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Non-Hypoellipticity for Degenerate Elliptic Operators

Dedicated to the memory of Professor C. Goulaouic

By

Yoshinori MORIMOTO*

Introduction

Let L_∞ be a partial differential operator in R^3 of the form

$$L_\infty = D_x^2 + \phi(x)^2 D_y^2 + D_t^2, \quad D_x = \frac{1}{i}\frac{\partial}{\partial x}, \cdots, \cdots,$$

where $\phi \in C^\infty$, $\phi(0)=0$, $\phi(x)>0$ $(x \neq 0)$, $\phi(x)=\phi(-x)$ and ϕ is non-decreasing in $[0, \infty)$. In the recent paper [6], Kusuoka-Strook have shown that L_∞ is C^∞-hypoelliptic in R^3 if and only if $\phi(x)$ satisfies

$$(1) \qquad \lim_{x \downarrow 0} x \log \phi(x) = 0.$$

If $\phi(x) = \exp(-1/|x|^\sigma)$ for $\sigma>0$ the condition (1) means $\sigma<1$. This result was given as an application of the Malliavin calculus (, which is a theory about stochastic differential equations). The purpose of the present paper is to show the necessity of the condition (1) for C^∞-hypoellipticity of L_∞ by another simple method.

The method used here is analogous to the one of Bouendi-Goulaouic [1], where nonanalytic hypoellipticity of the operator $L_1 = D_x^2 + x^2 D_y^2 + D_t^2$ was proved. In [1], a solution u of $L_1 u=0$ was constructed in the form

$$(2) \qquad u(x, y, t) = \sum_{N=0}^{\infty} t^{2N} A(x, D_x, D_y)^N w(x, y)/(2N)!,$$

where $A = L_1 - D_t^2$ and $w(x, y)$ $(=u(x, y, 0))$ is nonanalytic C^∞-function defined in a neighborhood W of the origin in $R^2_{x,y}$ and satisfies for any integer $N>0$

$$(3) \qquad \|A(x, D_x, D_y)^N w(x, y)\|_{L^2(W)} \leqq C^{N+1}(2N)!.$$

Here C is a positive constant independent of N. The estimate (3) implies that u is well-defined as an $L^2(W)$-valued analytic function with respect to $t \in (-\delta, \delta)$

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* Department of Engineering Mathematics, Faculty of Engineering, Nagoya University, Nagoya 464, Japan.

for a small $\delta>0$.

In the present paper, assuming that the condition (1) is not fulfilled we construct a solution $u\notin C^\infty$ of $L_\infty u=0$ which has the same form as (2). To find a convenient function $w(x, y)\notin C^\infty$ satisfying (3) we consider an eigenvalue problem in an interval $[-1, 1]$ with the Dirichlet boundary condition for an ordinary differential operator $A_\infty(x, D_x, \eta)=-\frac{d^2}{dx^2}+\phi(x)^2\eta^2$ with a parameter $\eta\neq 0$. This point of view permits to extend the result of [1]. Namely, we can also show nonanalytic hypoellipticity of operators $L_k=D_x^2+x^{2k}D_y^2+D_t^2$, $k=2, 3, \cdots$ (cf. [9]). We remark that the method of the present paper is applicable to show non-hypoellipticity of degenerate elliptic operators of higher order than 2, differing from that of [6].

As to the operator L_∞, it should be noted that an operator $A_\infty=D_x^2+\phi(x)^2D_y^2$ is C^∞-hypoelliptic in $R^2_{x,y}$ without the condition (1) (Fediĭ [3], cf. [10]). We remark that L_∞ and A_∞ with infinite degeneracy do not satisfy Hörmander's sufficient condition for C^∞-hypoellipticity in R^3 and R^2, respectively ([4]).

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§1. Main Results

Let L be a differential operator in R^3 of the form

$$L=D_x^2+g(x)D_y^2+D_t^2\,, \tag{4}$$

where $g(x)\in C^\infty$ satisfies $g(x)\geqq 0$ and $g(0)=0$.

Theorem 1. *Assume that* $g(x)$ *satisfies*

$$\liminf_{x\to 0}|x\log g(x)|\neq 0\,, \tag{5}$$

or

$$\begin{cases}\limsup_{x\downarrow 0}|x\log g(x)|\neq 0\,,\\ g(x)=g(-x) \text{ and } g(x) \text{ is non-decreasing in } [0, \infty).\end{cases} \tag{5$'$}$$

Then L is not C^∞-hypoelliptic in R^3. More precisely, one can find a function u defined in some neighborhood V of the origin, belonging to $L^2(V)$, not to C^∞ and such that $Lu=0$.

