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Remarks on the Feynman Representation

By

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Abstract

It is shown that there exists a complete space $\partial L^1(M_t^K, M_t^P)$ of integrable functions such that for any potential V with zero H_0-bound relative to the free Hamiltonian operator H_0 of a finite non-relativistic quantum system, the function $\exp[-i\int_0^t V\circ X_s\, ds]$ belongs to $\partial L^1(M_t^K, M_t^P)$, and the Feynman representation $e^{-i(H_0+V)t}=\int_\Omega \exp[-i\int_0^t V\circ X_s ds]\, dM_t^F$ is valid.

§ 0. Introduction

Suppose that Δ is the selfadjoint extension on $L^2(\boldsymbol{R}^d)$, $d=1,2,\ldots$ of the Laplacian operator $\partial^2/\partial x_1^2+\ldots+\partial^2/\partial x_d^2$ acting on all smooth functions of compact support on $\boldsymbol{R}^d$. Then for the appropriate choice of the dimension d, the free Hamiltonian operator of a finite non-relativistic quantum system is equivalent to the operator $H_0=-\frac{1}{2}\Delta$.

If $V:\boldsymbol{R}^d\to\boldsymbol{R}$ is a Borel measurable function representing the potential describing the interactions in the system, then under suitable conditions, $H=H_0+V$ is defined and selfadjoint on the domain of H_0, and it is equivalent to the Hamiltonian operator of the system.

The Feynman representation [2]

$$e^{-iHt}=\int_\Omega \exp[-i\int_0^t V\circ X_s ds]dM_t^F \tag{1}$$

has recently been established for a large class of potentials V, including, for example, Coulomb interactions. For more singular V, such as the attractive $1/r^2$ potential, the definite integrals seem to give a good description of the dynamics of the system [8]. A further

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extension has been considered for magnetic fields [3].

It was pointed out in [2] that the approach to integration with respect to the relevant operator valued set functions developed there is deficient in the sense that the space $L^1(M_t^K, M_t^D)$ of (equivalence classes of) M_t^K-M_t^D-integrable functions is not complete in its natural topology. Examples of non-convergent Cauchy sequences in $L^1(M_t^K, M_t^D)$ are provided by Johnson and Skoug [5] page 264.

The purpose of this note is to show that a slightly stronger integration process is sufficient to produce a complete space $\partial L^1(M_t^K, M_t^D)$ of integrable functions, at the expense of diminishing the class $\mathscr{P}$ of potentials V for which $\exp[-i\int_0^t V\circ X_s ds]$ is integrable for each $t>0$. For example, the Feynman-Nelson approximating sequence will not converge in $\partial L^1(M_t^K, M_t^D)$ for the attractive $1/r^2$ potential [8].

However, the potentials with zero H_0-bound [6] page 190 belong to $\mathscr{P}$, so the Feynman representation (1) is still valid for the stronger integrals, and the convergence of the integral is closer to Feynman's original approach; namely, for each $t>0$, the operator $\int_\Omega \exp[-i\cdot\int_0^t V\circ X_s ds]dM_t^F$ belongs to the closure in the strong operator topology of the family $\{\int_\Omega s\, dM_t^F:\ s\in \mathrm{sim}(\mathscr{S}_t)\}$ of bounded linear operators on $L^2(\mathbb{R}^d)$. In particular, $\mathscr{P}$ still contains Coulomb potentials.

In Section 1, slight modifications of convergence results for one-parameter semigroups of continuous linear operators on a Banach space [9], [10] are optained to cover the case of the uniform convergence of the limit of operators in the strong operator topology, as a certain parameter varies over compact sets.

The result is applied in Section 2 to the Feynman representation. An outline of integration with respect to closable set functions is given in the appendix.

§ 1. A Uniform Product Formula

The following lemma is probably well known, but because it is used repeatedly in what follows (and in [2] Lemma 4.3) it seems worthwhile to state and prove it explicitly.

Lemma 1. *Let E be a topological vector space. Let $\{T_\tau(a):\ \tau\in\Upsilon$,*

$a\in A\}$ be an equicontinuous family of linear operators on E such that for each $x\in E$, $\lim_{\tau\in\Upsilon}T_\tau(a)x$ converges in E uniformly for $a\in A$.

Then $\lim_{\tau\in\Upsilon}T_\tau(a)x$ converges in E uniformly for $a\in A$ and as x ranges over any precompact subset of E.

Proof. Let K be a precompact subset of E. Let U be a closed neighbourhood of zero in E. Then there exists a finite subset $\{x_i\colon i=1,\dots,n\}$ of K such that $K\subset\bigcup_{i=1}^{n}(x_i+V)$ and V is a neighbourhood of zero in E such that

$$\bigcup_{\substack{a\in A\\ \tau\in\Upsilon}} T_\tau(a)V\subset U,$$

by virtue of equicontinuity.

