$$

and so are the $\bar{R}(c_n) = \sum \bar{R}(b_i) a_{n-i}$. For instance, $\bar{R}(c_3) \equiv 2a_2a_1 + 2a_1^3$ mod $(4, 2t)$, which implies $\bar{R}(g_3) \equiv a_2a_1 + a_1^3$ mod $(2, t)$ since $H^6(B\tilde{T}; \boldsymbol{Z}_{(2)})$ is torsion free. In other words, in $H^*(B\tilde{T})$

(3.4)
$$\bar{R}(\rho(g_3)) \equiv \rho(a_2)\rho(a_1) + \rho(a_1)^3 \bmod (t).$$

Similar calculations give $\bar{R}(\rho(g_j))$ mod(t) for $j=5$ and 9. On the other hand, since $\gamma_{2i+1} = Sq^{2i}\gamma_{i+1}$ and $\bar{R}$ commutes with Sq^i, $\bar{R}(\gamma_j)$ $(j = 5, 9, 17)$ are derived from (3.4) by use of the Wu formula.

The results are given in the following table, where for simplicity the symbol ρ is omitted again, and the a_i $(i \geq 1)$ is abbreviated as i; e.g., 321^2 is the abbreviation of $a_3a_2a_1^2$ (0 denotes not a_0 but the null):

x	x mod (t)	$\bar{R}(x)$ mod (t)
c_1	0	1
c_4	$4+2^2+21^2+1^4$	$31+1^4$
c_6	$42+3^2+21^4$	$321+21^4$
c_7	$43+421+321^2+31^4+2^21^3+21^5$	$421+31^4+21^5$
c_8	$431+41^4+3^21^2+321^3+31^5+21^6$	$421^2+41^4+321^3+31^5+2^21^4+21^6$
$\gamma_3=g_3$	g_3	$21+1^3$
g_5	g_5	$g_31^2+41+21^3$
g_9	g_9	$g_531+g_3(321+21^4)+41^5+3^221+32^21^2+321^4+2^21^5$
γ_5	γ_5	$31^2+2^21+21^3+1^5$
γ_9	γ_9	$321^4+2^41+2^21^5+1^9$
γ_{17}	γ_{17}	$3^31^8+3^221^9+32^31^8+32^21^{10}+2^81+$ $2^41^9+2^31^{11}+1^{17}$

Remark 3. 2. The disappearance of 5 $(=a_5)$ from the table results from the relation $5\equiv 41+31^2+21^3 \bmod (t)$, which implies $41\equiv 31^2+21^3 \bmod (t)$ for $l=6$, 7 and $31^2\equiv 21^3 \bmod (t)$ for $l=6$, since i $(=a_i=\sigma_i(t_4,\dots,t_l))$ vanishes if $i>l-3$.

§4. Invariants of the Weyl Group and the Image of $\tilde{\lambda}^*$

By (3. 2) with $X=G$ and (2. 5), we see that

(4. 1) $$\mathrm{Im}\,\tilde{\lambda}^*\subset H^*(B\tilde{C})^R=H^*(B\tilde{C})\cap \mathrm{Ker}\,\bar{R}.$$

The case of $l=8$. In this case it is easily seen that

Lemma 4. 1. *In* $H^*(B\tilde{T})/(t)$ *the monomials in* γ_i *and* a_j $(i=3,5,9,17;\ j=1,2,3,4)$ *are linearly independent over* F_2.

Consider the map $H^*(B\tilde{C})\to H^*(B\tilde{T})/(t)$ induced by $\bar{R}$. Using the table in the previous section we have

Lemma 4. 2. $H^n(B\tilde{C})^R=0$ *for* $0<n<16$.

Since $\bar{R}(Sq^2\gamma_3)=Sq^2\bar{R}(\gamma_3)=Sq^2(a_2a_1+a_1^3)=a_3a_1+a_1^4=\bar{R}(c_4)$ by the Wu formula, we see that $Sq^2\gamma_3=c_4$ by the previous lemma, which is sufficient to make the following table by use of the Adem relations:

(4. 2)

	γ_3	γ_5	γ_9	γ_{17}
Sq^2	c_4	γ_3^2	γ_5^2	γ_9^2
Sq^4	γ_5	c_7'	0	0
Sq^8	0	γ_9	$c_7'c_6$	0
Sq^{16}	0	0	γ_{17}	$c_7'^3c_4+c_7'c_6^3$