Remark 1. The condition (1) is equivalent to $\limsup_{x\downarrow 0}|x\log\phi(x)|=0$. The operator L of (4) is more general than L_∞ because g is not always expressed in the form $g=\phi^2$ for a non-negative C^∞-function ϕ (see Remark 2 of Theorem 1.1 of [10]).

Remark 2. The solution u will be constructed as an $L^2(W)$-valued analytic function with respect to $t\in(-\delta, \delta)$, where $W=(-1, 1)\times R_y^1$ and δ is a small positive constant.

Let $\gamma^{(s)}(\Omega)$ for real $s\geqq 1$ denote a class of Gevrey function of order s defined in an open set Ω. (Here $\gamma^{(1)}(\Omega)$ denotes a class of analytic functions in Ω). We say that a differential operator L is $\gamma^{(s)}$-hypoelliptic in R^3 if and only if for any open set Ω of R^3 we have

$$u\in\mathcal{D}'(\Omega), \quad Lu\in\gamma^{(s)}(\Omega)\Rightarrow u\in\gamma^{(s)}(\Omega).$$

Theorem 2. *Assume that $g(x)$ equals x^{2k}, $k=1, 2, \cdots$, that is, the operator $L=L_k$ ($k=1, 2, \cdots$). Then L is not $\gamma^{(s)}$-hypoelliptic in R^3 for any s such that $1\leqq s<k+1$ (, and hence L is not analytic hypoelliptic). More precisely, for any $1\leqq s<k+1$ one can find a function u defined in some neighborhood V of the origin, belonging to $\gamma^{(k+1)}(V)$, not to $\gamma^{(s)}(V)$ and such that $Lu=0$.*

Remark 3. It is well-known that $A_k=D_x^2+x^{2k}D_y^2$ is analytic hypoelliptic in R^2 for any $k=1, 2, \cdots$ ([7]). Recently, Matsuzawa [8] has shown that L_k is $\gamma^{(k+1)}$-hypoelliptic in R^3, more precisely, L_k is partially $\gamma^{(k+1)}$-hypoelliptic with respect to y variable and partially analytic hypoelliptic with respect to x and t variables (cf. Derridj-Zuily [2]).

Remark 4. Métivier [9] independently proved non $\gamma^{(k+1)}$-hypoellipticity of L_k in more general form (see Theorem 3.5 and Corollary 3.7 of [9]). In [9], the existence of $w(x, y)$ satisfying (3) is reduced to the subelliptic estimate instead of the eigenvalue problem.

Theorem 3. *Let l, m and n be positive integers and let $\tilde{L}$ be a differential operator of the form*

$$\tilde{L}=D_x^{2l}+g(x)D_y^{2m}+D_t^{2n},$$

where $g(x)\in C^\infty$ satisfies $g\geqq 0$ and $g(0)=0$. If $g(x)$ satisfies

$$\liminf_{x\to 0}|x^{l/n}\log g(x)|\neq 0 \tag{6}$$

or

$$\begin{cases} \limsup_{x\downarrow 0}|x^{l/n}\log g(x)|\neq 0, \\ g(x)=g(-x) \text{ and } g(x) \text{ is non-decreasing in } [0, \infty), \end{cases} \tag{6$'$}$$

then $\tilde{L}$ is not C^∞-hypoelliptic in R^3.

Remark 5. If $g(x)$ equals $\exp(-1/|x|^\sigma)$ then the condition (6)$'$ means $\sigma\geqq l/n$. We remark that an operator $D_x^{2l}+\exp(-1/|x|^\sigma)D_y^{2m}$ is C^∞-hypoelliptic in R^2 for any $\sigma>0$ (see [10]).