If the limit of $T_\tau(a)$, $\tau\in\Upsilon$ is denoted by $T(a)$ for each $a\in A$, then for $\tau\in\Upsilon$ sufficiently large, $T_\tau(a)x_i-T(a)x_i\in U$ for $i=1,\dots,n$ and every $a\in A$, by the uniform convergence.

Therefore, for any $x\in K$,

$$[T_\tau(a)-T(a)]x\in U+U-U$$

for every $a\in A$, giving the result.

The domain of an operator T is denoted by $\mathscr{D}(T)$. The space of all continuous linear operators on a Banach space X is written as $\mathscr{L}(X)$.

Lemma 2. *Let X be a Banach space. Let Υ be a non-empty set and $B\colon\mathscr{D}(B)\to X$ a linear operator on X.*

Suppose that there exist a number $M>0$ and linear operators $A(\tau)\colon\mathscr{D}(A(\tau))\to X$, $\tau\in\Upsilon$ such that for each $\tau\in\Upsilon$ the operator $A(\tau)+B$ defined on $\mathscr{D}(A(\tau))\cap\mathscr{D}(B)$ is closable, and its closure $T(\tau)$ generates a C_0-semigroup $e^{T(\tau)t}$, $t>0$ such that

$$\|e^{T(\tau)t}\|\leq M$$

for every $0\leq t\leq 1$.

Suppose also that for each $y\in Y=\bigcap_{\tau\in\Upsilon}\mathscr{D}(A(\tau))\cap\mathscr{D}(B)$, the set $\{A(\tau)y\colon \tau\in\Upsilon\}$ is a relatively compact subset of X, and Y is dense in X.

Then for each $y\in Y$,

$$(e^{T(\tau)s}-I)y/s\to(A(\tau)+B)y$$

as $s \downarrow 0$, *uniformly for* $\tau \in \Upsilon$.

Proof. For each $f \in Y$, $\tau \in \Upsilon$

$$e^{T(\tau)t} f = f + \int_0^t e^{T(\tau)s}(A(\tau) + B) f \, ds$$

for each $t > 0$. The integral is the limit of Riemann sums.

By virtue of the assumed uniform boundedness of $e^{T(\tau)s}$, $\tau \in \Upsilon$ about $s = 0$, it follows that $e^{T(\tau)s} f \to f$ uniformly for $\tau \in \Upsilon$ as $s \to 0$, so $e^{T(\tau)s} x \to x$ uniformly for $\tau \in \Upsilon$ as $s \to 0$, for each $x \in X$ by equicontinuity. We have used the fact that $\{A(\tau)f : \tau \in \Upsilon\}$ is a bounded subset of X.

Moreover

$$(e^{T(\tau)t} - I) f/t - (A(\tau) + B) f = \int_0^t (e^{T(\tau)s} - I)(A(\tau) + B) f \, ds/t,$$

so the result follows from the precompactness of $\{A(\tau)f : \tau \in \Upsilon\}$ and Lemma 1.

The uniform product formula for contraction semigroups follows from a minor variation of Nelson's proof [8] of the Trotter product formula.

Theorem 1. *Let* X *be a Banach space. Let* B *be a bounded operator on* X. *For each* $\tau \in \Upsilon$, *let* $A(\tau)$ *be the generator of a* C_0*-contraction semigroup such that the set* $Y = \bigcap_{\tau \in \Upsilon} \mathscr{D}(A(\tau))$ *is a core for each* $A(\tau)$, $\tau \in \Upsilon$, *and* $\tau \mapsto A(\tau) y$, $\tau \in \Upsilon$ *is continuous for each* $y \in Y$, *with* Υ *metrizable and precompact. Then for each* $t > 0$

$$e^{(A(\tau)+B)t} = \lim_{n \to \infty} [e^{Bt/n} e^{A(\tau)t/n}]^n$$

in the strong operator topology on $\mathscr{L}(X)$, *uniformly for* $\tau \in \Upsilon$.

Proof. It can be assumed from the outset that $\|e^{Bt}\| \leq 1$ for all $t > 0$, since B can be replaced by $B - \|B\|_\infty$ if necessary.

Let $R^\tau(t) = e^{(A(\tau)+B)t}$, $S^\tau(t) = e^{A(\tau)t}$, $T(t) = e^{Bt}$ and $U^\tau(t) = T(t) S^\tau(t)$ for each $t > 0$ and $\tau \in \Upsilon$. For $\tau \in \Upsilon$ and $s > 0$, set $x_s^\tau = R^\tau(s) x$ for each $x \in X$. Then for $t > 0$

$$\|R^\tau(t) x - [U^\tau(t/n)]^n x\| = \|\sum_{j=0}^{n-1} [U^\tau(t/n)]^j (R^\tau(t/n) - U^\tau(t/n)) [R^\tau(t/n)]^{n-j-1} x\|$$

$$\leq \sup_{0\leq s\leq t} n\|(R^\tau(t/n)-U^\tau(t/n))x_s^\tau\| \qquad (2)$$

for each $x\in X$.