$(c_7'=c_7+c_6c_1)$

(For $Sq^{32}\gamma_{17}$, see 5. 6)

Remark 4. 3. In the similar way we have the following relations:

$$\gamma_5=g_5+g_3t^2+c_4t,$$
$$\gamma_9=g_9+g_5(c_4+t^4)+g_3(c_6+c_4t^2+t^6)+c_7t^2+c_4^2t+c_4t^5.$$

Now define polynomials $I_k\in H^{2k}(B\tilde{C})$ $(k=8, 12, 14, 15, 17, 18, 20, 24)$ as follows:

$$I_8=c_8+c_6c_1^2+c_4^2+c_4c_1^4+c_1^8$$
$$I_{12}=Sq^8I_8=c_8c_4+c_6^2+c_6c_4c_1^2+c_4^2c_1^4+c_4c_1^8$$
$$I_{14}=Sq^4I_{12}=c_8c_6+c_7'^2+c_6^2c_1^2+c_6c_4c_1^4+c_6c_1^8$$

$$I_{15}=Sq^2I_{14}=c_8c_7'+c_7'c_6c_1^2+c_7'c_4c_1^4+c_7'c_1^8$$
$$I_{17}=\gamma_{17}+\gamma_9I_8+\gamma_5I_{12}+\gamma_3I_{14}+c_7'c_6c_4$$
$$I_{18}=Sq^2I_{17}=\gamma_9^2+\gamma_5^2I_8+\gamma_3^2I_{12}+\gamma_3I_{15}+I_{14}c_4+c_7'^2c_4$$
$$I_{20}=Sq^4I_{18}=\gamma_5^4+\gamma_5I_{15}+\gamma_3^4I_8+\gamma_3^2I_{14}+I_{14}c_6+I_{12}c_4^2+c_7'^2c_6$$
$$I_{24}=Sq^8I_{20}=\gamma_9I_{15}+\gamma_5^2I_{14}+\gamma_3^4I_{12}+\gamma_3^8+I_{14}c_6c_4+I_{12}c_6^2+I_8c_4^4+c_7'^2c_6c_4.$$

Then the main results in this section are stated as follows:

Theorem 4.4. *For E_8 we have*

(i) $H^*(B\tilde{C})^R=\boldsymbol{F}_2[I_k,\ v_{33};\ k=8,12,14,15,17]$ *for* $*\leq 34$.

(ii) $$\tilde{\lambda}^*(y_j)=\begin{cases} I_{j/2} & (j=16,24,28,30,34,36,40,48)\\ v_j & (j=2^i+1 \text{ with } i\geq 5)\\ 0 & (j=31)\end{cases}$$

Proof. Denote by $T^*(m)$ (resp. $C^*(m)$) the subalgebra of $H^*(B\tilde{T})$ generated by $t_1,\dots,t_8$ (resp. $c_1,\dots,c_8$) and the γ_j with $j\leq m$. From the table in the previous section follow

(4.3) $$R(C^*(m))\subset T^*(m) \text{ and } \bar{R}(\gamma_i)\in T^*(0).$$

First we shall show $H^n(B\tilde{C})^R\subset\boldsymbol{F}_2[I_k,v_{33};k=8,12,14,15,17]$ inductively on n. By Lemma 4.2 this holds for $n<16$.

Let $x\in H^{16}(B\tilde{C})^R$ and write it in the form

$$x=\gamma_5p_3+\gamma_3^2q_2+\gamma_3q_5+q_8 \quad (p_3\in C^6(3),\ q_i\in C^{2i}(0)).$$

Applying $\bar{R}$ on both sides, we see, in view of the formula $\bar{R}(XY)=X\bar{R}(Y)+\bar{R}(X)R(Y)$ and (4.3), that $0\equiv\gamma_5\bar{R}(p_3)$ mod $T^*(3)$. This implies $\bar{R}(p_3)=0$, whence $p_3=0$ since $H^6(B\tilde{C})^R=0$. Then

$$0\equiv\gamma_3^2\bar{R}(q_2)+\gamma_3\bar{R}(q_5) \text{ mod } T^*(0)$$

which implies $\bar{R}(q_i)=0$, whence $q_i=0$ since $H^{2i}(B\tilde{C})^R=0$ $(i=2,5)$. Thus $x\in C^{16}(0)$, and after some calculations we see that $x=\alpha I_8$ $(\alpha\in\boldsymbol{F}_2)$ using Lemma 4.1.