§ 2. Proofs

We begin with the proof of Theorem 1 in the case that $g(x)$ satisfies the condition (5). Note that $\log g(x)$ is negative for $|x|$ small enough. The condition (5) implies that there exists a constant $\delta_0>0$ such that

$$g(x)\leqq\exp(-\delta_0/|x|) \tag{7}$$

if x belongs to a small neighborhood of the origin. Consider an eigenvalue problem in an interval $I_a=(-a, a)\subset R_x^1$ $(a>0)$

$$\begin{cases} A(x, D_x, \eta)v(x, \eta)\equiv(-d^2/dx^2+g(x)\eta^2)v(x, \eta)=\lambda v(x, \eta), \\ v(a, \eta)=v(-a, \eta)=0 \end{cases} \tag{8}$$

where η is the dual variable of y and considered as a parameter for a while. Since A is a selfadjoint operator that is bounded from below, it follows from Theorem XIII. 1 of [11] that the minimal eigenvalue $\lambda_0(a, \eta)$ is given by the formula

$$\lambda_0(a, \eta)=\inf_{\substack{f\in C_0^\infty(I_a),\\ \|f\|_{L^2}=1}}(Af, f)_{L^2}>0\,. \tag{9}$$

In view of (9), it is clear that $\lambda_0(a, \eta)\leqq\lambda_0(a_0, \eta)$ if $a\geqq a_0$. Set $a_0=\delta_0/2\log|\eta|$ and assume $|\eta|$ large enough. Then it follows from (7) that $g(x)\eta^2\leqq 1$ for $x\in I_{a_0}$. Let $\tilde\lambda_0(a)$ denote the minimal eigenvalue of the eigen value problem (8) with A replaced by $-d^2/dx^2+1$. Comparing (9) and a similar formula for $\tilde\lambda_0(a_0)$ we have $\lambda_0(a_0, \eta)\leqq\tilde\lambda_0(a_0)$ for large $|\eta|$. Since $\tilde\lambda_0(a_0)$ equals $C'a_0^{-2}$ for a constant C' independent of η we have

$$0<\lambda_0(1, \eta)\leqq C''(\log|\eta|)^2 \qquad \text{for large } |\eta|, \tag{10}$$

where C'' is a constant independent of η. Let $v_0(x, \eta)$ be an eigenfunction associated with $\lambda_0(1, \eta)$ such that $\|v_0(x, \eta)\|_{L^2(I_1)}=1$. Take a function $\phi(y)\in L^2$ satisfying for a constant $c_0>0$

$$c_0\langle\eta\rangle^{-2}\leqq|\hat\phi(\eta)|\leqq c_0^{-1}\langle\eta\rangle^{-2}, \quad \langle\eta\rangle=\sqrt{1+|\eta|^2}, \tag{11}$$

where $\hat\phi(\eta)$ denotes the Fourier transform of $\phi(y)$. Set

$$w(x, y)=\int\exp(iy\cdot\eta)v_0(x, \eta)\hat\phi(\eta)d\eta/2\pi\,.$$

Then it is clear that $w(x, y)\notin C^\infty$. Furthermore, we see that w satisfies the estimate (3) with $W=(-1, 1)\times R_y^1$ for any $N=1, 2, \cdots$. In fact, it follows from $A(x, D_x, \eta)^N v_0(x, \eta)=\lambda_0(1, \eta)^N v_0(x, \eta)$ that the estimate

$$(12)\qquad \begin{aligned}\|A(x, D_x, D_y)^N w(x, y)\|_{L^2(I_1\times R_y)} &= \|\lambda_0(1, \eta)^N v_0(x, \eta)\hat{\phi}(\eta)\|_{L^2(I_1\times R_\eta)}\\ &= \|\lambda_0(1, \eta)^N \hat{\phi}(\eta)\|_{L^2(R_\eta)}\\ &\leqq C_1^N \|(\log\langle\eta\rangle)^{2N}\langle\eta\rangle^{-2}\|_{L^2(R_\eta)}\\ &\leqq C_2^{N+1}(2N)\,!\end{aligned}$$

holds for constants C_1 and C_2 independent of N. Here in order to get the last inequality, we used an elementary inequality $s^{2N}e^{-s/2}\leqq 4^N(2N)\,!$ $(s\geqq 0)$ by setting $s=\log\langle\eta\rangle$. Define a function u by the formula (2) with $A=D_x^2+g(x)D_y^2$. Then the estimate (12) shows that u is well-defined as an $L^2((-1, 1)\times R_y)$-valued analytic function with respect to $t\in(-\delta, \delta)$ for a small $\delta>0$. Since $u(x, y, 0)=w(x, y)$ we see $u\notin C^\infty$. This concludes the proof of Theorem 1 when $g(x)$ satisfies the condition (5).