Let $x\in Y$. By Lemma 2

$$(R^\tau(s)-I)x/s\rightarrow(A(\tau)+B)x$$

as $s\downarrow 0$, uniformly for $\tau\in\Upsilon$.

The continuity of $R^\tau(\cdot)$ and $S^\tau(\cdot)$ is obviously uniform for $\tau\in\Upsilon$, because it is for a dense set of vectors, and the families of operators are equicontinuous. Now appealing to Lemma 1,

$$\begin{aligned}(U^\tau(s)-I)x/s&=T(s)A(\tau)x+T(s)[(S^\tau(s)-I)x/s-A(\tau)x]\\&\quad+(T(s)-I)x/s\rightarrow A(\tau)x+0+Bx, \qquad s\downarrow 0\end{aligned}$$

uniformly for $\tau\in\Upsilon$. Here we use the fact that $\{A(\tau)x\colon \tau\in\Upsilon\}$ is a precompact subset of X, and so is $\{(S^\tau(s_n)-I)x/s_n\colon n=1,2,\ldots, \tau\in\Upsilon\}$ for $s_n\rightarrow 0$ as $n\rightarrow\infty$. The last assertion follows from strong resolvent convergence [9] by the continuity of $A(\cdot)y$, $y\in Y$, and an elementary topological argument, given explicitly, for example, in [4] Lemma 4.1. Therefore, $\lim_{n\rightarrow\infty} n[R^\tau(t/n)-U^\tau(t/n)]x=0$ in X, uniformly for $\tau\in\Upsilon$.

Let Z be the space of continuous functions $\tau\mapsto z^\tau$, $\tau\in\Upsilon$ from Υ into X such that $z^\tau\in\mathscr{D}(A(\tau))$ for each $\tau\in\Upsilon$ and the function $\tau\mapsto A(\tau)z^\tau$, $\tau\in\Upsilon$ is continuous. Equip Z with the norm

$$\|z\|_Z=\sup_{\tau\in\Upsilon}\|z^\tau\|+\sup_{\tau\in\Upsilon}\|A(\tau)z^\tau\|, \qquad z\in Z.$$

Then Z is a Banach space because $A(\tau)$ is closed for each $\tau\in\Upsilon$.

The same argument as in Lemma 2 and as above shows that for each $z\in Z$

$$\lim_{n\rightarrow\infty} n[R^\tau(t/n)-U^\tau(t/n)]z^\tau=0$$

in X uniformly for $\tau\in\Upsilon$. By the uniform boundedness principle, there exists $C>0$ such that

$$\sup_{\tau\in\Upsilon}\|n[R^\tau(t/n)-U^\tau(t/n)]z^\tau\|\leq C\|z\|_Z$$

for all $n=1,2,\ldots$ and $z\in Z$. The convergence as $n\rightarrow\infty$ is uniform as z varies over compact subsets of Z.

Again, $A(\cdot)y$, $y\in Y$ is continuous, so $\tau\rightarrow R^\tau(s)$, $\tau\in\Upsilon$ is continuous for each $s>0$ by strong resolvent convergence. The uniform continuity of $R^\tau(\cdot)$ for $\tau\in\Upsilon$ on $[0,t]$ shows that

$$\{x_s^\cdot\colon 0\leq s\leq t\}$$

is a compact subset of Z.

The right-hand side of (2) therefore goes to zero, uniformly for $\tau\in\Upsilon$ and for t in compact subsets of $[0,\infty[$.

The notion of strong resolvent convergence needs to be extended to uniform convergence as a parameter varies over a set. In this case, the argument of Kato [6] IX. 2. 16 requires less modification than does Nelson's proof of the Trotter product formula.

Let E_a, $a\in A$ be dense subspaces of the topological vector space E. The collection $\{E_a: a\in A\}$ is said to be *uniformly dense* in E if for each $x\in E$ there exists a directed set Z and $x_\zeta^a\in E_a$, $a\in A$, $\zeta\in Z$ such that $\lim_{\zeta\in Z}x_\zeta^a=x$ in E uniformly for $a\in A$.

The condition is clearly satisfied whenever $\bigcap_{a\in A}E_a$ is dense in E.

Theorem 2. *Let $T(a)$, $T_n(a)$, $n=1,2,\dots$, $a\in A$ be the generators of C_0-semigroups. Suppose that there exist $M,\beta>0$ such that*

$$\|e^{T_n(a)t}\|\le Me^{\beta t}$$

for all $t>0$, $a\in A$, $n=1,2,\dots$.

Suppose also that A is precompact and metrizable, and the operator valued function $a\mapsto T(a)$, $a\in A$ is continuous in the sense of strong resolvent convergence.