Continuing this procedure yields the inclusion mentioned above for $n\leq 34$.

Next consider the Serre spectral sequence for the fibering $E_8/T\to B\tilde{T}\xrightarrow{\tilde{\lambda}}B\tilde{E}_8$. According to Bott the odd dimensional part of $H^*(E_8/T)$ vanishes, and by Theorem 1.1 so dose that of $H^*(B\tilde{E}_8)$ for $*\leq 30$. Therefore for $p\leq 30$ we have $E_2^{2p,0}=E_\infty^{2p,0}$, which implies that $\tilde{\lambda}^*$: $H^{2p}(B\tilde{E}_8)\to H^{2p}(B\tilde{T})$ is a monomorphism. In particular $\tilde{\lambda}^*(y_{16})$ and

$\tilde{\lambda}^*(y_{34})$ do not vanish. Then the theorem follows from (4. 1) and (1. 1).

Next we consider the case of $l=6$ and 7. As is well known, there is a sequence of inclusions $E_6 \subset E_7 \subset E_8$. We may assume that the maximal tori T^l of E_l $(l=6,7,8)$ are chosen so that $T^6 \subset T^7 \subset T^8$. The inclusions induce maps φ_l: $BT^{l-1} \to BT^l$ $(BE_{l-1} \to BE_l)$ and $\tilde{\varphi}_l$: $B\tilde{T}^{l-1} \to B\tilde{T}^l$ $(B\tilde{E}_{l-1} \to B\tilde{E}_l)$ $(l=7,8)$ such that

(4. 4) $$\tilde{\varphi}_l \circ f = f, \quad \varphi_l \circ g = g \circ \tilde{\varphi}_l \quad \text{and} \quad \tilde{\varphi}_l \circ \tilde{\lambda} = \tilde{\lambda} \circ \tilde{\varphi}_l.$$

We may assume, in addition, that the systems of the simple roots $\{\alpha_i;\ i=1,2,\dots,l\}$ are chosen so that $\alpha_i | T^{l-1} = \alpha_i$ $(i<l)$, $=0$ $(i=l)$. Then the corresponding systems of the fundamental weights are in the similar relation, from which and the commutativity (4. 4) it follows that

$$\tilde{\varphi}_l^*(c_i) = c_i \ (i<l), \ =0 \ (i=l); \ \tilde{\varphi}_l^*(\gamma_i) = \gamma_i \text{ and } \tilde{\varphi}_l^*(v_j) = v_j,$$

for each l. Moreover, we have the following:

Lemma 4. 5. *$\tilde{\varphi}_l^*(y_{16}) = y_{16}$ and $\tilde{\varphi}_l^*(y_{33}) = y_{33}$ for each l.*

Proof. Consider the Serre spectral sequence for the fibering $E_l/E_{l-1} \to B\tilde{E}_{l-1} \to B\tilde{E}_l$, where the cohomology of the fibre is given by

$$H^*(E_8/E_7) = \Delta(x_{12}, x_{20}, x_{24}, x_{29}, x_{30}), \quad H^*(E_7/E_6) = \Lambda(x_{10}, x_{18}, x_{27}).$$

It follows that $E_2^{p,0} = E_\infty^{p,0}$ $(p=16, 33)$, which implies $\tilde{\varphi}_l^*(y_p) \neq 0$, and the lemma follows since $\dim H^p(B\tilde{E}_{l-1}) = 1$.

The case of $l=7$. Here the relation $a_4 a_1 \equiv a_3 a_1^2 + a_2 a_1^3 \bmod (t)$ (see 3. 2) yields an invariant in dimension 12. To be precise, we have

Lemma 4. 6. $H^{12}(B\tilde{C})^R \subset \boldsymbol{F}_2 \cdot (\gamma_3^2 + c_4 c_1^2 + c_1^6)$.