If $g(x)$ satisfies the condition (5)′ there exist a $\delta_1>0$ and a decreasing sequence of positive numbers $\{a_j\}_{j=1}^\infty$ such that $\lim_{j\to\infty} a_j=0$ and $-a_j\log g(a_j)\geqq\delta_1$. Since $g(x)=g(-x)$ and $g(x)$ is non-decreasing in $[0, \infty)$, it follows that

$$g(x)\eta^2\leqq g(a_j)\eta^2\leqq 1 \qquad \text{for} \quad x\in[-a_j, a_j] \quad \text{if} \quad |\eta|\leqq\exp(\delta_1/2a_j)\,.$$

Replacing a_0 in the above by a_j we see that the minimal eigenvalue $\lambda_0(1, \eta)$ satisfies

$$\lambda_0(1, \eta)\leqq\tilde{\lambda}_0(a_j)=C'a_j^{-2} \qquad \text{if} \quad |\eta|\leqq\exp(\delta_1/2a_j)\,,$$

where C' is the absolute constant. Therefore, we get

$$(10)' \qquad \lambda_0(1, \eta)\leqq\tilde{C}(\log|\eta|)^2 \qquad \text{if} \quad \exp(\delta_1/3a_j)\leqq|\eta|\leqq\exp(\delta_1/2a_j)\,,$$

where $\tilde{C}$ is a constant independent of $|\eta|$ and j. Take a function $\phi(y)\in L^2$ satisfying

$$(11)' \qquad \begin{cases} c_1\langle\eta\rangle^{-2}\leqq\hat{\phi}(\eta)\leqq c_1^{-1}\langle\eta\rangle^{-2} \qquad (c_1>0) \\ \qquad\qquad \text{if} \quad \exp(\delta_1/3a_j)\leqq|\eta|\leqq\exp(\delta_1/2a_j)\,, \\ \hat{\phi}(\eta)=0 \qquad \text{otherwise.}\end{cases}$$

Using (10)′ and (11)′ in place of (10) and (11), respectively, we have the estimate (12), which completes the proof of Theorem 1 in the case that $g(x)$ satisfies the condition (5)′.

Theorem 2 can be proved by the same way as in the proof of Theorem 1. If we set $a_0=|\eta|^{-1/(k+1)}$ we have $g(x)\eta^2=x^{2k}\eta^2\leqq|\eta|^{2/(k+1)}$ for $x\in I_{a_0}$. Considering a "majorant" eigenvalue problem in I_{a_0} for an operator $-(d/dx)^2+|\eta|^{2/(k+1)}$, we also see that the minimal eigenvalue of (8) satisfies

$$(13) \qquad 0<\lambda_0(1, \eta)\leqq C_3|\eta|^{2/(k+1)} \qquad \text{for large} \quad |\eta|\,,$$

where C_3 is a constant independent of η. For a fixed s satisfying $1\leqq s<k+1$ take a function $\phi(y)\in\gamma^{(k+1)}\cap C_0^\infty$ such that $\phi(y)\notin\gamma^{(s)}$. Since $\phi\in\gamma^{(k+1)}\cap C_0^\infty$ it

follows that

$$|\hat{\phi}(\eta)| \leqq \varepsilon^{-1} \exp(-\varepsilon\langle\eta\rangle^{1/(k+1)}) \tag{14}$$

for a constant $\varepsilon>0$ independent of η. Using (13) and (14) in place of (10) and (11), respectively, we obtain the estimate (3), which shows the existence of the desired solution u of $L_k u=0$.

The proof of Theorem 3 is also parallel to that of Theorem 1. For the proof it suffices to consider an eigenvalue problem with the Dirichlet boundary condition for an operator $\tilde{A}(x, D_x, \eta)=(-d^2/dx^2)^l+g(x)\eta^{2m}$ instead of (8) and to replace (2) and (3) by

$$u(x, y, t)=\sum_{N=0}^{\infty} t^{2nN}((-1)^{n-1}\tilde{A}(x, D_x, D_y))^N w(x, y)/(2nN)!$$

and

$$\|\tilde{A}(x, D_x, D_y)^N w(x, y)\|_{L^2(W)} \leqq C^{N+1}(2nN)!,$$

respectively. The detail of the proof of Theorem 3 is left to the reader.

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