If $\{\mathscr{D}(T(a)): a\in A\}$ is uniformly dense in E, and for some $\lambda\in\boldsymbol{C}$ with $\mathrm{Re}\lambda>\beta$,

$$(\lambda I-T_n(a))^{-1}\to(\lambda I-T(a))^{-1}$$

uniformly for $a\in A$, in the strong operator topology as $n\to\infty$, then

$$e^{T_n(a)t}\to e^{T(a)t}$$

in the strong operator topology, uniformly for $a\in A$ and as t ranges over any compact subset of $[0,\infty]$, as $n\to\infty$.

Conversely, if for each $t>0$,

$$e^{T_n(a)t}\to e^{T(a)t}$$

uniformly for $a\in A$, in the strong operator topology as $n\to\infty$, then

$$(\lambda I-T_n(a))^{-1}\to(\lambda I-T(a))^{-1}$$

in the strong operator topology, uniformly for $a\in A$, and for λ in compact subsets of $\{z\in\boldsymbol{C}: \mathrm{Re}z>\beta\}$.

Proof. The first statement follows by systematically applying Lemma 1 to the proof of [6] IX. 2. 16. The converse follows by noting that

$$\sup_{a\in A}\|[(\lambda I-T_n(a))^{-1}-(\lambda I-T(a))^{-1}]x\|\leq\int_0^\infty e^{-\alpha t}\sup_{a\in A}\|[e^{T_n(a)t}-e^{T(a)t}]x\|dt$$

for every $\lambda\in\mathbb{C}$ with $\mathrm{Re}\lambda=\alpha>\beta$, and every vector x. Dominated convergence gives the result.

§2. Boundary Integration

The notation adopted in the appendix will be adhered to in this section. The lack of completeness of the space $L^1(M_t^K, M_t^P)$ introduced in [2] is a consequence of the space $H_c(D)$ being incomplete; this is the space of continuous functions on D, analytic in the interior of D, and equipped with the topology of uniform convergence on compact subsets of the interior of D. In some ways, this is a natural space to use because we are dealing with a boundary-value problem for holomorphic functions—solutions are constructed in the interior of D so that boundary-values are taken on continuously—a time-honoured technique in analysis.

One way to ensure that boundary-values are taken on continuously is to approximate by holomorphic functions which have this property, uniformly on compact subsets of D. It is to be expected that such solutions have better stability properties than those which are constructed by approximation in the interior of D.

Let $\mathscr{C}(D)$ denote the family of all compact subsets of D. For each $C\in\mathscr{C}(D)$ and $\phi\in L^2(\mathbb{R}^d)$, set $\Xi_t^{C,\phi}=\{(M_t^z\phi,\psi): z\in C, \|\psi\|_2\leq 1\}$ and $\Xi_t=\{\Xi_t^{C,\phi}: \|\phi\|_2\leq 1, C\in\mathscr{C}(D)\}$, for each $t>0$.

For each $t>0$, the increasing family of sub-semi-algebras is the usual one, $\mathscr{S}_J$, $J\subset]0,t]$ finite. The set functions $(M_t^z\phi,\psi)$, $\phi,\psi\in L^2(\mathbb{R}^d)$, $z\in D$ are viewed as elements of $\overleftarrow{\mathrm{ba}}(\mathscr{S}_J,\mathbb{C})$.

It is easily checked, as in [2] Theorem 3.4, that for each $t>0$, Ξ_t is Γ_t-closable. Because the space $\overleftarrow{\mathrm{ba}}(\mathscr{S}_J,\mathbb{C})$ is complete, every function (identifying a function with its equivalence class, as usual) belonging to the domain $\mathscr{D}(\bar{I}_{\Gamma_t\Xi_t})$ of the closed linear map $\bar{I}_{\Gamma_t\Xi_t}$ is Γ_t-Ξ_t-integrable. Although this is a simplification, the task of verifying that a function is Γ_t-Ξ_t-integrable is more difficult.

A Γ_t-Ξ_t-integrable function is said to be *boundary-M_t^K-M_t^D-integrable*, or briefly, *∂-M_t^K-M_t^D-integrable.* The space of (equivalence classes of) ∂-M_t^K-M_t^D-integrable functions equipped with the coarsest topology for which both the inclusion of $\mathrm{L}^1(\Gamma_t, \Xi_t)$ in $\mathrm{L}^1(M_t^K)$ and the map $\bar{I}_{\Gamma_t \Xi_t}$ are continuous is denoted by $\partial\mathrm{L}^1(M_t^K, M_t^D)$. Because the map $\bar{I}_{\Gamma_t \Xi_t}$ is closed, the space $\partial\mathrm{L}^1(M_t^K, M_t^D)$ is a complete locally convex Hausdorff topological vector space. The completeness of the space of ∂-M_t^K-M_t^D-integrable functions is obviously a desirable property.