Define polynomials $I'_k \in H^{2k}(B\tilde{C})$ $(k=6, 8, 10, 12, 14, 17, 18)$ by

$$I'_6 = \gamma_3^2 + c_4 c_1^2 + c_1^6, \ I_{10} = Sq^8 I'_6 = \gamma_5^2 + c_6 c_1^4 + c_4^2 c_1^2 + c_1^{10};$$

and $$I'_j = \tilde{\varphi}_8^*(I_j) \quad (j=8, 12, 14, 17, 18).$$

Then Theorem 4. 4 together with 4. 5, 4. 6 and (1. 1) implies:

Corollary 4. 7. *For E_7 we have*

$$\tilde{\lambda}^*(y_j) = \begin{cases} I'_{j/2} & (j=12, 16, 20, 24, 28, 34, 36) \\ v_j & (j=2^i+1 \text{ with } i \geq 5) \end{cases}$$

The case of $l=6$. Here the relation $a_3 a_1^2 \equiv a_2 a_1^3 \bmod (t)$ yields $H^{10}(B\tilde{C})^R \subset \boldsymbol{F}_2 \cdot (\gamma_5 + c_4 c_1 + c_1^5)$. Define polynomials $I''_k \in H^{2k}(B\tilde{C})$ $(k=5, 6,$

8, 9, 12, 17) by

$$I_5''=\gamma_5+c_4c_1+c_1^5,\ I_9''=Sq^8I_5''=\gamma_9+c_4^2c_1+c_1^9;$$

and $$I_j''=\tilde{\varphi}_7^*(I_j')\quad (j=6,8,12,17).$$

Corollary 4.8. *For* E_6 *we have*

$$\tilde{\lambda}^*(y_j)=\begin{cases} I_{j/2}'' & (j=10,12,16,18,24,34)\\ v_j & (j=2^i+1 \text{ with } i\geq 5)\end{cases}$$

§ 5. Squaring Operations on the y_j

Now we are ready to compute Sq^iy_j. First we shall consider to what extent they are decidable by use of the Adem relations and the algebra structure of $H^*(B\tilde{G})$. We use the Adem relations not only in the usual form but in the following forms:

(5.1) for $a>2b$, $Sq^aSq^b=Sq^{2b}Sq^{a-b}+\sum_{j=0}^{b-1}\binom{a-b-1-j}{2b-2j}Sq^{a+b-j}Sq^j$;

(5.2) for $r=1$ and 2^{m-1}, $Sq^{2^mk+r}=Sq^{2^{m-1}}Sq^{2^mk+r-2^{m-1}}$
$+\sum_{j=0}^{m-2}Sq^{2^mk+r-2^j}Sq^{2^j}$; etc.

Lemma 5.1. *For* $G=E_8$,

(i) $Sq^iy_{16}=y_{16+i}$ $(i=8,12,14,15)$, $=y_{16}^2$ $(i=16)$ *and* $=0$ *otherwise.*

(ii) $Sq^{16}y_{24}=y_{24}y_{16}$.

Proof. (i) This follows from (1.1), (5.2) and the structure of $H^*(B\tilde{E}_8)$.

(ii) We may put $Sq^{16}y_{24}=\varepsilon y_{40}+\varepsilon' y_{24}y_{16}$ $(\varepsilon,\varepsilon'\in F_2)$. Applying Sq^8 we have $Sq^{20}y_{28}+y_{24}^2=\varepsilon y_{48}+\varepsilon'(y_{24}^2+(Sq^8y_{24})y_{16})$. But $Sq^{20}y_{28}=Sq^{20}Sq^{12}y_{16}$ $=(Sq^{22}Sq^{10}+Sq^{23}Sq^9)y_{16}=0$ and $Sq^8y_{24}=Sq^8Sq^8y_{16}=(Sq^{12}Sq^4+Sq^{14}Sq^2+Sq^{15}Sq^1)y_{16}=0$ both by (i), which imply $\varepsilon=0$ and $\varepsilon'=1$.

Lemma 5.2. *For* $G=E_8$

(i) $Sq^iy_{33}=y_{34}$ $(i=1)$, $=0$ $(i=2,4,8)$, $=y_{33}y_{16}$ $(i=16)$, $=y_{65}$ $(i=32)$.

(ii) $Sq^{16}y_{48}=y_{40}y_{24}+y_{36}y_{28}+y_{34}y_{30}+y_{33}y_{31}$.