The cardinality of the topology of $\partial\mathrm{L}^1(M_t^K, M_t^D)$ is the same as the cardinality of the continuum, because we are using the collection of all *finite* subsets of the interval $]0, t]$; the topology may or may not be viewed as large, depending on whether or not one believes in the continuum hypothesis. In any case, $\partial\mathrm{L}^1(M_t^K, M_t^D)$ is surely not a Fréchet space.

Because $\partial\mathrm{L}^1(M_t^K, M_t^D) \subset \mathrm{L}^1(M_t^K, M_t^D)$, the integrals $fM_t^z: \mathcal{S}_t \to \mathcal{L}(\mathrm{L}^2(\boldsymbol{R}^d))$ of ∂-M_t^K-M_t^D-integrable functions are defined in exactly the same way as for M_t^K-M_t^D-integrable functions [2] Theorem 3.5.

Theorem 3. *Let $V: \boldsymbol{R}^d \to \boldsymbol{R}$ be a Borel measurable function such that the domain of the operator of multiplication by V contains $\mathcal{D}(H_0)$, and for each $a>0$, there exists $b>0$ such that*

$$\|Vf\|_2 \leq a\|H_0 f\|_2 + b\|f\|_2$$

for every $f \in \mathcal{D}(H_0)$.

Then for each $t>0$, the function

$$\exp\left[-\mathrm{i}\int_0^t V \circ X_s \, ds\right]: \omega \to \exp\left[-\mathrm{i}\int_0^t V \circ X_s(\omega)\, ds\right], \quad \omega \in \Omega_t$$

is defined on a set Ω_t of full M_t^K-measure and it is ∂-M_t^K-M_t^D-integrable.

Furthermore, $H_0 + V$ is selfadjoint on $\mathcal{D}(H_0)$ and

$$\mathrm{e}^{-\mathrm{i}(H_0+V)t} = \int_\Omega \exp\left[-\mathrm{i}\int_0^t V \circ X_s \, ds\right] dM_t^F.$$

Proof. First suppose that V is continuous and bounded. Define

$$f_n(\omega) = \prod_{j=1}^n \exp[-\mathrm{i}V(\omega(jt/n))t/n]$$

for $\omega \in \Omega$, and $n = 1, 2, \ldots$. It is easy to see that each function f_n,

$n=1,2,\dots$ is ∂-M_t^K-M_t^D-integrable by appealing to Lemma 1, and

$$M_t^z(f_n)=[e^{iVt/n}e^{-iH_0t/zn}]^n,\qquad n=1,2,\dots$$

For $z\in K$, the Riemann sums converge everywhere on Ω, and by dominated convergence, $f_n\to\exp[-i\int_0^t V\circ X_s ds]$ in $L^1(M_t^K)$ as $n\to\infty$, and

$$\lim_{n\to\infty} M_t^z(f_n)=\int_\Omega \exp[-i\int_0^t V\circ X_s ds]dM_t^z.$$

By virtue of Theorem 1, the left hand side converges uniformly for z in compact subsets of D.

Now fix $J=\{t_1,\dots,t_j\}\subset]0,t]$ and let $B_1,\dots,B_j$ be Borel subsets of $\boldsymbol{R}^d$. We suppose $0<t_1<\dots<t_j<t$ and that n is so large that $t/n<\min\{t_1,t_2-t_1,\dots,t_j-t_{j-1},t-t_j\}$.

Let $n_kt/n<t_k\leq(n_k+1)t/n$, $k=1,2,\dots,j$. Then

$$\begin{aligned}
f_nM_t^z\{X_{t_1}\in B_1,\dots,X_{t_j}\in B_j\} &= e^{-iVt/n}S^z(t/n)\dots e^{-iVt/n}S^z(t/n)e^{-iVt/n}\\
&S^z((n_j+1)t/n-t_j)B_jS^z(t_j-n_jt/n)e^{-iVt/n}S^z(t/n)\dots e^{-iVt/n}S^z(t/n)e^{-iVt/n}\\
&S^z((n_{j-1}+1)t/n-t_{j-1})B_{j-1}S^z(t_{j-1}-n_{j-1}t/n)\dots\\
&\vdots\\
&S^z(t_1-n_1t/n)e^{-iVt/n}S^z(t/n)\dots e^{-iVt/n}S^z(t/n)
\end{aligned}$$

By [2] Lemma 4.3, it follows that $f_nM_t^z\{X_{t_1}\in B_1,\dots,X_{t_j}\in B_j\}$, $n=1,2,\dots$ converges in the strong operator topology to

$$e^{-i(H_0/z+V)(t-t_j)}B_je^{-i(H_0/z+V)(t_j-t_{j-1})}B_{j-1}\dots e^{-i(H_0/z+V)(t_2-t_1)}B_1 e^{-i(H_0/z+V)t_1}$$

uniformly for $B_1,\dots,B_j$ and for z in compact subsets of D. Therefore f_n, $n=1,2,\dots$ converges in $\partial L^1(M_t^K,M_t^D)$, and (1) holds.

For V bounded, but not necessarily continuous, we can take a regularization V_n, $n=1,2,\dots$ of V by smooth functions, such that $|V_n|\leq\|V\|_\infty$, $n=1,2,\dots$ and $V_n\to V$ a. e. as $n\to\infty$.

Then $e^{-i(H_0/z+V_n)t}\to e^{-i(H_0/z+V)t}$ for each $z\in D$ and $t>0$ by strong resolvent convergence. To see that the convergence is uniform for z in compact subsets of D, we apply Theorem 2 and the argument of [6] IX. 2.4; the convergence of the second Neumann series is uniform for z in compact subsets of D, and for each perturbation V_n, $n=1,\dots$, V. Another application of [2] Lemma 4.3 yields the convergence of $\exp(-i\int_0^t V_n\circ X_s ds)$, $n=1,2,\dots$ to $\exp(-i\int_0^t V\circ X_s ds)$ in $\partial L^1(M_t^K,M_t^D)$

and (1).

Now truncate the positive and negative parts of V to obtain the bounded functions $V_{n,m}$, $n, m=1, 2, \ldots$. The same argument is enough to establish the convergence in $\partial L^1(M_t^K, M_t^D)$ of

$$\exp(-i\int_0^t V_{n,m}\circ X_s ds) \in \partial L^1(M_t^K, M_t^D)$$

as $n\to\infty$ and then $m\to\infty$; namely, [6] IX, 2.4 and the relative boundedness of V with respect to H_0, Theorem 2, and [2] Lemma 4.3. It then follows that the Feynman representation (1) is valid.

Remark. The corresponding result for complex potentials was proved in [2] 4.8. It is not possible to control the convergence of the approximating sequences for complex potentials on the boundary of D, so it is unreasonable to expect convergence of the integrals in $\partial L^1(M_t^K, M_t^D)$.

Appendix: Integration with Respect to Closable Set Functions

A semi-algebra of subsets of a set Ω is a semi-ring [1] containing the set Ω. Let E be a locally convex space with a fundamental system $\mathscr{P}$ of seminorms defining the topology of E.

The space $\mathrm{ba}(\mathscr{E}, E)$ of bounded additive [1] set functions $m: \mathscr{E}\to E$ on the semi-algebra $\mathscr{E}$ is equipped with the semivariation topology; that is, for any seminorm $p\in\mathscr{P}$, $p_{\mathscr{E}}$: $\mathrm{ba}(\mathscr{E}, E)\to[0,\infty[$ is defined by $p_{\mathscr{E}}(m)=\sup p(m(\mathscr{E}))$ for each $m\in\mathrm{ba}(\mathscr{E}, E)$ —the collection $\{p_{\mathscr{E}}: p\in\mathscr{P}\}$ then defines the semivariation topology on $\mathrm{ba}(\mathscr{E}, E)$.

Let Z be a directed set and $\langle\mathscr{S}_\zeta\rangle_{\zeta\in Z}$ an increasing family of semi-algebras. Set $\mathscr{S}=\bigcup_{\zeta\in Z}\mathscr{S}_\zeta$ and let $\underleftarrow{\mathrm{ba}}(\mathscr{S}_\zeta, E)$ be the projective limit of the spaces $\mathrm{ba}(\mathscr{S}_\zeta, E)$, $\zeta\in Z$ linked by the restriction maps. Then $\underleftarrow{\mathrm{ba}}(\mathscr{S}_\zeta, E)$ is naturally identified with a space of additive set functions on the semi-algebra $\mathscr{S}$ which are locally bounded.

Let W_0, W_1 be index sets and let Γ be a collection of families Γ_ξ, $\xi\in W_0$ of measures $\mu:\sigma(\mathscr{S})\to[0,\infty[$ on the σ-algebra $\sigma(\mathscr{S})$ generated by $\mathscr{S}$ such that for each $\xi\in W_0$, $\sup\{\mu(\Omega): \mu\in\Gamma_\xi\}<\infty$.

Let Λ be a collection of families Λ_ξ, $\xi\in W_1$ of E-valued additive set functions $\mu\in\underleftarrow{\mathrm{ba}}(\mathscr{S}_\zeta, E)$ such that for each $\xi\in W_1$, Λ_ξ is a bounded subset of $\overleftarrow{\mathrm{ba}}(\mathscr{S}_{\zeta}, E)$.

The space of finite linear combinations of characteristic functions of sets belonging to $\mathscr{S}$ is denoted by $\mathrm{sim}(\mathscr{S})$. If $s \in \mathrm{sim}(\mathscr{S})$ and $m \in \overleftarrow{\mathrm{ba}}(\mathscr{S}_{\zeta}, E)$, then $sm: \mathscr{S} \to E$ is the indefinite integral of s with respect to m, defined in the obvious way; clearly $sm \in \overleftarrow{\mathrm{ba}}(\mathscr{S}_{\zeta}, E)$.