Proof. (i) From (1.1) and the structure of $H^*(B\tilde{E}_8)$ follow all but the case of $i=16$. We may put $Sq^{16}y_{33}=\varepsilon y_{33}y_{16}$ $(\varepsilon\in F_2)$. Apply Sq^1 and use $Sq^1Sq^{16}=Sq^2Sq^{15}+Sq^{16}Sq^1$ from (5.2). Then $Sq^{16}y_{34}=$

$\varepsilon y_{34}y_{16}$. Applying $\tilde{\lambda}^*$, which is injective at dimension 50, we see that $\varepsilon=1$ since $Sq^{16}I_{17}=I_{17}I_8$ holds.

(ii) Since $Sq^{16}I_{24}=I_{20}I_{12}+I_{18}I_{14}+I_{17}I_{15}$ and the kernel of $\tilde{\lambda}^*:H^{64}(B\tilde{E}_8)\to H^{64}(B\tilde{T}^8)$ is spanned by $y_{33}y_{31}$, we may put $Sq^{16}y_{48}=y_{40}y_{24}+y_{36}y_{28}+y_{34}y_{30}+\delta y_{33}y_{31}$ $(\delta\in F_2)$. Apply Sq^1 and use $Sq^1Sq^{16}y_{48}=Sq^1y_j=0$ $(j=34,36,40)$. Then we see that $\delta=1$.

These data together with (1.1) are sufficient to determine Sq^iy_j for $G=E_8$ by use of the Adem relations.

For $G=E_6$ and E_7, we need

(5.3) $$Sq^4I'_6=I'_8,\ Sq^{16}I'_{10}=I'_{18}+I'_{12}I'_6+I'_{10}I'_8;$$
$$Sq^2I''_5=I''_6,\ Sq^{16}I''_9=I''_{17}+I''_{12}I''_5+I''_9I''_8.$$

(5.4) $$\tilde{\varphi}_8^*I_{15}=0,\ \tilde{\varphi}_8^*I_{20}=I'_{14}I'_6+I'_{12}I'_8+I'^2_{10}+I'_8I'^2_6,$$
$$\tilde{\varphi}_8^*I_{24}=I'_{14}I'_{10}+I'^2_{12}+I'_{12}I'^2_6+I'^3_8+I'^4_6;$$
$$\tilde{\varphi}_7^*I'_{10}=I''^2_5,\ \tilde{\varphi}_7^*I'_{14}=0,\ \tilde{\varphi}_7^*I'_{18}=I''_{12}I''_6+I''^2_9+I''_8I''^2_5.$$

Apply $(\tilde{\lambda}^*)^{-1}$ to these equalities in view of 4.7, 4.8 and (4.4). The results are as follows:

Theorem 5.3. *In $H^*(B\tilde{E}_l)$ $(l=6,7,8)$ Sq^iy_j are given by*

$j \backslash i$	1	2	4	8	16	32
16	0	0	0	y_{24}	y_{16}^2	0
24	0	0	y_{28}	0	$y_{24}y_{16}$	0
28	0	y_{30}	0	0	$y_{28}y_{16}$	0
30	y_{31}	0	0	0	$y_{30}y_{16}$	0
31	0	0	0	0	$y_{31}y_{16}$	0
33	y_{34}	0	0	0	$y_{33}y_{16}$	y_{65}
34	0	y_{36}	0	0	$y_{34}y_{16}$	$y_{36}y_{30}+y_{33}^2$
36	0	0	y_{40}	0	$y_{36}y_{16}$	$y_{40}y_{28}+y_{34}^2$
40	0	0	0	y_{48}	$y_{40}y_{16}$	$y_{48}y_{24}+y_{36}^2$
48	0	0	0	0	$y_{40}y_{24}+y_{36}y_{28}+y_{34}y_{30}+y_{33}y_{31}$	*
12	0	0	y_{16}	y_{20}	0	0
20	0	0	y_{12}^2	y_{28}	$y_{36}+y_{24}y_{12}+y_{20}y_{16}$	0
10	0	y_{12}	0	y_{18}	0	0
18	0	y_{10}^2	0	0	$y_{34}+y_{24}y_{10}+y_{18}y_{16}$	0