Two topologies τ_{Γ} and τ_{Λ} are defined on $\mathrm{sim}(\mathscr{S})$. The first, τ_{Γ}, is defined by the family of seminorms $s \mapsto \sup_{\mu \in \Gamma_{\xi}} \mu(|s|)$, $s \in \mathrm{sim}(\mathscr{S})$ as ξ ranges over W_0, and the second is coarsest such that for each $\xi \in W_1$, $s \mapsto sm$, $s \in \mathrm{sim}(\mathscr{S})$ is an equicontinuous family of linear maps from $\mathrm{sim}(\mathscr{S})$ to $\overleftarrow{\mathrm{ba}}(\mathscr{S}_{\zeta}, E)$ as m ranges over Λ_{ξ}.

The topologies $\tau_{\Gamma}, \tau_{\Lambda}$ may not be Hausdorff, so let $\mathrm{sim}_{\Gamma}(\mathscr{S})$, $\mathrm{sim}_{\Lambda}(\mathscr{S})$ be their respective quotient spaces. In addition, it is supposed that the identity map $I: \mathrm{sim}(\mathscr{S}) \to \mathrm{sim}(\mathscr{S})$ factors into a map $I_{\Gamma\Lambda}: \mathrm{sim}_{\Gamma}(\mathscr{S}) \to \mathrm{sim}_{\Lambda}(\mathscr{S})$.

Now let $\mathrm{L}^1(\Gamma)$ be the space of (equivalence classes of) Γ-integrable functions introduced by Kluvánek [7] page 40. If $\mathrm{L}^1(\Gamma)$ is complete, then Γ is said to be a *closed* system of measures [2], and in this case, the completion $\overline{\mathrm{sim}_{\Gamma}}(\mathscr{S})$ of $\mathrm{sim}_{\Gamma}(\mathscr{S})$ may be identified with a closed subspace of $\mathrm{L}^1(\Gamma)$, which in practice is all of $\mathrm{L}^1(\Gamma)$ (for example, Γ_{ξ} is uniformly countably additive for each $\xi \in W_0$).

If Γ is closed and the map $I_{\Gamma\Lambda}: \mathrm{sim}_{\Gamma}(\mathscr{S}) \to \mathrm{sim}_{\Lambda}(\mathscr{S})$ is a closable linear map from $L^1(\Gamma)$ into the completion $\overline{\mathrm{sim}_{\Lambda}}(\mathscr{S})$ of $\mathrm{sim}_{\Lambda}(\mathscr{S})$, then Λ is *Γ-closable.*

The integration map $s \mapsto sm$, $s \in \mathrm{sim}(\mathscr{S})$ is clearly continuous for τ_{Λ} into $\overleftarrow{\mathrm{ba}}(\mathscr{S}_{\zeta}, E)$, so a function $f: \Omega \to \mathbb{C}$ is called *Γ-Λ-integrable* if f belongs to the domain $\mathscr{D}(\bar{I}_{\Gamma\Lambda})$ of the closure $\bar{I}_{\Gamma\Lambda}$ of $I_{\Gamma\Lambda}$, and the image of f via the (continuous extension of) the integration map $\cdot m$ belongs to $\overleftarrow{\mathrm{ba}}(\mathscr{S}_{\zeta}, E)$ for all $m \in \cup \Lambda$.

If E is complete, then this last condition holds whenever $f \in \mathscr{D}(\bar{I}_{\Gamma\Lambda})$. The uniquely defined image of f by $\cdot m$ is denoted, of course, by fm; it is the *indefinite integral* of f with respect to m.

A convergence theorem for these indefinite integrals can be read straight off the closedness property of the map $\bar{I}_{\Gamma\Lambda}$ [2] Theorem 2.5.

To apply the definition to Schrödinger semigroups, set

$$K = \{ai:\ a > 0\}\ ;\ D = \{z \in \mathbb{C}:\ \mathrm{Im}z \geq 0, \quad z \neq 0\}$$

$$S^z(t) = e^{i\Delta t/2z}$$

for each $z \in D$ and $t > 0$. The operator Δ is the self-adjoint extension of the Laplacian $\partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_d^2$ on $L^2(\boldsymbol{R}^d)$. The exponential is defined by the operational calculus for self-adjoint operators.

Let Ω be the set of all continuous functions $\omega: [0, \infty] \to \boldsymbol{R}^d$, and set $X_t(\omega) = \omega(t)$, $\omega \in \Omega$, $t > 0$. The Borel σ-algebra of $\boldsymbol{R}^d$ is denoted by Σ. A set will sometimes be identified with its characteristic function, and a Borel measurable function will also be identified with the operator on $L^2(\boldsymbol{R}^d)$ it defines.