$(*=y_{48}y_{16}^2+y_{40}^2+y_{40}y_{24}y_{16}+y_{36}y_{28}y_{16}+y_{34}y_{30}y_{16}+y_{33}y_{31}y_{16})$

where the y_j *with* $j \notin J_l$ *must be read as*

$$y_{30}=0,\ y_{40}=y_{28}y_{12}+y_{24}y_{16}+y_{20}^2+y_{16}y_{12}^2 \quad \text{for } l=7;$$
$$y_{20}=y_{10}^2,\ y_{28}=0,\ y_{36}=y_{24}y_{12}+y_{18}^2+y_{16}y_{10}^2 \quad \text{for } l=6.$$

For $G=F_4$ and G_2, recall that the inclusions $G_2 \subset F_4 \subset E_6$ induce two fiberings $E_6/F_4 \to B\tilde{F}_4 \xrightarrow{\tilde{\varphi}_6} B\tilde{E}_6$ and $F_4/G_2 \to B\tilde{G}_2 \xrightarrow{\tilde{\varphi}_4} B\tilde{F}_4$. From $H^*(E_6/F_4)=\Lambda(x_9, x_{17})$ and $H^*(F_4/G_2)=\Lambda(x_{15}, x_{23})$ (deg $x_i=i$) it follows that $\tilde{\varphi}_6^* y_{10}=y_{10}$, $\tilde{\varphi}_6^* y_{18}=y_9^2$ and $\tilde{\varphi}_4^* y_9=y_9$. Thus we have

Corollary 5.4. *In* $H^*(B\tilde{F}_4)$ *and* $H^*(B\tilde{G}_2)$ $Sq^i y_j$ $(j=10, 12, 16, 24)$ *are given by those in* $H^*(B\tilde{E}_6)$ *with* y_{18} *replaced by* y_9^2, *and*

$$Sq^1 y_9=y_{10},\ Sq^2 y_9=Sq^4 y_9=0,\ Sq^8 y_9=y_{17}.$$

Remark 5.5. The $\tilde{\varphi}_l^*$ send the y_j in the following way:

(i) $\tilde{\varphi}_8^* y_{30}=\tilde{\varphi}_8^* y_{31}=0$, $\tilde{\varphi}_8^* y_{40}=y_{28}y_{12}+y_{24}y_{16}+y_{20}^2+y_{16}y_{12}^2$,
$\tilde{\varphi}_8^* y_{48}=y_{28}y_{20}+y_{24}^2+y_{24}y_{12}^2+y_{16}^3+y_{12}^4$

(ii) $\tilde{\varphi}_7^* y_{20}=y_{10}^2$, $\tilde{\varphi}_7^* y_{28}=0$, $\tilde{\varphi}_7^* y_{36}=y_{24}y_{12}+y_{18}^2+y_{16}y_{12}^2$

(iii) $\tilde{\varphi}_6^* y_{18}=y_9^2$, $\tilde{\varphi}_6^* y_{33}=y_{33}+y_{24}y_9+y_{17}y_{16}$, $\tilde{\varphi}_6^* y_{34}=y_{24}y_{10}+y_{17}^2+y_{16}y_9^2$,
$\tilde{\varphi}_6^* y_{2^i+1}=Sq^{2^{i-1}}\tilde{\varphi}_6^* y_{2^{i-1}+1}$ $(i\geq 6)$

(iv) $\tilde{\varphi}_4^* y_{16}=\tilde{\varphi}_4^* y_{24}=0$

and for the rest $\tilde{\varphi}_l^* y_j=y_j$ holds.

Most of them follow from 4.7, 4.8 and (4.4). For $\tilde{\varphi}_6^* y_{33}$ put $\tilde{\varphi}_6^* y_{33}=\varepsilon_1 y_{33}+\varepsilon_2 y_{24}y_9+\varepsilon_3 y_{17}y_{16}+\varepsilon_4 y_{12}^2 y_9$ $(\varepsilon_k \in \boldsymbol{F}_2)$, and apply $Sq^i, j^*: H^*(B\tilde{G}) \to H^*(K(\boldsymbol{Z}, 3))$, etc.

Remark 5.6. From $Sq^{32}y_{34}=y_{36}y_{30}+y_{33}^2$ it follows that

$$Sq^{32}\gamma_{17}=v_{33}^2+\gamma_{17}I_8^2+\gamma_9 I_{12}^2+\gamma_5 I_{14}^2+\gamma_3 I_{15}^2+(I_8^2 c_6 c_4+(c_7'^2+c_6 c_4^2)(c_6^2+c_4^3))c_7',$$

which completes the table (4.2).

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