For each $t > 0$, $z \in D$, define

$$M_t^z\{X_{t_1} \in B_1, \ldots, X_{t_k} \in B_k\} = S^z(t - t_k) B_k S^z(t_k - t_{k-1}) \ldots$$
$$B_2 S^z(t_2 - t_1) B_1 S^z(t_1)$$

for all $0 < t_1 < \ldots < t_k \leq t$, $B_1, \ldots, B_k \in \Sigma$, $k = 1, 2, \ldots$. Then M_t^z: $\mathscr{S}_t \to \mathscr{L}(L^2(\boldsymbol{R}^d))$ is an operator-valued set function on the semi-algebra of sets of the form $\{X_{t_1} \in B_1, \ldots, X_{t_k} \in B_k\}$, $0 < t_1, \ldots, t_k \leq t$, $B_1, \ldots, B_k \in \Sigma$, $k = 1, 2, \ldots$.

For each $z \in K$, M_t^z is the restriction to $\mathscr{S}_t$ of a unique $\mathscr{L}(L^2(\boldsymbol{R}^d))$-valued measure, also denoted by M_t^z, on $\sigma(\mathscr{S}_t)$. This follows by representing M_t^z in terms of the Wiener process [2].

Our space E will be the space $H_c(D)$ of continuous functions on D which are analytic in the interior D° of D, equipped with the topology of uniform convergence on compact subsets of D° (it is not complete).

For each $\phi, \psi \in L^2(\boldsymbol{R}^d)$, $(M_t^D \phi, \psi)$ represents the $H_c(D)$-valued set function defined by

$$(M_t^D \phi, \psi)(A)(z) = (M_t^z(A) \phi, \psi),$$

$A \in \mathscr{S}_t$, $\phi, \psi \in L^2(\boldsymbol{R}^d)$, $z \in D$ and $t > 0$.

Finally, for each $t > 0$, our increasing family of semi-algebras is the family $\langle \mathscr{S}_J \rangle_{J \in \mathscr{F}_t}$ of semi-algebras $\mathscr{S}_J$ of sets of the form $\{X_{t_1} \in B_1, \ldots, X_{t_j} \in B_j\}$, $B_1, \ldots, B_j \in \Sigma$, $J = \{t_1, \ldots, t_j\} \subset]0, t]$. The set $\mathscr{F}_t$ is the collection of all finite sets $J \subset]0, t]$ directed by inclusion.

Put $\Gamma_t^{a,\phi} = \{|(M_t^{a i} \phi, \psi)| : \psi \in L^2(\boldsymbol{R}^d), \|\psi\|_2 \leq 1\}$ and $\Gamma_t = \{\Gamma_t^{a,\phi} : a > 0, \phi \in L^2(\boldsymbol{R}^d), \|\phi\|_2 \leq 1\}$ for each $t > 0$. Here $|\cdot|$ denotes the variation (measure) of a complex measure on the σ-algebra $\sigma(\mathscr{S}_t)$.

For each $t > 0$, $\Lambda_t^\phi = \{(M_t^D \phi, \psi) : \psi \in L^2(\boldsymbol{R}^d), \|\psi\|_2 \leq 1\}$, and

$\Lambda_t = \{\Lambda_t^\phi : \phi \in L^2(\boldsymbol{R}^d),\ \|\phi\|_2 \leq 1\}$.

A Γ_t-Λ_t-integrable function is said to be M_t^K-M_t^D-integrable. A Γ_t-integrable function [7] III. 1 is said to be M_t^K-integrable. For each M_t^K-M_t^D-integrable function f, the additive operator-valued set functions

$$fM_t^z : \mathscr{S}_t \to \mathscr{L}(L^2(\boldsymbol{R}^d)), \quad z \in D$$

can be read off from the definitions in the obvious way [2].

These are our integrals. For $z \in K$, they correspond to the usual integrals with respect to an operator-valued measure [7], and they are analytic continuations of these off K; that is, for each $t>0$ and $A \in \mathscr{S}_t$, $fM_t^\cdot(A)$ is continuous for the weak operator topology on all of D, and analytic in the interior of D.

The space $L^1(\Gamma_t)$ is written as $L^1(M_t^K)$ for each $t>0$. The space $L^1(M_t^K, M_t^D)$ of all (equivalence classes of) M_t^K-M_t^D-integrable functions is equipped with the coarsest topology for which both the inclusion of $L^1(M_t^K, M_t^D)$ in $L^1(M_t^K)$ and the map $\bar{I}_{\Gamma_t \Lambda_t}$ are continuous. Unfortunately, $L^1(M_t^K, M_t^D)$ is not complete because $H_c(D)$ is not complete; we shall learn to live with this fact.

Expressions such as "M_t^K-a. e.", "M_t^K-null" have the obvious meanings attached to them in [2]. The set function M_t^1 is written as M_t^F, $t>0$.

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